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Walk on the Wild Side: Temporarily Unstable Paths and Multiplicative Sunspots†

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We propose a generalization of the rational expectations framework to allow for temporarily unstable paths. Our approach introduces multiplicative sunspot shocks and it yields drifting parameters and stochastic volatility. Then, we provide an econometric strategy to estimate this generalized model on the data. The methodology allows the data to choose between different possible alternatives: determinacy, indeterminacy, and temporary instability. We apply our methodology to US inflation dynamics in the 1970s through the lens of a simple New Keynesian model. When temporarily unstable paths are allowed, the data unambiguously select them to explain the stagflation period in the 1970s. (*JEL* D84, E12, E31, E32, E52)

The vast majority of modern dynamic macroeconomics has relied on models of rational expectations (RE) with a unique stable equilibrium, following the methodology in Blanchard and Kahn (1980). This somewhat limits the ability of the models to analyze unstable behavior in the data, which is an important issue in macroeconomics, especially after the Great Financial Crisis. One option is to make RE models more flexible to allow temporarily explosive paths. This work provides a novel theoretical framework that considers a broader class of solutions to allow for temporary instability. Moreover, it provides an empirical strategy to take this theoretical framework to the data. Our contribution is therefore both theoretical and empirical.

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From a theoretical perspective, our framework generalizes RE solutions to accommodate temporarily unstable paths. Macroeconomic models typically feature forward-looking behavior, where expectations of future variables affect the dynamics of the models. In this case, the RE assumption is consistent with an infinite number of trajectories. Therefore, after Muth's (1961) seminal contribution, the literature faced the problem of how to select an equilibrium out of many possible ones.[1] Dynamic macro models are usually derived from underlying dynamic optimization problems. Since explosive paths would generally violate the transversality conditions associated with these problems, the literature concurred that stability is a general consistency requirement to impose on an infinite horizon RE agents model. Under some condition (i.e., saddle path dynamics), there is a unique stable RE equilibrium and thus, the stability criterion is enough to pin down a unique admissible RE path. Blanchard and Kahn (1980) formalized this idea and conceptualized the solution algorithm on which dynamic macroeconomics is based.

Macroeconomic time series, on the other hand, often display irregular patterns that are difficult to describe relying on stable dynamics. Hence, RE solutions have a hard time explaining unstable behavior in the data, such as periods of high inflation, or boom and bust episodes in asset markets. This makes us reconsider the stability criterion as an equilibrium selection device.

There are two main steps in defining our approach. First, we parameterize all admissible solutions under RE such that the value of a parameter selects a particular solution among the infinite number of admissible ones. Second, we then assume that this parameter varies with time according to an exogenous stochastic process. Thus, we can build solutions that randomly jump between the admissible RE paths. Time variation in the solution allows temporary walks on unstable trajectories, provided that the system eventually converges to the stable one. The stability criterion regards the long-run convergence of a solution, so it just requires that our time-varying solution converges to the unique stable one in the limit. Hence, our framework could consider a class of solutions where RE paths are temporarily unstable, but stable in the long run. As we will discuss, under some conditions, allowing for unstable paths requires a minimal relaxation of the RE constraint to comply with the transversality conditions. The main insight, however, is that while unstable paths are usually ruled out by imposing the stability criterion to select equilibria, the time variation in the solution opens up the possibility for temporary instability, not necessarily in contrast with RE and stability in the long run.

Furthermore, we show that this time-varying parameter has an appealing economic interpretation, because it relates the different infinite RE solutions to the way agents form their expectations or, more precisely, it determines the way agents weight past data to calculate their RE. The economic interpretation of our proposed solution, hence, is that the economy randomly switches among the infinite RE solutions, because agents change the way they are forming their RE.

[1] As expressed by Blanchard and Watson (1982, footnote 1, p. 27): "This indeterminacy arises […] in all models in which expectations of future variables affect current decisions. It is the subject of much discussion currently in macroeconomics, under the label of 'non-uniqueness.'" Sargent and Wallace (1973), Brock (1974), Phelps and Taylor (1977), Taylor (1977), Blanchard (1979), Blanchard and Kahn (1980), and Flood and Garber (1980) are some examples of this compelling debate in the literature that followed Muth's contribution. See also the discussion in Burmeister, Flood, and Garber (1983).

Finally, our approach is similar in spirit to the sunspot one. A sunspot solution features exogenous changes in agents' expectations that make the economy switch between an infinite number of *stable* RE paths (e.g., Benhabib and Farmer 1999, Lubik and Schorfheide 2004). Our "sunspot" is related to expectations too, because it is the shock to the exogenous process governing the time-varying parameter. There are two main differences. First, our approach also allows temporary switches among unstable paths. Second, and more technically, our sunspot shock enters nonlinearly in the solution, so it is a *multiplicative sunspot*, instead of an additive sunspot, which has so far been considered by the literature. It follows that the time variation also affects the parameters and the variance of the solution. Therefore, drifting parameters and stochastic volatility emerge in the reduced form of a structural model without departing from RE, or without assuming time variation in the structural parameters. This may help rationalize the evidence in favor of drifting parameters and stochastic volatility as important features in the empirical analysis of many macroeconomic time series (see Cogley and Sargent 2005, Primiceri 2005, Justiniano and Primiceri 2008).

From an empirical perspective, we develop an econometric strategy suited to our framework. Given that our sunspots are multiplicative and imply stochastic volatility, the likelihood is not Normal and we cannot use Gaussian methods. The model parameters and the latent states are estimated using a Bayesian approach based on sequential Monte Carlo methods. In particular, we build an econometric strategy for parameter learning that combines the approach of Carvalho et al. (2010) and the particle filter of Liu and West (2001).[2] The econometric strategy allows for the cases of determinacy, indeterminacy, or explosiveness, without imposing them a priori. The methodology lets the data choose the preferred equilibria among all the possible ones, and thus to test the empirical validity of temporarily unstable paths. By the same token, our approach could be seen as checking the validity of the stability criterion as usually imposed on the RE solutions.

To show its potential, we apply our methodology to the US inflation dynamics in the postwar sample. In an extremely influential article, Clarida, Galí, and Gertler (2000) suggest that the change in the response of monetary policy to inflation explains the different inflation behavior between the Great Inflation and the Great Moderation. According to a simple New Keynesian model, an insufficient response of the interest rate to inflation generates indeterminacy of equilibria (i.e., an infinite number of stable RE equilibrium paths) that could explain the aggregate instability of the 1970s through shifts in self-fulfilling agents' beliefs due to sunspot shocks. In a seminal contribution about the econometrics of indeterminate RE equilibria, Lubik and Schorfheide (2004)—henceforth, LS—estimate a standard three-equations New Keynesian model under both determinacy and indeterminacy. Their results provide support for the original Clarida, Galí, and Gertler (2000) result in a multivariate context (see also other subsequent papers in the literature, e.g., Boivin and Giannoni

[2] Fernández-Villaverde and Rubio-Ramírez (2007) present pioneering work on the estimation of nonlinear or non-Gaussian dynamic stochastic general equilibrium (DSGE) models, based on particle filtering within a Markov chain Monte Carlo scheme. The use of Sequential Monte Carlo methods is less common in the literature. Exceptions are Creal (2007); Chen, Petralia, and Lopes (2010); and Herbst and Schorfheide (2014). Our approach differs somewhat from the latter as explained in Section II.

2006; Benati and Surico 2009; Mavroeidis 2010; Castelnuovo, Greco, and Raggi 2014; Castelnuovo and Fanelli 2015; Lubik and Matthes 2016).[3]

Therefore, the New Keynesian literature appeals to indeterminacy, induced by a dovish monetary policy, to explain the apparently explosive behavior of inflation during the Great Inflation period, and to a hawkish monetary policy to explain the Great Moderation. However, this has the rather paradoxical implication of resorting to a stable system to generate instability, as well as to an unstable system to ensure stability. From a theoretical perspective, indeterminacy has an infinite number of stable trajectories, so it is a stable dynamic system. Indeterminacy, however, opens up the possibility of rationalizing an explosive behavior by randomizing among all these possible stable trajectories thanks to a sunspot shock. In contrast, a saddle path describes an unstable dynamic system, because there are infinite unstable trajectories while only one, which thus has measure zero, is stable. Nonetheless, a central bank that does not respect the Taylor principle is certain that the economy is on stable dynamics, though subject to self-fulfilling beliefs, while on the contrary satisfying the Taylor principle is potentially highly risky, because the probability of being on the unique stable path (among an infinite number of unstable ones) is practically zero. Macroeconomists generally assume that agents are able to select this unique stable solution.

It seems to us that it would be more natural to associate the unstable behavior of inflation in the data to an unstable trajectory in the model. We thus apply our framework to ask the following question: is there any evidence that inflation is described by temporarily unstable equilibria in the 1970s?

The seminal paper of LS is the natural benchmark against which to compare our results, so we use both their econometric model and their data. Our econometric strategy recovers results that are practically identical to those in LS, when we impose the stability criterion on the estimation, that is when, as in LS, we just allow for determinacy or indeterminacy, ruling out temporary instability. Our main result, however, is to provide evidence that the high inflation during the 1970s is better explained by temporarily unstable dynamics. The data favor a temporarily unstable equilibrium path to explain the Great Inflation period, rather than a randomization over stable trajectories as suggested by the indeterminacy literature. Intuitively, to explain the rapid increase in inflation in the 1970s, a standard indeterminacy model needs to rely on persistent and successive sunspot shocks in the same direction. The data assign a low likelihood to such a sequence of shocks and favor a model that presents inherent temporarily explosive dynamics.

However, the solution also features stochastic volatility, so that one might conjecture that allowing for the possibility of large shocks, rather than for temporarily explosive dynamics, is what makes the model outperform the indeterminacy model. In Section V, we compare our framework to one with stochastic volatility and a unique stable trajectory (i.e., determinacy), as in Justiniano and Primiceri (2008).

[3] Alternative possible explanations for the Great Inflation period put forward in the literature are stochastic volatility of the shocks (e.g., Justiniano and Primiceri 2008; Fernández-Villaverde, Guerrón-Quintana, and Rubio-Ramírez 2010) or escape dynamics (e.g., Sargent 1999; Cho, Williams, and Sargent 2002; Sargent, Williams, and Zha 2006; Carboni and Ellison 2009).

Once more, the estimation favors a model with intrinsic temporarily unstable dynamics.

The paper proceeds as follows. Section I explains our approach by means of a simple model. Section II explains our econometric strategy. Section III presents the application of the approach to the New Keynesian model in LS. Section IV discusses the empirical results. Section V presents a comparison with a stochastic volatility model and Section VI concludes.

I. Multiplicative Sunspots and Unstable Paths

We use a simple example to illustrate our approach. We proceed in two steps. First, we introduce multiplicative sunspots by allowing agents to switch between all possible fundamental solutions under stability. Second, we discuss asymptotic stability and we examine the possibility of temporarily unstable paths. Finally, we generalize our simple example to a multivariate model.

A. *A Simple Example*

Consider the following expectational difference equation (as in LS, Section II)

$$y_t = \frac{1}{\theta} E_t y_{t+1} + \varepsilon_t, \tag{1}$$

where ε_t is an i.i.d. shock $\sim N(0, \sigma_\varepsilon^2)$ and $\theta \in \Theta = [0,2]$. The term $E_t y_{t+1} = E(y_{t+1} | \mathcal{J}_t)$ is the expected value of y at $t+1$ conditional on the information set available at time t.[4] A forward-looking equation as (1) implies that the expectation regarding the value of y in the following period determines the equilibrium value of y at t. Here lies a fundamental degree of freedom: the way agents form their expectations about future values of y pins down the equilibrium value today.[5] Equation (1) naturally has an infinite number of solutions, because one can find an infinite number of pairs $(y_t, E_t y_{t+1})$ that satisfy it.

Muth's (1961) RE seminal idea restricts the way agents form their expectation to be coherent with the economic system, so that the expected forecast error should be zero. Defining the forecast error as $\eta_t = y_t - E_{t-1} y_t$, thus, $E_{t-1}(\eta_t) = 0$. The RE requirement, however, is generally not enough to pin down a unique solution. This is easy to see by simply rewriting (1) using conditional expectations $\xi_t \equiv E_t(y_{t+1})$, as in Sims (2002),

$$\xi_t = \theta \xi_{t-1} - \theta \varepsilon_t + \theta \eta_t. \tag{2}$$

Any process η_t such that $E_{t-1}(\eta_t) = 0$ defines a different solution to (2), so that the solution is characterized up to an arbitrary martingale process.[6] Let ζ_t be a mean

[4] The set $\mathcal{J}_t$ contains all the relevant information: all the present and past values of the endogenous and exogenous variables, and the structure of the model with its parameters.

[5] See the discussion in Section 2.4 in Woodford (2003), especially pages 127–28.

[6] If m_t is a martingale, then $\Delta m_t = m_t - m_{t-1}$ is a martingale difference process, and $E_{t-1}(\Delta m_t) = E_{t-1}(m_t - m_{t-1}) = 0$. So one could interpret the error of expectations η_t as a martingale difference process, and

zero non-fundamental disturbance, uncorrelated with the fundamental one; then any forecast error of the form

$$\eta_t = (1 + M)\varepsilon_t + \zeta_t \tag{3}$$

yields a RE solution (the literature calls ζ_t a sunspot shock).[7]

The early literature on RE (see footnote 1) agreed on considering as admissible RE solutions only the *stable* ones. Stability is a general consistency requirement to impose on a model of infinite horizons RE agents, because of the set of transversality conditions associated with the agents' dynamic optimization problems in the underlying model. However, whether or not the stability criterion is sufficient to select a unique solution depends on the stability properties of the expectational difference equation (1), or (2), that is on the value of θ. If $\theta > 1$, deviations of ξ_t from 0 in (2) explode with time and thus, stability requires $\xi_t = 0, \forall t$. Hence, the stability criterion imposes a restriction on the forecast error (3): $\eta_t = \varepsilon_t$ and $M = \zeta_t = 0, \forall t$, pinning down the unique RE solution to (2) that does not violate the stability criterion. Blanchard and Kahn (1980) generalized this idea to a multivariate RE linear system with backward- and forward-looking variables, and conceptualized the well-known solution algorithm that is a cornerstone of dynamic macroeconomics.

If $\theta \leq 1$, however, the model is indeterminate, because any deviation of ξ_t from 0 will not lead ξ_t to explode over time. All infinite RE solutions of (2) are stable. Hence, the stability criterion imposes no restriction on the forecast error (3) and thus, it does not solve the problem of selecting a unique equilibrium. The indeterminacy literature then assumes that the economy will choose randomly among these infinite stable solutions. This randomization is usually done by adding an exogenous non-fundamental sunspot shock, ζ_t, in (3) for a given value for M (on which the system dynamics put no restrictions).[8]

B. *Indeterminacy: Multiplicative Sunspots and a Generalized Time-Varying Solution*

This section considers the case of indeterminacy (i.e., $\theta \leq 1$). Equation (3) suggests another possible source of multiplicity other than the additive sunspot ζ_t: M. Note that M parameterizes all possible fundamental solutions, where the expectation error is just a function of the structural shock, so that $\eta_t = (1 + M)\varepsilon_t$. Thinking along the lines of the Benhabib and Farmer (1999) quotation in footnote 8 suggests a different way of introducing sunspot shocks by randomizing over the fundamental solutions, i.e., randomizing over M, rather than adding ζ_t. This approach introduces a multiplicative sunspot shock, rather than an additive one.

the requirement of a zero expected error simply implies that the solution is characterized up to an arbitrary martingale (see Pesaran 1987).

[7] Plugging (3) into (1) yields $y_t = \theta y_{t-1} - \theta\varepsilon_{t-1} + (1 + M)\varepsilon_t + \zeta_t$, which is a way of writing all the possible fundamental solutions to (1) parameterized by M plus the sunspot shock.

