GENERALIZED SAMUEL MULTIPLICITIES OF MONOMIAL IDEALS AND VOLUMES

R. ACHILLES AND M. MANARESI

ABSTRACT. We describe conjecturally the generalized Samuel multiplicities $c_0, \dots, c_{d-1}$ of a monomial ideal $I \subset K[x_1, \dots, x_d]$ in terms of its Newton polyhedron $\Gamma(I)$. More precisely, we conjecture that c_i equals the sum of the normalized $(d-i)$-volumes of pyramids over the projections of the $(d-i-1)$-dimensional compact faces of $\Gamma(I)$ along the infinite-directions of i-unbounded facets in which they are contained. For c_0 proofs are known (Guibert, Jeffries and Montaño) and for c_{d-1} a proof will be given.

1. INTRODUCTION

In this paper, based on computations with the free softwares Germenes by A. Montesinos [11] and REDUCE [12] by A. C. Hearn [8] and REDUCE developers, we give a conjecture that in the case of monomial ideals links the generalized multiplicities defined algebraically in [3] with volumes derived from the Newton polyhedra of the ideals, thus extending a result of B. Teissier [14].

In 1988, B. Teissier [14, p. 131] proved that for an $\mathfrak{m}$-primary monomial ideal I of a local ring A the Samuel multiplicity is equal to the normalized volume of the complement of the Newton polyhedron of the ideal I. In 1999, G. Guibert [7] generalized Teissier's result. Precisely, Guibert defines the local Segre class of an ideal generated by a set of germs of holomorphic functions and, under a non-degeneracy condition, he describes such a class by Minkowski mixed volumes of polytopes. As a special case he obtains that for a certain class of monomial ideals the local Segre class is a normalized volume of the simplex generated by the origin and the vertices of the Newton polyhedron, see [7, 4.2]. By [4], the local Segre class is the so called j-multiplicity of the ideal. In 2013, J. Jeffries and J. Montaño [9] gave a different proof that the

2010 *Mathematics Subject Classification.* Primary 13H15; Secondary 13F20, 52B20.

Key words and phrases. Monomial ideal, generalized Samuel multiplicity, Newton polyhedron.

j-multiplicity of a monomial ideal is the normalized volume of the pyramid of the ideal.

The j-multiplicity of an ideal is different from zero if and only if its analytic spread is maximal, that is, equal to the Krull-dimension d of A. A result of C. Biviá-Ausina [5] states that the analytic spread diminished by one is the maximum of the dimensions of compact faces of the Newton polyhedron of I.

According to [3] the j-multiplicity is only the first coordinate of the generalized Samuel multiplicity vector $c(I) = (c_0(I), \dots, c_d(I))$. Here we present and illustrate a conjecture which expresses the other components of $c(I)$ in terms of the Newton polyhedron of I. Our conjecture extends the known result of G. Giubert and J. Jeffries and J. Montaño regarding $c_0(I)$, but we shall prove it here only for $c_{d-1}(I)$.

2. Generalized Samuel multiplicities

This section is a quick review of a generalization of Samuel's multiplicity by a sequence of numbers, the so-called generalized Samuel multiplicity, which we have introduced in [3].

Let A be a d-dimensional Noetherian local ring $(A, \mathfrak{m})$ with unique maximal ideal $\mathfrak{m}$ or a standard graded algebra $A = \bigoplus_{i \geq 0} A_i$ such that A_0 is a field and $\mathfrak{m} = (A_1)A$ is the unique homogeneous maximal ideal of A. Let $I \subset A$ be an arbitrary ideal (not necessarily $\mathfrak{m}$-primary).

In order to define the generalized Samuel multiplicity $c(I)$, consider $G_I(A) := \bigoplus_{j \geqslant 0} I^j/I^{j+1}$, the associated graded ring of A with respect to I and the bigraded ring

$$T = \bigoplus_{i,j \geq 0} T_{ij} = G_{\mathfrak{m}}(G_I(A)) = \bigoplus_{i,j \geq 0} \frac{\mathfrak{m}^i I^j + I^{j+1}}{\mathfrak{m}^{i+1} I^j + I^{j+1}} ,$$

where $T_{00} = A/\mathfrak{m} = K$ is a field.

Let $H^{(0,0)}(i,j) := \dim_K T_{ij}$ be the Hilbert function of the bigraded ring T and let

$$H^{(1,1)}(i,j) := \sum_{q=0}^{j} \sum_{p=0}^{i} H^{(0,0)}(p,q)$$

its twofold sum transform. For both $i, j \gg 1$ this function becomes a polynomial in (i,j), which can be written in the form

$$\sum_{k+l \leqslant d} a_{k,l}^{(1,1)} \binom{i+k}{k} \binom{j+l}{l} .$$

Following [3] define the *generalized Samuel multiplicity* to be the vector

$$\left(a_{0,d}^{(1,1)}, a_{1,d-1}^{(1,1)}, \ldots, a_{d,0}^{(1,1)}\right) =: (c_0(T), c_1(T), \ldots, c_d(T)) =: c(T)$$
$$=: (c_0(I), c_1(I), \ldots, c_d(I)) =: c(I)\,.$$

The first coefficient $c_0(I)$ plays an important role as an intersection number and was introduced in [2]. It is called the *j-multiplicity* $j(I) := c_0(I)$.

The generalized Samuel multiplicities depend only on the highest dimensional components of T, see [15] or [3, Proposition 1.2]:

Proposition 1. *With the preceding notation,*

$$c(I) = c(T) = \sum_P \operatorname{length}(T_P) \cdot c(T/P)\,,$$

where P runs through all highest dimensional prime ideals of T.

By analogy with the application of $c(I)$ to intersection theory, we shall call $c_i(T/P) \neq 0$ a *movable contribution* to $c_i(I)$ if there is an integer $k > i$ such that $c_k(T/P) \neq 0$.

3. A conjecture and some results

Let I be an ideal in $R = K[x_1, \ldots, x_d] = K[\mathbf{x}]$ (K a field) minimally generated by the monomials

$$\mathbf{x}^{v_1} := x_1{}^{v_1(1)} \cdots x_d{}^{v_1(d)}, \ \ldots, \ \mathbf{x}^{v_r} := x_1{}^{v_r(1)} \cdots x_d{}^{v_r(d)},$$

that is, $v_1 = (v_1(1), \ldots, v_1(d)), \ldots, v_r = (v_r(1) \ldots v_r(d))$ are the points of $\mathbb{Z}^d_{\geqslant 0}$ corresponding to the exponents of the generators of I.

The *Newton polyhedron* $\Gamma(I)$ of I is defined as the convex hull of $\{v \in \mathbb{Z}^d_{\geqslant 0} \mid \mathbf{x}^v\} \in I\}$ in $\mathbb{R}^d$, that is,

$$\Gamma(I) := \operatorname{conv}(\{v \in \mathbb{Z}^d_{\geqslant 0} \mid x_1^{v(1)} \cdots x_d{}^{v(d)} \in I\})$$
$$= \operatorname{conv}(\{v_1, \ldots, v_r\}) + \mathbb{R}^d_{\geqslant 0},$$

where $+$ denotes the Minkowski sum (for the equality see [13, Lemma 4.3]).

