IMA Journal of Numerical Analysis (2010) Page 1 of 22
doi:10.1093/imanum/drn000

Isogeometric regular discretization for the Stokes problem

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[Received on 22 January 2010]

The inf sup stability and optimal convergence of an isogeometric C^1 discretization for the Stokes problem are shown. In this discretization the velocities are the push forward through the geometrical map of cubic C^1 NURBS functions, and the pressures are the push forward of quadratic C^1 NURBS. This paper follows the work in Bazilevs et al. (2006) where the authors showed the numerical result of this discretization and proved the inf sup-stability for C^0 NURBS functions. The use of more regular functions is useful to decrease the degrees of freedom and thus the computational cost. The analysis is performed by means of the Verfürth trick, the macro-element technique, some approximation properties and the inf sup condition for tensor products of B-spline spaces.

Keywords: Stokes problem; isogeometric analysis; incompressible flows.

1. Introduction

The Stokes problem is a simplified model of the equations used to describe incompressible fluid flows and elastic deformations in solids. Its mathematical formulation is: find a velocity field $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$ and a pressure $p : \mathbb{R}^n \to \mathbb{R}$ such that

$$-\nu\triangle\mathbf{v} + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{1.1a}$$
$$\nabla\cdot\mathbf{v} = 0 \qquad \text{in } \Omega, \tag{1.1b}$$
$$\mathbf{v} = 0 \qquad \text{on } \partial\Omega, \tag{1.1c}$$

where:

$\Omega \subset \mathbb{R}^n,\ n = 2,3$ is a bounded domain with Lipschitz boundary,

$\mathbf{f} : \Omega \to \mathbb{R}^n,$ is a given force vector,

$\nu > 0,$ is a constant viscosity.

The corresponding variational form is: find $\mathbf{v} \in H_0^1(\Omega)^n$, $p \in L_0^2(\Omega)$ such that

$$\nu\langle\nabla\mathbf{v}, \nabla\mathbf{w}\rangle - \langle p, \nabla\cdot\mathbf{w}\rangle = \langle\mathbf{f}, \mathbf{w}\rangle \qquad \forall\mathbf{w} \in H_0^1(\Omega)^n, \tag{1.2a}$$
$$\langle\nabla\cdot\mathbf{v}, q\rangle = 0 \qquad \forall q \in L^2(\Omega). \tag{1.2b}$$

For each $\mathbf{f} \in H^{-1}(\Omega)^n$ the system has unique solutions $\mathbf{v} \in H_0^1(\Omega)^n$ and $p \in L_0^2(\Omega)$, which continuously depend upon the datum force vector $\mathbf{f}$ (see Girault & Raviart (1986)).

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As usual, $L^2(\Omega)$ is the Hilbert space of the square integrable functions defined on Ω, $H^1(\Omega)$ is the subspace of $L^2(\Omega)$ of the functions whose first order partial derivatives are in $L^2(\Omega)$, $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ of the functions with zero trace on the boundary $\partial\Omega$, $L_0^2(\Omega)$ is the subspace of $L^2(\Omega)$ of the functions with zero mean value and $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$. The symbol $L^\infty(\Omega)$ denotes the space of bounded functions on Ω, the L^∞ norm of a vector field is the sup in Ω of its euclidean norm, and that of a tensor field is the the sup in Ω of its matrix norm.

The isogeometric analysis was born to help integration between design and numerical simulation in engineering. A deep analysis of the motivations is found in Hughes et al. (2005). The main motivation for the development of the isogeometric method is that CAD and simulation tools use different descriptions of the geometry (polynomial vs. NURBS). This arose from the different development paths of the two disciplines and causes the need for complex software that creates and refines meshes from CAD data. In the isogeometric method, the CAD geometry is used directly to eliminate that complexity.

The main features of this method are

- exact description of the geometry, thus there is no error due to geometry approximation,
- mesh refinement is simplified,
- NURBS spaces with a given global regularity are easily built.

The possibility to easily control the regularity is interesting since it permits the construction of methods that are both efficient and accurate Evans et al. (2009). In fact the behavior of the approximation error with respect to the mesh size depends on the degree of the NURBS functions but not on the number of degrees of freedom Bazilevs et al. (2006). Thus, all the NURBS spaces of the same degree give the same convergence, but the most regular ones requires the smallest computational cost. In particular, a new refinement strategy has been developed, called k-refinement, that consists in both order elevation and mesh refinement. This technique gave good results in numerical simulations Hughes et al. (2005, 2008).

The isogeometric method is being applied in many fields. Good examples are elastic deformations Cottrell et al. (2006, 2007) and fluid mechanics Bazilevs & Hughes (2008). In particular there is great interest in fluid-structure interaction for applications in medicine Bazilevs et al. (2008), Calo et al. (2008), Bazilevs et al. (2009).

The aim of this article is to show the stability and optimal convergence of methods based on regular NURBS spaces subject only to mesh regularity and size. This analysis is done for the C^1 for which numerical results are known Bazilevs et al. (2006) (pages 1080, 1081), but the difference for the general case are minimal. The first section summarizes the isogeometric framework: the description of the geometry, the discrete spaces and their approximation properties. The second section contains the discrete formulation of the problem and the proof of its stability and error estimates.

2. Isogeometric framework

The following subsections provide a basic background on spline, NURBS, geometry description, meshes and discrete spaces.

2.1 *B-Splines*

A spline space over a real interval $I = [b_0, b_s]$ is a piecewise polynomial function space. Let $b_1 < \cdots < b_{s-1}$ be the desired junction points belonging to $]b_o, b_s[$, then a spline space is described by the

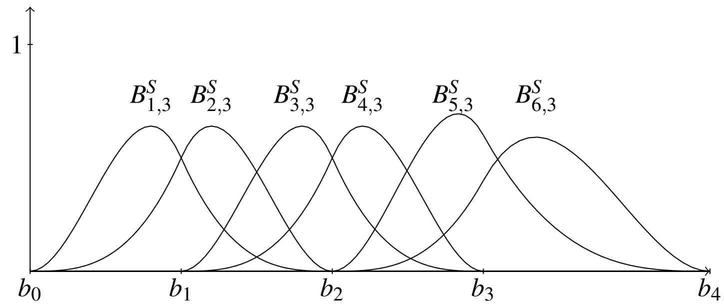

Figure 1. Canonical base of $S_{\Xi,3}$, where $\Xi = (b_0, b_0, b_1, b_1, b_2, b_2, b_3, b_3, b_4, b_4)$.

(maximum) degree of the polynomials d and an ordered knot vector $\Xi = (\xi_0 = b_0, \dots, \xi_n = b_s)$ of junction points that codify trough repetition the regularity of the functions. If b_i is repeated k_i times in Ξ then the functions have at least $r_i = d - k_i$ continuous derivatives in b_i: if $r_i = -1$ then jumps are admitted; r_i is called regularity in the knot b_i. At the boundary points, continuity is intended with the null function outside of I so the regularity is the number of derivatives that are null on the boundary. The space described by the knot vector Ξ and the degree d is denoted $S_{\Xi,d}$. The space $S_{\Xi,d}$ has a canonical base $\{B^S_{i,d}\}$ defined recursively over the degree by

$$B^S_{i,0}(x) = \begin{cases} 1 & x \in [\xi_{i-1}, \xi_i[, \\ 0 & \text{otherwise}, \end{cases}$$
$$B^S_{i,d}(x) = \frac{x - \xi_i}{\xi_{i+d} - \xi_i} B^S_{i,d-1}(x) - \frac{x - \xi_{i+d+1}}{\xi_{i+d+1} - \xi_{i+1}} B^S_{i+1,d-1}(x).$$

On a Cartesian product of intervals $\Theta = I_1 \times \cdots \times I_n$, spline spaces are described by n degrees $d_1, \dots, d_n$ and n knot vectors $\Xi_1, \dots, \Xi_n$ (one for each dimension), and are the tensor products of the corresponding one dimensional spaces

$$S_{(\Xi_1,\dots,\Xi_n),(d_1,\dots,d_n)} = S_{\Xi_1,d_1} \otimes \cdots \otimes S_{\Xi_n,d_n}. \tag{2.1}$$

In this case, the canonical basis is $\{B^S_{\mathbf{i},\mathbf{d}}\}_{\mathbf{i}}$ where $\mathbf{d} = (d_1, \dots, d_n)$ and $\mathbf{i} = (i_1, \dots, i_n)$

$$B^S_{\mathbf{i},\mathbf{d}}(x_1, \dots, x_n) = B^S_{i_1,d_1}(x_1) \dots B^S_{i_n,d_n}(x_n). \tag{2.2}$$

In this article only spaces where $d_1 = d_2 = \cdots = d_n$ and the regularity is the same in all the on knots (except for those on $\partial\Theta$) are used. These spaces are uniquely identified by the degree, the regularity and the knots. Let $b_{i,0}, \dots, b_{i,s_i} \in I_i$ be the junction points in the i-th dimension and

$$S_{d,r}(I_i) = \{\text{spline} : r_{i,0}, r_{i,s} = -1 \ \wedge \ r_{i,j} = r \text{ for } j = 1, \dots, s_i - 1\}, \tag{2.3a}$$
$$S^0_{d,r}(I_i) = \{\text{spline} : r_{i,j} = r \text{ for } j = 0, \dots, s_i\}. \tag{2.3b}$$

In higher dimensions, set

$$S_{d,r}(\Theta = I_1 \times \cdots \times I_n) = S_{d,r}(I_1) \otimes \cdots \otimes S_{d,r}(I_n), \tag{2.4a}$$
$$S^0_{d,r}(\Theta = I_1 \times \cdots \times I_n) = S^0_{d,r}(I_1) \otimes \cdots \otimes S^0_{d,r}(I_n). \tag{2.4b}$$

Using spline spaces, it is possible to construct maps from a parametric domain $\Theta = I_1 \times \cdots \times I_n$ to $\mathbb{R}^n$, called spline-maps. They are identified by a spline space S and $\dim S$ control points in $\mathbb{R}^n$: each control point $\bar{\mathbf{x}}_{\mathbf{i}} \in \mathbb{R}^n$ is associated with an element of the canonical basis $\{B^S_{\mathbf{i},\mathbf{d}}\}$ of S, and the corresponding map is

$$F(x_1, \ldots, x_n) = \sum_{\mathbf{i}} B^S_{\mathbf{i},\mathbf{d}}(x_1, \ldots, x_n)\bar{\mathbf{x}}_{\mathbf{i}}. \tag{2.5}$$

2.2 *NURBS*

Non Uniform Rational B-Splines were born to extend spline maps and allow the exact representation of useful geometries such as circles and ovals. The main idea is to map Θ in $\mathbb{R}^{n+1}$ with a spline map $\bar{F}$ such that $\gcd(\bar{F}_1, \ldots, \bar{F}_n) = 1$ and to project the result on the plain $\{x_{n+1} = 1\}$ by lines through the origin. Let $\widehat{F}$ be the spline map from Θ to $\mathbb{R}^n$ given by the first n components of $\bar{F}$, and w be the last component that is called *weight function*. The expression of the composition of $\bar{F}$ with the projection is

$$F(x_1, \ldots, x_n) = \frac{1}{w(x_1, \ldots, x_n)}\widehat{F}(x_1, \ldots, x_n), \tag{2.6}$$

so it is a piecewise quotient of polynomials. From this expression, it is clear that $w(x)$ cannot be 0 in any point, so it is assumed $w(x) > 0\ \forall x \in \Theta$. Usually, but it is not a requirement, by construction of the geometry,

$$w = \sum_{\mathbf{i}} w_{\mathbf{i}} B^S_{\mathbf{i},\mathbf{d}} \qquad\qquad w_{\mathbf{i}} \geqslant 1, \tag{2.7}$$

so that $w \geqslant 1$.