[8] "Sunspot equilibria can often be constructed by randomizing over multiple equilibria of a general equilibrium model, and models with indeterminacy are excellent candidates for the existence of sunspot equilibria since there are many equilibria over which to randomize" (Benhabib and Farmer 1999, p. 390).

To illustrate our approach, we first show that M parameterizes all fundamental RE solutions, and we provide an economic interpretation of M. Second, we introduce time variation in M. Then, we show that the RE condition restricts the type of admissible time variation processes, and we analyze the nature of these solutions.

Parameterization.—Consider only fundamental solutions where $\zeta_t = 0, \forall t$, in (3). Substituting in (2) and iterating backward yields

$$\xi_t = M\theta\sum_{i=0}^{t-1}\theta^i\varepsilon_{t-i} = M\sum_{i=1}^{t}\theta^i\varepsilon_{t+1-i}, \tag{4}$$

assuming that there exists a whatever distant period 0 where the economy was in a steady state: $\varepsilon_{-i} = \xi_{-i} = 0, \forall i \geq 0$. All possible fundamental solutions are thus parameterized by $M \in (-\infty, +\infty)$, because a particular value of M defines a particular solution. Among the infinite number of solutions, two important solutions are often considered in the literature: (i) the pure forward-looking solution corresponding to $M = 0$: $\eta_t = \varepsilon_t$, $\xi_t^F = 0$, $y_t^F = \varepsilon_t$; and (ii) the pure backward-looking solution, corresponding to $M = -1$: $\eta_t = 0$, $\xi_t^B = -\theta\sum_{i=0}^{t-1}\theta^i\varepsilon_{t-i}$, $y_t^B = \xi_{t-1}^B = -\sum_{i=1}^{t-1}\theta^i\varepsilon_{t-i}$.[9]

Notice that M has a very natural interpretation: it defines the way agents form their expectations. More precisely, it determines if and how agents are going to use past observations in forming their expectations. In the simple case of equation (1), Muth (1961) shows that the expectation of y_{t+1} at time t can be written as an exponentially weighted average of past observations (assuming $M \neq -1$; see Appendix A for the derivation),[10]

$$\xi_t \equiv E_t y_{t+1} = M\sum_{i=1}^{t}\left(\frac{\theta}{1+M}\right)^i y_{t+1-i}. \tag{5}$$

Here, $E_t y_{t+1}$ is the product of two terms. First, M measures how much the past matters in forming expectations in absolute terms: if $M = 0$, then past data do not matter. This is the forward-looking solution. Second, the weights $[\theta/(1+M)]^i$ tell us how much agents relatively weight the past data.

Time Variation.—Following Muth's RE original formulation, we just argued that M can be interpreted as pinning down the infinite number of ways agents could combine past data to form their expectations. Our proposed class of solutions simply generalizes the standard one, (4), by letting M be a random variable that can change over time, so that

$$\xi_t = M_t\theta\sum_{i=0}^{t-1}\theta^i\varepsilon_{t-i} \equiv -M_t\xi_t^B. \tag{6}$$

[9] It is easy to rewrite (4) as a linear combination of the forward- and backward-looking solutions as: $\xi_t = \theta M\varepsilon_t + M\theta\sum_{i=1}^{t-1}\theta^i\varepsilon_{t-i} = \theta M(y_t^F - y_t^B)$. It follows that one can also write the solution for y as a weighted average of the backward- and the forward-looking solution (see Blanchard 1979).

[10] One of the purposes of Muth's (1961) original paper is to write the expectation at time t as an exponentially weighted average of past observations, because a previous paper (i.e., Muth 1960) demonstrated that this is the optimal estimator under some assumptions.

A key feature of our approach is the particular way we introduce time variation in the solution. In each period t, the solution (6) only depends on the current realization of M_t and not on its past values. We impose this restriction on our class of solutions. Hence, we are not considering all possible solutions under time variation. For example, we do not assume a time-varying M in (3), i.e., $\eta_t = (1 + M_t)\varepsilon_t$. This would yield a different solution with respect to ours: $\xi_t = \theta \sum_{i=0}^{\infty} \theta^i M_{t-i}\varepsilon_{t-i}$, where a change in M_t only affects the weight in period t, while in our framework it affects all weights in (6). This feature of our proposed solution is important for the way we model temporarily unstable paths, as discussed in Section IC.

RE implies $E_{t-1}(\eta_t) = 0$. Plugging the proposed solution (6) into the original equation (2) and solving for η_t yields

$$\eta_t = \left(1 + M_t\right)\varepsilon_t + \left(M_t - M_{t-1}\right)\left(\sum_{i=1}^{t-1} \theta^i \varepsilon_{t-i}\right), \tag{7}$$

which gives the forecast error implied by our proposed solution. For it to be a RE solution it must be: $E_{t-1}(\eta_t) = 0$. Thus, M_t must satisfy the following two conditions: (i) $E_{t-1}(M_t) = M_{t-1}$, $\forall t$, that is, M_t must be a martingale process; and (ii) $E_{t-1}[(1 + M_t)\varepsilon_t] = 0$, that is, M_t must be uncorrelated with ε_t.

Implications.—Our approach has a number of implications. First, from the point of view of the economic interpretation, our approach simply allows for agents to change the weights they assign to past shocks or past data in forming their expectations over time. The solution has the same form as (5) but with M being time-varying. Under indeterminacy, a given M_t selects one of the infinite stable RE paths, and then agents randomly shift from one to another.

Second, our approach allows to recover the minimum state variable solution extremely easily, simply by putting $M_t = 0$. This remains true in a more general model (see Section ID) and hence in the empirical implementation, where the data could choose the usual minimum state variable RE solution by supporting the estimate of $M_t = 0$.

Third, by introducing a multiplicative sunspot shock, rather than an additive one, our solution features time-varying parameters and stochastic volatility. From (1), (6), and (7), we can write our solution as

$$y_t = \alpha_t y_{t-1} - \alpha_t \varepsilon_{t-1} + \left(1 + M_t\right)\varepsilon_t \quad \text{if and only if} \quad M_{t-1} \neq 0, \tag{8}$$

with $\alpha_t = \theta M_t / M_{t-1}$.[11] The random variation of M_t causes both a different structural dependence of ξ_t (or y_t) from its lagged value and a different reaction of the system to the current shock. Drifting parameters naturally arise because agents change how they form their expectations formation process in each period, thus changing the equilibrium trajectory and thus the intrinsic dynamics of the model (i.e., α_t). Stochastic volatility arises because the reaction to the current fundamental shock

[11] Given the two conditions above (i.e., $E_{t-1}(M_t) = M_{t-1}$ and $E_{t-1}[(1 + M_t)\varepsilon_t] = 0$), it follows that (8) satisfies the original equation (1). Moreover, it can also be written as a dynamic formulation of Blanchard (1979): $y_t = -M_t y_t^B + (1 + M_t) y_t^F$. See Appendix A.

depends on the current realization of M_t through the term $(1 + M_t)\varepsilon_t$, possibly amplifying the effects of ε_t on the economy. These two important properties of our solution are evident in the expression for the forecast error (7) which is the sum of two terms. The first term is the interaction term between the innovation in M_t and the structural shock. The second term arises from the change in M_t which leads agents to respond differently to past shocks, putting the system on a different RE path. Our approach has the potential for an economic explanation of drifting parameters and stochastic volatility, without departing from the RE hypothesis. The empirical research (Cogley and Sargent 2005, Primiceri 2005, Justiniano and Primiceri 2008, and related literature) considers these as important features in explaining the dynamics of macroeconomic variables.

C. *Modeling Temporarily Unstable Paths*

Consider now the case where $\theta > 1$: the solution (6) is unstable. If explosiveness is allowed in the model, as could be the case for an asset pricing model or a model with only nominal variables, then the only restriction comes from the RE requirement that constrains M_t to be: (i) a martingale; and (ii) uncorrelated with the fundamental shock ε_t.

In the more general case where the model needs to satisfy the stability criterion, the only stable solution when M is a constant is the forward-looking solution. The stability criterion, however, relates to the asymptotic behavior of the solution. Hence, it does not rule out "bubbly," but temporary, trajectories, featuring unstable dynamics that are temporarily explosive, but stable in the long run. In this section, we show how our approach could consider this broader class of solutions, because of the flexibility provided by the key assumption of time variation in M_t. To allow for equilibrium paths that are temporarily unstable, but exhibit the same asymptotic behavior as the one selected by the stability criterion, we need a minimal relaxation of the requirement of RE. The great advantage, however, is that our approach then provides an econometric procedure to assess the empirical relevance of these temporarily unstable paths in the data.

On the one hand, the stability criterion usually induced by transversality conditions in optimization problems imposes a restriction on the *current expectation* of the asymptotic behavior of the solution, requiring that $\lim_{i\to\infty} E_t(y_{t+i}) = 0$. On the other hand, the RE requirement restricts the admissible time-varying processes for M_t to be a martingale. The martingale requirement implies that if $\theta > 1$ and $M_t \neq 0$, the economy is expected to remain forever on the unstable path selected by M_t, so that the transversality condition would be violated.[12] To allow for temporarily unstable paths in this case, we need to relax the martingale assumption and thus, the RE assumption. This deviation, however, could be minimal without practical implications when the

[12]Note that this would be true even if the instability is only temporary, that is even if the process for M_t will eventually converge to 0 with probability 1, as for example in the case of *asymptotically equal to a stationary processes* suggested by Gourieroux, Laffont, and Monfort (1982). In this case, $\lim_{i\to\infty} M_{t+i} = 0$ such that $\lim_{i\to\infty} \xi_{t+i} = \lim_{i\to\infty} E_{t+i}(y_{t+1+i}) = 0$. However, this does not imply: $\lim_{i\to\infty} E_t(y_{t+i}) = 0$, and the transversality condition will be violated, because even if the economy is *actually* converging with probability 1, the martingale assumption requires that agents *expect*, *conditionally on M_t*, the economy to explode.

model is taken to the data. In our empirical analysis in Section IV, we will assume that M_t follows

$$M_t = N_t \mathbf{1}_{\{\|y_{t-1}^B\| < \bar{U}\}}, \tag{9}$$

where N_t is a martingale and $\mathbf{1}_{\{\cdot\}}$ is the indicator function. The latter is equal to 1 if the norm of the backward-looking solution at time $t-1$ is less than a certain scalar $\bar{U} < \infty$, and is equal to 0 otherwise. Since $\theta > 1$, there exists a random date $\bar{T}$ in which $\| y_{\bar{T}-1}^B \|$ becomes greater than $\bar{U}$ and the indicator function will be equal to 0 for all $t > \bar{T}$. After this date, the stochastic process M_t will also be equal to 0, and the dynamics will coincide with the unique stable solution (i.e., the forward-looking solution). The indicator function in (9) is a random variable where the realization is known at time t, since it depends on the past value of the backward-looking solution. Then, in general (i.e., except in period $\bar{T}$) $E_t M_{t+1} = M_t$, which implies that the expected value of the forecast error is equal to 0. However, $\lim_{i\to\infty} E_t M_{t+i} = 0$, so that the transversality condition also holds. The presence of the indicator function is a simple expedient to capture the idea that the transversality condition is an asymptotic one (i.e., $\bar{T}$ is really large). In other words, the probability that the RE requirement may be violated in any near future is zero. However, it will be violated in a finite future, whatever far, depending on how large is $\bar{U}$. In order to allow for temporarily unstable paths, we implicitly assume that this possibility in the very distant future is disregarded by the agents.[13] In the practical implementation of the econometric procedure, one can choose a $\bar{U}$ so large that the martingale condition is satisfied by any estimate or simulated impulse response paths.

Implications.—There are three main differences between our approach and the standard sunspot literature. First, in the standard additive sunspot approach, M is constant and thus it can be different from 0 only if $\theta \leq 1$. Hence, sunspots are allowed only if $\theta \leq 1$, i.e., under indeterminacy. Our approach, instead, allows temporarily unstable paths, even if $\theta > 1$, because M_t varies with time, and stability can be imposed asymptotically on the process for M_t. Our proposed solution makes it possible to consider an infinite number of possible asymptotically stable solutions: the stability criterion is no longer enough to select a unique equilibrium even if $\theta > 1$. Hence, the model is not "determinate": *indeterminacy*, *in the sense of an infinite number of admissible paths*, *is the natural case*.

Second, as already stressed, multiplicative sunspots imply stochastic volatility and drifting parameters within a RE framework.

Last but not least, we provide a way of taking our framework to the data. If M is time-varying, theoretically it is harder to rule out equilibria that are only temporarily unstable. We simply acknowledge that the empirical analysis should allow for the possibility of temporary "walks along unstable paths." Hence, we are not taking a stand a priori on the possible equilibria in our estimation strategy, by allowing for all

[13] As expressed by Blanchard and Watson (1982, p. 8), the argument that rules out this possibility "may be pushing rationality too far. […] the probability […] may be so small, and the future time so far as to be considered nearly rationally irrelevant for market participants."

possible cases: determinacy ($\theta > 1; M_t = 0$), indeterminacy ($\theta \leq 1; M_t \neq 0$), and temporary instability ($\theta > 1; M_t \neq 0$). We then propose a methodology to let the data choose the preferred equilibria by estimating the latent process for M_t. At the very least, our approach could be seen as a test of RE, or of the transversality conditions, as normally applied. This is what we turn to next, explaining our proposed methodology in a more general context.

D. *Implementation: The General Solution*

To implement our proposed solution in the simple case, recursively define the solution using the backward-looking solution $\xi_t^B = -\theta \sum_{i=0}^{t-1} \theta^i \varepsilon_{t-i}$ and (6) to write

$$\xi_t^B = -\theta \sum_{i=0}^{t-1} \theta^i \varepsilon_{t-i} = -\theta \varepsilon_t + \theta \xi_{t-1}^B, \tag{10}$$

$$\xi_t = M_t \xi_t^B, \tag{11}$$

plus a given stochastic process for M_t. The solution for the expectation error in (7) is $\eta_t = (1 + M_{t-1})\varepsilon_t - (M_t - M_{t-1})\xi_t^B/\theta$.

