Weak vorticity formulation of 2D Euler equations with white noise initial condition

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Abstract

The 2D Euler equations with random initial condition distributed as a certain Gaussian measure are considered. The theory developed by S. Albeverio and A.-B. Cruzeiro in [1] is revisited, following the approach of weak vorticity formulation. A solution is constructed as a limit of random point vortices. This allows to prove that it is also limit of L^∞-vorticity solutions. The result is generalized to initial measures that have a continuous bounded density with respect to the original Gaussian measure.

1 Introduction

We consider the 2D Euler equations on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, formulated in terms of the vorticity ω

$$\partial_t \omega + u \cdot \nabla \omega = 0 \tag{1}$$

where u is the velocity, divergence free vector field such that $\omega = \partial_2 u_1 - \partial_1 u_2$. The classical theory (see for instance [13], [27], [28], [30]) includes the following results:

1. existence and uniqueness of weak solutions of class $L^\infty\left([0,T]\times\mathbb{T}^2\right)\cap C\left([0,T];L^p\left(\mathbb{T}^2\right)\right)$ for every $p\in[1,\infty)$, satisfying

$$\langle \omega_t, \phi\rangle = \langle \omega_0, \phi\rangle + \int_0^t \langle \omega_s, u_s\cdot\nabla\phi\rangle\, ds \tag{2}$$

   for every $\phi\in C^\infty\left(\mathbb{T}^2\right)$, when the initial condition ω_0 is of class $L^\infty\left(\mathbb{T}^2\right)$ ([43], [44], [30]);

2. existence of weak solutions of class $C\left([0,T];L^p\left(\mathbb{T}^2\right)\right)$, satisfying (2), when the initial condition ω_0 is of class $L^p\left(\mathbb{T}^2\right)$, for some $p\in[1,\infty)$;

3. existence of measure-valued solutions $\omega_t(dx)$, of class $L^\infty\left(0,T;\mathcal{M}\left(\mathbb{T}^2\right)\cap H^{-1}\left(\mathbb{T}^2\right)\right)$, satisfying for every $\phi\in C^\infty\left(\mathbb{T}^2\right)$ the so called weak vorticity formulation

$$\langle\omega_t,\phi\rangle=\langle\omega_0,\phi\rangle+\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}H_\phi(x,y)\,\omega_s(dx)\,\omega_s(dy)\,ds \tag{3}$$

   where

$$H_\phi(x,y):=\frac{1}{2}K(x-y)\left(\nabla\phi(x)-\nabla\phi(y)\right)$$

   and $K(x)$ is Biot-Savart kernel on $\mathbb{T}^2$, when the initial condition is a measure of class $H^{-1}\left(\mathbb{T}^2\right)$ with a certain condition of preference for a single sign, see [19], [37], [17]; here we have denoted by $\mathcal{M}\left(\mathbb{T}^2\right)$ the space of finite signed measures and by $H^\alpha\left(\mathbb{T}^2\right)$ the classical Sobolev spaces of order $\alpha\in\mathbb{R}$ defined in Section 1.1;

4. existence and uniqueness of a measure-valued solution of the form $\omega_t(dx)=\sum_{i=1}^N\xi_i\delta_{X_t^i}$, fulfilling (3), when the initial condition has the form $\omega_0(dx)=\sum_{i=1}^N\xi_i\delta_{X_0^i}$, with real valued intensities $\xi_1,...,\xi_N$, and $\left(X_0^1,...,X_0^N\right)$ belonging to a set of full Lebesgue measure in $\left(\mathbb{T}^2\right)^N$, see [30].

Obviously there are many other results, reported in the references above and other works, including counterexamples to uniqueness like [35]. The previous choice has been made to illustrate the attempt to include weaker and weaker concepts of solutions. Very important for result n. 3 has been the symmetrization step from (2) to (3): the kernel $H_\phi(x,y)$ is bounded, smooth outside the diagonal, discontinuous along the diagonal; hence a fine analysis of the concentration of $\omega_t(dx)$ around the diagonal is important but at least the singularity of order $\frac{1}{|x|}$ of Biot-Savart kernel $K(x)$ has been removed.

In the present paper we discuss a probabilistic result for Euler equations, interpreted in the form (3). We like to state it first in purely deterministic terms, here in the introduction, for the sake of comparison with the "scale" of results above. Then, in the rest of the paper, the probabilistic side will be stressed more. Denote by $H^{-1-}\left(\mathbb{T}^2\right)$ the space $\bigcap_{\epsilon>0}H^{-1-\epsilon}\left(\mathbb{T}^2\right)$, with the topology described in Section 1.1 and notice that $\mathcal{M}\left(\mathbb{T}^2\right)\subset H^{-1-}\left(\mathbb{T}^2\right)$, because by Sobolev embedding $H^{1+\epsilon}\left(\mathbb{T}^2\right)\subset C\left(\mathbb{T}^2\right)$. Moreover, denote by K_ϵ the smooth approximations of K given by (7) below and, given a sequence $\epsilon_n\to0$, set $H_\phi^n(x,y):=\frac{1}{2}K_{\epsilon_n}(x-y)\left(\nabla\phi(x)-\nabla\phi(y)\right)$; by classical distribution theory, $s\mapsto\left\langle\omega_s\otimes\omega_s,H_\phi^n\right\rangle$ is well defined and continuous when $\omega\in C\left([0,T];H^{-1-}\left(\mathbb{T}^2\right)\right)$.