NURBS spaces are identified by a spline space $S_{(\Xi_1,\ldots,\Xi_n),(d_1,\ldots,d_n)}$ and a weight function $0 < w \in S_{(\Xi_1,\ldots,\Xi_n),(d_1,\ldots,d_n)}$, and are defined by

$$N_{(\Xi_1,\ldots,\Xi_n),(d_1,\ldots,d_n),w} = \{\frac{f}{w} :\ f \in S_{(\Xi_1,\ldots,\Xi_n),(d_1,\ldots,d_n)}\}. \tag{2.8}$$

The degree and the regularity in the junctions of a NURBS space are, by construction, those of the corresponding spline space, moreover the regularity is yet the number of continuous derivatives. NURBS spaces have a canonical basis whose elements are

$$B^N_{\mathbf{i},\mathbf{d}} = \frac{w_{\mathbf{i}} B^S_{\mathbf{i},\mathbf{d}}}{w}. \tag{2.9}$$

As for spline spaces, only NURBS spaces with $d_1 = d_2 = \cdots = d_n$ and the same regularity in all internal junctions are considered:

$$N_{d,r,w}(\Theta = I_1 \times \cdots \times I_n) = \{\frac{f}{w} :\ f \in S_{d,r}(\Theta)\}, \tag{2.10a}$$

$$N^0_{d,r,w}(\Theta = I_1 \times \cdots \times I_n) = \{\frac{f}{w} :\ f \in S^0_{d,r}(\Theta)\}. \tag{2.10b}$$

NURBS-maps from Θ to $\mathbb{R}^n$ are built as spline-maps: choose a NURBS space N and select a control point $\bar{\mathbf{x}}_{\mathbf{i}} \in \mathbb{R}^n$ for each $B^N_{\mathbf{i},\mathbf{d}}$, then the map is

$$F(x_1, \ldots, x_n) = \sum_{\mathbf{i}} B^N_{\mathbf{i},\mathbf{d}}(x_1, \ldots, x_n)\bar{\mathbf{x}}_{\mathbf{i}}. \tag{2.11}$$

2.3 *Geometry and discrete spaces*

In the isogeometric method, the domain Ω is parametrized over a rectangular (or cuboid) domain $\Theta = I_1 \times \cdots \times I_n$ by a NURBS map F. Let w be the piecewise polynomial denominator of F, then each discrete space $\mathscr{V}_{(\Xi_1,\dots,\Xi_n),(d_1,\dots,d_n)}(\Omega)$ is the push-forward through F of a NURBS space $N_{(\Xi_1,\dots,\Xi_n),(d_1,\dots,d_n),w}(\Theta)$ whose weight function is the denominator of F, w:

$$\mathscr{V}_{(\Xi_1,\dots,\Xi_n),(d_1,\dots,d_n)}(\Omega) = \{f : f \circ F \in N_{(\Xi_1,\dots,\Xi_n),(d_1,\dots,d_n),w}(\Theta)\}. \tag{2.12}$$

The canonical basis of the discrete spaces is the set of

$$B^{\mathscr{V}}_{\mathbf{i},\mathbf{d}} = B^{N}_{\mathbf{i},\mathbf{d}} \circ F^{-1}. \tag{2.13}$$

Remark that both the map F and the weight function w are determined by the geometry of Ω so are common to all discrete spaces defined on Ω.

The knot vectors of a discrete space naturally define a mesh for the parametric domain Θ. Let $b_{i,j}$ be the j^{th} junction point in the i^{th} dimension. The induced mesh for Θ is the set of the elements $K_{\Theta,\mathbf{j}}$

$$K_{\Theta,\mathbf{j}} = [b_{1,\mathbf{j}_1-1}, b_{1,\mathbf{j}_1}] \times \cdots \times [b_{n,\mathbf{j}_n-1}, b_{n,\mathbf{j}_n}]. \tag{2.14}$$

A corresponding mesh for Ω is the set of the elements

$$K_{\mathbf{j}} = F(K_{\Theta,\mathbf{j}}). \tag{2.15}$$

Vice versa, giving the degree, the regularity and the junction points in each dimension of Θ, a unique discrete space is identified. Thus the correspondence between meshes and discrete spaces with given degree and regularity is one to one. To simplify the notation, and avoid carrying around the junction points, $\mathscr{T}_h$ is used to denote a generic mesh for Ω whose maximum diameter of the elements is less than h; the corresponding discrete spaces of degree d, and regularity r are:

$$\mathscr{V}_{d,r,h}(\Omega) = \{f : f \circ F^{-1} \in N_{d,r,w}(\Theta)\}, \tag{2.16a}$$

$$\mathscr{V}^0_{d,r,h}(\Omega) = \{f : f \circ F^{-1} \in N^0_{d,r,w}(\Theta)\}. \tag{2.16b}$$

2.3.1 *Regularity.* There are two regularity requirements:

- the regularity of the domain, which is expressed by the regularity of F; in particular F must be invertible and

$$F \in C^1(\bar{\Theta}), \tag{2.17}$$

$$F^{-1} \in C^1(\bar{\Omega}), \tag{2.18}$$

- the regularity of the meshes

$$\exists \zeta : \forall h,\ \mathscr{T}_h, K \in \mathscr{T}_h \qquad \frac{h_K}{\rho_K} \leqslant \zeta, \tag{2.19}$$

  where h_K is the diameter of the element K and ρ_K is the maximum diameter of a contained circle. This condition implies both shape regularity of the elements and local quasi uniformity of the mesh. Note that this condition (assuming domain regularity) is equivalent to

$$\exists \zeta_\Theta : \forall h,\ \mathscr{T}_h, K \in \mathscr{T}_h \qquad \frac{h_{K_\Theta}}{\rho_{K_\Theta}} \leqslant \zeta_\Theta. \tag{2.20}$$

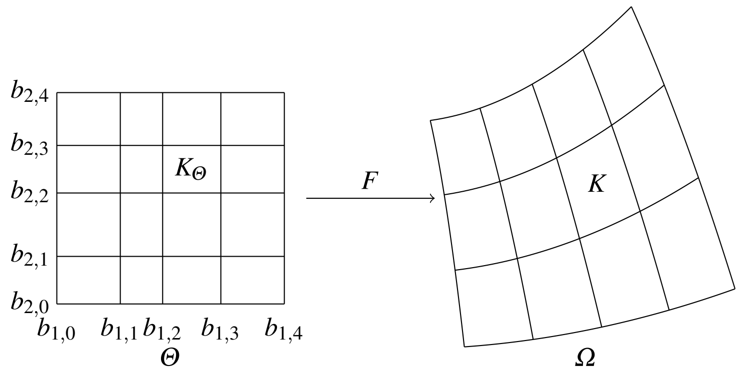

Figure 2. Scheme of the geometrical setup.

2.4 *Projections and approximation properties*

In this subsection, some of the results presented in Bazilevs et al. (2006) are summarized.

The projection operator $\pi_{\mathscr{V}_h}$ from $L^2(\Omega)$ to $\mathscr{V}_{d,r,h}(\Omega)$ is defined in terms of an auxiliary operator π_{S_h} from $L^2(\Theta)$ to $S_{d,r,h}(\Theta)$:

$$\pi_{S_h} f = \sum_{\mathbf{i}} \langle B_{\mathbf{i}}^{S*}, f \rangle B_{\mathbf{i}}^S, \tag{2.21}$$

where $\{B_{\mathbf{i}}^S\}$ is the canonical basis of $S_{d,r,h}(\Theta)$ and $\{B_{\mathbf{i}}^{S*}\}$ is a dual basis defined in $L^2(\Theta)$: i.e. $B_{\mathbf{i}}^{S*} \in L^2(\Theta)$ and

$$\langle B_{\mathbf{i}}^{S*}, B_{\mathbf{j}}^S \rangle = \delta_{\mathbf{i},\mathbf{j}}.$$

The definition of the projector is

$$\pi_{\mathscr{V}_h} f = \frac{\pi_{S_h}(w f \circ F)}{w} \circ F^{-1}. \tag{2.22}$$

It is possible to define an analogous operator $\pi^0_{\mathscr{V}_h}$ from $H^1_0(\Omega)$ to $\mathscr{V}_{d,r,h}(\Omega) \cap H^1_0(\Omega)$ by restricting the definition (2.21) to a the basis $\{B_{\mathbf{i}}\}$ of $S_{d,r}(\Theta) \cap H^1_0(\Theta)$:

$$\pi^0_{S_h} f = \sum_{\mathbf{i}} \langle B_{\mathbf{i}}{}^*, f \rangle B_{\mathbf{i}},$$

$$\pi^0_{\mathscr{V}_h} f = \frac{\pi^0_{S_h}(w f \circ F)}{w} \circ F^{-1}.$$

Define $\widetilde{K}$, the *support extension* of K, as

$$\widetilde{K} = \bigcup_{\mathbf{i}:\ \operatorname{Supp} B^{\mathscr{V}}_{\mathbf{i},\mathbf{d}} \supset K} \operatorname{Supp} B^{\mathscr{V}}_{\mathbf{i},\mathbf{d}}, \tag{2.23}$$

where $\operatorname{Supp} f$ is the support of f. Then the following approximation property holds: $\forall f \in H^l(\widetilde{K})$, $\forall 0 \leqslant k \leqslant l \leqslant d+1$,

$$\left| f - \pi_{\mathscr{V}_h} f \right|_{H^k(K)} \leqslant h_K^{l-k} C_{shape} \sum_{i=0}^{l} \|\nabla F\|^{i-l}_{L^\infty(F^{-1}(\widetilde{K}))} \left| f \right|_{H^i(\widetilde{K})}, \tag{2.24}$$

where C_{shape} is dependent on the shape of K but not on its diameter h_K. The estimate (2.24) holds also for $\pi^0_{\mathscr{V}_h}$, provided that $f \in H^l(\widetilde{K}) \cap H^1_0(\Omega)$. In particular, for f in $H^1_0(\Omega)$,

$$h_K^{-1} \left\| f - \pi^0_{\mathscr{V}_h} f \right\|_{L^2(K)} \leqslant C_{shape} \max\{1, \|\nabla F\|_{L^\infty(\Theta)}\} \|f\|_{H^1(\widetilde{K})} . \tag{2.25}$$