The multivariate case is a relatively straightforward extension of the simple case. The online Appendix describes it in detail, following similar steps as above, involving: (i) parameterizing the system using $\mathbf{M}$ (now a matrix); (ii) introducing time variation in $\mathbf{M}$; and (iii) imposing stability. As in LS, we follow the approach of Sims (2002) and we write a general linear RE system as

$$\mathbf{y}_t = \boldsymbol{\Gamma}_1^* \mathbf{y}_{t-1} + \boldsymbol{\Psi}^* \boldsymbol{\varepsilon}_t + \boldsymbol{\Pi}^* \boldsymbol{\eta}_t, \tag{12}$$

where $\mathbf{y}_t$ is the vector of the n endogenous variables (including the expectations as in (2)), $\boldsymbol{\varepsilon}_t$ is the vector of the j exogenous fundamental shocks, and $\boldsymbol{\eta}_t$ is the vector of the $k \leq n$ RE forecast errors. As usual, we need to partition the system. Use Jordan decomposition to diagonalize $\boldsymbol{\Gamma}_1^* = \mathbf{J}\boldsymbol{\Lambda}\mathbf{J}^{-1}$ and define the vector of transformed variables $\tilde{\mathbf{y}}_t = \mathbf{J}^{-1}\mathbf{y}_t$. We depart from Sims (2002) and LS by partitioning the system according to the number of forward-looking variables/expectation errors, rather than the number of explosive eigenvalues. Then,

$$\tilde{\mathbf{y}}_t = \begin{bmatrix} \underset{((n-k)\times(n-k))}{\boldsymbol{\Lambda}_1} & \underset{((n-k)\times k)}{\mathbf{0}} \\ \underset{(k\times(n-k))}{\mathbf{0}} & \underset{(k\times k)}{\boldsymbol{\Lambda}_2} \end{bmatrix} \tilde{\mathbf{y}}_{t-1} + \begin{bmatrix} \underset{(n-k)\times n}{\mathbf{J}_{\mu 1}} \\ \underset{(k\times n)}{\mathbf{J}_{\mu 2}} \end{bmatrix} [\boldsymbol{\Psi}^* \boldsymbol{\varepsilon}_t + \boldsymbol{\Pi}^* \boldsymbol{\eta}_t]. \tag{13}$$

Let m be the number of explosive eigenvalues (i.e., such that $\lambda_i \geq 1$). As usual, we assume that the number of explosive eigenvalues is smaller or equal to the number of forecast errors, i.e., $m \leq k$. This case is the usual one in the literature where one can have either determinacy $(m = k)$ or indeterminacy $(m < k)$.[14] This means that in our partition, the first $(n-k)$ rows only contain stable eigenvalues, while the last k rows contain both $(k-m)$ stable and m unstable eigenvalues. Hence, we do not need to impose any stability condition on the first block of the system (13). However, we will do so on the second block of the system (13),

$$\tilde{\mathbf{y}}_{k,t} = \boldsymbol{\Lambda}_2 \tilde{\mathbf{y}}_{k,t-1} + \mathbf{J}_{\mu 2}[\boldsymbol{\Psi}^* \varepsilon_t + \boldsymbol{\Pi}^* \boldsymbol{\eta}_t], \tag{14}$$

where $\tilde{\mathbf{y}}_{k,t}$ denotes a vector of dimension k.

Define $\mathbf{M}_t$ as a $(k \times k)$ diagonal matrix whose elements on the principal diagonal are changing over time. Generalizing (10) and (11), the online Appendix shows that the solution to the system of disconnected difference equations (14) can be written recursively using the backward-looking variable $\tilde{\mathbf{y}}^B_{k,t}$ as

$$\tilde{\mathbf{y}}^B_{k,t} = \boldsymbol{\Lambda}_2 \tilde{\mathbf{y}}^B_{k,t-1} + \mathbf{J}_{\mu 2} \boldsymbol{\Psi}^* \varepsilon_t, \tag{15}$$

$$\tilde{\mathbf{y}}_{k,t} = -\mathbf{M}_t \tilde{\mathbf{y}}^B_{k,t} = -\mathbf{M}_t\left(\boldsymbol{\Lambda}_2 \tilde{\mathbf{y}}^B_{k,t-1} + \mathbf{J}_{\mu 2} \boldsymbol{\Psi}^* \varepsilon_t\right), \tag{16}$$

and the expectation error is equal to

$$\boldsymbol{\eta}_t = \left(\mathbf{J}_{\mu 2} \boldsymbol{\Pi}^*\right)^{-1}\left[-\left(\mathbf{I} + \mathbf{M}_{t-1}\right)\mathbf{J}_{\mu 2} \boldsymbol{\Psi}^* \varepsilon_t - \left(\mathbf{M}_t - \mathbf{M}_{t-1}\right)\tilde{\mathbf{y}}^B_{k,t}\right], \tag{17}$$

assuming that the $(k \times k)$ matrix $\mathbf{J}_{\mu 2}\boldsymbol{\Pi}^*$ is invertible.

Note that if all the elements $M_{i,t}$ on the principal diagonal of $\mathbf{M}_t$ are equal to 0 for all t, then $\tilde{\mathbf{y}}_{k,t} = 0$, so that we recover the forward-looking solution. As in the simple model, we impose stability by only allowing particular processes for $M_{i,t}$. In general, if we have k non-predetermined variables, the cardinality of the set of solutions is infinite to the power of k. However, the stability requirement imposes a restriction on the $M_{i,t}$ that correspond to eigenvalues of the system that are outside the unit circle. As in the simple example, we will restrict the processes for these $M_{i,t}$ to randomly converge to the stationary forward-looking solution in finite time, such that $M_{i,t} = 0, \forall t > \bar{T}$, where $\bar{T}$ is a random variable. The stability condition, instead, does not impose any restrictions on the stochastic processes governing the $(k-m)$ elements of $\mathbf{M}_t$ corresponding to stable eigenvalues.

The online Appendix shows that the solution for the original variables is

$$\begin{bmatrix} \mathbf{y}_t \\ \mathbf{y}^B_t \end{bmatrix} = \begin{bmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} \mathbf{G}^*_t \begin{bmatrix} \mathbf{J}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{t-1} \\ \mathbf{y}^B_{t-1} \end{bmatrix} + \begin{bmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} \mathbf{H}^*_t \varepsilon_t, \tag{18}$$

where

$$\mathbf{G}^*_t = \begin{bmatrix} \boldsymbol{\Lambda}_1 & \mathbf{0} & \mathbf{0} & -\mathbf{B}_{t,t-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{M}_t \boldsymbol{\Lambda}_2 \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}_2 \end{bmatrix}, \quad \mathbf{H}^*_t = \begin{bmatrix} \mathbf{A}_t \\ -\mathbf{M}_t \mathbf{J}_{\mu 2} \boldsymbol{\Psi}^* \\ \mathbf{J}_{\mu 1} \boldsymbol{\Psi}^* \\ \mathbf{J}_{\mu 2} \boldsymbol{\Psi}^* \end{bmatrix}, \tag{19}$$

[14] We rule out the case $m > k$, where the number of unstable eigenvalues, m, is bigger than the number of forward-looking variables, k. In this case, a stable solution does not exist.

and $\mathbf{A}_t$ is an $(n-k) \times l$ matrix and $\mathbf{B}_{t,t-1}$ is an $(n-k) \times k$ matrix equal to, respectively,

$$\mathbf{A}_t = \mathbf{J}_{\mu 1}\left[\boldsymbol{\Psi}^* - \boldsymbol{\Pi}^*\left(\mathbf{J}_{\mu 2}\boldsymbol{\Pi}^*\right)^{-1}\left(\mathbf{I} + \mathbf{M}_t\right)\mathbf{J}_{\mu 2}\boldsymbol{\Psi}^*\right], \tag{20}$$

$$\mathbf{B}_{t,t-1} = \mathbf{J}_{\mu 1}\boldsymbol{\Pi}^*\left(\mathbf{J}_{\mu 2}\boldsymbol{\Pi}^*\right)^{-1}\left(\mathbf{M}_t - \mathbf{M}_{t-1}\right)\boldsymbol{\Lambda}_2. \tag{21}$$

Note that $\mathbf{M}_t = \mathbf{0}, \forall t$, implies $\mathbf{A}_t = \mathbf{J}_{\mu 1}[\boldsymbol{\Psi}^* - \boldsymbol{\Pi}^*(\mathbf{J}_{\mu 2}\boldsymbol{\Pi}^*)^{-1}\mathbf{J}_{\mu 2}\boldsymbol{\Psi}^*]$ and $\mathbf{B}_{t,t-1} = \mathbf{J}_{\mu 1}\boldsymbol{\Pi}^*(\mathbf{J}_{\mu 2}\boldsymbol{\Pi}^*)^{-1}(\mathbf{M}_t - \mathbf{M}_{t-1})\boldsymbol{\Lambda}_2 = \mathbf{0}$, so that the system no *longer* depends on $\mathbf{y}^B_{k,t}$, and the solution coincides with

$$\mathbf{y}_t = \mathbf{J}\begin{bmatrix} \underset{((n-k)\times(n-k))}{\boldsymbol{\Lambda}_1} & \underset{((n-k)\times k)}{\mathbf{0}} \\ \underset{(k\times(n-k))}{\mathbf{0}} & \underset{(k\times k)}{\mathbf{0}} \end{bmatrix}\mathbf{J}^{-1}\mathbf{y}_{t-1} + \mathbf{J}\begin{bmatrix} \underset{(n-k)\times l}{\mathbf{J}_{\mu 1}\left[\boldsymbol{\Psi}^* - \boldsymbol{\Pi}^*\left(\mathbf{J}_{\mu 2}\boldsymbol{\Pi}^*\right)^{-1}\mathbf{J}_{\mu 2}\boldsymbol{\Psi}^*\right]} \\ \underset{(k\times l)}{\mathbf{0}} \end{bmatrix}\varepsilon_t, \tag{22}$$

which is the usual Blanchard-Kahn solution in case of a determinate system or the minimum state variable solution for an indeterminate system.

II. Econometric Strategy

In this section, we take a Bayesian approach to make inference regarding the parameters and the latent processes of a DSGE model when considering the class of solutions (18). The presence of stochastic volatility in the reduced form of the model, related to the time-varying characteristic of the latent state $\mathbf{M}_t$, leads to a non-Gaussian, analytically intractable likelihood function. In such situations, when estimating nonlinear or non-Gaussian DSGE models, a well-known approach proposed by Fernández-Villaverde and Rubio-Ramírez (2007) is performed in two steps. In the first step, the integrated likelihood of the parameters is approximated through the implementation of a particle filter. Then, in the second step, one uses the approximated likelihood within a Markov Chain Monte Carlo (MCMC) scheme that samples from the posterior distribution of the parameters.[15] We depart from this tradition and suggest the use of an efficient particle filtering strategy directly to approximate the joint posterior distribution of both the parameters and the latent state variables such as $\mathbf{M}_t$. In particular, we follow Chen, Petralia, and Lopes (2010), who introduced a particle strategy for DSGE models that

[15] Recent papers in which this approach is implemented are Carvalho et al. (2017) and Gust et al. (2017).

combines the Sequential Monte Carlo (SMC) algorithms of Liu and West (2001) and Carvalho et al. (2010).

In what follows, we first show how to write the solution (18) in a convenient state space form, and then we illustrate and discuss the main aspects of our particle filtering strategy, referring to the online Appendix for an in-depth description. Finally, we motivate our choice by comparing it with possible alternatives.

A. *The State Space Form*

In the class of solutions (18), we need to keep track of the pure backward-looking solution $\mathbf{y}_t^B$. This vector contains both endogenous and exogenous variables, and since the latter do not depend on the agent's expectations, their evolution will be the same as the analogous variables in $\mathbf{y}_t$. When estimating the model, it is convenient to rewrite the solution (18) so that the exogenous variables appear only once, that is, using a compact notation,

$$\mathbf{l}_t = \mathbf{G}_t \mathbf{l}_{t-1} + \mathbf{H}_t \boldsymbol{\varepsilon}_t \tag{23}$$

with

$$\mathbf{l}_t = \begin{bmatrix} \mathbf{y}_t \\ \mathbf{y}_t^{B,E} \end{bmatrix},$$

where $\mathbf{y}_t^{B,E}$ is a vector with the endogenous variables in the pure backward-looking solution. The matrices $\mathbf{G}_t$ and $\mathbf{H}_t$ are appropriate transformations of the matrices in (18).

At each time t, we observe a vector of data, which will be simply denoted by $\mathbf{D}_t$. Then, the solution of model (12) has the following state space representation:

$$\begin{cases} \mathbf{D}_t = \mathbf{c} + \mathbf{F}\mathbf{l}_t + \mathbf{v}_t & \mathbf{v}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_v) \\ \mathbf{l}_t = \mathbf{G}_t \mathbf{l}_{t-1} + \mathbf{H}_t \boldsymbol{\varepsilon}_t & \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon) \end{cases}, \tag{24}$$

where $\mathbf{c}$ is a vector of constants, $\mathbf{F}$ is a matrix with appropriate dimensions, and $\mathbf{v}_t$ is a vector of measurement errors.

B. *The Particle Filter*

The parameters in $\mathbf{c}$, $\mathbf{F}$, $\mathbf{G}_t$, $\mathbf{H}_t$, $\boldsymbol{\Sigma}_v$, $\boldsymbol{\Sigma}_\varepsilon$ are collected in the vector $\boldsymbol{\theta}$. As already mentioned, let $\mathbf{D}_t$ be the vector of observed data at time t, and $\mathbf{D}_{m:n}$ denote the set of all observations from $t = m$ to $t = n$ for $m \leq n$. We perform posterior Bayesian inference via Monte Carlo methods to approximate the joint posterior distribution of parameters and latent states of the model by a sufficiently large number of sample draws, or particles. More precisely, our econometric strategy is based on Bayesian sequential learning via particle filtering: at time $t-1$, we start with a particle set $\left\{(\mathbf{l}_{t-1}, \mathbf{M}_{t-1}, \boldsymbol{\theta})^{(i)}\right\}_{i=1}^N$ and associated particle weights $\left\{w_{t-1}^{(i)}\right\}_{i=1}^N$ that summarize, via Monte Carlo, the full joint posterior of states $(\mathbf{l}_{t-1}, \mathbf{M}_{t-1})$ and parameters

$\boldsymbol{\theta}$, i.e., $p(\mathbf{l}_{t-1}, \mathbf{M}_{t-1}, \boldsymbol{\theta} | \mathbf{D}_{1:t-1})$. The goal is to arrive at the end of time t with a similar set of particles $\{(\mathbf{l}_t, \mathbf{M}_t, \boldsymbol{\theta})^{(i)}\}_{i=1}^{N}$ and weights $\{w_t^{(i)}\}_{i=1}^{N}$ representing the joint posterior distribution

$$p(\mathbf{l}_t, \mathbf{M}_t, \boldsymbol{\theta} | \mathbf{D}_{1:t}). \tag{25}$$

Loosely speaking, a particle filter is a sampling importance resampling (SIR) scheme implemented iteratively over time: since it is not possible to extract the particles directly from the posterior distribution, we draw from another distribution, say $q(\mathbf{l}_t, \mathbf{M}_t, \boldsymbol{\theta} | \mathbf{D}_{1:t})$, commonly referred to as an importance distribution, and we approximate the target density (25), assigning appropriate weights to each particle. The reweighting of a particle from the importance distribution gives that particle the "status" of an actual draw from the posterior distribution.[16] If the support of the target $p(\cdot)$ is included in the support of proposal $q(\cdot)$, then for each particle i, the appropriate weight is given by

$$w_t^{(i)} = \frac{p\left(\mathbf{l}_t^{(i)}, \mathbf{M}_t^{(i)}, \boldsymbol{\theta}^{(i)} | \mathbf{D}_{1:t}\right)}{q\left(\mathbf{l}_t^{(i)}, \mathbf{M}_t^{(i)}, \boldsymbol{\theta}^{(i)} | \mathbf{D}_{1:t}\right)}. \tag{26}$$

The essence of a particle filter ultimately depends on the design of the importance distribution $q(\mathbf{l}_t, \mathbf{M}_t, \boldsymbol{\theta} | \mathbf{D}_{1:t})$. Our choice is tailored on the peculiar aspects related to the set of solutions we analyze through equation (18).

The most important peculiarity is that, conditionally on $\mathbf{M}_t$, the state space form (24) is linear and Gaussian, which implies that, given a set of particles for $\mathbf{M}_t$, both the predictive likelihood and the full conditional distribution of the other latent states are analytically available through the standard Kalman filter recursion. This practice increases the efficiency of our particle filter through analytical integration, as it follows from the Rao-Blackwell theorem (see Lopes et al. 2011 for further details).

The posterior distribution of the parameters can be updated sequentially combining two different methodologies for parameter learning. In particular, it is useful to divide the parameters into two sets: one with the variances and covariances of the exogenous disturbances, and one with all the other structural parameters. For the variances and covariances, we assume that the prior distributions are Inverse Gamma or Inverse Wishart. Then, we are able to characterize the posterior distribution analytically (up to a normalizing constant), using sufficient statistics computed as functions of the data and the latent processes of the model. This is the idea of the Particle Learning approach introduced by Carvalho et al. (2010). The posterior distribution of the other parameters is, in general, not available analytically. It can be approximated using mixtures of Gaussian densities, as in the flexible and general particle filtering with parameter learning algorithm proposed by Liu and West (2001).[17]

[16] See, for instance, Cappe, Godsill, and Moulines (2007) and Lopes and Tsay (2011) (and the references therein) for a review of particle methods for Bayesian inference.