Theorem 1 *There exist $\epsilon_n\to0$ and a large set*

$$\mathcal{IC}_0\subset H^{-1-}\left(\mathbb{T}^2\right)\setminus\left(H^{-1}\left(\mathbb{T}^2\right)\cup\mathcal{M}\left(\mathbb{T}^2\right)\right)$$

of initial conditions such that for all $\omega_0\in\mathcal{IC}_0$ the following properties hold.

i) there exists $\omega \in C\left([0,T];H^{-1-}\left(\mathbb{T}^2\right)\right)$ *such that, for every* $\phi \in C^\infty\left(\mathbb{T}^2\right)$*, the sequence of functions* $s \mapsto \left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle$ *is a Cauchy sequence in* $L^2(0,T)$ *and, denoted by* $s \mapsto \langle \omega_s \otimes \omega_s, H_\phi \rangle$ *its limit, one has the analog of (3), namely*

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle \, ds \tag{4}$$

ii) there is a sequence $\left\{\omega^{(n)}\right\}$ *of solutions of Euler equations of class* $L^\infty\left([0,T]\times\mathbb{T}^2\right) \cap C\left([0,T];L^p\left(\mathbb{T}^2\right)\right)$ *for every* $p \in [1,\infty)$ *(those of point 1 above) such that* $\left\langle \omega_t^{(n)}, \phi \right\rangle \to \langle \omega_t, \phi \rangle$ *uniformly in* $t \in [0,T]$*, for every* $\phi \in C^\infty\left(\mathbb{T}^2\right)$*.*

Remark 2 *How large is the set of initial conditions, it is clarified below in Section 6. It is a full measure set with respect to the Gaussian measure* μ *introduced in Section 2.1*

Remark 3 *In fact the set of initial conditions given by this theorem is included in a more regular space* $H^{-1-,\infty}\left(\mathbb{T}^2\right)$*, where also the solutions live, defined in Section 2.1 below. We have not used* $H^{-1-,\infty}\left(\mathbb{T}^2\right)$ *in place of* $H^{-1-}\left(\mathbb{T}^2\right)$ *because* $\mathcal{M}\left(\mathbb{T}^2\right) \not\subseteq H^{-1-,\infty}\left(\mathbb{T}^2\right)$ *and thus the statement would be less clear. Moreover* $H^{-1}\left(\mathbb{T}^2\right)$ *and* $\mathcal{M}\left(\mathbb{T}^2\right)$ *are not included one in the other, which again explains the statement.*

Part (i) of Theorem 1 is a deterministic reformulation of Theorem 24 below, which states that Euler equations, interpreted in the form (4), has a stochastic solution, a stationary stochastic process with time marginal given by the so called *white noise* on $\mathbb{T}^2$, defined in Section 2.1 below. This probabilistic result is due to Sergio Albeverio and Ana Bela Cruzeiro [1]. Here we provide, with respect to that seminal work, the so called weak vorticity formulation (4) (opposite to a Fourier formulation), which fits more nicely in the scheme of results 1-4 above; and we prove the existence of a solution as a limit of random point vortices, a suitable random version of point 4 above. Opposite to other schemes that can be used to prove existence, based on *approximated* equations (like the Galerkin scheme of [1], or a Leray type scheme), point vortices are *true solutions* of Euler equations (see Section 22) and thus establish a bridge between L^∞-vorticity solutions, those of point 1 above, and Albeverio-Cruzeiro solution, via the result of approximation of point vortices by vortex patches of Marchioro and Pulvirenti [29]. Nicolai Tzvetkov suggested to investigate question (ii) of Theorem 1, which is similar to a question solved (in a stronger sense) for nonlinear wave equations, see [42].

The point vortex approximation provides an interesting interpretation of the white noise solution of Albeverio and Cruzeiro, as a limit of randomly distributed vortices with positive and negative random vorticities. Under the viewpoint of the weak vorticity formulation, having in mind the deep discussions of the deterministic literature on concentration of solutions of Euler equations (see for instance the works of Delort [19], Schochet [36], [37], Poupaud [32], Di Perna and Majda [17]), a natural question is why the solution found here with white noise distribution does not "concentrate on the diagonal", in the double integration of the weak vorticity formulation, where the function $H_{\phi}(x,y)$ is discontinuous. For the white noise solution the absence of concentration is encoded in the results of Section 2.4, which show the power of Gaussian analysis but may still look obscure. However, the approximation by point vortices provides a clear intuition about the lack of concentration: at every time, vortices are distributed at random uniformly in space, independently one of the other.

Among the reasons to reconsider Albeverio-Cruzeiro theory today, there is the clear success of randomization of initial conditions in solving dispersive equations, see for instance [10], [11], [12], [33], [34] [31] (the last two, for instance, describe another PDE that leaves a Gaussian measure invariant) and in particular the review of N. Tzvetkov [42] on nonlinear wave equation where Theorems 2.6, 2.7 are devoted to prove that solutions with poor regularity (constructed for a.e. initial condition with respect to a Gaussian measure) are the limit of more regular solutions belonging to the classical theory. As a technical remark, the approximation result above in Theorem 1 is definitely weaker than Theorem 2.6 of [42], where any reasonable smooth approximation of initial conditions leads to convergent solutions; it is more in the spirit of Theorem 2.7, where particular approximations are considered. As a general remark, it is not reasonable to expect for Euler equations the richness of results obtained in dispersive equations, but nevertheless it may be of interest to make little improvements. Another source of inspiration for the present work have been the striking recent theories for certain stochastic nonlinear equations having Gaussian measures invariant, see for instance [25], [23], [24]; however, the difficulties for such equations are much greater than those solved here, although Gaussian analysis is a common core.