A hyperplane

$$H = \{v \in \mathbb{R}^d \,|\, \langle v, a \rangle = b\} \quad (\text{with } a \in \mathbb{R}^d_{\geqslant 0},\ b \in \mathbb{R})$$

is called a *supporting hyperplane* of the Newton polyhedron $\Gamma(I)$ if

$$\Gamma(I) \subset H^+ = \{v \in \mathbb{R}^d \,|\, \langle v, a \rangle \geqslant b\} \text{ and } \Gamma(I) \cap H \neq \emptyset.$$

A subset $F \subset \Gamma(I)$ is called a *proper face* of $\Gamma(I)$ if there exists a supporting hyperplane H of $\Gamma(I)$ such that $F = \Gamma(I) \cap H$. The boundary of $\Gamma(I)$ is a set of faces of dimension $d-1$, called *facets* of $\Gamma(I)$, some of them compact.

The zero-dimensional faces are called *vertices* of $\Gamma(I)$. We shall denote the set of vertices by $\mathrm{vert}(I)$. Note that the monomials corresponding to the points in $\mathrm{vert}(I)$ are part of the set of minimal generators of I, so by renumbering we will assume that

$$\mathrm{vert}(I) = \{v_1, \ldots, v_s\} \text{ with some } s \leq r,$$

hence

$$\Gamma(I) = \mathrm{conv}(\{v_1, \ldots, v_r\}) + \mathbb{R}^d_{\geqslant 0} = \mathrm{conv}(\{v_1, \ldots, v_s\}) + \mathbb{R}^d_{\geqslant 0}.$$

Any face F can be described using its vertices and infinite-directions. Let e_j denote the unit vector with non-zero jth component, let H be a supporting hyperplane such that $F = \Gamma(I) \cap H$ and let a be a normal vector to H. Then the *infinite-directions* of F are given by those e_j such that the jth component of a is zero. If $v_{i_1}, \ldots, v_{i_s}$ are the vertices of F, then

$$F = \mathrm{conv}(\{v_{i_1}, \ldots, v_{i_s}\}) + \sum_{j\colon a(j)=0} \mathbb{R}_{\geq 0}\, e_j.$$

Of course, the compact faces are precisely those that do not have infinite directions e_j.

By the Minkowski-Weyl Theorem for convex polyhedra, there are uniquely determined finitely many closed half spaces

$$H_i^+ = \{v \in \mathbb{R}^d \mid \langle v, a_i \rangle \geqslant b_i\} \text{ (with } a_i \in \mathbb{Z}^d_{\geqslant 0},\ b_i \in \mathbb{Z}_{\geqslant 0}),\ i = 1, \ldots, t,$$

such that

$$\Gamma(I) = H_1^+ \cap \cdots \cap H_t^+.$$

Then $F_i := H_i \cap \Gamma(I)$, $i = 1, \ldots, t$, are the facets of $\Gamma(I)$. We will assume that $H_1, \ldots, H_r$ are the hyperplanes corresponding to the unbounded facets and that $H_{r+1}, \ldots, H_t$ are those corresponding to the compact facets.

To each bounded facet $F = \mathrm{conv}(\{v_{i1}, \ldots, v_{is}\})$ of $\Gamma(I)$ we associate the polytope (or pyramid)

$$\hat{F} := \mathrm{conv}(0, F) = \mathrm{conv}(\{0, v_{i1}, \ldots, v_{is}\})$$

and denote by $\mathrm{vol}_d(\hat{F})$ its d-dimensional volume and by

$$\mathrm{Vol}_d(\hat{F}) := d!\, \mathrm{vol}_d(\hat{F})$$

its *normalized volume.*

A facet $F \subset \Gamma(I)$ is called an *h-unbounded facet* if the normal vector a to its supporting hyperplane has at least $h > 0$ coordinates $a(j)$ which are zero, that is, the facet has h infinite-directions e_j.

Let $\mathcal{F}(k)$ be the set of all $(d-(k+1))$-unbounded facets containing at least one k-dimensional compact face F^k, $0 \leq k \leq d-2$. We define $\mathcal{F}(d-1)$ to be the set of all compact or bounded facets of $\Gamma(I)$.

To each couple (F^k, F^{d-1}), with F^k a k-dimensional compact face and $F^{d-1} \in \mathcal{F}(k)$ containing F^k, we associate a $(k+1)$-dimensional normalized volume $\mathrm{Vol}(F^k, F^{d-1})$ in the following way. A normal vector to the facet F^{d-1} lies on at least $d-(k+1)$ coordinate hyperplanes. We project F^k on all linear subspaces $\mathbb{R}^{k+1} \subseteq \mathbb{R}^d$ obtained by intersecting $d-(k+1)$ of these coordinate hyperplanes, that is, we project F^k along all possible choices of $d-(k+1)$ infinite-directions of the facet F^{d-1}. We obtain polytopes of dimension at most k. We consider only the k-dimensional polytopes $\mathrm{pr}_{\mathbb{R}^{k+1}}(F^k) \subset \mathbb{R}^{k+1}$ obtained by the aforementioned projections and set

$$\hat{F}^k := \mathrm{conv}(\{0, \mathrm{pr}_{\mathbb{R}^{k+1}}(F^k)\}),$$

which has dimension k or $k+1$. The volume associated to the couple (F^k, F^{d-1}) is

$$\mathrm{Vol}(F^k, F^{d-1}) := \min_{\mathbb{R}^{k+1}} \mathrm{Vol}_{k+1}(\hat{F}^k).$$

Conjecture 1. *For each $k = 0, \dots, d-1$, the generalized Samuel multiplicity of a monomial ideal I is*

$$c_{d-(k+1)}(I) = \sum_{F^{d-1} \in \mathcal{F}(k)} \min_{F^k} \left\{ \mathrm{Vol}(F^k, F^{d-1}) \right\},$$

where the minimum is taken over all compact faces F^k of $\Gamma(I)$ that are contained in the facet F^{d-1}.

Conjecture 2. *Each summand in the formula of Conjecture 1 corresponds, in the sense of Proposition 1, to the contribution of a highest dimensional primary component of $T = G_{\mathfrak{m}}(G_I(R))$ to $c_{d-(k+1)}(I)$.*

In particular, the number of compact facets of $\Gamma(I)$ is equal to the the number of d-dimensional associated prime ideals of T that contain $\mathfrak{m} = (x_1, \dots, x_d)R$.

Note that in general the zero-ideal $\mathfrak{n}$ of $T \cong K[x_1, \dots, x_d, y_1, \dots, y_r]/\mathfrak{n}$ is a binomial but not a monomial ideal, see [6].

Our conjectures are confirmed by many examples, but so far we do not have a proof except for Conjecture 1 in the extremal cases $k = 0$ and $k = d-1$, as it is stated in the following two theorems.