[17] This methodology builds on the resample propagation scheme of the Auxiliary Particle Filter proposed by Pitt and Shephard (1999).

The use of SMC methods to approximate the posterior distribution of the parameters of a DSGE model is not very common in the literature: Creal (2007); Chen, Petralia, and Lopes (2010); and Herbst and Schorfheide (2014) are exceptions to the usual practice based on MCMC. Nevertheless, the particle filtering approach is the most suited to our framework, given the peculiarity of the class of solutions we are considering. First of all, a time-varying $\mathbf{M}_t$ makes the model non-Gaussian, and it is well known that in this context, MCMC methods may have serious limitations due to the high time-dependence of the latent variables. In particular, the convergence of the Markov chain generated through MCMC to the posterior distribution can be very slow and difficult to achieve.

Moreover, within the class of solutions we analyze, the case corresponding to $\mathbf{M}_t = \mathbf{0}$ implies a completely different reduced form compared to all the other cases: as shown in the previous section, this is the minimum state variable solution, characterized by a simpler lag structure. The shape of the likelihood function, conditional on $\mathbf{M}_t = \mathbf{0}$, may be substantially different from that under $\mathbf{M}_t \neq \mathbf{0}$, making the MCMC-based inference more complicated: the Markov chain accurately explores the parameter space around the mode of the distribution but, in practice, it is less able to approximate the posterior when the latter is not well-shaped, or has multiple modes. This is not a mere technicality, because $\mathbf{M}_t = \mathbf{0}$ is a very important case: it is the unique stable solution when the Blanchard-Kahn conditions are satisfied, and it characterizes the dynamics implied by the model under the case that the literature labels determinacy. In order to deal with this issue, LS run the posterior sampler over determinacy and indeterminacy separately. Our suggestion is to estimate the model considering all the relevant cases simultaneously using particle filters instead of MCMC. In general, SMC methods are more appropriate when the posterior distribution displays irregular patterns. We show the ability of our econometric strategy to deal with this specific problem in the empirical application described in the next section.

Another advantage of particle filters is computational: the use of multi-core processors makes it possible to increase the speed and the accuracy of the estimation through parallel computing. The gains one can achieve are substantial for SMC, while they are limited for MCMC even if parallelization is implemented in an efficient way, as in the prefetching approach described by Strid (2010).[18]

There are two approaches in the SMC literature to estimate the "static" parameters of a model. One uses all data available in the sample in each iteration of the SMC to approximate a sequence of distributions, starting from a very simple case (i.e., the prior distribution), and ending with the posterior distribution of interest. We follow a second alternative: we construct particle approximations to the posterior distribution augmenting, at each iteration, the sample data we use. In this case, each step of the SMC corresponds to an additional observation, as if new data become available sequentially. We prefer this second technique because it gives us the possibility to study how the inference on the unknowns evolves over time. In the empirical application below, we show how this "learning" perspective unveils additional information on the role of sunspots and temporarily unstable paths in describing

[18] For a discussion, see also Herbst and Schorfheide (2014).

the data. Moreover, this approach makes it simpler for us to deal with the filtering problem related to the estimation of $\mathbf{M}_t$.

III. Multiplicative Sunspots and Unstable Paths at Work: The Great Inflation and the New Keynesian Model

We apply our new methodology to inflation dynamics. We employ the same New Keynesian model (and notation) as in the seminal paper by LS, which is the natural paper to compare the results of our methodology:

$$x_t = E_t(x_{t+1}) - \tau\big(R_t - E_t(\pi_{t+1})\big) + g_t, \tag{27}$$

$$\pi_t = \beta E_t(\pi_{t+1}) + \kappa(x_t - z_t), \tag{28}$$

$$R_t = \rho_R R_{t-1} + (1 - \rho_R)\big(\psi_1 \pi_t + \psi_2(x_t - z_t)\big) + \varepsilon_{R,t}, \tag{29}$$

where x is output, π is inflation, and R is the nominal interest rate. Values of π and R are expressed in deviation from the steady state, and x in deviation from the steady-state trend path. The model admits three shocks: (i) a demand shock, g, that can be interpreted as a time-varying government spending shock or a preference shock; (ii) a shock to the marginal costs of production, z; and (iii) a monetary policy shock, ε_R. The first equation is the New Keynesian IS curve (NKIS), which relates the dynamics of the output x_t to the real interest rate. The New Keynesian Phillips curve (NKPC), (28), describes the dynamics of the inflation rate π_t. A standard Taylor rule, (29), where the central bank reacts to the deviations of inflation and the output gap from their targets plus inertia, closes the model. As in LS, we assume autocorrelation in the shocks in the NKIS and in the NKPC,

$$g_t = \rho_g g_{t-1} + \varepsilon_{g,t}; \qquad z_t = \rho_z z_{t-1} + \varepsilon_{z,t}, \tag{30}$$

and a nonzero correlation, ρ_{gz}, between $\varepsilon_{g,t}$ and $\varepsilon_{z,t}$. Variables σ_g, σ_z, and σ_R, denote the standard deviations of the zero-mean innovations $\varepsilon_{g,t}, \varepsilon_{z,t}$, and $\varepsilon_{R,t}$, respectively.

The parameters of the model are also standard: $\beta \in (0,1)$ is the households' subjective discount factor, τ is the elasticity of intertemporal substitution in consumption, κ is the slope of the NKPC, which ultimately depends on the degree of nominal price stickiness and the labor supply elasticity, ρ_R is the inertial parameter in the Taylor rule while ψ_1 and ψ_2 measure the response of the nominal interest rate to inflation and output, respectively.

The model has five variables: three predetermined (R_t, g_t, and z_t) and two non-predetermined (x_t, π_t). Then, the matrix $\mathbf{M}_t$ has dimension 2. We also know that among the five eigenvalues of the dynamic system, three of them are inside the unit circle (because ρ_g, ρ_z, and ρ_R are less than 1 in absolute value), and one is always outside the unit circle (for sensible values of the parameters, see Bullard and Mitra 2002). The remaining eigenvalue can be inside or outside the unit circle, depending on the following condition (i.e., the Taylor principle):

$$\psi_1 > 1 - \frac{1-\beta}{\kappa}\psi_2. \tag{31}$$

The literature usually imposes the stability criterion to select valid equilibria and thus, it distinguishes two possible cases. If (31) holds, the model has two eigenvalues greater than 1 in absolute value. This is the determinacy case: there is a unique *stable* RE equilibrium, i.e., the forward-looking one. Otherwise, if (31) does not hold, there will be an infinite number of *stable* RE equilibria and this case is normally labeled indeterminacy.

However, note that in both cases, there is an infinite number of *unstable* RE equilibria associated with the unstable eigenvalue, which the literature usually does not consider because of the stability criterion. Our framework, instead, imposes stability in the long run, but it admits temporary walks on these unstable paths by allowing for time variation in the way agents are setting their expectations. We can test the validity of our framework in a particular sample comparing the relative performance of the New Keynesian model, under different hypotheses on the set of admissible solutions. Hence, we distinguish two cases: one in which we impose the stability criterion, so that the economy needs to be on a stable trajectory at any point in time; and one in which we also consider temporarily unstable solutions. The aim is to let the data speak about their preferred assumption.

Model M_S (Stable Solutions).—We exclude unstable solutions. We label this case model M_S, and the matrix $\mathbf{M}_t$ is

$$(32) \qquad \mathbf{M}_t = \begin{bmatrix} M_{1,t} & 0 \\ 0 & 0 \end{bmatrix},$$

$$(33) \qquad M_{1,t} = \begin{cases} 0 & \text{if } \psi_1 > 1 - \dfrac{1-\beta}{\kappa}\psi_2 \\ M_{1,t-1} + \zeta_t, \quad \zeta_t \sim N\big(0, \sigma_\zeta^2\big) & \text{otherwise.} \end{cases}$$

The south east element in $\mathbf{M}_t$ is imposed to be zero because there is always one explosive eigenvalue. For the first element, $M_{1,t}$, instead, we distinguish the two cases usually considered in the literature. In the case of determinacy, when the Taylor principle is satisfied, the corresponding eigenvalue is outside the unit circle and we need to select the forward-looking solution ($M_{1,t} = 0$) for all t. In the case of indeterminacy, when the Taylor principle is not satisfied, there is an infinite number of stable solutions and the stability condition poses no restrictions on $M_{1,t}$. Element $M_{1,t}$ needs to be a martingale, so we will assume that it follows a random walk driven by a sunspot shock.

Model M_U (Temporarily Unstable Solutions).—In this case, we define the matrix $\mathbf{M}_t$ as

$$(34) \qquad \mathbf{M}_t = M_{1,t}\mathbf{I},$$

where $\mathbf{I}$ is the identity matrix, and we thus assume that the elements in the main diagonal of $\mathbf{M}_t$ are the same. We do not impose (as we did for the stable model M_S)

that $M_{1,t}$ is equal to 0 when the Taylor principle is satisfied. Note that $M_{1,t}$ is also associated with the unstable eigenvalue, which is always outside the unit circle. Thus, the dynamics are unstable when $M_{1,t} \neq 0$, irrespective of the Taylor principle being satisfied or not. However, this model still has the possibility to select stable dynamics, because it admits a unique stable solution that corresponds to $M_{1,t} = 0$. Hence, expectations (i.e., $M_{1,t}$) determine the presence of unstable solutions, not the policy parameters.

The process $M_{1,t}$ is as in (9), where we assume that the martingale process N_t multiplies the indicator function $\mathbf{1}_{\{\|\mathbf{y}_{t-1}^B\| < \bar{U}\}}$ for the transversality condition to hold. We assume the following process for N_t:

$$N_t = \begin{cases} N_{t-1}/\gamma + \zeta_t & \text{with probability } \gamma \\ 0 & \text{with probability } 1 - \gamma, \end{cases} \tag{35}$$

where $\zeta_t \sim N(0, \sigma_\zeta^2)$. Without the shock ζ_t, the process for N_t would have 0 as absorbing state so that the stable solution would be forever active once selected. The presence of the shock makes it possible for an economy that is on the stable path to jump on an explosive trajectory. This "walk on unstable paths" will only be temporary, either because with probability $(1 - \gamma)$ the solution will become stable again or because of the indicator. As discussed in Section IC, under this hypothesis, RE may be violated in the future time period $\bar{T}$. However, if $\bar{U}$ is very big, the probability that this can happen in the near future is approximately zero, and in order to allow for temporarily unstable paths, we need to assume that it is disregarded by the agents. When we estimate the model under M_U, we choose $\bar{U} = 10^{300}$, and this ensures that the indicator function is equal to 1 for all the draws and the times considered in our sample.

IV. Empirical Results

A. *Data and Subsamples*

To compare our results with the seminal work by LS, we estimate the New Keynesian model (27)–(30) on the same quarterly postwar data for inflation, output, and nominal interest rate used by LS, as available from the *AER* website. Inflation and interest rates are annualized, and the HP filter is used to get a measure of the output gap.[19]

Figure 1 plots the inflation series. From the mid-1960s until the end of the 1970s, the United States experienced a period of price instability, also known as the Great Inflation. Then, the Volcker disinflation took place and inflation became low, as did the volatility of prices and other macroeconomic variables. In contrast to the previous period, these times are known as the Great Moderation. One popular explanation of this change through the lens of the New Keynesian model (e.g., Clarida, Galí, and Gertler 2000) ascribes it to the shift from a passive (i.e., (31) not satisfied)

[19] As from LS (footnote 9, p. 202): (i) output is log real per capita GDP HP detrended over the period 1955:I to 1998:IV; (ii) inflation is the annualized percentage change of CPI-U; and (iii) the nominal interest rate is the average Federal Funds Rate in percent.

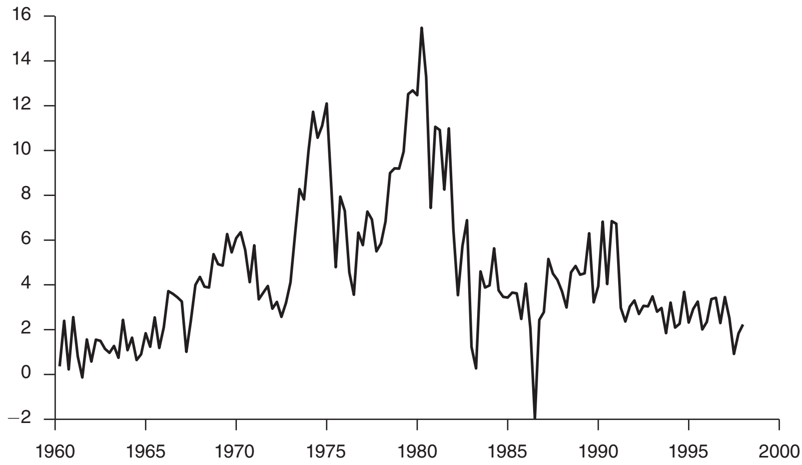

Figure 1. CPI Inflation, Quarterly Data, 1960:I–1997:IV

to an active (i.e., (31) satisfied) monetary policy. This interpretation excludes a priori unstable paths, even though inflation reached 15 percent. Here, we want to answer the following question: would the data prefer an explanation of the Great Inflation based on a stable system plus sunspot shocks, as in LS, or one based on unstable dynamics? Again we closely follow LS in considering two subsamples: the pre-Volcker period, from 1960:I to 1979:II, and a post-1982 period from 1982:IV to 1997:IV.[20]

B. *Priors*

Table 1 collects the prior distributions for the parameters, again in accordance with LS. Differently from LS, we specify the prior for the variance covariance matrix of the shock $\varepsilon_{g,t}$ and $\varepsilon_{z,t}$ as an Inverse Wishart with scale matrix and degrees of freedom as in Table 1. The Inverse Wishart prior allows us to update the posterior of the parameters using sufficient statistics, as in the Particle Learning approach described above. This is a big advantage in terms of the efficiency of our particle filter. Our choice is, however, very similar to the one of LS in terms of mean and variances of the three parameters involved (σ_g, σ_z, and ρ_{gz}).

The standard deviation of our sunspot shock is distributed as an Inverse Gamma with mean equal to 0.1 and standard deviation equal to 0.05. This value is lower than that in LS because our sunspot shock enters in a multiplicative way.

Under model M_U, we estimate the probability that the economy is on a temporarily unstable path, that is the parameter γ, for which the prior density is a Beta distribution with mean 0.8 and standard deviation 0.15. A mean of 0.8 implies that when an unstable trajectory is selected, this temporary situation is expected to last 5 quarters. Since this is a new parameter, we try different values both for the mean and for the standard deviation: the results are robust, and some are discussed below.

[20]As in LS, we exclude the Volcker disinflation period where monetary policy is characterized by nonborrowed-reserve targeting rather than by an interest rate rule.

Table 1—Prior Distributions

Parameter	Density	Mean	SD
ψ_1	Gamma	1.1	0.5
ψ_2	Gamma	0.25	0.15
ρ_R	Beta	0.5	0.2
π^*	Gamma	4	2
r^*	Gamma	2	1
κ	Gamma	0.5	0.2
τ^{-1}	Gamma	2	0.5
ρ_g	Beta	0.7	0.1
ρ_z	Beta	0.7	0.1
γ	Beta	0.8	0.15
σ_R	Inverse Gamma	0.31	0.16
σ_ζ	Inverse Gamma	0.1	0.05
Variance covariance	Density	Scale	Degrees of freedom
Σ_{gz}	Inverse Wishart	$5\begin{bmatrix} 0.38^2 & 0 \\ 0 & 1 \end{bmatrix}$	8

Finally, the process $M_{1,t}$ at $t = 0$ is supposed to be Normally distributed, with mean 0, and standard deviation 0.1, as the prior of the standard deviation of the sunspot shock.