We prove existence of a stochastic solution also when the initial condition is a random distribution with law that have a continuous bounded density with respect to the original Gaussian measure. The solution has a density also at time t, that satisfies a continuity equation; the results proved here in this direction are quite elementary corollaries of the main results on white noise solutions but we think it is of interest to state them for future investigations in connection with deeper theories on continuity equations in infinite dimensions, see for instance [7], [8], [9], [15], [16], [18], [20]. The case with a density with respect to white noise arises an open question, described in Section 7.2, concerning the approximation of smooth solutions by white noise ones, a sort of dual problem to the one discussed above.

Let us finally mention several other works related to Gaussian invariant measures for 2D Euler equations: see [5], [2], [3], [4], [14], [41]. Several elements of these works may deserve further analysis.

1.1 Notations

We denote by $\{e_n\}$ the complete orthonormal system in $L^2\left(\mathbb{T}^2;\mathbb{C}\right)$ given by $e_n(x) = e^{2\pi i n\cdot x}$, $n\in\mathbb{Z}^2$. Given a distribution $\omega\in C^\infty\left(\mathbb{T}^2\right)'$ and a test function $\phi\in C^\infty\left(\mathbb{T}^2\right)$, we denoted by $\langle \omega, \phi \rangle$ the duality between ω and ϕ (namely $\omega(\phi)$), and we use the same symbol for the inner product of $L^2(\mathbb{T}^2)$. We set $\widehat{\omega}(n) = \langle \omega, e_n \rangle$, $n \in \mathbb{Z}^2$ and we define, for each $s \in \mathbb{R}$, the space $H^s(\mathbb{T}^2)$ as the space of all distributions $\omega \in C^\infty(\mathbb{T}^2)'$ such that

$$\|\omega\|_{H^s}^2 := \sum_{n \in \mathbb{Z}^2} \left(1 + |n|^2\right)^s |\widehat{\omega}(n)|^2 < \infty.$$

We use similar definitions and notations for the space $H^s(\mathbb{T}^2, \mathbb{C})$ of complex valued functions. In the space $H^{-1-}(\mathbb{T}^2) = \bigcap_{\epsilon > 0} H^{-1-\epsilon}(\mathbb{T}^2)$ we consider the metric

$$d_{H^{-1-}}(\omega, \omega') = \sum_{n=1}^{\infty} 2^{-n} \left(\|\omega - \omega'\|_{H^{-1-\frac{1}{n}}} \wedge 1 \right).$$

Convergence in this metric is equivalent to convergence in $H^{-1-\epsilon}(\mathbb{T}^2)$ for every $\epsilon > 0$. The space $H^{-1-}(\mathbb{T}^2)$ with this metric is complete and separable. We denote by $\mathcal{X} := C([0,T]; H^{-1-}(\mathbb{T}^2))$ the space of continuous functions with values in this metric space; a function is in $\mathcal{X}$ if and only if it is in $C([0,T]; H^{-1-\epsilon}(\mathbb{T}^2))$ for every $\epsilon > 0$. The distance in $C([0,T]; H^{-1-}(\mathbb{T}^2))$ is given by $d_{\mathcal{X}}(\omega_\cdot, \omega'_\cdot) = \sup_{t \in [0,T]} d_{H^{-1-}}(\omega_t, \omega'_t)$, which makes $\mathcal{X}$ a Polish space.

For $s > 0$, the spaces $H^s(\mathbb{T}^2)$ and $H^{-s}(\mathbb{T}^2)$ are dual each other. By $H^{s+}(\mathbb{T}^2)$ we shall therefore mean the space $\bigcup_{\epsilon > 0} H^{s+\epsilon}(\mathbb{T}^2)$. We shall use this notation in the case of the space $H^{2+}(\mathbb{T}^2 \times \mathbb{T}^2)$, which is similarly defined.

2 White noise vorticity distribution and the nonlinear term in the weak vorticity formulation

2.1 White noise

We start recalling the well known notion of white noise, reviewing some of its main properties used in the sequel.

White noise on $\mathbb{T}^2$ is by definition a Gaussian distributional-valued stochastic process $\omega : \Xi \to C^\infty(\mathbb{T}^2)'$, defined on some probability space $(\Xi, \mathcal{F}, \mathbb{P})$, such that

$$\mathbb{E}[\langle \omega, \phi \rangle \langle \omega, \psi \rangle] = \langle \phi, \psi \rangle \tag{5}$$

for all $\phi, \psi \in C^\infty(\mathbb{T}^2)$ (Gaussian means that the real valued r.v. $\langle \omega, \phi \rangle$ is Gaussian, for every $\phi \in C^\infty(\mathbb{T}^2)$). We have denoted by $\langle \omega(\theta), \phi \rangle$ the duality between the distribution $\omega(\theta)$ (for some $\theta \in \Xi$) and the test function $\phi \in C^\infty(\mathbb{T}^2)$. These properties uniquely characterize the law of ω. In more heuristic terms, as it is often written in the Physics literature,

$$\mathbb{E}\left[\omega\left(x\right)\omega\left(y\right)\right]=\delta\left(x-y\right)$$

since double integration of this identity against $\phi\left(x\right)\psi\left(y\right)$ gives (5). White noise exists: it is sufficient to take the complete orthonormal system $\left\{e_n\right\}_{n\in\mathbb{Z}^2}$ of $L^2\left(\mathbb{T}^2,\mathbb{C}\right)$ introduced in Section 1.1, a probability space $\left(\Xi,\mathcal{F},\mathbb{P}\right)$ supporting a sequence of independent standard Gaussian variables $\left\{G_n\right\}_{n\in\mathbb{Z}^2}$, and consider the series