Summing over all elements, gives the important approximation property

$$\Big(\sum_{K \in \mathscr{T}_h} h_K^{-2} \left\| f - \pi^0_{\mathscr{V}_h} f \right\|^2_{L^2(K)} \Big)^{\frac{1}{2}} \leqslant C_{approx} \|f\|_{H^1(\Omega)} , \tag{2.26}$$

where $C_{approx} = C_{shape} C_{space}^{\frac{1}{2}} \max\{1, \|\nabla F\|_{L^\infty(\Theta)}\}$ and C_{space} is the maximum number of the $\widetilde{K}_i$'s that contains a given element K. For $\mathscr{V}_{d,r,h}$, $C_{space} = \lceil \frac{d+1}{d-r} \rceil^n$. In the same way, the continuity of $\pi^0_{\mathscr{V}_h}$ is obtained:

$$\left\| \pi^0_{\mathscr{V}_h} f \right\|_{H^1(\Omega)} \leqslant C_{cont} \|f\|_{H^1(\Omega)} . \tag{2.27}$$

3. Discretization and theoretical analysis

3.1 *Discrete problem*

Let Θ be the parametric domain, Ω be the domain, $F \in C^1(\bar{\Theta})$ be the parametrization map, $\mathscr{T}_h$ be a mesh such that the regularity condition (2.19) holds, and

$$V_h = \mathscr{V}_{3,1,h}(\Omega)^n \cap H^1_0(\Omega)^n, \tag{3.1}$$

$$P_h = \mathscr{V}_{2,1,h}(\Omega) \cap L^2_0(\Omega). \tag{3.2}$$

The discrete problem corresponding to $\mathscr{T}_h$ is: find $\mathbf{v}_h \in V_h$ and $p_h \in P_h$ such that

$$\nu \langle \nabla \mathbf{v}_h, \nabla \mathbf{w} \rangle - \langle p_h, \nabla \cdot \mathbf{w} \rangle = \langle \mathbf{f}_h, \mathbf{w} \rangle \qquad \forall \mathbf{w} \in V_h, \tag{3.3a}$$

$$\langle \nabla \cdot \mathbf{v}_h, q \rangle = 0 \qquad \forall q \in P_h. \tag{3.3b}$$

Sufficient conditions for well-posedness, stability and continuous dependence of $\mathbf{v}_h$ and p_h upon $\mathbf{f}_h$ are (see Brezzi & Fortin (1991)):

- coercivity: $\exists C_{coerc} : \ \forall h, \mathscr{T}_h, \mathbf{w} \in V_h$

$$\langle \nabla \mathbf{w}, \nabla \mathbf{w} \rangle \geqslant C_{coerc} \|\mathbf{w}\|^2_{H^1(\Omega)} , \tag{3.4}$$

- inf sup condition: $\exists C_{\inf\sup} > 0 : \ \forall h, \mathscr{T}_h$

$$\inf_{q \in P_h} \sup_{\mathbf{w} \in V_h} \frac{\langle \nabla \cdot \mathbf{w}, q \rangle}{\|\mathbf{w}\|_{H^1(\Omega)^n}} \geqslant C_{\inf\sup} \|q\|_{L^2(\Omega)} . \tag{3.5}$$

From these conditions, the following error estimate can be derived (see Brezzi & Fortin (1991)):

$$\|\mathbf{v} - \mathbf{v}_h\|_{H^1(\Omega)^n} + \|p - p_h\|_{L^2(\Omega)} \leqslant C_{solution} \Big(\inf_{\mathbf{w} \in V_h} \|\mathbf{v} - \mathbf{w}\|_{H^1(\Omega)^n} + \inf_{q \in P_h} \|p - q\|_{L^2(\Omega)} \Big). \tag{3.6}$$

The first condition is satisfied on all $H^1_0(\Omega)$, as a consequence of the Poincaré inequality. The inf sup condition is proved in the next subsections.

3.2 *Verfürth trick*

It is known (see Girault & Raviart (1986)) that $\exists \mathring{C}_{\inf\sup} > 0$:

$$\inf_{q \in L^2_0(\Omega)} \sup_{\mathbf{w} \in H^1_0(\Omega)^n} \frac{\langle \nabla \cdot \mathbf{w}, q \rangle}{\|\mathbf{w}\|_{H^1(\Omega)^n}} \geqslant \mathring{C}_{\inf\sup} \|q\|_{L^2(\Omega)} . \tag{3.7}$$

Thus, for each $q \in P_h$, there exists $\bar{\mathbf{w}} \in H^1_0(\Omega)^n$ such that

$$\langle \nabla \cdot \bar{\mathbf{w}}, q \rangle \geqslant \mathring{C}_{\inf\sup} \|q\|^2_{L^2(\Omega)} , \tag{3.8}$$

$$\|\bar{\mathbf{w}}\|_{H^1(\Omega)^n} = \|p\|_{L^2(\Omega)} . \tag{3.9}$$

The projection $\pi^0_{\mathscr{V}_h} \bar{\mathbf{w}}$ of $\bar{\mathbf{w}}$ in V_h can be decomposed as $\pi^0_{\mathscr{V}_h} \bar{\mathbf{w}} = \bar{\mathbf{w}} - (\bar{\mathbf{w}} + \pi^0_{\mathscr{V}_h} \bar{\mathbf{w}})$. Using this decomposition, equation (3.8) and integration by parts gives

$$\begin{aligned} \langle \nabla \cdot \pi^0_{\mathscr{V}_h} \bar{\mathbf{w}}, q \rangle &= \langle \nabla \cdot \bar{\mathbf{w}}, q \rangle + \langle \nabla \cdot (\pi^0_{\mathscr{V}_h} \bar{\mathbf{w}} - \bar{\mathbf{w}}), q \rangle \\ &\geqslant \mathring{C}_{\inf\sup} \|q\|^2_{L^2(\Omega)} + \langle \pi^0_{\mathscr{V}_h} \bar{\mathbf{w}} - \bar{\mathbf{w}}, \nabla q \rangle. \end{aligned} \tag{3.10}$$

Moreover, the second term on the right side of (3.10) can be written as a sum over all elements and bounded by

$$\begin{aligned} \left| \langle \pi^0_{\mathscr{V}_h} \bar{\mathbf{w}} - \bar{\mathbf{w}}, \nabla q \rangle \right| &\leqslant \sum_{K \in \mathscr{T}_h} \int_K \left| (\pi^0_{\mathscr{V}_h} \bar{\mathbf{w}} - \bar{\mathbf{w}}) \cdot \nabla q \right| d\mathbf{x} \\ &\leqslant \sum_{K \in \mathscr{T}_h} h_K^{-1} \left\| \pi^0_{\mathscr{V}_h} \bar{\mathbf{w}} - \bar{\mathbf{w}} \right\|_{L^2(K)^n} h_K \|\nabla q\|_{L^2(K)^n} \\ &\leqslant \Big(\sum_{K \in \mathscr{T}_h} h_K^{-2} \left\| \pi^0_{\mathscr{V}_h} \bar{\mathbf{w}} - \bar{\mathbf{w}} \right\|^2_{L^2(K)^n} \Big)^{\frac{1}{2}} \Big(\sum_{K \in \mathscr{T}_h} h_K^2 \|\nabla q\|^2_{L^2(\Omega)^n} \Big)^{\frac{1}{2}}. \end{aligned}$$

From the approximation properties of $\pi^0_{\mathscr{V}_h}$ (2.26) and (3.9), the estimate

$$\begin{aligned} \left| \langle \pi^0_{\mathscr{V}_h} \bar{\mathbf{w}} - \bar{\mathbf{w}}, \nabla q \rangle \right| &\leqslant C_{approx} \|\bar{\mathbf{w}}\|_{H^1(\Omega)} \Big(\sum_{K \in \mathscr{T}_h} h_K^2 \|\nabla q\|^2_{L^2(\Omega)^n} \Big)^{\frac{1}{2}} \\ &\leqslant C_{approx} \|q\|_{L^2(\Omega)} \Big(\sum_{K \in \mathscr{T}_h} h_K^2 \|\nabla q\|^2_{L^2(\Omega)^n} \Big)^{\frac{1}{2}} \end{aligned}$$

follows. Since $P_h \subset L^2_0(\Omega)$, the expression $\left(\sum_{K \in \mathscr{T}_h} h_K^2 \|\nabla q\|^2_{L^2(K)^n} \right)^{\frac{1}{2}}$ defines a norm for P_h, hereafter called $\|\cdot\|_{P_h}$:

$$\|q\|_{P_h} = \Big(\sum_{K \in \mathscr{T}_h} h_K^2 \|\nabla q\|^2_{L^2(K)^n} \Big)^{\frac{1}{2}}. \tag{3.11}$$

Inserting these results within (3.10) gives

$$\langle \nabla \cdot \pi^0_{\mathscr{V}_h} \bar{\mathbf{w}}, q \rangle \geqslant \mathring{C}_{\inf\sup} \|q\|^2_{L^2(\Omega)} - C_{approx} \|q\|_{L^2(\Omega)} \|q\|_{P_h} , \tag{3.12}$$

from which, using (2.27) and (3.9), it follows

$$\sup_{\mathbf{w} \in V_h} \frac{\langle \nabla \cdot \mathbf{w}, q \rangle}{\|\mathbf{w}\|_{H^1(\Omega)}} \geqslant \frac{\mathring{C}_{\inf\sup}}{C_{cont}} \|q\|_{L^2(\Omega)} - \frac{C_{approx}}{C_{cont}} \|q\|_{P_h} . \tag{3.13}$$

The Verfürth trick Verfürth (1984) consists in reducing the inf sup condition (3.5) to the validity of (3.13) and of the following property: $\exists C_{\mathrm{Verf}}: \ \forall h, \ \mathscr{T}_h, \ q \in P_h$

$$\sup_{\mathbf{w}\in V_h} \frac{\langle \nabla\cdot\mathbf{w}, q\rangle}{\|\mathbf{w}\|_{H^1(\Omega)}} \geqslant C_{\mathrm{Verf}} \|q\|_{P_h}. \tag{3.14}$$

Indeed, suppose that (3.14) holds and call $t = \frac{\|q\|_{P_h}}{\|q\|_{L^2(\Omega)}}$ then combining (3.13) and (3.14) gives

$$\sup_{\mathbf{w}\in V_h} \frac{\langle \nabla\cdot\mathbf{w}, q\rangle}{\|\mathbf{w}\|_{H^1(\Omega)}} \geqslant \min_{t>0}\Big(\max\{C_{\mathrm{Verf}}\, t, \frac{\mathring{C}_{\inf\sup}}{C_{cont}} - \frac{C_{approx}}{C_{cont}} t\}\Big) \|q\|_{L^2(\Omega)}, \tag{3.15}$$

thus

$$\inf_{q\in P_h} \sup_{\mathbf{w}\in V_h} \frac{\langle \nabla\cdot\mathbf{w}, q\rangle}{\|\mathbf{w}\|_{H^1(\Omega)}} \geqslant C_{\inf\sup} \|q\|_{L^2(\Omega)},$$

where $C_{\inf\sup} = \frac{C_{\mathrm{Verf}}\mathring{C}_{\inf\sup}}{C_{\mathrm{Verf}}C_{cont}+C_{approx}}$.