Theorem 1 (Jeffries and Montaño, [9])**.** *If $I \subset K[x_1, \ldots, x_d]$ is a monomial ideal and $F_{r+1}, \ldots, F_t$ are the compact facets of the Newton polyhedron $\Gamma(I)$, then*

$$c_0(I) = \sum_{i=r+1}^{t} d! \operatorname{vol}(\hat{F}_i) = \sum_{i=r+1}^{t} \operatorname{Vol}(\hat{F}_i).$$

Theorem 2. *Let I be a monomial ideal in $R = K[x_1, \ldots, x_d]$ generated by ${x_1}^{v_1(1)} \cdots {x_d}^{v_1(d)}, \ldots, {x_1}^{v_r(1)} \cdots {x_d}^{v_r(d)}$ and $m_j = \min\{v_1(j), \ldots, v_r(j)\}$, $j = 1, \ldots, d$. Then*

$$c_{d-1}(I) = m_1 + \cdots + m_d.$$

Proof. By [3, Proposition 2.3], $c_{d-1}(I) \neq 0$ if and only if $\dim R/I = d-1$. If $\dim R/I < d-1$, then none of the variables x_j appears in all monomials generating I, hence $m_j = 0$ for all j, and the result is true. If $\dim R/I = d-1$, then again by [3, Proposition 2.3],

$$c_{d-1}(I) = \sum_P e\left(IR_P\right) \cdot e(R/P),$$

where P runs through all $(d-1)$-dimensional associated prime ideals of R/I, that is, prime ideals of the form (x_j) for some j, see [10, Satz 9]. Therefore $IR_P = (x_j^{m_j})R_P$ and $e(IR_P) = m_j$. By [10] the $(d-1)$-dimensional part of the primary decomposition of I is $(x_1^{m_1}) \cap (x_2^{m_2}) \cap \cdots \cap (x_d^{m_d})$, which is of degree $m_1 + \cdots + m_d$. □

The following corollary of Theorem 2 states that the Conjecture 1 is true for $k = 0$.

Corollary 3. *Using the preceding notations, for $j = 1, \ldots, d$ set*

$$F_j := \operatorname{conv}(\{v \in \operatorname{vert}(I) \mid v(j) = m_j\}) + \sum_{1 \le i \le d, i \neq j} \mathbb{R}_{\geqslant 0} e_i$$

and $\operatorname{vert}(F_j) := \operatorname{vert}(I) \cap F_j$.

Then $\mathcal{F}(0) = \{F_1, \ldots, F_d\}$. If $v \in \operatorname{vert}(F_j)$, then $\operatorname{Vol}(v, F_j) = m_j$ and it holds

$$c_{d-1}(I) = \sum_{j=1}^{d} \min_{v \in \operatorname{vert}(F_j)} \left\{\operatorname{Vol}(v, F_j)\right\}.$$

Proof. Since each $v \in \Gamma(I)$ is the sum of a convex combination of the vertices $v_1, \ldots, v_s$ of $\Gamma(I)$ and of some $w \in \mathbb{R}^d_{\geqslant 0}$, we have

$$v(j) \geq \min\{v_1(j), \ldots, v_s(j)\} + w(j) \geq \min\{v_1(j), \ldots, v_s(j)\},$$

hence

$$m_j := \min\{v_1(j), \ldots, v_r(j)\} = \min\{v_1(j), \ldots, v_s(j)\} = \min_{v\in\Gamma(I)}\{v(j)\}.$$

It follows that $F_1, \ldots, F_d$ are precisely the $(d-1)$-unbounded facets of $\Gamma(I)$, that is, $\mathcal{F}(0) = \{F_1, \ldots, F_d\}$.

Since the projection of $\mathbb{R}^d \to \mathbb{R}$ with center $\sum_{1\le i\le d, i\ne j} \mathbb{R}e_i$ sends $v \in \mathrm{vert}(F_j)$ to $v(j) = m_j$, we have $\mathrm{Vol}(v, F_j) = v(j) = m_j$. Then, by Theorem 2, we obtain the desired formula for c_{d-1}, which finishes the proof. □

4. Examples

We illustrate the theorems and the conjecture by examples of monomial ideals $I \subset R = K[x_1, \ldots, x_d]$, K an arbitrary field. We set $\mathfrak{m} := (x_1, \ldots, x_d)R$ and $T := G_{\mathfrak{m}}(G_I(R))$. All the examples will show a closed relation between the summands in the formula of Conjecture 1 and the highest dimensional primary components of T.

We begin with the simplest case of a monomial ideal generated by one monomial in two variables. The first two examples are covered by Theorems 1 and 2.

Example 1 (Figure 1)**.** Let $I = (x^3y^2) \subset R = K[x, y]$. We have

$$c(I) = (c_0(I), c_1(I), c_2(I)) = (0, 5, 0) = 2 \cdot (0, 1, 0) + 3 \cdot (0, 1, 0),$$

where the summands are the contributions of the components of the bigraded ring $G_{\mathfrak{m}}(G_I(R))$, see Proposition 1.

The Newton polyhedron $\Gamma(I)$ has only one vertex $v = (3, 2)$ and two (unbounded) facets $F_1 = v + R_{\ge 0}e_2$ and $F_2 = v + R_{\ge 0}e_1$ (see Figure 1), hence $\mathcal{F}(1) = \emptyset$ and $c_0(I) = 0$. We have $\mathcal{F}(0) = \{F_1, F_2\}$ and $\mathrm{Vol}(v, F_1) = 3$ and $\mathrm{Vol}(v, F_2) = 2$, hence $c_1(I) = 5$.

Example 2 (Figure 2)**.** Let $I = (x^2y^5, x^3y^4, x^4y^2, x^6y) \subset R = K[x, y]$. We have

$$\begin{aligned} c(I) &= (c_0(I), c_1(I), c_2(I)) \\ &= (24, 3, 0) = 16 \cdot (1, 0, 0) + 8 \cdot (1, 0, 0) + 2 \cdot (0, 1, 0) + (0, 1, 0), \end{aligned}$$

where the summands are the contributions of the components of the bigraded ring $G_{\mathfrak{m}}(G_I(R))$, see Proposition 1.

The Newton polyhedron $\Gamma(I)$ has three vertices $v_1 = (2, 5)$, $v_3 = (4, 2)$, $v_4 = (6, 1)$, two (unbounded) facets $F_1 = v_1 + R_{\ge 0}e_2$, $F_2 = v_4 +$

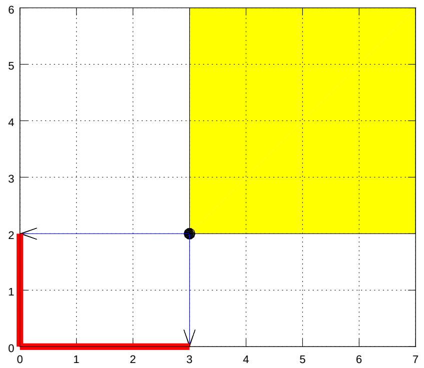

FIGURE 1. Projection along the infinite-directions of the facets gives $c_1(I)$, which is the red distance.