$$\omega=\sqrt{2}\operatorname{Re}\sum_{n\in\mathbb{Z}^2}G_n e_n.$$

The partial sums $\omega_N^{\mathbb{C}}\left(\theta,x\right)=\sum_{|n|\leq N}G_n\left(\theta\right)e_n\left(x\right)$ are well defined complex valued random fields with square integrable paths, $\omega_N:\Xi\to L^2\left(\mathbb{T}^2,\mathbb{C}\right)$. For every $\epsilon>0$, $\left\{\omega_N^{\mathbb{C}}\right\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^2\left(\Xi;H^{-1-\epsilon}\left(\mathbb{T}^2,\mathbb{C}\right)\right)$, because

$$\mathbb{E}\left[\left\|\omega_N^{\mathbb{C}}\left(\theta,x\right)-\omega_M^{\mathbb{C}}\left(\theta,x\right)\right\|_{H^{-1-\epsilon}}^2\right]=\mathbb{E}\left[\sum_{M<|n|\leq N}\left(1+|n|^2\right)^{-1-\epsilon}|G_n|^2\right]=\sum_{M<|n|\leq N}\left(1+|n|^2\right)^{-1-\epsilon}.$$

The limit $\omega^{\mathbb{C}}$ in $L^2\left(\Xi;H^{-1-\epsilon}\left(\mathbb{T}^2,\mathbb{C}\right)\right)$ thus exists, and $\omega=\sqrt{2}\operatorname{Re}\omega^{\mathbb{C}}$ is a white noise because (doing rigorously the computation on the partial sums and then taking the limit) it is centered and for $\phi,\psi\in C^\infty\left(\mathbb{T}^2\right)$,

$$\begin{aligned}\mathbb{E}\left[\left\langle\omega,\phi\right\rangle\left\langle\omega,\psi\right\rangle\right]&=\operatorname{Re}\mathbb{E}\left[\left\langle\omega_C,\phi\right\rangle\left\langle\overline{\omega_C},\psi\right\rangle\right]=\operatorname{Re}\sum_{n,m\in\mathbb{Z}^2}\left\langle e_n,\phi\right\rangle\overline{\left\langle e_m,\psi\right\rangle}\mathbb{E}\left[G_nG_m\right]\\&=\operatorname{Re}\sum_{n\in\mathbb{Z}^2}\left\langle e_n,\phi\right\rangle\overline{\left\langle e_n,\psi\right\rangle}=\left\langle\phi,\psi\right\rangle.\end{aligned}$$

[One obtains the same result by taking $\omega=\sum_{n\in\mathbb{Z}^2}G_ne_n$ where $\mathbb{Z}^2\backslash\left\{0\right\}$ is partitioned as $\mathbb{Z}^2=\Lambda\cup\left(-\Lambda\right)$, G_n are i.i.d. $N\left(0,1\right)$ on $\Lambda\cup\left\{0\right\}$ and $G_{-n}=\overline{G_n}$ for $n\in\Lambda$.] The law μ of the measurable map $\omega:\Xi\to H^{-1-\epsilon}\left(\mathbb{T}^2\right)$ is a Gaussian measure (it is sufficient to check that $\left\langle\omega,\phi\right\rangle$ is Gaussian for every $\phi\in C^\infty\left(\mathbb{T}^2\right)$, and this is true since $\left\langle\omega,\phi\right\rangle$ is the $L^2\left(\Xi\right)$-limit of the Gaussian variables $\sum_{|n|\leq N}G_n\left\langle e_n,\phi\right\rangle$). The measure μ is supported by $H^{-1-}\left(\mathbb{T}^2\right)$ but not by $H^{-1}\left(\mathbb{T}^2\right)$, namely we have

$$\mu\left(H^{-1}\left(\mathbb{T}^2\right)\right)=0.$$

It follows from

$$\mathbb{E}\left[\left\|\omega^{\mathbb{C}}\right\|_{H^{-1}}^2\right]=\sum_{n\in\mathbb{Z}^2}\left(1+|n|^2\right)^{-1}=+\infty.$$

The measure μ is sometimes denoted heuristically as

$$\mu(d\omega) = \frac{1}{Z} \exp\left(-\frac{1}{2}\int_{\mathbb{T}^2} \omega^2 dx\right) d\omega$$

and called the *enstrophy measure*. The notation "$d\omega$" has no meaning (unless interpreted as a limit of measures on finite dimensional Euclidean spaces), just reminds the structure of centered nonsingular Gaussian measures in $\mathbb{R}^n$, that is $\mu_n(d\omega_n) = \frac{1}{Z_n}\exp\left(-\frac{1}{2}\left\langle Q_n^{-1}\omega_n, \omega_n\right\rangle_{\mathbb{R}^n}\right) d\omega_n$ where $d\omega_n$ is Lebesgue measure in $\mathbb{R}^n$ and Q_n is the covariance matrix. The notation $\int_{\mathbb{T}^2} \omega^2 dx$ alludes to the fact that μ, heuristically considered as a Gaussian measure on $L^2(\mathbb{T}^2)$ (this is not possible, $\mu\left(L^2(\mathbb{T}^2)\right) = 0$), has covariance equal to the identity: if $Q = Id$, then $\left\langle Q^{-1}\omega, \omega\right\rangle_{L^2} = \int_{\mathbb{T}^2} \omega^2 dx$. The fact that in $L^2(\mathbb{T}^2)$ the covariance operator Q, heuristically defined as

$$\langle Q\omega, \omega\rangle_{L^2} = \mathbb{E}\left[\langle \omega, \phi\rangle_{L^2} \langle \omega, \psi\rangle_{L^2}\right]$$

is the identity in the case of the law μ of white noise, is a simple "consequence" (the argument is not rigorous ab initio) of the definition (5) of white noise.