Summarizing, the existence of C_{Verf} such that (3.14) holds is sufficient to get the inf sup condition (3.5), and thus to get stability and convergence of the method.

To simplify the next subsections, it is possible to reduce (3.14) to: $\exists C^*_{\mathrm{Verf}}: \ \forall h, \ \mathscr{T}_h, \ q \in P_h$

$$\sup_{\mathbf{w}\in V_h} \frac{\langle \nabla\cdot\mathbf{w}, q\rangle}{|\mathbf{w}|_{H^1(\Omega)}} \geqslant C^*_{\mathrm{Verf}} \|q\|_{P_h}, \tag{3.16}$$

Indeed by the Poincaré inequality: $\|\mathbf{w}\|_{H^1(\Omega)^n} \leqslant (1 + C_P(\Omega)) |\mathbf{w}|_{H^1(\Omega)^n}$, equation (3.16) implies

$$\sup_{\mathbf{w}\in V_h} \frac{\langle \nabla\cdot\mathbf{w}, q\rangle}{\|\mathbf{w}\|_{H^1(\Omega)}} \geqslant C^*_{\mathrm{Verf}}(1 + C_P(\Omega))^{-1} \|q\|_{P_h}. \tag{3.17}$$

3.3 *Macro-element technique*

The proof of (3.16) is based on the macro-element technique (see Stenberg (1984) and Stenberg (1990)) that consists in reducing it to the validity of the same inf sup condition on suitable macro-elements.

A macro-element $\mathscr{M}$ is a subset of $\mathscr{T}_h$ such that each contained element is connected to the union of the others elements by at least a face. Each macro-element $\mathscr{M}$ naturally defines a domain $M = \bigcup_{K\in\mathscr{M}} K$ and a corresponding macro-element on the parametric domain $\mathscr{M}_\Theta = \{K_\Theta : K \in \mathscr{M}\}$. On each macro-element $\mathscr{M}$ consider the local discrete spaces:

$$V_M = \{\mathbf{f}|_M : \mathbf{f} \in V_h, \ \mathrm{Supp}\,\mathbf{f} \subset M\}, \tag{3.18}$$

$$P_M = \{p|_M : p \in P_h, \ \int_M p\, d\mathbf{x} = 0\}. \tag{3.19}$$

The functions on V_M are identified with their zero extension to the domain Ω, and their norm is $\|\cdot\|_{H^1(M)^n} = \|\cdot\|_{H^1(\Omega)^n}$. On P_M define the norm

$$\|q\|_{P_M} = \Big(\sum_{K\in\mathscr{M}} h_K^2 \|\nabla q\|^2_{L^2(K)^n}\Big)^{\frac{1}{2}}. \tag{3.20}$$

There is a natural projection π_{P_M} from P_h to P_M given by

$$\pi_{P_M} f = f|_M - |M|^{-1} \int_M f \, d\mathbf{x}, \tag{3.21}$$

where $|M|$ is the measure of M.

For each mesh $\mathscr{T}_h$, let $\mathfrak{M}_h$ be the set of all macro-elements $\mathscr{M}$ whose $\mathscr{M}_\Theta$ contains 4^n elements laid out in a “hypercube” of side 4. Then, for sufficiently fine meshes, this choice guarantees the conditions:

- $\forall \mathscr{T}_h, \forall K \in \mathscr{T}_h$, there exists a macro-element in $\mathfrak{M}_h$ containing K,

- $\forall \mathscr{T}_h, \forall K \in \mathscr{T}_h$, there are at most $C_{overlap}$ macro-elements in $\mathfrak{M}_h$ containing K, with $C_{overlap} = 4^n$,

- $\forall \mathscr{T}_h, \forall \mathscr{M} \in \mathfrak{M}_h$, $\mathscr{M}$ contains (at most) C_{elem} elements, with $C_{elem} = 4^n$.

Suppose that $\exists C_{macro} : \ \forall \mathscr{T}_h, \ \forall \mathscr{M} \in \mathfrak{M}_h$

$$\inf_{q \in P_M} \sup_{\mathbf{w} \in V_M} \frac{\langle \mathbf{w}, \nabla q \rangle}{|\mathbf{w}|_{H^1(\Omega)^n}} \geqslant C_{macro} \, ||q||_{P_M} \, , \tag{3.22}$$

then (3.16) holds. In fact, let q be a function of P_h and, for each macro-element $\mathscr{M}$ of $\mathfrak{M}_h$, let $\mathbf{w}_M \in V_M$ be such that

$$\begin{aligned} &\langle \mathbf{w}_M, \nabla \pi_{P_M} q \rangle \geqslant C_{macro} \, ||\pi_{P_M} q||^2_{P_M} \, , \\ &|\mathbf{w}_M|_{H^1(\Omega)^n} = ||\pi_{P_M} q||_{P_M} \, , \end{aligned}$$

then

$$\begin{aligned} \sup_{\mathbf{w} \in V_h} \langle \mathbf{w}, \nabla q \rangle &\geqslant \langle \sum_{\mathscr{M} \in \mathfrak{M}_h} \mathbf{w}_M, \nabla q \rangle = \sum_{\mathscr{M} \in \mathfrak{M}_h} \langle \mathbf{w}_M, \nabla q \rangle \\ &\geqslant \sum_{\mathscr{M} \in \mathfrak{M}_h} C_{macro} \, ||\pi_{P_M} q||^2_{P_M} \\ &= C_{macro} \sum_{\mathscr{M} \in \mathfrak{M}_h} \sum_{K \in \mathscr{M}} h_K^2 \, ||\nabla q||^2_{L^2(K)^n} \\ &\geqslant C_{macro} \sum_{K \in \mathscr{T}_h} h_K^2 \, ||\nabla q||^2_{L^2(K)^n} = C_{macro} \, ||q||^2_{P_h} \, , \end{aligned}$$

and

$$\begin{aligned}
\Big|\sum_{\mathcal{M}\in\mathfrak{M}_h} \mathbf{w}_M\Big|^2_{H^1(\Omega)} &= \sum_{K\in\mathcal{T}_h}\Big|\sum_{\mathcal{M}\in\mathfrak{M}_h} \mathbf{w}_M\Big|^2_{H^1(K)} \\
&\leqslant \sum_{K\in\mathcal{T}_h}\Big(\sum_{\substack{\mathcal{M}\in\mathfrak{M}_h\\ K\in\mathcal{M}}} |\mathbf{w}_M|_{H^1(K)}\Big)^2 \\
&\leqslant C_{overlap}\sum_{K\in\mathcal{T}_h}\sum_{\substack{\mathcal{M}\in\mathfrak{M}_h\\ K\in\mathcal{M}}} |\mathbf{w}_M|^2_{H^1(K)} \\
&= C_{overlap}\sum_{K\in\mathcal{T}_h}\sum_{\substack{\mathcal{M}\in\mathfrak{M}_h\\ K\in\mathcal{M}}} \|\pi_{P_M} q\|^2_{P_M} \\
&\leqslant C_{overlap}C_{elem}\sum_{\mathcal{M}\in\mathfrak{M}_h}\sum_{K\in\mathcal{M}} h_K^2\,\|\nabla q\|^2_{L^2(K)^n} \\
&\leqslant C^2_{overlap}C_{elem}\,\|q\|^2_{P_h}.
\end{aligned}$$

Combining the above estimates gives (3.16) with

$$C^*_{\mathrm{Verf}} = C_{macro}C^{-1}_{overlap}C^{-\frac{1}{2}}_{elem}. \tag{3.23}$$

The proof of the existence of C_{macro} such that (3.22) holds is composed of three logical steps:

1. proof for the case when F is the identity map and thus the discrete spaces are spline spaces,
2. proof of the existence of $\bar{h} : \forall h < \bar{h}, \forall \mathcal{T}_h$ there exists C_{macro} in the case of NURBS spaces on the parametric domain,
3. proof of the existence of $\bar{\bar{h}} : \forall h < \bar{h}, \forall \mathcal{T}_h$ there exists C_{macro} in the case of isogeometric spaces on the physical domain.

Each step corresponds to a subsection, one more section is put between the first and the second step which contains the study of the relations between the used norms.

3.4 *Spline spaces on Θ*

Let $\mathcal{F}$ be the family of all the "abstract macro-elements" $\mathcal{M}_a$ containing the 4^n elements defined by

$$K_\alpha = \bigotimes_{i=1}^{n} K^i_{\alpha_i}, \qquad \alpha = (\alpha_1,\dots,\alpha_n),\ 1 \leqslant \alpha_i \leqslant 4,\ i = 1,\dots,n, \tag{3.24}$$

where

$$\begin{aligned}
&\left|K^i_j\right| = l_{i,j} > 0, && i = 1,\dots,n,\\
&(K^i_1, K^i_2, K^i_3, K^i_4) \text{ form a partition of } [0, \textstyle\sum_{j=1}^{4} l_{i,j}], && i = 1,\dots,n,\\
&\forall \mathcal{M}_a \in \mathcal{F}, \forall K \in \mathcal{M}_a \quad \frac{h_K}{\rho_K} \leqslant \zeta_\Theta, && \text{see (2.20).}
\end{aligned}$$

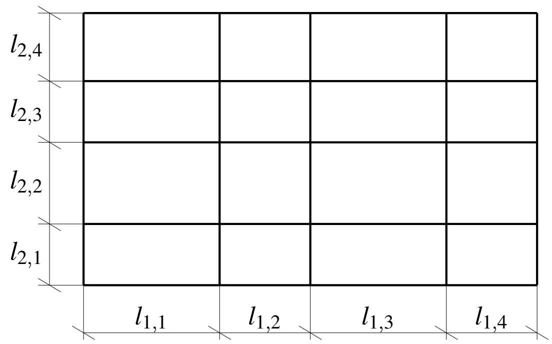

Figure 3. An "abstract macro-element" in $\mathscr{F}$ and its coordinates, $n = 2$.