C. *Estimation Results*

Table 2 reports the estimates of the parameters in the two subsamples. For each subsample, Table 2 shows the estimates for both the stable (M_S) and the unstable (M_U) model and, for comparison, the correspondent estimates in LS (see Table 3, p. 206).

Great Inflation Subsample

The Model under Stability (M_S).—Under stability, our methodology allows to consider contemporaneously determinate and indeterminate equilibria, letting the data choose which one to select during the estimation. Table 2 shows that under stability (model M_S), our methodology recovers results very similar to LS. This is particularly true for the crucial policy rule parameters. Figure 2 displays our prior and posterior distributions and the 90 percent intervals in LS for these parameters. It shows that our estimation method yields posterior distributions, which are very close and statistically indistinguishable from those in LS.[21] We interpret this finding as corroborating our estimation methodology.

In accordance with most of the literature, our method also points to indeterminacy as the most plausible explanation for the Great Inflation period when temporarily unstable paths are excluded. It suggests that the Fed did not respect the Taylor principle, and thus movements in inflation (and output) were due to shifts in

[21] The 90 percent intervals do not overlap only for the slope of the Phillips curve, κ, and for the inverse of the elasticity of intertemporal substitution, τ^{-1}.

Table 2—Posterior Estimates

	Pre-Volcker 1960:I–1979:II			Post-82 1982:IV–1997:IV			Sample: 1960:I–1997:IV
Parameter	M_S	M_U	LS	M_S	M_U	LS	Stochastic Volatility
ψ_1	0.80 [0.66 0.92]	0.76 [0.61 0.91]	0.77 [0.64 0.91]	2.18 [1.53 3.07]	2.32 [1.44 3.58]	2.19 [1.38 2.99]	1.25 [1.12 1.39]
ψ_2	0.16 [0.11 0.20]	0.20 [0.10 0.34]	0.17 [0.04 0.30]	0.17 [0.06 0.38]	0.23 [0.07 0.66]	0.30 [0.07 0.51]	0.21 [0.11 0.41]
ρ_R	0.68 [0.65 0.71]	0.60 [0.53 0.68]	0.60 [0.42 0.78]	0.86 [0.81 0.90]	0.85 [0.80 0.89]	0.84 [0.79 0.89]	0.75 [0.70 0.80]
π^*	1.90 [1.62 2.25]	1.73 [1.31 2.47]	4.28 [2.21 6.21]	3.28 [2.73 3.82]	3.25 [2.82 3.73]	3.43 [2.84 3.99]	2.88 [2.41 3.41]
r^*	1.40 [1.29 1.58]	1.22 [0.93 1.73]	1.13 [0.63 1.62]	2.81 [2.17 3.59]	3.00 [2.40 3.69]	3.01 [2.21 3.80]	2.11 [1.69 2.59]
κ	0.14 [0.10 0.18]	0.10 [0.07 0.14]	0.77 [0.39 1.12]	0.30 [0.22 0.39]	0.48 [0.30 0.81]	0.58 [0.27 0.89]	0.36 [0.26 0.51]
τ^{-1}	3.41 [2.65 4.51]	3.02 [2.46 3.74]	1.45 [0.85 2.05]	2.56 [1.97 3.37]	1.69 [1.20 2.45]	1.86 [1.04 2.64]	2.07 [1.54 2.78]
ρ_g	0.64 [0.59 0.69]	0.68 [0.63 0.74]	0.68 [0.54 0.81]	0.76 [0.69 0.81]	0.78 [0.71 0.84]	0.83 [0.77 0.89]	0.80 [0.77 0.83]
ρ_z	0.76 [0.72 0.79]	0.75 [0.67 0.81]	0.82 [0.72 0.92]	0.72 [0.59 0.83]	0.73 [0.61 0.82]	0.85 [0.77 0.93]	0.81 [0.75 0.85]
ρ_{gz}	0.26 [0.19 0.37]	0.16 [0.06 0.25]	0.14 [−0.4 0.71]	0.03 [0.00 0.07]	0.04 [0.01 0.08]	0.36 [0.06 0.67]	0.14 [0.08 0.19]
γ	—	0.96 [0.85 0.99]	—	—	0.04 [0.01 0.12]	—	—
σ_R	0.22 [0.20 0.26]	0.19 [0.16 0.22]	0.23 [0.19 0.27]	0.16 [0.13 0.19]	0.16 [0.13 0.2]	0.18 [0.14 0.21]	$\delta_R = 0.11$ [0.09 0.13]
σ_g	0.35 [0.30 0.40]	0.31 [0.24 0.37]	0.27 [0.17 0.36]	0.20 [0.16 0.25]	0.21 [0.17 0.26]	0.18 [0.14 0.23]	$\delta_g = 0.012$ [0.010 0.014]
σ_z	1.11 [0.97 1.29]	1.00 [0.85 1.31]	1.13 [0.95 1.30]	0.67 [0.55 0.87]	0.63 [0.53 0.76]	0.64 [0.52 0.76]	$\delta_z = 0.02$ [0.016 0.026]
σ_ς	0.08 [0.07 0.10]	0.06 [0.05 0.08]	0.20 [0.12 0.27]	—	—	—	—

Note: 90 percent credibility interval in brackets.

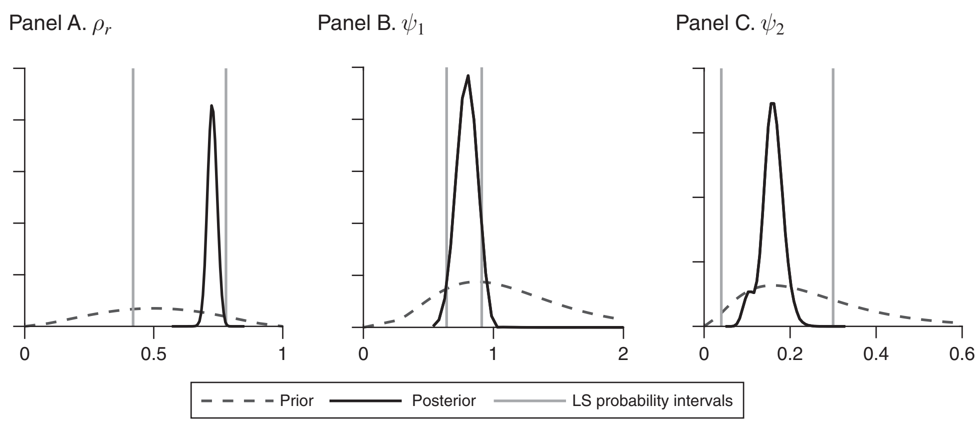

Figure 2. M_S: Comparison between the Posterior Distributions of the Policy Parameters and the Probability Intervals of LS

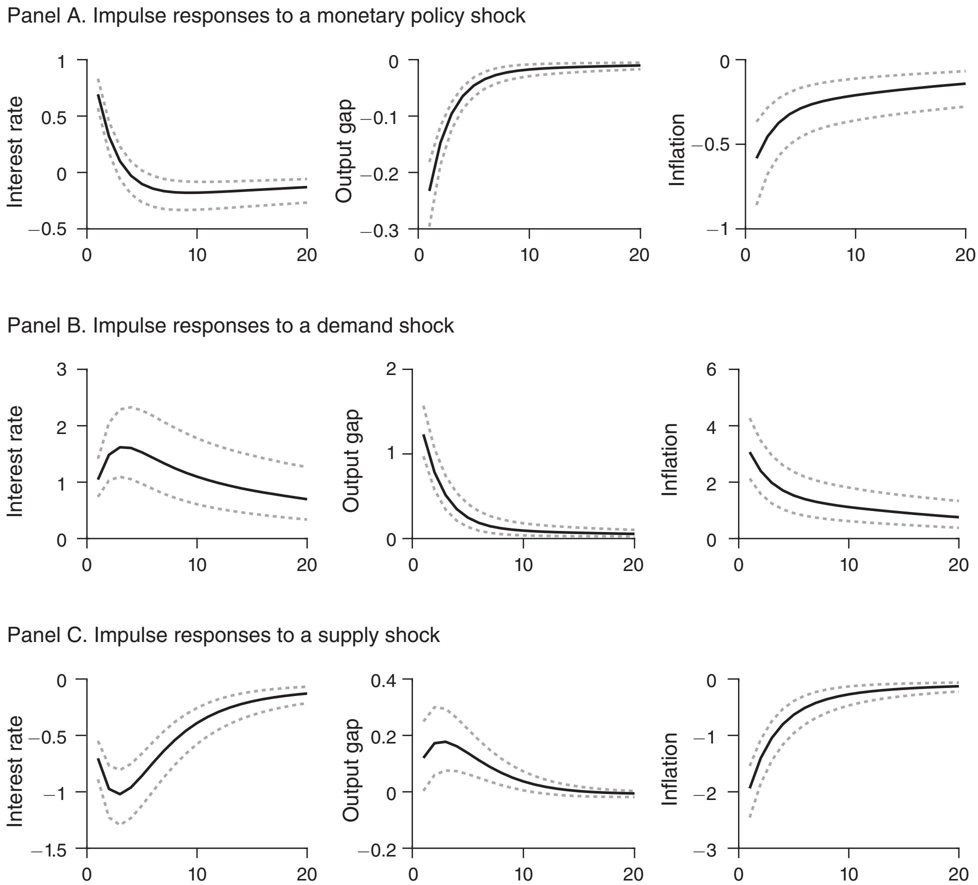

Figure 3. Generalized Impulse Response Function in the M_S Model Computed under the Posterior Distribution of $M_{1,t}$ in 1979:II

expectations due to sunspot shocks. The estimated standard deviation of the sunspot shock for M_S is lower than the one estimated by LS. However, these standard deviations are not really comparable because our sunspot shock is a multiplicative one that interacts with and amplifies the structural shocks, rather than an additive one as in LS's approach.

Figure 3 displays the transmission mechanism of the structural shocks. It shows the generalized impulse response functions (GIRFs) and the 90 percent intervals of R, x, and π to the monetary policy shock in the first row, to the demand shock in the second row and to the supply shock in the third row.[22] These GIRFs are very similar in shape to those of a determinate equilibrium, and to the IRFs in LS under their prior 2. Note that the technology shock is the only one that moves output and inflation in opposite directions, as required to explain the stagflation episode during the last part of the Great Inflation period.

[22] The GIRFs show the impulse responses to one standard deviation of each shock, and are computed conditioning on the distribution of $M_{1,t}$ at the end of the first subsample, that is 1979:II. The uncertainty of the GIRFs, summarized by the 90 percent probability interval, also reflects the posterior distribution of the parameters.

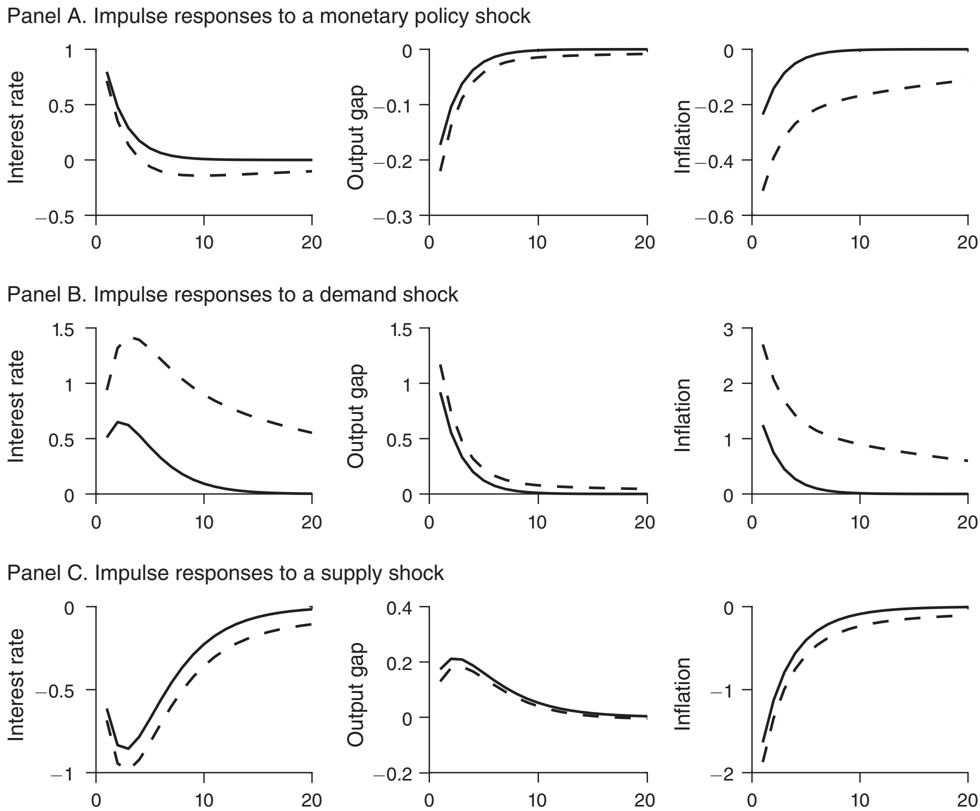

Figure 4. Generalized Impulse Response Function in the M_S Model

Note: Solid line: $M_{1,t} = 0$, dashed line: $M_{1,t} = 0.49$

Recall that the nonlinear multiplicative sunspot shock affects the model only in the presence of a structural shock. Hence, to understand how the sunspot shock affects the transmission mechanism of our model, we plot in Figure 4 the GIRFs for two different values of $M_{1,t}$: the solid line corresponds to the pure forward-looking solution $(M_{1,t} = 0)$, and it shows the impulse response functions estimated under the stable model before 1974:IV; the dashed line refers to $M_{1,t} = 0.49$, that is the value estimated in 1974:IV. The sunspot shock amplifies the effects of the structural shock. While it does not qualitatively change the response of the variables, it acts as a stochastic volatility shifter.

One of the most interesting aspects of our methodology is the estimated path for $M_{1,t}$ that measures how much expectations deviate from the standard forward-looking RE solution. Recall that expectations are selecting the forward-looking solution only when $M_{1,t} = 0$. Figure 5 shows the estimated path for $M_{1,t}$ in the case of M_S, and the corresponding sequential estimate of the policy parameter ψ_1. Figure 5 clearly depicts the challenge faced by the New Keynesian model in this subsample: to simultaneously explain the stable output and inflation paths in the first part of the subsample and the stagflation in the second part of the subsample. Until the first oil shock, the estimate of $M_{1,t}$ points toward expectations aligned on the "standard" forward-looking solution and, correspondingly, ψ_1 is estimated to satisfy the

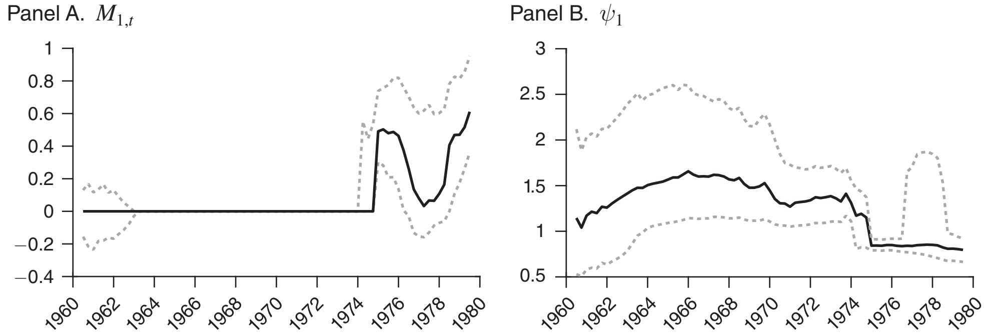

FIGURE 5. ESTIMATED PATH OF $M_{1,t}$ FOR THE STABLE MODEL M_S IN THE GREAT INFLATION SUBSAMPLE (PANEL A); SEQUENTIAL INFERENCE ON THE PARAMETER ψ_1 (PANEL B)

Taylor principle. Until that point, the data would favor a determinate stable model. However, such a model has hard times explaining the data in the second part of the subsample. Then, the data switch to favor the only alternative model available under stability: an indeterminate model with sunspot shocks. The extra degree of freedom provided by the sunspot makes the data choose the indeterminate model both in LS's and in our estimation. Indeed, $M_{1,t}$ drifts away from zero, when inflation starts to grow in the data.