White noise realizations are in fact more regular than $H^{-1-}(\mathbb{T}^2)$. The general idea, used several times in investigations of this kind, is that when a Gaussian field is L^2 it is also more regular, because higher order moments are simply related to second moments and Kolmogorov regularity theorem applies. Let us see this fact in the case of white noise ω. Given $\epsilon > 0$, we know that $\omega \in H^{-1-\epsilon}(\mathbb{T}^2)$ with probability one. Consider the random field

$$\psi(\theta, x) := \left((1+\Delta)^{-\frac{1+\epsilon}{2}} \omega(\theta)\right)(x) = \left\langle \omega(\theta), (1+\Delta)^{-\frac{1+\epsilon}{2}} \delta_x \right\rangle.$$

We have $\psi \in L^2(\mathbb{T}^2)$ with probability one. But ψ is a Gaussian field. We have in particular

$$\mathbb{E}\left[|\psi(x) - \psi(y)|^p\right] \le C_p \mathbb{E}\left[|\psi(x) - \psi(y)|^2\right]^{p/2}$$

and, denoting $(1+\Delta)^{-\frac{1+\epsilon}{2}} \delta_x$ and $(1+\Delta)^{-\frac{1+\epsilon}{2}} \delta_y$ respectively by f_x, f_y,

$$\begin{aligned}
&\mathbb{E}\left[|\psi(x) - \psi(y)|^2\right] \\
&= \mathbb{E}\left[\psi(x)\psi(x)\right] - 2\mathbb{E}\left[\psi(x)\psi(y)\right] + \mathbb{E}\left[\psi(y)^2\right] \\
&= \mathbb{E}\left[\langle \omega, f_x\rangle \langle \omega, f_x\rangle\right] - 2\mathbb{E}\left[\langle \omega, f_x\rangle \langle \omega, f_y\rangle\right] + \mathbb{E}\left[\langle \omega, f_y\rangle \langle \omega, f_y\rangle\right]
\end{aligned}$$

and now we use definition (5)

$$\begin{aligned}
&= \langle f_x, f_x\rangle - 2\langle f_x, f_y\rangle + \langle f_y, f_y\rangle \\
&= \|f_x - f_y\|_{L^2}^2 = \left\|(1+\Delta)^{-\frac{1+\epsilon}{2}} \delta_x - (1+\Delta)^{-\frac{1+\epsilon}{2}} \delta_y\right\|_{L^2}^2 \\
&\le C_\epsilon |x-y|^{\alpha(\epsilon)}
\end{aligned}$$

for a suitable number $\alpha(\epsilon) > 0$ and a constant $C_\epsilon > 0$; the last inequality can be proved as

$$\begin{aligned}\sup_{\|\phi\|_{L^2}\leq 1}\left|\left\langle (1+\Delta)^{-\frac{1+\epsilon}{2}}(\delta_x-\delta_y),\phi\right\rangle\right| &= \sup_{\|\phi\|_{L^2}\leq 1}\left|\left((1+\Delta)^{-\frac{1+\epsilon}{2}}\phi\right)(x)-\left((1+\Delta)^{-\frac{1+\epsilon}{2}}\phi\right)(y)\right| \\ &\leq \sup_{\|\phi\|_{L^2}\leq 1}\left\|(1+\Delta)^{-\frac{1+\epsilon}{2}}\phi\right\|_{C^{\alpha(\epsilon)}}|x-y|^{\alpha(\epsilon)} \\ &\leq C_\epsilon \sup_{\|\phi\|_{L^2}\leq 1}\|\phi\|_{L^2}|x-y|^{\alpha(\epsilon)}\end{aligned}$$

due to the fact that $H^{1+\epsilon}(\mathbb{T}^2)$ is embedded in a space of Hölder continuous functions. Therefore

$$\mathbb{E}\left[|\psi(x)-\psi(y)|^p\right] \leq C_p C_\epsilon^{p/2}|x-y|^{\frac{p\alpha(\epsilon)}{2}}.$$

Taking p so large that $\frac{p\alpha(\epsilon)}{2} > 2$, we may apply Kolmogorov regularity theorem and deduce that the random field $\psi(x)$ has a version with continuous paths. It means that, up to a modification, $(1+\Delta)^{-\frac{1+\epsilon}{2}}\omega \in C(\mathbb{T}^2)$ with probability one, not only $(1+\Delta)^{-\frac{1+\epsilon}{2}}\omega$ belongs to $L^2(\mathbb{T}^2)$. Let us summarize this fact by the notation

$$\mathbb{P}\left(\omega \in H^{-1-,\infty}(\mathbb{T}^2)\right) = 1.$$

In spite of this additional regularity, we are not in the realm of signed measures, that received so much attention in the case of the vorticity of 2D fluids. One has Among the properties, one has

$$\mu\left(\mathcal{M}(\mathbb{T}^2)\right) = 0.$$

(see [22], Proposition A2 for a concise proof). To understand this property, think to the analogy with the more classical 1-dimensional case, on $[0,\infty)$ instead of the torus. In such case, white noise is the distributional derivative of Brownian motion. It is well known that, with probability one, trajectories of a continuous version of Brownian motion are not of bounded variation (because they have finite non zero quadratic variation, and are continuous). Therefore their derivatives are not signed measures. The gap in regularity between white noise and signed measures is thus comparable to the gap between total variation and quadratic variation.