The $l_{i,j}$'s are a set of coordinates for $\mathscr{F}$ and induce the topology of $\mathbb{R}^{4n}$ on $\mathscr{F}$. For each "abstract macro-element" $\mathscr{M}_a \in \mathscr{F}$, define the discrete spaces:

$$V_{M_a} = S^0_{3,1}(M_a)^n, \tag{3.25}$$

$$P_{M_a} = S_{2,1}(M_a) \cap L^2_0(M_a), \tag{3.26}$$

with norms

$$|\cdot|_{H^1(M_a)^n}, \tag{3.27}$$

$$\|q\|^2_{P_{M_a}} = \sum_{K \in \mathscr{M}_a} h_K^2 \, \|\nabla q\|^2_{L^2(K)^n}, \tag{3.28}$$

respectively.

On $\mathscr{F}$, it is possible to study the positive function

$$C_S(\mathscr{M}_a) = \inf_{\substack{q \in P_{M_a} \\ q \neq 0}} \sup_{\substack{\mathbf{w} \in V_{M_a} \\ \mathbf{w} \neq 0}} \frac{\langle \mathbf{w}, \nabla q \rangle}{|\mathbf{w}|_{H^1(M_a)^n} \, \|q\|_{P_{M_a}}}. \tag{3.29}$$

Note that C_S is scaling invariant: if $\mathscr{M}_1 = \lambda \mathscr{M}_2$ i.e. $l_{i,j}(\mathscr{M}_1) = \lambda l_{i,j}(\mathscr{M}_2)$, then $C_S(\mathscr{M}_1) = C_S(\mathscr{M}_2)$. Indeed setting $\hat{f}(x) = f(\lambda x)$ gives

$$\begin{aligned} \langle \mathbf{w}, \nabla q \rangle &\geqslant C_S(\mathscr{M}_1) \, |\mathbf{w}|_{H^1(M_1)} \, \|q\|_{P_{M_1}} \\ \Rightarrow \lambda^{n-1} \langle \hat{\mathbf{w}}, \nabla \hat{q} \rangle &\geqslant C_S(\mathscr{M}_1) \lambda^{\frac{n-2}{2}} \, |\hat{\mathbf{w}}|_{H^1(M_2)} \, \lambda^{\frac{n}{2}} \, \|\hat{q}\|_{P_{M_2}} \\ \Rightarrow \langle \hat{\mathbf{w}}, \nabla \hat{q} \rangle &\geqslant C_S(\mathscr{M}_1) \, |\hat{\mathbf{w}}|_{H^1(M_2)} \, \|\hat{q}\|_{P_{M_2}} \\ \Rightarrow C_S(\mathscr{M}_1) &\leqslant C_S(\mathscr{M}_2), \end{aligned} \tag{3.30}$$

then by symmetry $C_S(\mathscr{M}_1) \geqslant C_S(\mathscr{M}_2)$ thus they are equal to each other.

Let $\mathfrak{S}$ be the subset of $\mathscr{F}$ of the "abstract macro-elements" having unitary diameter: $\operatorname{diam} \mathscr{M}_a = 1$, then $\mathfrak{S}$ is closed and bounded, thus compact. Moreover, C_S is continuous in the chosen topology, thus it admits a minimum on $\mathfrak{S}$, let it be C_S^-. This minimum is absolute due to the scaling invariance of C_S.

The positivity of C_S^- is equivalent to the following property: $\forall \mathscr{M}_a \in \mathfrak{S}$ if $q \in P_{M_a}$ is such that $\forall \mathbf{w} \in V_{M_a}$ $\langle \mathbf{w}, \nabla q \rangle = 0$, then $\nabla q = 0$ and thus $q = 0$. This property can be checked by introducing GP such that $GP \supset \{\nabla q :\ q \in P_{M_a}\}$ and showing that if $0 \neq \mathbf{g} \in GP$ then there exists $\mathbf{w} \in V_M$ such that $\langle \mathbf{w}, \mathbf{g} \rangle \neq 0$. Recalling that $P_{M_a} = S_{2,1}(M_a) \cap L_0^2(M_a)$, define GP as

$$\begin{aligned} GP &= \frac{\partial}{\partial x_1} S_{2,1}(M_a) \times \cdots \times \frac{\partial}{\partial x_n} S_{2,1}(M_a) \\ &= S_{1,0}([0, \sum_{i=1}^{4} l_{1,i}]) \otimes S_{2,1}([0, \sum_{i=1}^{4} l_{2,i}]) \otimes \cdots \otimes S_{2,1}([0, \sum_{i=1}^{4} l_{n,i}]) \times \\ &\quad \cdots \\ &\quad \times S_{2,1}([0, \sum_{i=1}^{4} l_{1,i}]) \otimes S_{2,1}([0, \sum_{i=1}^{4} l_{2,i}]) \otimes \cdots \otimes S_{1,0}([0, \sum_{i=1}^{4} l_{n,i}]), \end{aligned} \tag{3.31}$$

that has independent components. If $0 \neq \mathbf{g} \in GP$ then at least one of its components is non zero. Without loss of generality, suppose $\mathbf{g}_1 \neq 0$. We show that there exists $\mathbf{w} \in V_{M_a}$ with $\mathbf{w} = (\mathbf{w}_1, 0, \ldots, 0)$ such that $\langle \mathbf{w}, \mathbf{g} \rangle > 0$. Indeed, $\mathbf{g}_1$ admits a decomposition of the form

$$\mathbf{g}_1(x) = \sum_{\mathbf{i}} \beta_{\mathbf{i}} B_{\mathbf{i}}^{GP}(x) = \sum_{\mathbf{i}} \beta_{\mathbf{i}} \prod_{j=1}^{n} B_{\mathbf{i}_j}^{GP}(x_j). \tag{3.32}$$

If there were functions $B_i^V(x_j) \in S_{3,1}([0, \sum_{z=1}^4 l_{j,i}])$ such that $\langle B_i^{GP}(x_j), B_i^V(x_j) \rangle = \delta_{i,j}$, then choosing

$$\mathbf{w}_1 = \sum_{\mathbf{i}} \beta_{\mathbf{i}} \prod_{j=1}^{n} B_{\mathbf{i}_j}^{V}(x_j), \tag{3.33}$$

and applying the Fubini-Tonelli decomposition theorem, it would follow

$$\langle \mathbf{w}, \mathbf{g} \rangle = \sum_{\mathbf{i}} \beta_{\mathbf{i}}^2 > 0. \tag{3.34}$$

The existence of the $B_{\mathbf{i}}^V(x_j)$'s is implied by the fact that, for all space dimensions, and for every interval $I = [0, \sum_{i=1}^4 l_i]$, the ranks of the matrices associated with the L^2-scalar product between $S_{2,1}(I)$ and $S_{3,1}^0(I)$, and between $S_{1,0}(I)$ and $S_{3,1}^0(I)$ equal the dimension of $S_{2,1}(I)$. This is proved by calculating the determinant of the upper-leftmost minor of order $\dim S_{2,1}(I)$ of these matrices expressed in the canonical basis. The expressions of the determinants in terms of the lengths of the subsegments $l_1, \ldots, l_4$ are

$S_{3,1}^0(I)$ against $S_{2,1}(I)$

$$\begin{aligned} &\frac{l_1 l_4}{43200000(l_1+l_2)(l_2+l_3)(l_3+l_4)} \\ &[2l_2(l_2+l_3)+l_1(2l_2+l_3)] \cdot [2l_3(l_3+l_4)+l_2(2l_3+l_4)] \\ &[l_1(l_3(l_3+l_4)+l_2(2l_3+l_4))+l_2(2l_3(l_3+l_4)+l_2(2l_3+l_4))], \end{aligned} \tag{3.35}$$

$S_{3,1}^0(I)$ against $S_{1,0}(I)$

$$\frac{l_1 l_3 l_4^2}{640000(l_4+l_3)}[(9l_2+5l_1)l_3+(9l_2^2+9l_1l_2))]. \tag{3.36}$$

Since all the coefficients are positive, also the determinants are (assuming positive lengths), thus $C_S^- > 0$.

Concluding for every regular mesh $\mathscr{T}_h$, for all macro-element $\mathscr{M} \in \mathfrak{M}_h$ there is a translation $\mathscr{M}_a$ of $\mathscr{M}$ that is in $\mathscr{F}$. Leaving the translation implicit, $V_{M_a} \subset V_M$ and $P_{M_a} = P_M$ thus (3.22) holds with $C_{macro} = C_S^-$.

The spaces V_{M_a} and V_M differ only if $\partial M \cap \partial \Theta \neq \emptyset$ since the functions of V_{M_a} have null gradient on $\partial \Theta$, where those in V_M do not. The use of smaller spaces is a stricter condition, thus it is not necessary to threat differently the macro-elements that touch the boundary.

3.5 *Norm equivalences*

In the general isogeometric setting, it is possible to associate with each macro-element $\mathscr{M} \in \mathfrak{M}_h$ of any mesh $\mathscr{T}_h$ an abstract macro-element $\mathscr{M}_a \in \mathscr{F}$ by taking the unique translation of $\mathscr{M}_\Theta$ that is in $\mathscr{F}$. To simplify the notation $\mathscr{M}_a$ and $\mathscr{M}_\Theta$ are identified.

The relation between the discrete spaces on M and M_Θ require a deeper analysis than in the spline case. For each $\mathscr{T}_h$, $\mathscr{M} \in \mathfrak{M}_h$, $\mathbf{w} \in V_M$ and $q \in P_M$, set

$$\begin{aligned} \mathbf{w}_N &= \mathbf{w} \circ F, & \mathbf{w}_S &= w(\mathbf{w} \circ F), \\ q_N &= q \circ F, & q_S &= w(q \circ F), \end{aligned}$$

then the correspondences between $\mathbf{w}_S, \mathbf{w}_N$ and $\mathbf{w}$ and between q_S, q_N and q are one to one.