The literature suggested another plausible possibility to make a stable determinate model able to explain such behavior in the data: a stochastic volatility model where the standard deviation of technology shocks increases in the second part of the subsample (e.g., Justiniano and Primiceri 2008). We will consider such a model in Section V. Our multiplicative sunspot shock yields a similar effect, as explained above, but the sunspot shock only occurs if the model is indeterminate under M_S.

The Model under Instability (M_U).—The model M_U makes the data consider also temporarily unstable paths. The point estimates in Table 2 are very similar between the two models M_S and M_U. However, by now, it should be clear to the reader that the two cases imply very different dynamics. The M_U case does not imply indeterminacy as is usually intended in the literature, that is, an infinite number of *stable* RE trajectories. It does imply another sort of indeterminacy, in the sense that we let the data choose among an infinite number of *unstable*, *but temporary* trajectories, irrespective of whether the Taylor principle is satisfied. As explained in Section III, whatever the value of ψ_1, there is always an unstable eigenvalue. However, we do not force the model to the forward-looking solution with respect to this unstable eigenvalue in the M_U case. It follows that despite the parameter estimates being very similar between the M_S and M_U cases, M_U gives a completely different interpretation about the instability of that period. The dynamics of M_U are structurally unstable, independently of the Taylor rule parameters.

Figure 6 shows the GIRFs in this case. Once more, a supply shock generates stagflation. Most importantly, however, stagflation could now also be generated by a monetary policy shock. In particular, a contractionary monetary policy shock can be inflationary: inflation drops on impact but then starts rising and it is above steady

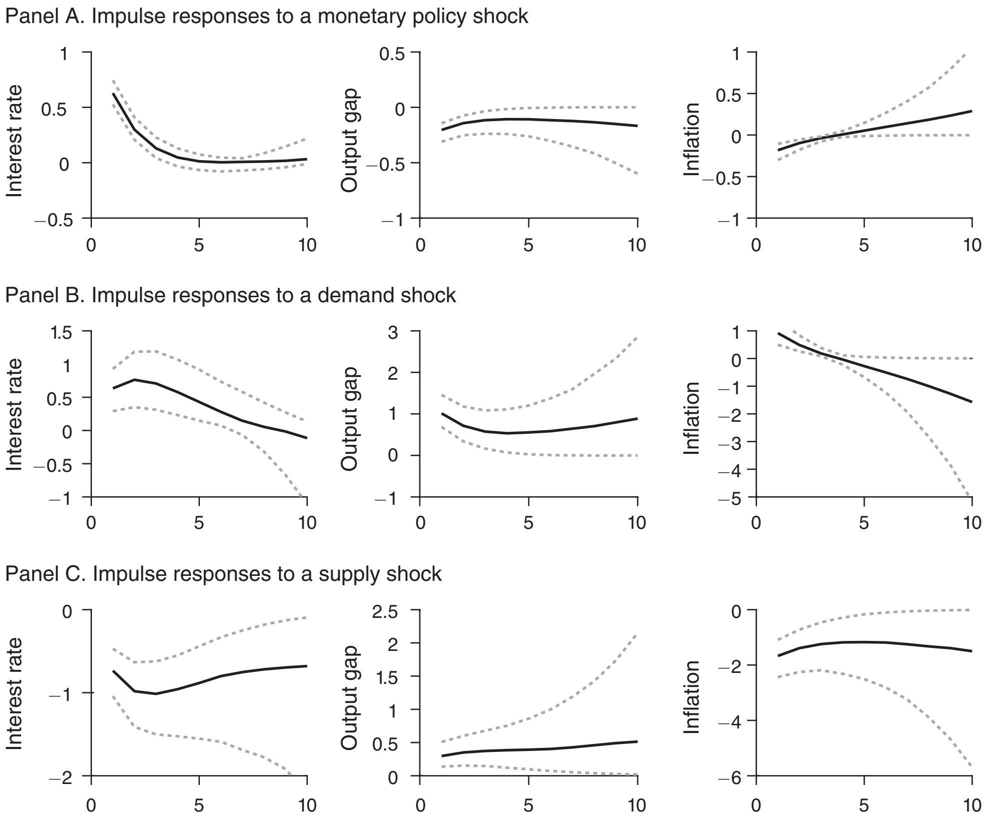

Figure 6. Generalized Impulse Response Function in the M_U Model Computed under the Posterior Distribution of $M_{1,1}$ in 1979:II

state from the fourth quarter onward. Interestingly, a somewhat similar behavior is highlighted in LS under their preferred prior 1: "an increase in the nominal interest rate can have a slightly inflationary effect" (p. 207, see Figure 3, p. 208 and the discussion at pp. 207–208 therein). They conclude that "before 1979 indeterminacy substantially altered the propagation of shocks" (LS, abstract).[23] Similarly, instability in our framework substantially alters the transmission mechanism. However, in our case, output remains below steady state, so that a monetary policy shock could generate an opposite response of output and inflation. In the LS case, instead, inflation and output move in the same direction after a monetary policy shock: after dropping on impact, they both become slightly positive. The same consideration also applies to the demand shock: both output and inflation increase on impact, but inflation turns negative in the fourth quarter. Our framework therefore seems to be able to provide a transmission mechanism more prone to accommodate stagflation under instability.

The transmission mechanism of the sunspot shock is also quite different in our case. In LS, the impulse response function to a sunspot shock under indeterminacy

[23] "This finding suggests that the fit of the model can be improved by deviating from the baseline solution and altering the propagation of the structural shocks" (LS, p. 205).

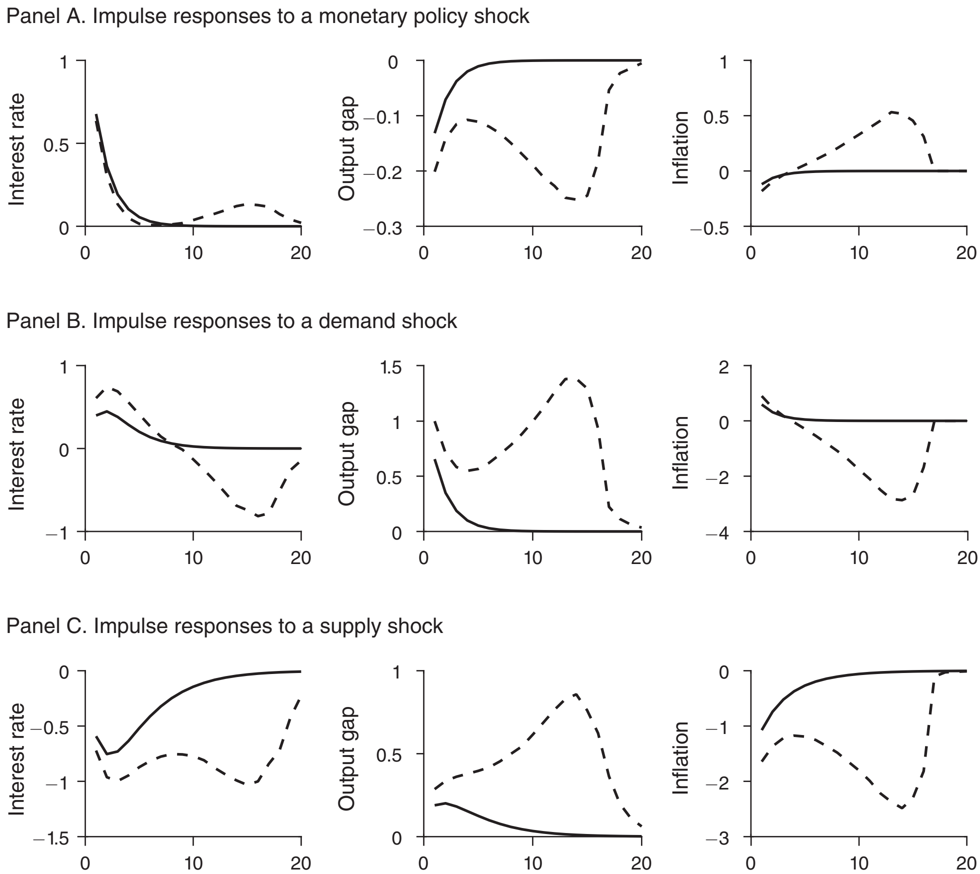

Figure 7. Generalized Impulse Response Function in the M_U Model

Note: Solid line: $M_{1,t} = 0$, dashed line: $M_{1,1} = 0.52$

does imply (again) that output and inflation move in the same direction (see LS, Figure 2, p. 207). Intuitively, if a sunspot shock leads to a self-fulfilling increase in inflation, then the real interest rate decreases, due to the passive monetary policy, and thus output increases, rather than decreases. Thus, the structural dynamics implied by an indeterminate stable model do not seem to be well suited to explain stagflation episodes after an additive sunspot shock. In our setup, instead, the nonlinear multiplicative sunspot shock amplifies the responses of the model to a structural shock. Similarly to Figure 4, Figure 7 shows the GIRFs for two different values of $M_{1,t}$ in the M_U case: the solid line corresponds to the pure forward-looking solution $(M_{1,t} = 0)$; the dashed line refers to $M_{1,1} = 0.52$, so that the initial value is the one estimated in 1974:IV. As in the M_S case, the sunspot shock amplifies the GIRFs, but the implied dynamics are very different in the M_U case. The amplification is similar on impact between the two cases, but then the unstable root induces an explosive dynamics such that initially the distance between the two lines increases over time. The economy is traveling on an explosive trajectory and it diverges away from the stable forward-looking solution. However, the walk on the unstable trajectory is temporary and the GIRFs exhibits a boom-bust type of behavior: given the assumed

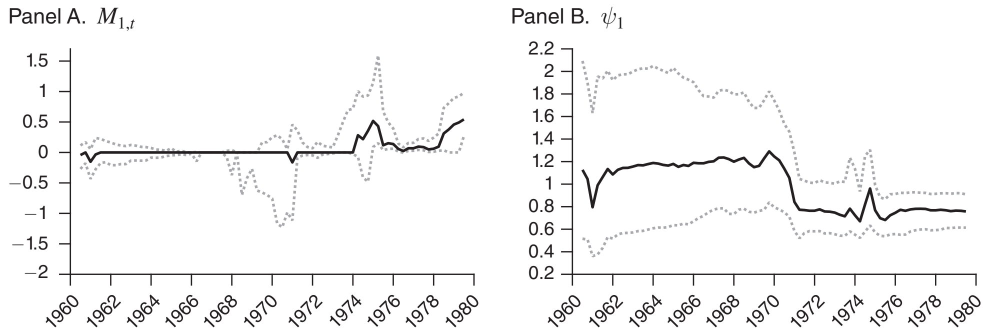

Figure 8. Estimated Path of $M_{1,t}$ for the Unstable Model M_U in the Great Inflation Subsample (Panel A); Sequential Inference on the Parameter ψ_1 (Panel B)

process for $M_{1,t}$, at a certain stochastic date, the economy converges back to the unique stable forward-looking solution.

The estimated path for the latent process $M_{1,t}$ in Figure 8 is again very instructive. Recall that we let the data choose: they could still choose a stable forward-looking solution when $M_{1,t}$ is estimated to equal 0. Similarly to the previous case, the estimate of $M_{1,t}$ is initially equal to 0, but then it moves away from 0 (the 90 percent interval exhibits a mass above 0), exactly when inflation starts increasing away from its steady-state value. If we allow for temporarily unstable paths, the estimation then unambiguously selects those to explain the data in this period.

It is possible to compare the relative fit of the stable (M_S) and unstable (M_U) models by computing the sequential Bayes factor as in West (1986). The Bayes factor is the model likelihood ratio,

$$L_t = \frac{p(\mathbf{D}_t | \mathbf{D}_{0:t-1}, M_S)}{p(\mathbf{D}_t | \mathbf{D}_{0:t-1}, M_U)}, \tag{36}$$

and measures the relative success of M_S and M_U in predicting the data: values of L_t lower than 1 indicate a worse predictive performance of M_S than the alternative M_U. West (1986) suggests to compute the Bayes factor sequentially as $W_t = L_t L_{t-1} \cdots L_1$. Here, W_t is called the *cumulative* Bayes factor and it assesses the relative fit of the two models by considering all observations sequentially. Figure 9 shows twice the natural logarithm of the cumulative Bayes factor W_t (as suggested by Kass and Raftery 1995) together with the path of inflation. Therefore, a value of 0 of the logarithm of the cumulative Bayes factor means that the two models have the same performance in terms of predictive likelihood; while a positive value means that M_S is preferred (and vice versa for negative values). The advantage of the cumulative Bayes factor, with respect to the conventional measures in Bayesian econometrics, is that we can compare two models over time. In our specific case, as expected, the unstable model is strongly preferred from the 1970s onward, when inflation increases and reaches high values. According to the Kass and Raftery (1995, p. 777) classification, there is "very strong" evidence in favor of M_U from the beginning of the 1970s. In particular, the cumulative Bayes factor reaches a very low level from 1974:I onward.

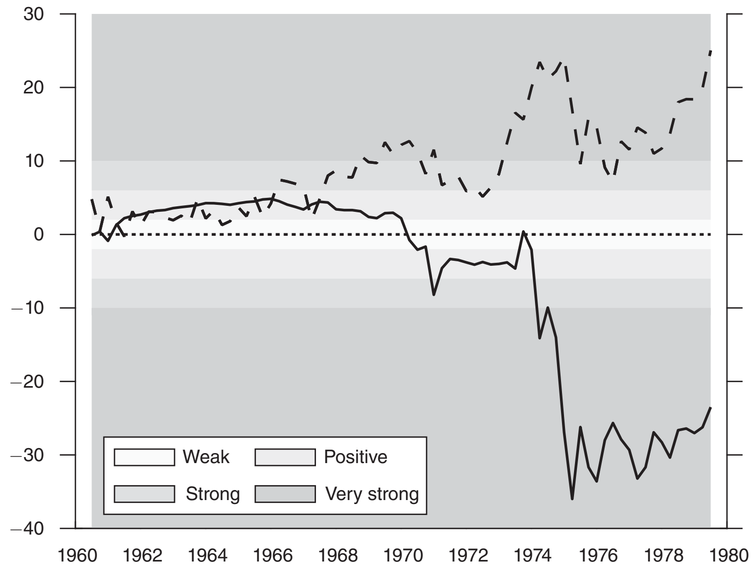

FIGURE 9. COMPARING M_S–M_U, GREAT INFLATION PERIOD

Note: The panel shows $2\ \ln(W_t)$ (solid line, scale on the left axis) and the inflation rate (dashed line, scale on the right axis).

To conclude, our methodology allows the data to choose between different possible alternatives: determinacy, indeterminacy and temporary instability. When the data are allowed this possibility, they unambiguously select the unstable model to explain the stagflation period in the 1970s.

Post-1982 Subsample.—In the second subsample, our estimates under stability again reproduce the same results as in LS (see Table 2). There is no statistically significant difference between our parameter estimates and those in LS, again signaling the reliability of our estimation methodology (see Figure 10). The Taylor principle is satisfied and hence the data choose the unique determinate forward-looking solution under M_S: there is no sunspot shock and the process for $M_{1,t}$ degenerates to the value of 0.