2.2 Colored noise

For technical reasons, sometimes it is convenient to consider a smooth approximation of white noise. A simple one is $\omega_N(\theta,x) = \mathrm{Re}\sum_{|n|\leq N} G_n(\theta)e_n(x)$ but, although the difference is really minor, for the PDE approach followed here the use of mollifiers looks a bit more natural. We set, for $\epsilon > 0$,

$$\omega_\epsilon(x) = \langle \omega, \theta_\epsilon(x-\cdot)\rangle$$

formally written also as $(\theta_\epsilon * \omega)(x) = \int_{\mathbb{T}^2} \theta_\epsilon (x-y)\, \omega(y)\, dy$, where $\theta_\epsilon(x) = \epsilon^{-2}\theta\left(\epsilon^{-1}x\right)$, and θ is a smooth probability density on $\mathbb{T}^2$ with a small support around $x = 0$. Assume θ symmetric. We have

$$\mathbb{E}\left[\langle \omega_\epsilon, \phi\rangle \langle \omega_\epsilon, \psi\rangle\right] = \mathbb{E}\left[\langle \omega, \theta_\epsilon * \phi\rangle \langle \omega, \theta_\epsilon * \psi\rangle\right] = \langle \theta_\epsilon * \phi, \theta_\epsilon * \psi\rangle$$

$$\begin{aligned}\mathbb{E}\left[\omega_\epsilon(x)\, \omega_\epsilon(y)\right] &= \mathbb{E}\left[\langle \omega, \theta_\epsilon(x-\cdot)\rangle \langle \omega, \theta_\epsilon(y-\cdot)\rangle\right] = \langle \theta_\epsilon(x-\cdot), \theta_\epsilon(y-\cdot)\rangle \\ &= \int_{\mathbb{T}^2} \theta_\epsilon(x-y-z)\, \theta_\epsilon(z)\, dz = (\theta_\epsilon * \theta_\epsilon)(x-y) =: \delta^\epsilon_{x-y}\end{aligned}$$

where we have used the notation δ^ϵ_a to denote $(\theta_\epsilon * \theta_\epsilon)(a)$ because it is an approximation of the Dirac delta distribution.

Notice that $\omega_\epsilon \in C^\infty\left(\mathbb{T}^2\right)$ with probability one. Moreover, since $\langle \omega_\epsilon, \phi\rangle = \langle \omega, \theta_\epsilon * \phi\rangle$ and $\theta_\epsilon * \phi \to \phi$ in $H^{1+\gamma}\left(\mathbb{T}^2\right)$ for every $\phi \in H^{1+\gamma}\left(\mathbb{T}^2\right)$ and given $\gamma > 0$, we have the following statement:

Lemma 4 *$\mathbb{P}$-almost surely, for every $\phi \in H^{1+\gamma}\left(\mathbb{T}^2\right)$ we have*

$$\lim_{\epsilon \to 0} \langle \omega_\epsilon, \phi\rangle = \langle \omega, \phi\rangle .$$

2.3 Weak vorticity formulation, preliminaries

Let us first recall the weak vorticity formulation in the case of measure-valued vorticities. First, one rewrites equation (1) against test functions $\phi \in C^\infty\left(\mathbb{T}^2\right)$, using $\operatorname{div} u = 0$:

$$\langle \omega_t, \phi\rangle = \langle \omega_0, \phi\rangle + \int_0^t \langle \omega_s, u_s \cdot \nabla\phi\rangle\, ds.$$

Then recall that Biot-Savart law gives us

$$u_t(x) = \int_{\mathbb{T}^2} K(x-y)\, \omega_t(dy)$$

where $K(x,y)$ is the Biot-Savart kernel; in full space it is given by $K(x-y) = \frac{1}{2\pi}\frac{(x-y)^\perp}{|x-y|^2}$; on the torus its form is less simple but we still have K smooth for $x \neq y$, $K(y-x) = -K(x-y)$,

$$|K(x-y)| \leq \frac{C}{|x-y|}$$

for small values of $|x-y|$. See for instance [36] for details. Thus we write the weak formulation in the more explicit form

$$\langle \omega_t, \phi\rangle = \langle \omega_0, \phi\rangle + \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} K(x-y)\, \nabla\phi(x)\, \omega_s(dx)\, \omega_s(dy)\, ds.$$

Since the double space integral, when we rename x by y and y by x, is the same (the renaming doesn't affect the value), and $K(y-x) = -K(x-y)$, we get (3). Identity (3) is the weak vorticity formulation of Euler equations. Depending on the assumptions on the measures ω_s (whether or not they have concentrated masses), one has to specify the value of $K(0)$, which is not given a priori, and thus the value of $H_\phi(x,x)$; in the analysis of point vortices, for instance, it is usually set equal to zero, to avoid self-interaction. The weak vorticity formulation of Euler equations proved to be a fundamental tool in the investigation of limits of solutions, especially in the context of measures. Below we shall follow a similar path in the case of white noise distributional solutions.

2.4 The nonlinear term for white noise vorticity

Our purpose now is to define

$$\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} H_\phi(x,y)\,\omega(x)\,\omega(y)\,dxdy$$

when $\omega : \Xi \to C^\infty\left(\mathbb{T}^2\right)'$ is a white noise.