The space $V_{M_\Theta} = V_{M_a}$ is contained in the space of $\{\mathbf{w}_S : \mathbf{w} \in V_M\}$. As previously noted they differ only if M touches on the boundary of Ω. We now prove that there exist $C_{V_M}^{V_{M_\Theta}}$, $C_{V_{M_\Theta}}^{V_M} > 0$ such that $\forall \mathscr{T}_h, \mathscr{M} \in \mathfrak{M}_h, \mathbf{w} \in V_M$:

$$C_{V_M}^{V_{M_\Theta}\,-1} |\mathbf{w}_S|_{H^1(M_\Theta)^n} \leqslant |\mathbf{w}|_{H^1(M)^n} \leqslant C_{V_{M_\Theta}}^{V_M} |\mathbf{w}_S|_{H^1(M_\Theta)^n}. \tag{3.37}$$

Indeed

$$
\begin{aligned}
|\mathbf{w}_S|^2_{H^1(M_\Theta)^n} &= \int_{M_\Theta} \|\nabla \mathbf{w}_S\|^2 \, d\mathbf{x} = \int_{M_\Theta} \|\nabla (w\,\mathbf{w}\circ F)\|^2 \, d\mathbf{x} \\
&= \int_{M_\Theta} \|\nabla w(\mathbf{w}\circ F) + w\nabla(\mathbf{w}\circ F)\|^2 \, d\mathbf{x} \\
&\leqslant 2\int_{M_\Theta} \|\nabla w(\mathbf{w}\circ F)\|^2 + \|w\nabla(\mathbf{w}\circ F)\|^2 \, d\mathbf{x} \\
&\leqslant 2\,\|\nabla w\|^2_{L^\infty(\Theta)} \int_{M_\Theta} \|\mathbf{w}\circ F\|^2 \, d\mathbf{x} + \\
&\qquad 2\,\|w\|^2_{L^\infty(\Theta)} \int_{M_\Theta} \|(\nabla \mathbf{w}\circ F)\nabla F\|^2 \, d\mathbf{x} \\
&= 2\,\|\nabla w\|^2_{L^\infty(\Theta)} \int_{M} \|\mathbf{w}\|^2 \left|\det F^{-1}\right| \, d\mathbf{x} + \\
&\qquad 2\,\|w\|^2_{L^\infty(\Theta)} \int_{M} \left\|\nabla \mathbf{w}(\nabla F\circ F^{-1})\right\|^2 \left|\det F^{-1}\right| \, d\mathbf{x} \\
&\leqslant 2\left\|\det F^{-1}\right\|_{L^\infty(\Omega)} \left(\|\nabla w\|^2_{L^\infty(\Theta)} \|\mathbf{w}\|^2_{L^2(M)^n} + \right. \\
&\qquad \left. \|w\|^2_{L^\infty(\Theta)} \|\nabla F\|^2_{L^\infty(\Theta)} |\mathbf{w}|^2_{H^1(M)^n}\right) \\
&\leqslant 2\left\|\det F^{-1}\right\|_{L^\infty(\Omega)} \left(\|\nabla w\|^2_{L^\infty(\Theta)} C_P(\Omega)^2 + \right. \\
&\qquad \left. \|w\|^2_{L^\infty(\Theta)} \|\nabla F\|^2_{L^\infty(\Theta)}\right) |\mathbf{w}|^2_{H^1(M)^n} \\
&= {C_{V_M}^{V_{M_\Theta}}}^2 |\mathbf{w}|^2_{H^1(M)^n},
\end{aligned}
\tag{3.38}
$$

where $M \subset \Omega$ assures that the Poincaré constants satisfy $C_P(M) \leqslant C_P(\Omega)$. The other inequality follows by the same steps using $C_P(\Theta)$ instead of $C_P(\Omega)$.

In the general case P_{M_Θ} and $\{q_S : q \in P_M\}$ are distinct spaces since $\int_{M_\Theta} q_S \, d\mathbf{x}$ can be different from 0. Anyway it is possible to define a one to one correspondence between the functions $q_a \in P_{M_\Theta}$ and $q \in P_M$ by

$$
q_a = q_S - |M_\Theta|^{-1} \int_{M_\Theta} q_S \, d\mathbf{x}, \qquad q = \frac{q_a}{w}\circ F^{-1} - |M|^{-1}\int_M \frac{q_a}{w}\circ F^{-1} \, d\mathbf{x}.
$$

As for the velocities the norm of the associated functions are equivalents, but the proof requires some technical properties of the Poincaré-Wirtinger inequality: let D be a domain then there exists $C_{PW}(D) : \forall f \in H^1(D)\cap L^2_0(D)$

$$
\|f\|_{L^2(D)} \leqslant C_{PW}(D)\,|f|_{H^1(D)}\,. \tag{3.39}
$$

It is known that $C_{PW}(D)$ is the square root of the inverse of the second eigenvalue of the Neumann Laplacian, thus if $D = \bigotimes_{i=1}^n [0, l_i]$ and D satisfy the regularity condition (2.20) then

$$
C_{PW}(D) = \left(\sum_{i=0}^{n} \frac{\pi^2}{l_i^2}\right)^{-\frac{1}{2}} = \pi^{-1}\frac{\prod_{i=1}^n l_i}{\sqrt{\sum_{i=1}^n \prod_{j\neq i} l_j^2}} \leqslant \pi^{-1}\frac{l_{\max}^n}{\sqrt{n\frac{l_{\max}^{2n-2}}{\zeta_\Theta^{2n-2}}}} \leqslant \frac{\zeta_\Theta^{n-1} l_{\max}}{\pi\sqrt{n}}, \tag{3.40}
$$

where l_{max} is the longest edge of D.

If D is a subset of Θ, then it is possible to associate with each function $f \in H^1(D) \cap L_0^2(D)$ a function $\hat{f} \in H^1(F(D)) \cap L_0^2(F(D))$ defined by $\hat{f} = f \circ F^{-1} - |F(D)|^{-1} \int_{F(D)} f \circ F^{-1}\, d\mathbf{x}$. Then for all $f \in H^1(D) \cap L_0^2(D)$

$$\begin{aligned}\left\|\hat{f}\right\|^2_{L^2(F(D))} &\leqslant 2\left\|f \circ F^{-1}\right\|^2_{L^2(F(D))} + 2|F(D)|^{-1}\left(\int_{F(D)} f \circ F^{-1}\, d\mathbf{x}\right)^2 \\ &\leqslant 2\left\|f \circ F^{-1}\right\|^2_{L^2(F(D))} + 2|F(D)|^{-1}\left\|f \circ F^{-1}\right\|^2_{L^2(F(D))}\|1\|^2_{L^2(F(D))} \\ &\leqslant 2\left\|f \circ F^{-1}\right\|^2_{L^2(F(D))} + 2\left\|f \circ F^{-1}\right\|^2_{L^2(F(D))} \\ &\leqslant 4\left\|f \circ F^{-1}\right\|^2_{L^2(F(D))} \leqslant 4\|\det F\|_{L^\infty(\Theta)}\|f\|^2_{L^2(D)},\end{aligned} \tag{3.41}$$

and

$$|f|_{H^1(D)} \leqslant \left\|\det F^{-1}\right\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\nabla F\|_{L^\infty(\Theta)}\left|\hat{f}\right|_{H^1(F(D))}. \tag{3.42}$$

Combining (3.41) and (3.42) gives

$$C_{PW}(F(D)) \leqslant 2\left\|\det F^{-1}\right\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\det F\|^{\frac{1}{2}}_{L^\infty(\Theta)}\|\nabla F\|_{L^\infty(\Theta)} C_{PW}(D). \tag{3.43}$$

We now prove that there exist $C_{P_{M_\Theta}}^{L^2(M_\Theta)}$ and $C_{P_M}^{L^2(M)}$ such that $\forall \mathscr{T}_h, \mathscr{M} \in \mathfrak{M}_h, q \in P_M$

$$\|q_a\|_{L^2(M_\Theta)} \leqslant C_{P_{M_\Theta}}^{L^2(M_\Theta)}\|q_a\|_{P_{M_\Theta}}, \tag{3.44}$$

$$\|q\|_{L^2(M)} \leqslant C_{P_M}^{L^2(M)}\|q_a\|_{P_M}. \tag{3.45}$$

Indeed using (3.40) $\forall \mathscr{T}_h, \mathscr{M} \in \mathfrak{M}_h, q_a \in P_{M_\Theta}$

$$\|q_a\|^2_{L^2(M_\Theta)} \leqslant C_{PW}(M_\Theta)^2|q_a|^2_{H^1(M_\Theta)} \leqslant \frac{\zeta_\Theta^{2n-2}l^2_{\max}}{n\pi^2}\sum_{K_\Theta \in \mathscr{M}_\Theta}\int_{K_\Theta}\|\nabla q_a\|^2\, d\mathbf{x},$$

then by noting that the ratio between the longest edge of an abstract macro-element and the diameter of one of its elements is less than $4\zeta_\Theta$ it follows

$$\begin{aligned}\|q_a\|^2_{L^2(M_\Theta)} &\leqslant \frac{\zeta_\Theta^{2n-2}}{n\pi^2}\sum_{K_\Theta \in \mathscr{M}_\Theta}(4\zeta_\Theta h_{K_\Theta})^2\int_{K_\Theta}\|\nabla q_a\|^2\, d\mathbf{x} \\ &\leqslant \frac{\zeta_\Theta^{2n-2}16\zeta_\Theta^2}{n\pi^2}\sum_{K_\Theta \in \mathscr{M}_\Theta}h^2_{K_\Theta}\int_{K_\Theta}\|\nabla q_a\|^2\, d\mathbf{x} \\ &= {C_{P_{M_\Theta}}^{L^2(M_\Theta)}}^2\|q_a\|^2_{P_{M_\Theta}}.\end{aligned} \tag{3.46}$$

A similar argument, based on (3.43), gives the existence of $C_{P_M}^{L^2(M)}$.