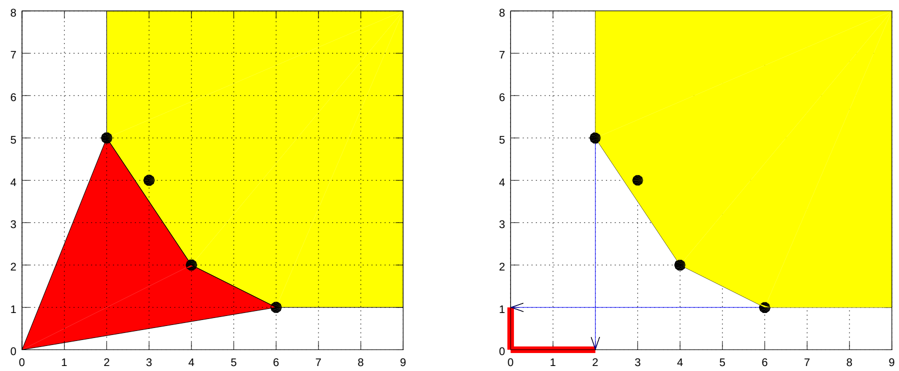

FIGURE 2. The red area is $c_0(I)/2$, the red distance $c_1(I)$.

$R_{\geq 0}e_1$ and two bounded facets: the line segments $F_3 = \mathrm{conv}(v_1, v_3)$, $F_4 = \mathrm{conv}(v_3, v_4)$ (see Figure 2), hence $\mathcal{F}(1) = \{F_3, F_4\}$ and

$$c_0(I) = \mathrm{Vol}(\mathrm{conv}(0, F_3)) + \mathrm{Vol}(\mathrm{conv}(0, F_4)) = \begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} + \begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 16 + 8.$$

We have $\mathcal{F}(0) = \{F_1, F_2\}$ and

$$c_1(I) = \mathrm{Vol}(v_1, F_1) + \mathrm{Vol}(v_4, F_2) = 2 + 1 = 3.$$

In the following example some of the compact faces of $\Gamma(I)$ do not contribute to the generalized Samuel multiplicity $c(I)$. In this example there is also a movable contribution, therefore the number of the highest dimensional components of T is one less than the number of summands in the conjectured formula.

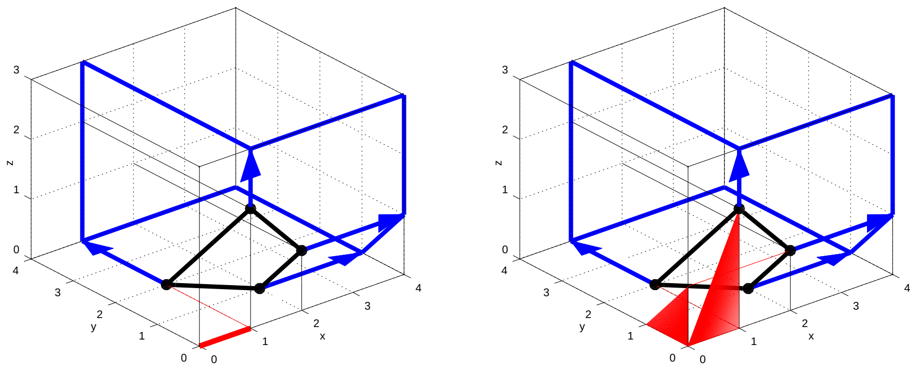

FIGURE 3. Infinite-directions (blue arrows) of the unbounded facets, $c_2(I)$ (red distance) and $c_1(I)/2$ (red area).

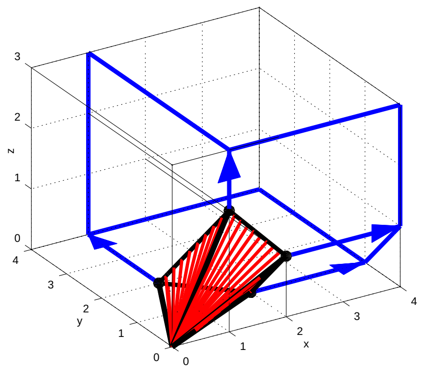

FIGURE 4. The volume of the red pyramid is $c_0/6$.

Example 3 (Figures 3, 4 and 5)**.** Let $I = (x^2y,\ x^2z,\ xy^2,\ xz^2) \subset K[x,y,z]$. By a computer computation (using [1]) we have

$$\begin{aligned} c(I) &= (c_0(I), c_1(I), c_2(I), c_3(I)) \\ &= (9,3,1,0) = 3\cdot(3,0,0,0) + (0,1,0,0) + (0,2,1,0), \end{aligned}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T, see Proposition 1. The contribution 2 in the last vector is a movable contribution to $c_1(I)$. This can be read off also from the Newton polyhedron $\Gamma(I)$, see Figure 6.

According to the program Germenes [11], the compact faces of $\Gamma(I)$ are the vertices $v_1 = (2,1,0)$, $v_2 = (2,0,1)$, $v_3 = (1,2,0)$, $v_4 = (1,0,2)$, the line segments v_1v_2, v_1v_3, v_2v_4, v_3v_4 and the quadrilateral facet

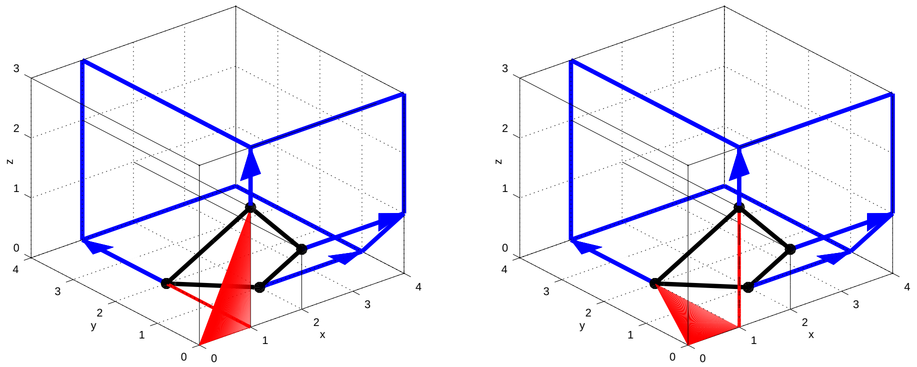

FIGURE 5. A movable contribution (to $c_1(I)/2$, red area) can be realized by at least two projections.

$v_1v_2v_4v_3$. The unbounded facets are

$$F_1 = v_3v_4 + \mathbb{R}_{\geq 0}\, e_2 + \mathbb{R}_{\geq 0}\, e_3, \quad F_2 = v_2v_4 + \mathbb{R}_{\geq 0}\, e_1 + \mathbb{R}_{\geq 0}\, e_3,$$
$$F_3 = v_1v_3 + \mathbb{R}_{\geq 0}\, e_1 + \mathbb{R}_{\geq 0}\, e_2, \quad F_4 = v_1v_2 + \mathbb{R}_{\geq 0}\, e_1.$$

We observe that the set of bounded facets is $\mathcal{F}(2) = \{v_1v_2v_4v_3\}$ and

$$c_0(I) = \mathrm{Vol}(\mathrm{conv}(0, v_1, v_2, v_4, v_3)) = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 3 + 6 = 9,$$

see Figure 4.