Also in the case of model M_U, the estimation yields results similar to LS. Panel A in Figure 11 shows that $M_{1,t}$ is estimated to be equal to 0 for the whole period, meaning that the estimation chooses the standard MSV solution under determinacy, rather than a temporarily unstable path (i.e., $M_{1,t} \neq 0$). Coherently, the sequential estimate of the parameter γ in panel B of Figure 11 implies a negligible probability that the economy travels on a temporarily unstable path.[24] In the Great Moderation sample, the final point estimate of γ is extremely low (0.04 in Table 2), despite the fact that the prior was set to 0.8, as in the Great Inflation sample (where the posterior point estimate is 0.96). The data are thus, in this case, extremely informative, and strongly

[24] From (35), γ represents the probability of $M_{1,t}$ being different from 0.

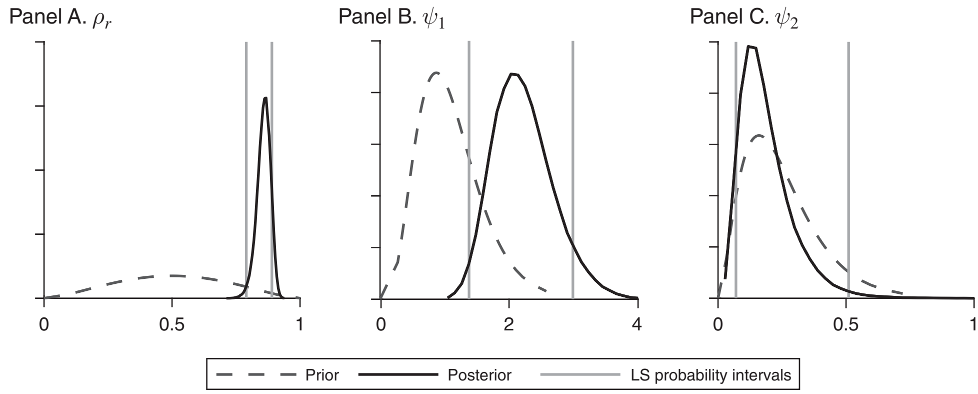

FIGURE 10. M_S: COMPARISON BETWEEN THE POSTERIOR DISTRIBUTIONS OF THE POLICY PARAMETERS AND THE PROBABILITY INTERVALS OF LS

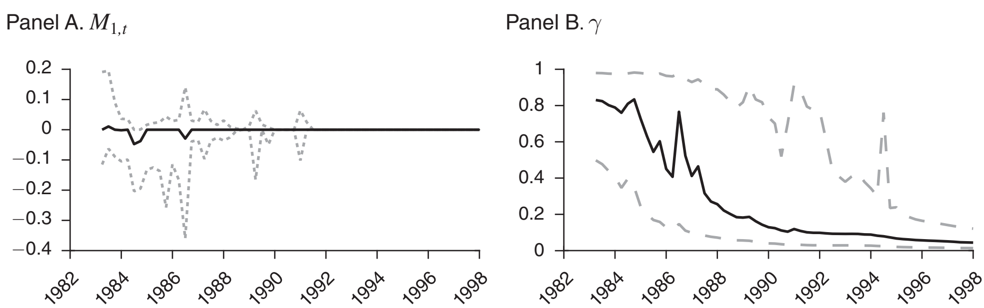

FIGURE 11. ESTIMATED PATH OF $M_{1,t}$ FOR THE UNSTABLE MODEL M_U IN THE GREAT MODERATION SUBSAMPLE (PANEL A); SEQUENTIAL INFERENCE ON THE PARAMETER γ (PANEL B)

point toward the stable MSV solution, which is the same as the one imposed by the standard RE methods.

Comparing the two models as in the previous case using the cumulative Bayes factor presents mild evidence in favor of the (less parameterized) stable model (see Figure 12). The evidence is not strong though, but "weak" until 1992 and then "positive," because the two models deliver very similar estimates.

Prior on γ.—In terms of point estimates, our results are very robust to changing the priors of our parameters. A prior of 0.9 for γ would deliver very similar results, and the cumulative Bayes factor would favor our benchmark choice even more strongly. A tighter prior on γ (i.e., the standard error prior equals 0.05 rather than 0.15) improves the fit of the model in the Great Inflation sample, because the sequential estimate of γ is very stable around 0.8. For the same reason, however, the estimation performs worse in the Great Moderation period. Notably, it is not able to recover the standard rational expectation MSV solution for that subsample, because the tighter prior does not allow the particles to sufficiently explore that region of the parameter space, and the sequential estimate of γ fluctuates quite tightly around 0.8. Hence, it is very important in our approach to allow for a sufficiently wide prior over

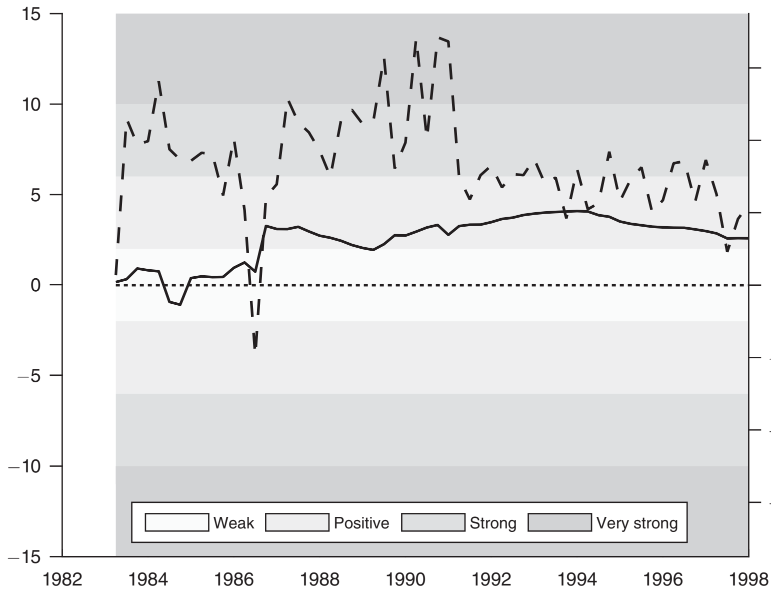

FIGURE 12. COMPARING M_S–M_U, GREAT MODERATION PERIOD

Note: The panel shows $2\ \ln(W_t)$ (solid line, scale on the left axis) and the inflation rate (dashed line, scale on the right axis).

the parameter γ to give the estimation a chance to adequately explore all different regions of the parameter space corresponding to the cases of determinacy, indeterminacy and temporarily explosive paths.

V. A Comparison with a Stochastic Volatility Model

A large empirical literature shows how stochastic volatility is an important feature of US macroeconomic variables in the sample we analyze. Cogley and Sargent (2005), Primiceri (2005), and Justiniano and Primiceri (2008) find evidence in favor of high volatility in the 1970s and a subsequent decrease during the Great Moderation.

On the one hand, our methodology rationalizes this evidence through the hypothesis of time variation in the agents' expectations formation process, as the estimated values of $M_{1,t}$ amplify the effects of structural shocks during the last part of the first subsample (see Figures 4 and 7). On the other hand, our framework imposes a strong link between the "walks on unstable trajectories" and stochastic volatility: under the unstable model M_U, stochastic volatility always occurs in the presence of temporarily unstable paths, and it is absent only when the unique stable solution is selected. This restriction may be too tight, and it might be the case that our M_U model is favored by the data because of the implied stochastic volatility rather than because of the intrinsic temporarily unstable dynamics. In other words, a model with stochastic volatility without unstable dynamics might be sufficient to adequately interpret the data.

To investigate this issue, we compare the fit of the unstable model M_U with a stochastic volatility model under determinacy during the Great Inflation. As we will see, the results show that while modeling the heteroskedasticity of shocks in a flexible way leads to some improvements, temporarily unstable paths remain a key feature to interpret the behavior of inflation, GDP and the interest rate during the 1970s.

Closely following Justiniano and Primiceri (2008), the logarithm of the standard error of each shock is described by a random walk process: $\log \sigma_{i,t} = \log \sigma_{i,t-1} + \nu_{i,t}$, where $\nu_{i,t} \sim N(0, \delta_i^2)$ and $i = g, z, R$. We impose determinacy, then $M_{1,t} = 0, \forall t$, and we only explore the region of the parameter space that satisfies the Taylor principle. Moreover, we estimate the model considering the entire sample from 1960:I to 1997:IV.

Inference on the parameters and the time-varying volatilities is performed using the same econometric strategy as above. Note that conditional on the values of the volatilities, the model is linear and Gaussian. Then, we simply proceed in analogy with the estimation of models M_S and M_U, and we treat the time variation in the variances in the same way as the time variation in $M_{1,t}$ (see the online Appendix for details).

The prior distributions on the parameters are the same as in Table 1, with the exception that now we only allow for determinacy. In practice, this is simply done by setting the particle weight equal to 0 whenever the parameters are such that the Taylor principle is not satisfied. For the variances of the shocks to the volatilities, we assume an Inverse Gamma distribution with mean equal to 0.02 and 3 degrees of freedom.[25] Finally, we assume that the standard deviations at time 0 have the same prior distribution as in the time invariant case, reported in Table 1.

The last column of Table 2 displays the posterior distribution of the parameters, and Figure 13 shows the estimated pattern of the time-varying standard deviations of the different shocks. With respect to Justiniano and Primiceri (2008), we work with a smaller model and a shorter sample period. Nevertheless, we find very similar results. First, the model accounts for the reduction in the volatility of the US macroeconomic variables during the Great Moderation due to a substantial decrease in the volatility of exogenous disturbances. Second, the degree of stochastic volatility is not the same for all shocks. As in Justiniano and Primiceri (2008), the disturbance to monetary policy, which is the unique directly comparable shock, exhibits the largest variation in the standard deviation. Moreover, the pattern of stochastic volatility is remarkably similar to that in Justiniano and Primiceri (2008). Finally, for the two other shocks, we find a decline of roughly one-third in the last part of the sample, again in line with the results in Justiniano and Primiceri (2008).

In Table 3, we compare the overall fit of this model during the Great Inflation period, with both the stable model M_S and the unstable model M_U. The model with determinacy and stochastic volatility is favored by the Bayes factor when compared to the stable model M_S. In M_S, the variations in the variances are all related to one common component, that is $M_{1,t}$, while the standard deviation of the monetary

[25] Justiniano and Primiceri (2008) set the prior mean equal to 0.01, one-half of what we assume. In our model we find that this specification restricts the time variation in the standard deviations too much, penalizing the model with determinacy and stochastic volatility. Under our prior, instead, we find results that are very similar to those of Justiniano and Primiceri (2008), as described below.

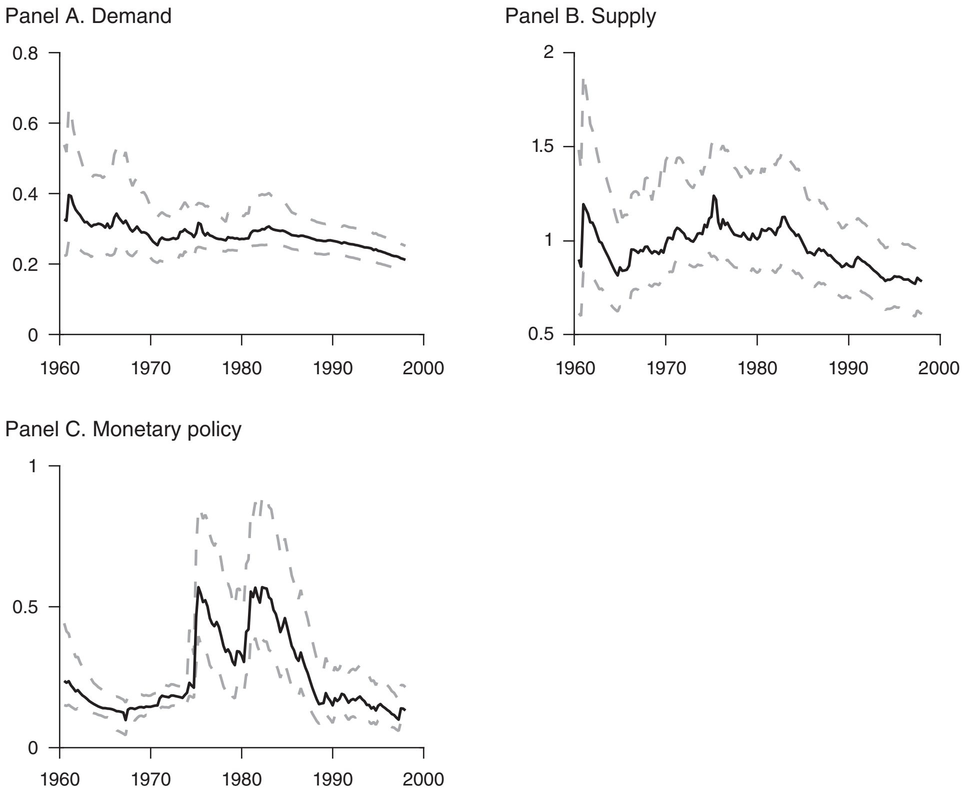

Figure 13. Time-Varying Standard Deviation of Each Shock: Model with Determinacy and Stochastic Volatility

Table 3—Model Comparison with Determinacy and Stochastic Volatility, 1960:I–1979:II

Alternative model	2 log(Bayes factor)
M_S	−7.2611
M_U	16.2346

Note: A positive value means evidence in favor of the alternative model.

policy shock behaves differently with respect to the other two, when more flexibility is allowed. This finding does not necessarily imply that the restrictions imposed by our method are in general too tight. The size of the model we consider is small, allowing for only one element in the matrix $\mathbf{M}_t$ to be time-varying (i.e., only indeterminacy of order one), when stability is imposed.

In estimating the model under instability, we choose to limit ourselves to the case of only one degree of freedom, setting the elements in the main diagonal of the matrix $\mathbf{M}_t$ to the same stochastic process. Then, also the unstable model M_U penalizes the variability of the variances in the same way as model M_S. Despite this limit, Table 3 shows that the Bayes factor clearly favors the unstable model: the evidence for model M_U is labeled as "very strong" in the Kass and Raftery (1995) classification.

This result suggests that temporarily unstable paths are a key feature to describe the unstable pattern of the US macroeconomic variables during the Great Inflation period. Therefore, we conclude that stochastic volatility alone, without explosive dynamics, is not able to fully capture the unstable behavior of the data during the Great Inflation period.

VI. Conclusions

We propose a novel framework to consider a broader class of solutions to stochastic linear RE models.

Theoretically, we provide two main generalizations: our framework allows for the possibility of the economy walking on temporarily unstable paths and it generates time-varying parameter solutions and stochastic volatility.

Empirically, we propose an econometric methodology that allows the data to choose among the different RE alternatives: determinacy, indeterminacy, and temporary instability, without imposing them a priori in the estimation. This methodology can be used to test the empirical relevance of temporarily unstable dynamics.

Finally, we apply this approach to the data to explain US inflation dynamics in the Great Inflation and Great Moderation period. The empirical evidence suggests that the Great Inflation in the United States can be explained by temporarily unstable paths. The usual practice of excluding *a priori* unstable solutions does not seem to be supported by the data, which, if allowed, unambiguously select the unstable model to explain the stagflation period in the 1970s. Our framework provides a different interpretation of the Great Inflation from a policy perspective. Despite the fact that our estimates point to a passive monetary policy behavior in the 1970s, our framework implies that this is not the cause in itself of unstable inflation dynamics, which were instead due to drifting expectations, independently of the stance of monetary policy.