It is now possible to prove the existence of $C_{P_M}^{P_{M_\Theta}}$, $C_{P_{M_\Theta}}^{P_M} > 0$ such that $\forall \mathscr{T}_h, \mathscr{M} \in \mathfrak{M}_h, q \in P_M$:

$${C_{P_M}^{P_{M_\Theta}}}^{-1}\|q_a\|_{P_{M_\Theta}} \leqslant \|q\|_{P_M} \leqslant C_{P_{M_\Theta}}^{P_M}\|q_a\|_{P_{M_\Theta}}. \tag{3.47}$$

Indeed, reasoning as in (3.38) gives

$$\begin{aligned}\|q_a\|^2_{P_{M_\Theta}} &= \sum_{K_\Theta\in\mathscr{M}_\Theta} h^2_{K_\Theta}\int_{K_\Theta}\|\nabla q_a\|^2\,d\mathbf{x} = \sum_{K_\Theta\in\mathscr{M}_\Theta} h^2_{K_\Theta}\int_{K_\Theta}\|\nabla q_S\|^2\,d\mathbf{x}\\ &\leqslant 2\sum_{K\in\mathscr{M}} h^2_{K_\Theta}\left\|\det F^{-1}\right\|_{L^\infty(\Omega)}\Big(\|\nabla w\|^2_{L^\infty(\Theta)}\int_K\|q\|^2\,d\mathbf{x}+\\ &\qquad \|w\|^2_{L^\infty(\Theta)}\|\nabla F\|^2_{L^\infty(\Theta)}\int_K\|\nabla q\|^2\,d\mathbf{x}\Big)\\ &\leqslant 2\left\|\det F^{-1}\right\|_{L^\infty(\Omega)}\Big(\operatorname{diam}\Theta^2\|\nabla w\|^2_{L^\infty(\Theta)}\|q\|^2_{L^2(M)}+\\ &\qquad \|w\|^2_{L^\infty(\Theta)}\|\nabla F\|^2_{L^\infty(\Theta)}\sum_{K\in\mathscr{M}}\left\|\nabla F^{-1}\right\|^2_{L^\infty(\Theta)}h^2_K\int_K\|\nabla q\|^2\,d\mathbf{x}\Big).\end{aligned} \tag{3.48}$$

Inserting (3.46) in (3.48) gives the desired inequality

$$\begin{aligned}\|q_a\|^2_{P_{M_\Theta}} &\leqslant 2\left\|\det F^{-1}\right\|_{L^\infty(\Omega)}\Big(\operatorname{diam}\Theta^2\|\nabla w\|^2_{L^\infty(\Theta)}{C^{L^2(M)}_{P_M}}^2+\\ &\qquad \|w\|^2_{L^\infty(\Theta)}\|\nabla F\|^2_{L^\infty(\Theta)}\left\|\nabla F^{-1}\right\|^2_{L^\infty(\Omega)}\Big)\|q\|^2_{P_M}\\ &= {C^{P_{M_\Theta}}_{P_M}}^2\|q\|^2_{P_M}.\end{aligned} \tag{3.49}$$

The other inequality follows in a similar way.

Using these equivalences and some approximation properties the proof is completed in two steps:

- prove, by approximating the weight w, that $\exists\bar{h}, C^-_N>0:\forall h\leqslant\bar{h}, \mathscr{T}_h, \mathscr{M}\in\mathfrak{M}_h, q\in P_M, \exists\mathbf{w}\in V_M:$

$$\langle\mathbf{w}_N,\nabla q_N\rangle\geqslant C^-_N|\mathbf{w}|_{H^1(M)^n}\|q\|_{P_M}, \tag{3.50}$$

- prove, by approximating the map F, that $\exists\bar{\bar{h}}, C_{macro}>0:\forall h\leqslant\bar{\bar{h}}, \mathscr{T}_h, \mathscr{M}\in\mathfrak{M}_h, q\in P_M, \exists\mathbf{w}\in V_M:$

$$\langle\mathbf{w},\nabla q\rangle\geqslant C_{macro}|\mathbf{w}|_{H^1(M)^n}\|q\|_{P_M}. \tag{3.51}$$

3.6 *Proof of* (3.50)

In this subsection it is shown that from $0<C^-_S$ it follows that $\exists\bar{h}, C^-_N>0:\forall h\leqslant\bar{h}, \mathscr{T}_h, \mathscr{M}\in\mathfrak{M}_h, \forall q\in P_M, \exists\mathbf{w}\in V_M:$

$$\langle\mathbf{w}_N,\nabla q_N\rangle\geqslant C^-_N|\mathbf{w}|_{H^1(M)^n}\|q\|_{P_M}. \tag{3.52}$$

For all $q\in P_M$, it is possible to choose $\mathbf{w}_S\in V_{M_\Theta}$:

$$\begin{aligned}\langle\mathbf{w}_S,\nabla q_a\rangle = \langle\mathbf{w}_S,\nabla q_S\rangle &\geqslant C^-_S\|q_a\|_{P_{M_\Theta}}|\mathbf{w}_S|_{H^1(M_\Theta)^n}\\ &\geqslant C^-_S{C^{V_M}_{V_{M_\Theta}}}^{-1}{C^{P_M}_{P_{M_\Theta}}}^{-1}\|q\|_{P_M}|\mathbf{w}|_{H^1(M)^n}.\end{aligned} \tag{3.53}$$

Moreover, for all $\mathbf{w}_S$, it holds

$$\begin{aligned}|\langle \mathbf{w}_S, \nabla q_S\rangle| &= |\langle \mathbf{w}_S, \nabla q_a\rangle| = |\langle \nabla\cdot\mathbf{w}_S, q_a\rangle| \\ &\leqslant |\mathbf{w}_S|_{H^1(M_\Theta)^n} \|q_a\|_{L^2(M_\Theta)} \\ &\leqslant C_{V_M}^{V_{M_\Theta}} C_{P_{M_\Theta}}^{L^2(M_\Theta)} C_{P_M}^{P_{M_\Theta}} |\mathbf{w}|_{H^1(M)^n} \|q\|_{P_M} \\ &= C_S^+ |\mathbf{w}|_{H^1(M)^n} \|q\|_{P_M},\end{aligned} \tag{3.54}$$

and

$$\begin{aligned}|\langle \|\mathbf{w}_S\|_{\mathbb{R}^n}, q_S\rangle| &\leqslant \|\mathbf{w}_S\|_{L^2(M_\Theta)} \|q_S\|_{L^2(M_\Theta)} \\ &\leqslant C_P(M_\Theta) C_{V_M}^{V_{M_\Theta}} |\mathbf{w}|_{H^1(M)} \left\|\det F^{-1}\right\|^{\frac{1}{2}}_{L^\infty(\Omega)} \|w\|_{L^\infty(\Theta)} \|q\|_{L^2(M)} \\ &\leqslant \frac{4h}{\pi\sqrt{n}} \left\|\nabla F^{-1}\right\|_{L^\infty(\Omega)} C_{V_M}^{V_{M_\Theta}} |\mathbf{w}|_{H^1(M)} \\ &\qquad \left\|\det F^{-1}\right\|^{\frac{1}{2}}_{L^\infty(\Omega)} \|w\|_{L^\infty(\Theta)} C_{P_M}^{L^2(M)} \|q\|_{P_M} \\ &\leqslant h C_{\nabla w}^+ |\mathbf{w}|_{H^1(M)^n} \|q\|_{P_M},\end{aligned} \tag{3.55}$$

where in third line the following estimate of $C_P(D)$ for a rectangular domain D with longest edge $l_{\max}$ is used:

$$C_P(D) \leqslant \frac{l_{\max}}{\pi\sqrt{n}}. \tag{3.56}$$

By expressing the NURBS functions in terms of spline functions, it holds

$$\begin{aligned}\int_{M_\Theta} \mathbf{w}_N \cdot \nabla q_N \, d\mathbf{x} &= \int_{M_\Theta} \frac{\mathbf{w}_S}{w} \cdot \nabla \frac{q_S}{w} \, d\mathbf{x} \\ &= \int_{M_\Theta} \frac{\mathbf{w}_S \cdot \nabla q_S}{w^2} - \frac{q_S \, \mathbf{w}_S \cdot \nabla w}{w^3} \, d\mathbf{x}.\end{aligned} \tag{3.57}$$

Let $w_{m.v.}$ be the mean value of w in M_Θ; then, $\forall \mathscr{T}_h, \mathscr{M} \in \mathfrak{M}_h$, the approximation error of w by $w_{m.v.}$ can be bounded as

$$\begin{aligned}\|w - w_{m.v.}\|_{L^\infty(M_\Theta)} &\leqslant \operatorname{diam} M_\Theta \, \|\nabla w\|_{L^\infty(\Theta)} \\ &\leqslant 4h \left\|\nabla F^{-1}\right\|_{L^\infty(\Omega)} \|\nabla w\|_{L^\infty(\Theta)}.\end{aligned} \tag{3.58}$$

Using this approximation and equation (3.57) gives

$$\begin{aligned}\int_{M_\Theta} \mathbf{w}_N \cdot \nabla q_N \, d\mathbf{x} \geqslant& \int_{M_\Theta} \frac{\mathbf{w}_S \cdot \nabla q_S}{w_{m.v.}^2} \, d\mathbf{x} \\ &- \left| \int_{M_\Theta} \frac{\mathbf{w}_S \cdot \nabla q_S}{w^2} - \frac{\mathbf{w}_S \cdot \nabla q_S}{w_{m.v.}^2} \, d\mathbf{x} \right| \\ &- \left| \int_{M_\Theta} \frac{q_S \, \mathbf{w}_S \cdot \nabla w}{w^3} \, d\mathbf{x} \right|.\end{aligned} \tag{3.59}$$

By hypothesis for all $q \in P_M$ there is $\mathbf{w}_a \in V_{M_\Theta}$ such that :

$$\begin{aligned}\int_{M_\Theta} \frac{\mathbf{w}_S \cdot \nabla q_s}{w_{m.v.}^2} \, d\mathbf{x} &\geqslant C_S^- {C_{V_{M_\Theta}}^{V_M}}^{-1} {C_{P_{M_\Theta}}^{P_M}}^{-1} \left\|w^{-2}\right\|_{L^\infty(\Theta)} \|q\|_{P_M} |\mathbf{w}|_{H^1(M)^n} \\ &= C_1 \|q\|_{P_M} |\mathbf{w}|_{H^1(M)^n}.\end{aligned} \tag{3.60}$$