The set of 1-unbounded facets that contain a compact one-dimensional face is $\mathcal{F}(1) = \{F_1, F_2, F_3, F_4\}$, and we have

$$\mathrm{Vol}(v_1v_2, F_4) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

$$\mathrm{Vol}(v_1v_3, F_3) = \min\left\{ \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} \right\} = 0,$$

$$\mathrm{Vol}(v_2v_4, F_2) = \min\left\{ \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} \right\} = 0,$$

$$\mathrm{Vol}(v_3v_4, F_1) = \min\left\{ \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \right\} = 2$$

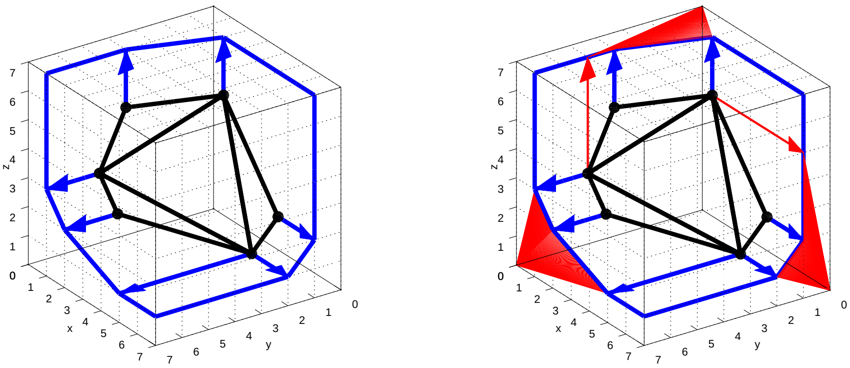

FIGURE 6. $\Gamma(I)$ with compact (black) and unbounded (blue) facets; projections of compact edges along infinite-directions (blue arrows) give $c_1(I)/2$ (red area).

(the last minimum is given by two different projections and is a movable contribution, see Figure 5), hence

$$c_1(I) = \mathrm{Vol}(v_1v_2, F_4) + \mathrm{Vol}(v_1v_3, F_3) + \mathrm{Vol}(v_2v_4, F_2) + \mathrm{Vol}(v_3v_4, F_1)$$
$$= 1 + 0 + 0 + 2 = 3.$$

The set of 2-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3\}$. We have $\mathrm{Vol}(v_3, F_1) = 1$, $\mathrm{Vol}(v_4, F_1) = 1$, $\mathrm{Vol}(v_4, F_2) = 0$, $\mathrm{Vol}(v_1, F_3) = 0$, $\mathrm{Vol}(v_3, F_3) = 0$, hence

$$c_2(I) = \min\{\mathrm{Vol}(v_3, F_1), \mathrm{Vol}(v_4, F_1)\} + \mathrm{Vol}(v_4, F_2) +$$
$$+ \min\{\mathrm{Vol}(v_1, F_3), \mathrm{Vol}(v_3, F_3)\} = 1 + 0 + 0 = 1.$$

Example 4 (Figure 6)**.** Let $I = (xy^4z^5,\ x^2y^5z^2,\ xy^5z^3,\ x^5yz^2,\ x^2yz^5,\ x^5y^2z) \subset K[x, y, z]$. By a computer computation we have

$$c(I) = (c_0(I), c_1(I), c_2(I), c_3(I)) = (168, 26, 3, 0) =$$
$$= 19 \cdot (1, 0, 0, 0) + 103 \cdot (1, 0, 0, 0) + 22 \cdot (1, 0, 0, 0)+$$
$$+ 24 \cdot (1, 0, 0, 0) + 7 \cdot (0, 1, 0, 0) + (0, 3, 1, 0)+$$
$$+ 8 \cdot (0, 1, 0, 0) + (0, 1, 0, 0) + 4 \cdot (0, 1, 0, 0)+$$
$$+ 3 \cdot (0, 1, 0, 0) + (0, 0, 1, 0) + (0, 0, 1, 0),$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T, see Proposition 1. The contribution 3 in the sixth vector is a movable contribution to $c_1(I)$.

The software Germenes [11] gives the following description of the Newton polyhedron $\Gamma(I)$. The compact faces of $\Gamma(I)$ are the 6 vertices

$v_1 = (1,4,5)$, $v_2 = (2,5,2)$, $v_3 = (1,5,3)$, $v_4 = (5,1,2)$, $v_5 = (2,1,5)$, $v_6 = (5,2,1)$, the 9 line segments v_4v_6, v_2v_6, v_5v_6, v_4v_5, v_3v_6, v_3v_2, v_3v_5, v_1v_5, v_1v_3 and the 4 triangles (bounded facets) $v_4v_5v_6$, $v_2v_3v_6$, $v_3v_5v_6$, $v_1v_3v_5$. There are 7 unbounded facets:

$$F_1 = v_1v_3 + \mathbb{R}_{\geq 0}\, e_2 + \mathbb{R}_{\geq 0}\, e_3, \quad F_2 = v_4v_5 + \mathbb{R}_{\geq 0}\, e_1 + \mathbb{R}_{\geq 0}\, e_3,$$
$$F_3 = v_6 + \mathbb{R}_{\geq 0}\, e_1 + \mathbb{R}_{\geq 0}\, e_2, \quad F_4 = v_1v_5 + \mathbb{R}_{\geq 0}\, e_3,$$
$$F_5 = v_2v_3 + \mathbb{R}_{\geq 0}\, e_2, \quad F_6 = v_2v_6 + \mathbb{R}_{\geq 0}\, e_2, \quad F_7 = v_4v_6 + \mathbb{R}_{\geq 0}\, e_1.$$

From the set of bounded facets $\mathcal{F}(2) = \{v_4v_5v_6, v_2v_3v_6, v_3v_5v_6, v_1v_3v_5\}$ we get

$$\begin{aligned} c_0(I) &= \mathrm{Vol}(\mathrm{conv}(0, v_4, v_5, v_6)) + \mathrm{Vol}(\mathrm{conv}(0, v_2, v_3, v_6)+ \\ &\quad + \mathrm{Vol}(\mathrm{conv}(0, v_3, v_5, v_6) + \mathrm{Vol}(\mathrm{conv}(0, v_1, v_3, v_5) = \\ &= \begin{vmatrix} 5 & 1 & 2 \\ 5 & 2 & 1 \\ 2 & 1 & 5 \end{vmatrix} + \begin{vmatrix} 2 & 5 & 2 \\ 1 & 5 & 3 \\ 5 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 5 & 3 \\ 2 & 1 & 5 \\ 5 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 4 & 5 \\ 2 & 1 & 5 \\ 1 & 5 & 3 \end{vmatrix} = \\ &= 24 + 22 + 103 + 19 = 168. \end{aligned}$$

We have $\mathcal{F}(1) = \{F_1, F_2, F_4, F_5, F_6, F_7\}$ and

$$\mathrm{Vol}(v_1v_3, F_1) = \min \left\{ \begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} \right\} = \min\{1, 2\} = 1,$$

$$\mathrm{Vol}(v_4v_5, F_2) = \min \left\{ \begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} \right\} = \min\{3, 3\} = 3$$