Preliminarily, notice that if $\omega \in C^\infty\left(\mathbb{T}^2\right)'$ is a distribution, we can define a distribution $\omega \otimes \omega \in C^\infty\left(\mathbb{T}^2 \times \mathbb{T}^2\right)'$ which satisfies

$$\langle \omega \otimes \omega, \phi \otimes \psi \rangle = \langle \omega, \phi \rangle \langle \omega, \psi \rangle$$

for all $\phi, \psi \in C^\infty\left(\mathbb{T}^2\right)$, where $\phi \otimes \psi$ denotes the function $(\phi \otimes \psi)(x,y) = \phi(x)\psi(y)$. The definition of $\omega \otimes \omega$ can be based on limits of test functions of the form $\sum_{i=1}^n \phi_i(x)\psi_i(y)$, or more directly on the following argument. Given $f \in C^\infty\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$, for each $x \in \mathbb{T}^2$ we have $f(x,\cdot) \in C^\infty\left(\mathbb{T}^2\right)$, hence $\langle \omega, f(x,\cdot)\rangle$ is well defined. The function $g(x) = \langle \omega, f(x,\cdot)\rangle$ belongs to $C^\infty\left(\mathbb{T}^2\right)$, as one can verify using the continuity properties of distributions on test functions. Then we can set

$$\langle \omega \otimes \omega, f \rangle = \langle \omega, g \rangle, \qquad \text{where } g(x) = \langle \omega, f(x,\cdot)\rangle. \tag{6}$$

If $\omega \in H^{-s}\left(\mathbb{T}^2\right)$ for some $s > 0$, one can check that $\omega \otimes \omega \in H^{-2s}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$.

Let us go back to white noise. First notice that, being $\omega \in H^{-1-}\left(\mathbb{T}^2\right)$ with probability one, we have at least

$$\omega \otimes \omega \in H^{-2-}\left(\mathbb{T}^2 \times \mathbb{T}^2\right) \text{ with probability one.}$$

Hence $\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} f(x,y)\,\omega(x)\,\omega(y)\,dxdy$, or more properly the duality

$$\langle \omega \otimes \omega, f \rangle$$

is well defined when $f \in H^{2+}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$. The question is: can we define

$$\langle \omega \otimes \omega, H_\phi \rangle$$

for the function H_ϕ, which is smooth outside the diagonal, and bounded, but discontinuous along the diagonal and thus not of class H^{2+}? We have the following results, over which all our analysis is based. The first result is concerned with the smooth approximations $\omega_\epsilon(x) = \langle \omega, \theta_\epsilon(x - \cdot) \rangle$, the second one with white noise.

Lemma 5 *i) If $\omega : \Xi \to C^\infty(\mathbb{T}^2)'$ is a white noise and f is bounded measurable on $\mathbb{T}^2 \times \mathbb{T}^2$, then for every $p \geq 1$ there is a constant $C_p > 0$ such that, for all $\epsilon > 0$,*

$$\mathbb{E}\left[|\langle \omega_\epsilon \otimes \omega_\epsilon, f \rangle|^p\right] \leq C_p \|f\|_\infty^p .$$

ii) We have $\mathbb{E}\left[\langle \omega_\epsilon \otimes \omega_\epsilon, f \rangle\right] = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \delta^\epsilon_{x-y} f(x,y)\, dxdy$.

iii) If f is symmetric, then

$$\mathbb{E}\left[|\langle \omega_\epsilon \otimes \omega_\epsilon, f \rangle - \mathbb{E}\left[\langle \omega_\epsilon \otimes \omega_\epsilon, f \rangle\right]|^2\right] = 2 \int_{(\mathbb{T}^2)^4} \delta^\epsilon_{x_1 - x_2} \delta^\epsilon_{y_1 - y_2} f(x_1, y_1) f(x_2, y_2)\, dx_1 dy_1 dx_2 dy_2.$$

Proof. i) It is sufficient to prove the claim for integer values of p. We have

$$\langle \omega_\epsilon \otimes \omega_\epsilon, f \rangle = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \omega_\epsilon(x) \omega_\epsilon(y) f(x,y)\, dxdy$$

$$\mathbb{E}\left[|\langle \omega_\epsilon \otimes \omega_\epsilon, f \rangle|^p\right] = \int_{(\mathbb{T}^2)^{2p}} \mathbb{E}\left[\prod_{i=1}^p (\omega_\epsilon(x_i) \omega_\epsilon(y_i))\right] \prod_{i=1}^p f(x_i, y_i)\, dx_1 dy_1 \cdots dx_p dy_p.$$

From Isserlis-Wick theorem,

$$\mathbb{E}\left[\prod_{i=1}^p (\omega_\epsilon(x_i) \omega_\epsilon(y_i))\right] = \sum_\pi \prod_{(a,b)\in\pi} \mathbb{E}\left[\omega_\epsilon(a) \omega_\epsilon(b)\right] = \sum_\pi \prod_{(a,b)\in\pi} \delta^\epsilon_{a-b}$$

where the sum is over all partitions π of $(x_1, y_1, ..., x_p, y_p)$ in pairs, generically denoted by (a,b). Therefore

$$\begin{aligned}
\mathbb{E}\left[|\langle \omega_\epsilon \otimes \omega_\epsilon, f \rangle|^p\right] &= \sum_\pi \int_{(\mathbb{T}^2)^{2p}} \prod_{(a,b)\in\pi} \delta^\epsilon_{a-b} \prod_{i=1}^p f(x_i, y_i)\, dx_1 dy_1 \cdots dx_p dy_p \\
&\leq \|f\|_\infty^p \sum_\pi \int_{(\mathbb{T}^2)^{2p}} \prod_{(a,b)\in\pi} \delta^\epsilon_{a-b} dx_1 dy_1 \cdots dx_p dy_p \\
&= \|f\|_\infty^p \sum_\pi \left(\int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \delta^\epsilon_{a-b} da db \right)^p \\
&= \|f\|_\infty^p \sum_\pi \left(\int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \langle \theta_\epsilon(a - \cdot), \theta_\epsilon(b - \cdot) \rangle\, da db \right)^p \\
&= \|f\|_\infty^p \sum_\pi \left|\mathbb{T}^2\right|^p =: C_p \|f\|_\infty^p
\end{aligned}$$

(the sum has $(2p)!/(2^p p!)$ terms).