For the second term on the right-hand side of (3.59), the estimate

$$\begin{aligned}\left|\int_{M_\Theta} \frac{\mathbf{w}_S \cdot \nabla q_S}{w^2} - \frac{\mathbf{w}_S \cdot \nabla q_S}{w_{m.v.}^2}\, d\mathbf{x}\right| &= \left|\int_{M_\Theta} \frac{(w^2 - w_{m.v.}^2)(\mathbf{w}_S \cdot \nabla q_S)}{w^2 w_{m.v.}^2}\, d\mathbf{x}\right| \\ &\leqslant \left\| \frac{(w - w_{m.v.})(w + w_{m.v.})}{w^2 w_{m.v.}^2} \right\|_{L^\infty(M_\Theta)} C_S^+ \, |\mathbf{w}|_{H^1(M)^n} \, \|q\|_{P_M} \\ &\leqslant hC_2 \, |\mathbf{w}|_{H^1(M)^n} \, \|q\|_{P_M}\end{aligned} \tag{3.61}$$

holds, where

$$C_2 = 8C_S^+ \, \|w\|_{L^\infty(\Theta)} \left\|w^{-4}\right\|_{L^\infty(\Theta)} \left\|\nabla F^{-1}\right\|_{L^\infty(\Omega)} \|\nabla w\|_{L^\infty(\Theta)} \,.$$

For the third term on the right-hand side of (3.59) the estimate

$$\begin{aligned}\left|\int_{M_\Theta} w^{-3}(q_S \mathbf{w}_S \cdot \nabla w)\, d\mathbf{x}\right| &\leqslant \left\|w^{-3}\nabla w\right\|_{L^\infty(M_\Theta)} \left|\int_{M_\Theta} \|\mathbf{w}_S\|_{\mathbb{R}^n} \, q_S \, d\mathbf{x}\right| \\ &\leqslant h \left\|w^{-3}\nabla w\right\|_{L^\infty(M_\Theta)} C_{\nabla w}^+ \, \|q\|_{P_M} \, |\mathbf{w}|_{H^1(M)^n} \\ &\leqslant hC_2 \, \|q\|_{P_M} \, |\mathbf{w}|_{H^1(M)^n}\end{aligned} \tag{3.62}$$

holds, where $C_2 = C_{\nabla w}^+ \left\|w^{-3}\nabla w\right\|_{L^\infty(\Theta)}$. Inserting (3.60), (3.61) and (3.62) in (3.59) gives

$$\langle \mathbf{w}_N, \nabla q_N \rangle \geqslant (C_1 - hC_2 - hC_3) \, \|q\|_{P_M} \, |\mathbf{w}|_{H^1(M)^n} \,. \tag{3.63}$$

So choosing $\bar{h} < C_1(C_2 + C_3)^{-1}$ gives

$$C_N^- := C_1 - \bar{h}C_2 - \bar{h}C_3 > 0. \tag{3.64}$$

3.7 *Proof of* (3.51)

In this subsection it is shown that from $\exists \bar{h}, C_N^- > 0$ such that (3.50) holds, it follows that $\exists \bar{\bar{h}}, \exists C_{macro} > 0$: $\forall h \leqslant \bar{\bar{h}}, \mathscr{T}_h, \mathscr{M} \in \mathfrak{M}_h, \forall q \in P_M, \exists \mathbf{w} \in V_M$:

$$\langle \mathbf{w}, \nabla q \rangle \geqslant C_{macro} |\mathbf{w}|_{H^1(M)^n} \, \|q\|_{P_M}. \tag{3.65}$$

Using (3.54), (3.55) and (3.57), it follows that $\forall h, \mathscr{T}_h, \mathscr{M} \in \mathfrak{M}_h, \forall \mathbf{w} \in V_M, q \in P_M$

$$\langle \mathbf{w}_N, \nabla q_N \rangle \leqslant C_N^+ \, |\mathbf{w}|_{H^1(M)} \, \|q\|_{P_M} \,, \tag{3.66}$$

with

$$C_N^+ := C_S^+ \left\|w^{-2}\right\|_{L^\infty(\Theta)} + C_{\nabla w}^+ \operatorname{diam} \Omega \left\|w^{-3} \, \|\nabla w\|\right\|_{L^\infty(\Theta)} \,. \tag{3.67}$$

The thesis is related to the hypothesis by the relation

$$\int_M \mathbf{w} \cdot \nabla q \, d\mathbf{x} = \int_{M_\Theta} \mathbf{w}_N \cdot \nabla F^{-t} \nabla q_N \, |\det \nabla F| \, d\mathbf{x}, \tag{3.68}$$

where ∇F^{-t} is the transpose of the inverse of ∇F.

The main assumption is the regularity of F in particular that $F \in C^1(\bar{\Theta})$ and $F^{-1} \in C^1(\bar{\Omega})$. Since both Ω and Θ are bounded their closures are compact. This mean that both ∇F and ∇F^{-1} are uniformly continuous i.e. $\forall \varepsilon > 0\ \exists \delta : \forall x, y : \|x-y\| \leqslant \delta$

$$\|\nabla F(x) - \nabla F(y)\|_{\mathscr{L}(\mathbb{R}^n,\mathbb{R}^n)} \leqslant \varepsilon, \tag{3.69}$$

$$\left\|\nabla F^{-1}(x) - \nabla F^{-1}(y)\right\|_{\mathscr{L}(\mathbb{R}^n,\mathbb{R}^n)} \leqslant \varepsilon, \tag{3.70}$$

$$|\det \nabla F(x) - \det \nabla F(y)| \leqslant \varepsilon. \tag{3.71}$$

Moreover, the determinant of ∇F is non zero everywhere so it possible to assume $\det \nabla F > 0$.

For all $\mathscr{T}_h$, $\mathscr{M} \in \mathfrak{M}_h$, choose a point $x \in M$ and let F_{app} be a linear approximation of F such that $\nabla F_{app} = \nabla F(x)$. Using this approximation, equation (3.68) can be written as a sum of three terms

$$\begin{aligned}\int_M \mathbf{w} \cdot \nabla q \, d\mathbf{x} = &\int_{M_\Theta} \mathbf{w}_N \cdot \nabla F_{app}^{-t} \nabla q_N \det \nabla F_{app} \, d\mathbf{x} \\ &+ \int_{M_\Theta} \mathbf{w}_N \cdot \nabla F_{app}^{-t} \nabla q_N (\det \nabla F - \det \nabla F_{app}) \, d\mathbf{x} \\ &+ \int_{M_\Theta} \mathbf{w}_N \cdot (\nabla F^{-t} - \nabla F_{app}^{-t}) \nabla q_N \det \nabla F \, d\mathbf{x}.\end{aligned} \tag{3.72}$$

By hypothesis for all $q \in P_M$ there is $\tilde{\mathbf{w}}$ such that

$$\int_{M_\Theta} \tilde{\mathbf{w}}_N \cdot \nabla q_N \, d\mathbb{R}^n \geqslant C_N^- \|q\|_{P_M} |\tilde{\mathbf{w}}|_{H^1(M)^n}. \tag{3.73}$$

Let $\mathbf{w} = \nabla F_{app} \tilde{\mathbf{w}}$ then

$$|\mathbf{w}|_{H^1(M)^n} \leqslant \|\nabla F\|_{L^\infty(\Theta)} |\tilde{\mathbf{w}}|_{H^1(M)^n}. \tag{3.74}$$

For such $\mathbf{w}$, for the first term on the right side of (3.72) it holds:

$$\begin{aligned}\int_{M_\Theta} \mathbf{w}_N \cdot \nabla F_{app}^{-t} \nabla q_N \det \nabla F_{app} \, d\mathbf{x} &= \int_{M_\Theta} \tilde{\mathbf{w}}_N \cdot \nabla q_N \det \nabla F_{app} \, d\mathbf{x} \\ &\geqslant C_N^- \inf_\Theta |\det \nabla F| \, |\tilde{\mathbf{w}}|_{H^1(M)^n} \|q\|_{P_M} \\ &\geqslant C_4 |\mathbf{w}|_{H^1(M)^n} \|q\|_{P_M}.\end{aligned} \tag{3.75}$$

where $C_4 = C_N^- \left\|\det \nabla F^{-1}\right\|_{L^\infty(\Omega)}^{-1} \|\nabla F\|_{L^\infty(\Theta)}^{-1}$. For the second term the following estimate holds:

$$\begin{aligned}&\left|\int_{M_\Theta} \mathbf{w}_N \cdot \nabla F_{app}^{-t} \nabla q_N (\det \nabla F - \det \nabla F_{app}) \, d\mathbf{x}\right| \\ &\qquad \leqslant C_N^+ \left\|\nabla F^{-t}\right\|_{L^\infty(\Omega)} \varepsilon \, |\mathbf{w}|_{H^1(M)^n} \|q\|_{P_M} \\ &\qquad \leqslant \varepsilon C_5 |\mathbf{w}|_{H^1(M)^n} \|q\|_{P_M}.\end{aligned} \tag{3.76}$$

For the third term on the right hand side of (3.72) it holds:

$$\begin{aligned}&\left|\int_{M_\Theta} \mathbf{w}_N \cdot (\nabla F^{-t} - \nabla F_{app}^{-t}) \nabla q_N \det \nabla F \, d\mathbf{x}\right| \\ &\quad \leqslant \left\|\nabla F^{-t} - \nabla F_{app}^{-t}\right\|_{\mathscr{L}(\mathbb{R}^n,\mathbb{R}^n)} \|\det \nabla F\|_{L^\infty(M_\Theta)} C_N^+ |\mathbf{w}|_{H^1(M)^n} \|\nabla q\|_{P_M} \\ &\quad \leqslant \varepsilon C_6 |\mathbf{w}|_{H^1(M)^n} \|\nabla q\|_{P_M}.\end{aligned} \tag{3.77}$$

Inserting (3.75), (3.76), (3.77) in (3.72) gives

$$\int_M \mathbf{w} \cdot \nabla q,\, d\mathbf{x} \geqslant (C_4 - \varepsilon C_5 - \varepsilon C_6)\, |\mathbf{w}|_{H^1(M)^n}\, \|q\|_{P_M}\,. \tag{3.78}$$

Thus, for $\bar{\varepsilon} < \frac{C_4}{C_5+C_6}$, and choosing $\bar{\bar{h}} = \delta(\bar{\varepsilon})$ it holds

$$C_{macro} := C_4 - \bar{\varepsilon} C_5 - \bar{\varepsilon} C_6 > 0.$$

4. Conclusion

The Stokes problem is a simplified model for both elastic deformations in solids and fluid-dynamics. In this article the inf sup stability and optimal convergence of an isogeometric C^1 discretization for the Stokes problem is proved. The problem of the inf sup stability of isogeometric discrete spaces (the push forward through the geometrical map of NURBS space on the parametric domain), is reduced to the inf sup stability of spline spaces on the parametric domain. In this case the multidimensional problem is reduced to two unidimensional problems. The one dimensional problems associated with cubic C^1 velocities and quadratic C^1 pressures are analyzed by symbolic computation. The case of higher regularity spaces is the subject of a forthcoming paper. The use of more regular functions is useful to decrease the degrees of freedom and thus the computational cost without affecting the convergence to zero of the error estimates.

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