(here the minimum is attained twice, that is, by two different projections which indicates a movable contribution),

$$\mathrm{Vol}(v_1v_5, F_4) = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7, \quad \mathrm{Vol}(v_2v_3, F_5) = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4,$$

$$\mathrm{Vol}(v_2v_6, F_6) = \begin{vmatrix} 5 & 1 \\ 2 & 2 \end{vmatrix} = 8, \quad \mathrm{Vol}(v_4v_6, F_7) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3,$$

hence

$$\begin{aligned} c_1(I) &= \mathrm{Vol}(v_1v_3, F_1) + \mathrm{Vol}(v_4v_5, F_2) + \mathrm{Vol}(v_1v_5, F_4)+ \\ &\quad + \mathrm{Vol}(v_2v_3, F_5) + \mathrm{Vol}(v_2v_6, F_6) + \mathrm{Vol}(v_4v_6, F_7) = \\ &= 1 + 3 + 7 + 4 + 8 + 3 = 26. \end{aligned}$$

We observe that the compact 1-dimensional faces v_5v_6, v_3v_6, v_3v_5, that is, the edges of the big triangle $v_3v_5v_6$, do not contribute to $c_1(I)$

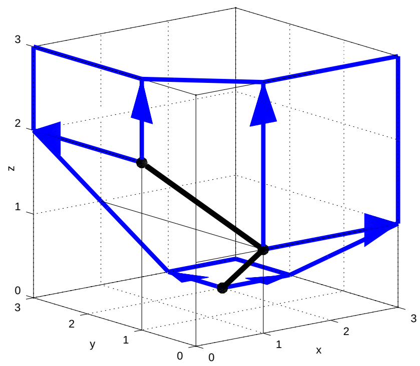

FIGURE 7. The triangle defined by 3 affinely independent vertices is not a compact facet.

since they lie on no 1-unbounded facets. Moreover, as in the previous example, there is a movable contribution, namely $\mathrm{Vol}(v_4v_5, F_2) = 3$.

The set of 2-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3\}$, and we have $\mathrm{Vol}(v_1, F_1) = 1$, $\mathrm{Vol}(v_3, F_1) = 1$, $\mathrm{Vol}(v_4, F_2) = 1$, $\mathrm{Vol}(v_5, F_2) = 1$, $\mathrm{Vol}(v_6, F_3) = 1$, hence

$$c_2(I) = \min\{\mathrm{Vol}(v_1, F_1), \mathrm{Vol}(v_1, F_1)\} + \min\{\mathrm{Vol}(v_4, F_2), \mathrm{Vol}(v_5, F_2)\} + \\ + \mathrm{Vol}(v_6, F_3) = 1 + 1 + 1 = 3.$$

Example 5 (Figure 7)**.** Let $I = (xz,\ x^2y^2,\ yz^2) \subset K[x, y, z]$. By a computer computation we have

$$c(I) = (c_0(I), c_1(I), c_2(I), c_3(I)) = (0, 7, 0, 0) = \\ = 2 \cdot (0, 1, 0, 0) + 2 \cdot (0, 1, 0, 0) + 2 \cdot (0, 1, 0, 0) + (0, 1, 0, 0),$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T, see Proposition 1.

A computation with the program Germenes [11] shows the the compact faces of $\Gamma(I)$ are the vertices $v_1 = (1, 0, 1)$, $v_2 = (2, 2, 0)$, $v_3 = (0, 1, 2)$ and the line segments v_1v_2, v_1v_3. There are no compact or bounded facets, but 6 unbounded facets:

$$F_1 = v_3 + \mathbb{R}_{\geq 0}\, e_2 + \mathbb{R}_{\geq 0}\, e_3, \quad F_2 = v_1 + \mathbb{R}_{\geq 0}\, e_1 + \mathbb{R}_{\geq 0}\, e_3,$$
$$F_3 = v_2 + \mathbb{R}_{\geq 0}\, e_1 + \mathbb{R}_{\geq 0}\, e_2, \quad F_4 = v_1v_2 + \mathbb{R}_{\geq 0}\, e_1,$$
$$F_5 = v_1v_3 + \mathbb{R}_{\geq 0}\, e_3, \quad F_6 = v_1v_2v_3 + \mathbb{R}_{\geq 0}\, e_2.$$

We observe that $\mathcal{F}(2) = \emptyset$, hence

$$c_0(I) = 0 \neq \mathrm{Vol}_3(\mathrm{conv}(\{0, v_1, v_2, v_3\})) = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 6.$$

This means that v_1, v_2, v_3 are affinely independent, but the local Segre class is zero and not equal to the normalized volume of the simplex generated by the origin and v_1, v_2, v_3 as claimed in [7, 4.2].

The set of 1-unbounded facets which contain a compact 1-dimensional face is $\mathcal{F}(1) = \{F_4, F_5, F_6\}$, and we have

$$\mathrm{Vol}(v_1v_2, F_4) = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2, \quad \mathrm{Vol}(v_1v_2, F_6) = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2,$$

$$\mathrm{Vol}(v_1v_3, F_5) = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2, \quad \mathrm{Vol}(v_1v_3, F_6) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

hence

$$\begin{aligned} c_1(I) &= \mathrm{Vol}(v_1v_2, F_4) + \mathrm{Vol}(v_1v_2, F_6) + \mathrm{Vol}(v_1v_3, F_5) + \mathrm{Vol}(v_1v_3, F_6) \\ &= 2 + 2 + 2 + 1 = 7. \end{aligned}$$

The set of 2-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3\}$ and we have

$$\mathrm{Vol}(v_3, F_1) = 0, \quad \mathrm{Vol}(v_1, F_2) = 0, \mathrm{Vol}(v_2, F_3) = 0,$$

hence $c_2(I) = 0$.

In the following example there is only one 1-dimensional compact face, but it lies on three 2-unbounded facets.

Example 6. Let $d = 4$, $I = (x_1^3x_2x_3x_4,\ x_1x_2x_3x_4^2)$. By a computer computation we have

$$\begin{aligned} c(I) &= (c_0(I), c_1(I), c_2(I), c_3(I), c_4(I)) = (0, 0, 7, 4, 0) = \\ &= 5 \cdot (0, 0, 1, 0, 0) + (0, 0, 1, 0, 0) + (0, 0, 1, 0, 0) + (0, 0, 0, 1, 0) + \\ &\quad + (0, 0, 0, 1, 0) + (0, 0, 0, 1, 0) + (0, 0, 0, 1, 0), \end{aligned}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T, see Proposition 1.