Our analysis therefore suggests that unstable paths can be empirically relevant. This result may call for a rethinking of the stability criterion as the selection mechanism, and for theoretically considering the possibility that RE could push the economy to temporarily walk along unstable paths.

This line of research is still in its infancy and can be expanded in many directions. A first important direction would be to endogenize the expectations formation process that drives the (exogenous) multiplicative sunspot and then estimating it on the data, in a spirit similar to the escape dynamics literature. Moreover, the estimation indicates a possible link between unstable paths and the monetary policy parameter, which is reminiscent of the debate about monetary policy and the anchoring of inflation expectations.

Second, extending the framework to nonlinear models and nonlinear solution methods is a second direction for future research. The linear approximation of a model could become unreliable if the system drifts too far away from the steady state by following a temporarily unstable path. The extension should be feasible because there are available methods to solve nonlinear models and the econometric strategy does not depend on the model being linear. An important application, then, would be to use a model with the zero lower bound (see, e.g., Gust et al. 2017) to investigate how the zero lower bound affects the process of expectations formation and hence the stability of the economy.

Third, one could modify the framework to allow a subset of variables to explode. Following the insights in Cochrane (2011), for example, nominal variables do not need to satisfy a transversality condition.

Finally, there are many potential applications of our framework, notably, but not exclusively, finance, where boom and bust episodes of asset prices (stock, houses, etc.) is a pervasive phenomenon.

Appendix A. Appendix to Section I

A1. *Derivation of Equation* (5)

Use equation (4) and write it recursively (or simply consider only fundamental solutions where $\zeta_t = 0, \forall t$, in (3) and substitute in (2))

$$\xi_t = M\sum_{i=1}^{t}\theta^i \varepsilon_{t+1-i} = \theta\xi_{t-1} + \theta M\varepsilon_t. \tag{A1}$$

Substitute $\varepsilon_t = y_t - \frac{1}{\theta}\xi_t$ from (1) to get (assuming $M \neq 1$)

$$\xi_t = \theta\xi_{t-1} + \theta M\varepsilon_t = \theta\xi_{t-1} + \theta M\left(y_t - \frac{1}{\theta}\xi_t\right) \Rightarrow$$
$$\xi_t = \frac{\theta M}{1+M}y_t + \frac{\theta}{1+M}\xi_{t-1}.$$

Roll backward

$$\begin{aligned}
\xi_t &= \frac{\theta M}{1+M}\,y_t + \frac{\theta}{1+M}\,\xi_{t-1} \\
&= \frac{\theta M}{1+M}\,y_t + \frac{\theta}{1+M}\left(\frac{\theta M}{1+M}\,y_{t-1} + \frac{\theta}{1+M}\,\xi_{t-2}\right) \\
&= \frac{\theta M}{1+M}\,y_t + M\left(\frac{\theta}{1+M}\right)^2 y_{t-1} + \left(\frac{\theta}{1+M}\right)^2 \xi_{t-2} \\
&= \cdots \\
&= \frac{\theta M}{1+M}\,y_t + M\left(\frac{\theta}{1+M}\right)^2 y_{t-1} + \cdots + M\left(\frac{\theta}{1+M}\right)^t y_1 + \left(\frac{\theta}{1+M}\right)^t \xi_0.
\end{aligned}$$

Assuming a period 0 where $\xi_0 = E_0(y_1) = 0$ yields equation (5) in the main text

$$\xi_t = M\sum_{i=1}^{t}\left(\frac{\theta}{1+M}\right)^i y_{t+1-i}. \tag{A2}$$

A2. *Derivation of Equation* (8)

From (1), (6), and (7), we can show that our solution for y_t implies time-varying parameters and stochastic volatility. From the definition of the forecast error: $\eta_t = y_t - E_{t-1}(y_t) \Rightarrow y_t = \eta_t + \xi_{t-1}$. Substitute η_t from (7) and ξ_{t-1} from (6) to get

$$\begin{aligned} y_t &= (1 + M_t)\varepsilon_t + (M_t - M_{t-1})\sum_{i=1}^{t-1}\theta^i \varepsilon_{t-i} + M_{t-1}\theta\sum_{i=0}^{t-2}\theta^i \varepsilon_{t-1-i} \\ &= (1 + M_t)\varepsilon_t + (M_t - M_{t-1})\sum_{i=1}^{t-1}\theta^i \varepsilon_{t-i} + M_{t-1}\sum_{i=1}^{t-1}\theta^i \varepsilon_{t-i} \\ &= (1 + M_t)\varepsilon_t + M_t\sum_{i=1}^{t-1}\theta^i \varepsilon_{t-i} = (1 + M_t)\varepsilon_t + \frac{M_t}{M_{t-1}}\,\xi_{t-1} \qquad \text{then by (1)} \\ &= (1 + M_t)\varepsilon_t + \frac{M_t}{M_{t-1}}\,(\theta y_{t-1} - \theta\varepsilon_{t-1}). \end{aligned}$$

Hence,

$$\text{(A3)} \qquad y_t = \alpha_t y_{t-1} - \alpha_t \varepsilon_{t-1} + (1 + M_t)\varepsilon_t \quad \text{if and only if} \quad M_{t-1} \neq 0,$$

with $\alpha_t = \theta M_t / M_{t-1}$.

Note that if $M_{t-1} = 0$, from (6) $\xi_{t-1} = M_{t-1}\theta\sum_{i=0}^{t-2}\theta^i \varepsilon_{t-1-i} = 0$, so we obtain the forward-looking solution for y_{t-1}, i.e., $y_{t-1} = \varepsilon_{t-1}$. From an economic point of view, this is a period where agents coordinate their expectations on the forward-looking solution, so the system has returned to the stable solution (recall that it is always an admissible solution). In this case, then from (2), in the next period we have $E_t(y_{t+1}) \equiv \xi_t = -\theta\varepsilon_t + \theta\eta_t$, where $\eta_t \equiv y_t - \xi_{t-1} = y_t$. Substitute η_t, rearrange so $y_t = \varepsilon_t + \frac{1}{\theta}\xi_t$, and then using (6),

$$\text{(A4)} \quad \eta_t = y_t = \varepsilon_t + \frac{1}{\theta}\xi_t = \varepsilon_t + \frac{1}{\theta}\left(M_t\theta\sum_{i=1}^{t-1}\theta^i\varepsilon_{t-i}\right) = (1+M_t)\varepsilon_t + M_t\sum_{i=1}^{t-1}\theta^i\varepsilon_{t-i},$$

which is coherent with (7) when $M_{t-1} = 0$.

Finally note that, since $y_t^F = \varepsilon_t$ and $y_t^B = -\sum_{i=1}^{t-1}\theta^i\varepsilon_{t-i}$, then the solution can be written as in Blanchard (1979): $y_t = (1 + M_t)\,y_t^F - M_t y_t^B$.

REFERENCES

Ascari, Guido, Paolo Bonomolo, and Hedibert F. Lopes. 2019. "Walk on the Wild Side: Temporarily Unstable Paths and Multiplicative Sunspots: Dataset." *American Economic Review.* https://doi.org/10.1257/aer.20160576.

Benati, Luca, and Paolo Surico. 2009. "VAR Analysis and the Great Moderation." *American Economic Review* 99 (4): 1636–52.

Benhabib, Jess, and Roger E. A. Farmer. 1999. "Indeterminacy and Sunspots in Macroeconomics." In *Handbook of Macroeconomics*, Vol. 1A, edited by John B. Taylor and Michael Woodford, 387–448. Amsterdam: Elsevier.

Blanchard, Olivier J. 1979. "Backward and Forward Solutions for Economies with Rational Expectations." *American Economic Review* 69 (2): 114–18.

Blanchard, Olivier Jean, and Charles M. Kahn. 1980. "The Solution of Linear Difference Models under Rational Expectations." *Econometrica* 48 (5): 1305–11.

Blanchard, Olivier J., and Mark W. Watson. 1982. "Bubbles, Rational Expectations and Financial Markets." NBER Working Paper 0945.

Boivin, Jean, and Marc P. Giannoni. 2006. "Has Monetary Policy Become More Effective?" *Review of Economics and Statistics* 88 (3): 445–62.

Brock, William A. 1974. "Money and Growth: The Case of Long Run Perfect Foresight." *International Economic Review* 15 (3): 750–77.

Bullard, James, and Kaushik Mitra. 2002. "Learning about Monetary Policy Rules." *Journal of Monetary Economics* 49 (6): 1105–29.

Burmeister, Edwin, Robert P. Flood, and Peter M. Garber. 1983. "On the Equivalence of Solutions in Rational Expectations Models." *Journal of Economic Dynamics and Control* 5: 311–21.

Cappe, Olivier, Simon J. Godsill, and Eric Moulines. 2007. "An Overview of Existing Methods and Recent Advances in Sequential Monte Carlo." *Proceedings of the IEEE* 95 (5): 899–924.

Carboni, Giacomo, and Martin Ellison. 2009. "The Great Inflation and the Greenbook." *Journal of Monetary Economics* 56 (6): 831–41.

Carvalho, Carlos, Stefano Eusepi, Emanuel Moench, and Bruce Preston. 2017. "Anchored Inflation Expectations." https://ssrn.com/abstract=3018198.

Carvalho, Carlos M., Michael S. Johannes, Hedibert F. Lopes, and Nicholas G. Polson. 2010. "Particle Learning and Smoothing." *Statistical Science* 25 (1): 88–106.

Castelnuovo, Efrem, and Luca Fanelli. 2015. "Monetary Policy Indeterminacy and Identification Failures in the U.S.: Results from a Robust Test." *Journal of Applied Econometrics* 30 (6): 924–47.

Castelnuovo, Efrem, Luciano Greco, and Davide Raggi. 2014. "Policy Rules, Regime Switches, and Trend Inflation: An Empirical Investigation for the United States." *Macroeconomic Dynamics* 18 (4): 920–42.

Chen, Hao, Francesca Petralia, and Hedibert F. Lopes. 2010. "Sequential Monte Carlo Estimation of DSGE Models." Unpublished.

Cho, In-Koo, Noah Williams, and Thomas J. Sargent. 2002. "Escaping Nash Inflation." *Review of Economic Studies* 69 (1): 1–40.

Clarida, Richard, Jordi Galí, and Mark Gertler. 2000. "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory." *Quarterly Journal of Economics* 115 (1): 147–80.

Cochrane, John H. 2011. "Determinacy and Identification with Taylor Rules." *Journal of Political Economy* 119 (3): 565–615.

Cogley, Timothy, and Thomas J. Sargent. 2005. "Drifts and Volatilities: Monetary Policies and Outcomes in the Post WWII US." *Review of Economic Dynamics* 8 (2): 262–302.

Creal, Drew D. 2007. "Sequential Monte Carlo Samplers for Bayesian DSGE Models." Unpublished.

Fernández-Villaverde, Jesús, Pablo A. Guerrón-Quintana, and Juan F. Rubio-Ramírez. 2010. "Fortune or Virtue: Time-Invariant Volatilities versus Parameter Drifting in U.S. Data." NBER Working Paper 15928.

Fernández-Villaverde, Jesús, and Juan F. Rubio-Ramírez. 2007. "Estimating Macroeconomic Models: A Likelihood Approach." *Review of Economic Studies* 74 (4): 1059–87.

Flood, Robert P., and Peter M. Garber. 1980. "Market Fundamentals versus Price-Level Bubbles: The First Tests." *Journal of Political Economy* 88 (4): 745–70.

Gourieroux, Christian, Jean-Jacques Laffont, and Alain Monfort. 1982. "Rational Expectations in Dynamic Linear Models: Analysis of the Solutions." *Econometrica* 50 (2): 409–25.

Gust, Christopher, Edward Herbst, David López-Salido, and Matthew E. Smith. 2017. "The Empirical Implications of the Interest-Rate Lower Bound." *American Economic Review* 107 (7): 1971–2006.

Herbst, Edward, and Frank Schorfheide. 2014. "Sequential Monte Carlo Sampling For DSGE Models." *Journal of Applied Econometrics* 29 (7): 1073–98.

Justiniano, Alejandro, and Giorgio E. Primiceri. 2008. "The Time-Varying Volatility of Macroeconomic Fluctuations." *American Economic Review* 98 (3): 604–41.

Kass, Robert E., and Adrian E. Raftery. 1995. "Bayes Factors." *Journal of the American Statistical Association* 90 (430): 773–95.

Liu, Jane, and Mike West. 2001. "Combined Parameter and State Estimation in Simulation-Based Filtering." In *Sequential Monte Carlo Methods in Practice*, edited by Arnaud Doucet and Nando de Freitas and Neil Gordon, 197–223. Berlin: Springer.

Lopes, Hedibert F., Michael S. Johannes, Carlos M. Carvalho, and Nicholas G. Polson. 2011. "Particle Learning for Sequential Bayesian Computation (with Discussion)." *Bayesian Statistics* 9: 317–60.

Lopes, Hedibert F., and Ruey S. Tsay. 2011. "Particle Filters and Bayesian Inference in Financial Econometrics." *Journal of Forecasting* 30 (1): 168–209.
Lubik, Thomas A., and Christian Matthes. 2016. "Indeterminacy and Learning: An Analysis of Monetary Policy in the Great Inflation." *Journal of Monetary Economics* 82: 85–106.
Lubik, Thomas A., and Frank Schorfheide. 2004. "Testing for Indeterminacy: An Application to U.S. Monetary Policy." *American Economic Review* 94 (1): 190–217.
Mavroeidis, Sophocles. 2010. "Monetary Policy Rules and Macroeconomic Stability: Some New Evidence." *American Economic Review* 100 (1): 491–503.
Muth, John F. 1960. "Optimal Properties of Exponentially Weighted Forecasts." *Journal of the American Statistical Association* 55 (290): 299–306.
Muth, John F. 1961. "Rational Expectations and the Theory of Price Movements." *Econometrica* 29 (3): 315–35.
Pesaran, M. Hashem. 1987. *The Limits to Rational Expectations*. Oxford: Basil Blackwell.
Phelps, Edmund S., and John B. Taylor. 1977. "Stabilizing Powers of Monetary Policy under Rational Expectations." *Journal of Political Economy* 85 (1): 163–90.
Pitt, Michael K., and Neil Shephard. 1999. "Filtering via Simulation: Auxiliary Particle Filters." *Journal of the American Statistical Association* 94 (446): 590–99.
Primiceri, Giorgio E. 2005. "Time Varying Structural Vector Autoregressions and Monetary Policy." *Review of Economic Studies* 72 (3): 821–52.
Sargent, Thomas J. 1999. *The Conquest of American Inflation.* Princeton, NJ: Princeton University Press.
Sargent, Thomas J., and Neil Wallace. 1973. "The Stability of Money and Growth with Perfect Foresight." *Econometrica* 41 (6): 1043–48.
Sargent, Thomas, Noah Williams, and Tao Zha. 2006. "Shocks and Government Beliefs: The Rise and Fall of American Inflation." *American Economic Review* 96 (4): 1193–1224.
Sims, Christopher A. 2002. "Solving Linear Rational Expectations Models." *Computational Economics* 20 (1–2): 1–20.
Strid, Ingvar. 2010. "Efficient Parallelisation of Metropolis-Hastings Algorithms Using a Prefetching Approach." *Computational Statistics & Data Analysis* 54 (11): 2814–35.
Taylor, John B. 1977. "On the Conditions for Unique Solutions in Stochastic Macroeconomic Models with Rational Expectations." *Econometrica* 45 (6): 1377–85.
West, Mike. 1986. "Bayesian Model Monitoring." *Journal of the Royal Statistical Society Series B* (48) 1: 70–78.
Woodford, Michael. 2003. *Interest and Prices: Foundations of a Theory of Monetary Policy.* Princeton, NJ: Princeton University Press.