The compact faces of the Newton polyhedron $\Gamma(I)$ are the vertices $v_1 = (3, 1, 1, 1)$, $v_2 = (1, 1, 1, 2)$ and the line segment v_1v_2. There are no compact facets, but 5 unbounded facets:

$$\begin{aligned}
F_1 &= v_2 + \mathbb{R}_{\geq 0}\, e_2 + \mathbb{R}_{\geq 0}\, e_3 + \mathbb{R}_{\geq 0}\, e_4,\\
F_2 &= v_1v_2 + \mathbb{R}_{\geq 0}\, e_1 + \mathbb{R}_{\geq 0}\, e_3 + \mathbb{R}_{\geq 0}\, e_4,\\
F_3 &= v_1v_2 + \mathbb{R}_{\geq 0}\, e_1 + \mathbb{R}_{\geq 0}\, e_2 + \mathbb{R}_{\geq 0}\, e_4,\\
F_4 &= v_1 + \mathbb{R}_{\geq 0}\, e_1 + \mathbb{R}_{\geq 0}\, e_2 + \mathbb{R}_{\geq 0}\, e_3,\\
F_5 &= v_1v_2 + \mathbb{R}_{\geq 0}\, e_2 + \mathbb{R}_{\geq 0}\, e_3.
\end{aligned}$$

Obviously $\mathcal{F}(3) = \mathcal{F}(2) = \emptyset$, hence $c_0(I) = c_1(I) = 0$. We have $\mathcal{F}(1) = \{F_2, F_3, F_5\}$ and

$$\begin{aligned}
\mathrm{Vol}(v_1v_2, F_2) &= \min\{\begin{vmatrix} 3 & 1\\ 1 & 1\end{vmatrix}, \begin{vmatrix} 1 & 1\\ 1 & 2\end{vmatrix}\} = \min\{2,1\} = 1,\\
\mathrm{Vol}(v_1v_2, F_3) &= \min\left\{\begin{vmatrix} 3 & 1\\ 1 & 1\end{vmatrix}, \begin{vmatrix} 1 & 1\\ 1 & 2\end{vmatrix}\right\} = \min\{2,1\} = 1,
\end{aligned}$$

$$\mathrm{Vol}(v_1v_2, F_5) = \begin{vmatrix} 3 & 1\\ 1 & 2\end{vmatrix} = 5,$$

We observe that in the computations of $\mathrm{Vol}(v_1v_2, F_2)$ and $\mathrm{Vol}(v_1v_2, F_3)$ the projection of the line segment v_1v_2 on the $\{x_2, x_3\}$-plane gives the point $(1,1)$ and must not be considered. We obtain $c_2(I) = \mathrm{Vol}(v_1v_2, F_2) + \mathrm{Vol}(v_1v_2, F_3) + \mathrm{Vol}(v_1v_2, F_5) = 1 + 1 + 5 = 7$.

The set of 3-unbounded facets is $\mathcal{F}(0) = \{F_1, F_2, F_3, F_4\}$, and we have

$$\begin{aligned}
&\mathrm{Vol}(v_2, F_1) = 1, \quad \mathrm{Vol}(v_1, F_2) = 1, \quad \mathrm{Vol}(v_2, F_2) = 1,\\
&\mathrm{Vol}(v_1, F_3) = 1, \quad \mathrm{Vol}(v_2, F_3) = 1, \quad \mathrm{Vol}(v_1, F_4) = 1,
\end{aligned}$$

hence

$$\begin{aligned}
c_3(I) = \mathrm{Vol}(v_2, F_1) &+ \min\{\mathrm{Vol}(v_1, F_2), \mathrm{Vol}(v_2, F_2)\} +\\
&+ \min\{\mathrm{Vol}(v_1, F_3), \mathrm{Vol}(v_2, F_3)\} + \mathrm{Vol}(v_1, F_3) =\\
&= 1 + 1 + 1 + 1 = 4.
\end{aligned}$$

We now present an example in dimension 5 in which several compact 1-dimensional faces lie on the same 3-unbounded facet.

Example 7. Let $d = 5$, $I = (x_1^3x_2x_3x_4x_5,\ x_1x_2^2x_3x_4x_5,\ x_1x_2x_3x_4x_5^5)$. By a computer computation we have

$$\begin{aligned} c(I) = (c_0(I), c_1(I), c_2(I), c_3(I), c_4(I), c_5(I)) = (0,0,26,6,5,0) = \\ = 22\cdot(0,0,1,0,0,0) + 2\cdot(0,0,1,0,0,0) + 2\cdot(0,0,1,0,0,0)+ \\ + 2\cdot(0,0,0,1,0,0) + (0,0,0,1,0,0) + (0,0,0,1,0,0)+ \\ + (0,0,0,1,0,0) + (0,0,0,1,0,0) + (0,0,0,0,1,0)+ \\ + (0,0,0,0,1,0) + (0,0,0,0,1,0) + (0,0,0,0,1,0)+ \\ + (0,0,0,0,1,0), \end{aligned}$$

where the summands are the contributions of the highest dimensional components of the bigraded ring T, see Proposition 1.

The software Germenes [11] shows that the compact faces of $\Gamma(I)$ are the vertices $v_1 = (3,1,1,1,1)$, $v_2 = (1,2,1,1,1)$, $v_3 = (1,1,1,1,5)$, the line segments v_1v_2, v_1v_3, v_2v_3 and the triangle $v_1v_2v_3$. The facets, all of them unbounded, are:

$$\begin{aligned} F_1 &= v_2v_3 + \mathbb{R}_{\geq 0}\, e_2 + \mathbb{R}_{\geq 0}\, e_3 + \mathbb{R}_{\geq 0}\, e_4 + \mathbb{R}_{\geq 0}\, e_5, \\ F_2 &= v_1v_3 + \mathbb{R}_{\geq 0}\, e_1 + \mathbb{R}_{\geq 0}\, e_3 + \mathbb{R}_{\geq 0}\, e_4 + \mathbb{R}_{\geq 0}\, e_5, \\ F_3 &= v_1v_2v_3 + \mathbb{R}_{\geq 0}\, e_1 + \mathbb{R}_{\geq 0}\, e_2 + \mathbb{R}_{\geq 0}\, e_4 + \mathbb{R}_{\geq 0}\, e_5, \\ F_4 &= v_1v_2v_3 + \mathbb{R}_{\geq 0}\, e_1 + \mathbb{R}_{\geq 0}\, e_2 + \mathbb{R}_{\geq 0}\, e_3 + \mathbb{R}_{\geq 0}\, e_5, \\ F_5 &= v_1v_2 + \mathbb{R}_{\geq 0}\, e_1 + \mathbb{R}_{\geq 0}\, e_2 + \mathbb{R}_{\geq 0}\, e_3 + \mathbb{R}_{\geq 0}\, e_4, \\ F_6 &= v_1v_2v_3 + \mathbb{R}_{\geq 0}\, e_3 + \mathbb{R}_{\geq 0}\, e_4. \end{aligned}$$

Obviously $\mathcal{F}(4) = \mathcal{F}(3) = \emptyset$, hence $c_0(I) = c_1(I) = 0$.
We have $\mathcal{F}(2) = \{F_3, F_4, F_6\}$ and

$$\begin{aligned} \mathrm{Vol}(v_1v_2v_3, F_3) = \mathrm{Vol}(v_1v_2v_3, F_4) = \min\left\{ \begin{vmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 5 \end{vmatrix} \right\} = \\ = \min\{2, 4\} = 2, \end{aligned}$$

$$\mathrm{Vol}(v_1v_2v_3, F_6) = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 5 \end{vmatrix} = 22.$$

Observe that in the computations of $\mathrm{Vol}(v_1v_2v_3, F_3)$ and $\mathrm{Vol}(v_1v_2v_3, F_4)$ four projections of the compact triangle $v_1v_2v_3$ give a line segment and must not be considered. Summing up we get

$$\begin{aligned} c_2(I) &= \mathrm{Vol}(v_1v_2v_3, F_3) + \mathrm{Vol}(v_1v_2v_3, F_4) + \mathrm{Vol}(v_1v_2v_3, F_6) \\ &= 2 + 2 + 22 = 26. \end{aligned}$$

The set of 3-unbounded facets containing 1-dimensional compact faces is $\mathcal{F}(1) = \{F_1, F_2, F_3, F_4, F_5\}$ and we have

$$\mathrm{Vol}(v_2v_3, F_1) = \min\left\{\begin{vmatrix}1 & 1\\1 & 2\end{vmatrix}, \begin{vmatrix}1 & 1\\1 & 5\end{vmatrix}\right\} = \min\{1, 4\} = 1,$$

$$\mathrm{Vol}(v_1v_3, F_2) = \min\left\{\begin{vmatrix}3 & 1\\1 & 1\end{vmatrix}, \begin{vmatrix}1 & 1\\1 & 5\end{vmatrix}\right\} = \min\{2, 4\} = 2,$$

$$\mathrm{Vol}(v_1v_2, F_3) = \mathrm{Vol}(v_1v_2, F_4) = \min\left\{\begin{vmatrix}3 & 1\\1 & 1\end{vmatrix}, \begin{vmatrix}2 & 1\\1 & 1\end{vmatrix}\right\} = \min\{2, 1\} = 1,$$

$$\mathrm{Vol}(v_1v_3, F_3) = \mathrm{Vol}(v_1v_3, F_4) = \min\left\{\begin{vmatrix}3 & 1\\1 & 1\end{vmatrix}, \begin{vmatrix}1 & 1\\1 & 5\end{vmatrix}\right\} = \min\{2, 4\} = 2,$$

$$\mathrm{Vol}(v_2v_3, F_3) = \mathrm{Vol}(v_2v_3, F_4) = \min\left\{\begin{vmatrix}2 & 1\\1 & 1\end{vmatrix}, \begin{vmatrix}1 & 1\\1 & 5\end{vmatrix}\right\} = \min\{1, 4\} = 1,$$

$$\mathrm{Vol}(v_1v_2, F_5) = \min\left\{\begin{vmatrix}3 & 1\\1 & 1\end{vmatrix}, \begin{vmatrix}2 & 1\\1 & 1\end{vmatrix}\right\} = \min\{2, 1\} = 1,$$

hence

$$\begin{aligned} c_3(I) = {}& \mathrm{Vol}(v_2v_3, F_1) + \mathrm{Vol}(v_1v_3, F_2) + \\ &+ \min\{\mathrm{Vol}(v_1v_2, F_3), \mathrm{Vol}(v_1v_3, F_3), \mathrm{Vol}(v_2v_3, F_3)\} + \\ &+ \min\{\mathrm{Vol}(v_1v_2, F_4), \mathrm{Vol}(v_1v_3, F_4), \mathrm{Vol}(v_2v_3, F_4)\} + \\ &+ \mathrm{Vol}(v_1v_2, F_5) = 1 + 2 + 1 + 1 + 1 = 6. \end{aligned}$$

From the list of the facets we see that there are five 4-unbounded facets, precisely $\mathcal{F}(0) = \{F_1, F_2, F_3, F_4, F_5\}$ and we have

$$\begin{aligned} &\mathrm{Vol}(v_2, F_1) = 1, \quad \mathrm{Vol}(v_3, F_1) = 1, \quad \mathrm{Vol}(v_1, F_2) = 1, \quad \mathrm{Vol}(v_3, F_2) = 1, \\ &\mathrm{Vol}(v_1, F_3) = 1, \quad \mathrm{Vol}(v_2, F_3) = 1, \quad \mathrm{Vol}(v_3, F_3) = 1, \quad \mathrm{Vol}(v_1, F_4) = 1, \\ &\mathrm{Vol}(v_2, F_4) = 1, \quad \mathrm{Vol}(v_3, F_4) = 1, \quad \mathrm{Vol}(v_1, F_5) = 1, \quad \mathrm{Vol}(v_2, F_5) = 1, \end{aligned}$$

$$\begin{aligned} c_4(I) = {}& \min\{\mathrm{Vol}(v_2, F_1), \mathrm{Vol}(v_3, F_1)\} + \min\{\mathrm{Vol}(v_1, F_2), \mathrm{Vol}(v_3, F_2)\} + \\ &+ \min\{\mathrm{Vol}(v_1, F_3), \mathrm{Vol}(v_2, F_3), \mathrm{Vol}(v_3, F_3)\} + \\ &+ \min\{\mathrm{Vol}(v_1, F_4), \mathrm{Vol}(v_2, F_4), \mathrm{Vol}(v_3, F_4)\} + \\ &+ \min\{\mathrm{Vol}(v_1, F_5), \mathrm{Vol}(v_2, F_5)\} = 1 + 1 + 1 + 1 + 1 = 5. \end{aligned}$$

References

[1] R. Achilles and D. Aliffi, Segre, a script for the Reduce package Cali, Bologna, 1999–2017, http://www.dm.unibo.it/~achilles/segre/.

[2] R. Achilles and M. Manaresi, *Multiplicity for ideals of maximal analytic spread and intersection theory*, J. Math. Kyoto Univ. **33** (1993), 569–578.

[3] R. Achilles and M. Manaresi, *Multiplicities of a bigraded ring and intersection theory*, Math. Ann. **309** (1997), 573–591.

[4] R. Achilles and S. Rams, *Intersection numbers, Segre numbers and generalized Samuel multiplicities*, Arch. Math. (Basel) **77** (2001), 391–398.

[5] C. Bivià-Ausina, *The analytic spread of monomial ideals*, Comm. Algebra **31** (2003), No. 7, 3487–3496

[6] D. Eisenbud and B. Sturmfels, *Binomial ideals*, Duke Math. J. **84** (1996), 1–45.

[7] G. Guibert, *Classe de Segre locale d'un idéal et frontières de Newton*, C. R. Acad. Sci. Paris Sér. I Math. **329**, (1999), 315–320.

[8] A. C. Hearn, Reduce: A user-oriented interactive system for algebraic simplification. In M. Klerer and J. Reinfelds, editors, Interactive Systems for Experimental Applied Mathematics, pages 79–90, New York, 1968. Academic Press.

[9] J. Jeffries and J. Montaño, *The j-multiplicity of monomial ideals*, Math. Res. Lett. **20** (2013), No. 4, 729–744.

[10] R. Kummer and B. Renschuch, *Potenzproduktideale.* I., Publ. Math. Debrecen **17** (1970), 81–98 (1971).

[11] A. Montesinos, Germenes, program available at http://www.uv.es/montesin/.

[12] Reduce: A portable general-purpose computer algebra system, https://reduce-algebra.sourceforge.io/

[13] L. Reid and M. A. Vitulli, *The weak subintegral closure of a monomial ideal*, Comm. Algebra **27** (1999), 5649–5667.

[14] B. Teissier, *Monômes, volumes et multiplicités*, Introduction à la théorie des singularités, II, 127–141, Travaux en Cours, 37, Hermann, Paris, 1988.

[15] B. L. van der Waerden, On Hilbert's function, series of composition of ideals and a generalisation of the theorem of Bezout, Proc. Roy. Acad. Amsterdam **31** (1929), 749–770.

Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy

E-mail address: rudiger.achilles@unibo.it, mirella.manaresi@unibo.it