ii) We simply have

$$\mathbb{E}[\langle\omega_\epsilon\otimes\omega_\epsilon,f\rangle]=\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}\mathbb{E}[\omega_\epsilon(x)\,\omega_\epsilon(y)]\,f(x,y)\,dxdy=\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}\delta^\epsilon_{x-y}f(x,y)\,dxdy.$$

iii) We just develop more carefully

$$\mathbb{E}\left[\langle\omega_\epsilon\otimes\omega_\epsilon,f\rangle^2\right]=\int_{(\mathbb{T}^2)^4}\mathbb{E}\left[\prod_{i=1}^2(\omega_\epsilon(x_i)\,\omega_\epsilon(y_i))\right]\prod_{i=1}^2 f(x_i,y_i)\,dx_1dy_1dx_2dy_2.$$

We have, again from Isserlis-Wick theorem,

$$\begin{aligned}\mathbb{E}\left[\prod_{i=1}^2(\omega_\epsilon(x_i)\,\omega_\epsilon(y_i))\right]&=\mathbb{E}[\omega_\epsilon(x_1)\,\omega_\epsilon(x_2)]\,\mathbb{E}[\omega_\epsilon(y_1)\,\omega_\epsilon(y_2)]\\&\quad+\mathbb{E}[\omega_\epsilon(x_1)\,\omega_\epsilon(y_2)]\,\mathbb{E}[\omega_\epsilon(y_1)\,\omega_\epsilon(x_2)]\\&\quad+\mathbb{E}[\omega_\epsilon(x_1)\,\omega_\epsilon(y_1)]\,\mathbb{E}[\omega_\epsilon(x_2)\,\omega_\epsilon(y_2)]\\&=\delta^\epsilon_{x_1-x_2}\delta^\epsilon_{y_1-y_2}+\delta^\epsilon_{x_1-y_2}\delta^\epsilon_{y_1-x_2}+\delta^\epsilon_{x_1-y_1}\delta^\epsilon_{x_2-y_2}.\end{aligned}$$

Hence, using the symmetry,

$$\begin{aligned}&\mathbb{E}\left[\langle\omega_\epsilon\otimes\omega_\epsilon,f\rangle^2\right]\\&=\int_{(\mathbb{T}^2)^4}\left(\delta^\epsilon_{x_1-x_2}\delta^\epsilon_{y_1-y_2}+\delta^\epsilon_{x_1-y_2}\delta^\epsilon_{y_1-x_2}+\delta^\epsilon_{x_1-y_1}\delta^\epsilon_{x_2-y_2}\right)f(x_1,y_1)\,f(x_2,y_2)\,dx_1dy_1dx_2dy_2\\&=2\int_{(\mathbb{T}^2)^4}\delta^\epsilon_{x_1-x_2}\delta^\epsilon_{y_1-y_2}f(x_1,y_1)\,f(x_2,y_2)\,dx_1dy_1dx_2dy_2+\left(\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}\delta^\epsilon_{x-y}f(x,y)\,dxdy\right)^2.\end{aligned}$$

We have found

$$\mathbb{E}\left[\langle\omega_\epsilon\otimes\omega_\epsilon,f\rangle^2\right]-\mathbb{E}[\langle\omega_\epsilon\otimes\omega_\epsilon,f\rangle]^2=2\int_{(\mathbb{T}^2)^4}\delta^\epsilon_{x_1-x_2}\delta^\epsilon_{y_1-y_2}f(x_1,y_1)\,f(x_2,y_2)\,dx_1dy_1dx_2dy_2.$$

■

Corollary 6 *i) If $\omega:\Xi\to C^\infty(\mathbb{T}^2)'$ is a white noise and $f\in H^{2+}(\mathbb{T}^2\times\mathbb{T}^2)$, then for every $p\geq 1$ there is a constant $C_p>0$ such that*

$$\mathbb{E}[|\langle\omega\otimes\omega,f\rangle|^p]\leq C_p\,\|f\|_\infty^p.$$

ii) We have $\mathbb{E}[\langle\omega\otimes\omega,f\rangle]=\int_{\mathbb{T}^2}f(x,x)\,dx$.

iii) If f is symmetric, then

$$\mathbb{E}\left[|\langle\omega\otimes\omega,f\rangle-\mathbb{E}[\langle\omega\otimes\omega,f\rangle]|^2\right]=2\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}f(x,y)^2\,dxdy.$$

Proof. Notice that f is continuous and thus bounded and uniformly continuous, on $\mathbb{T}^2$, by Sobolev embedding theorem. Thus we may apply the previous lemma to $\langle \omega_\epsilon \otimes \omega_\epsilon, f \rangle$; and we have

$$\lim_{\epsilon \to 0} \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \delta^\epsilon_{x-y} f(x,y)\, dxdy = \int_{\mathbb{T}^2} f(x,x)\, dx$$

$$\lim_{\epsilon \to 0} \int_{(\mathbb{T}^2)^4} \delta^\epsilon_{x_1-x_2} \delta^\epsilon_{y_1-y_2} f(x_1,y_1) f(x_2,y_2)\, dx_1 dy_1 dx_2 dy_2 = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} f(x_1,y_1)^2\, dx_1 dy_1.$$

From the identity

$$\langle \omega_\epsilon \otimes \omega_\epsilon, f \rangle = \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \omega_\epsilon(x)\, \omega_\epsilon(y)\, f(x,y)\, dxdy = \langle \omega \otimes \omega, (\theta_\epsilon \otimes \theta_\epsilon) * f \rangle$$

we see that $\mathbb{P}$-almost surely, for every $f \in H^{2+}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$ we have

$$\lim_{\epsilon \to 0} \langle \omega_\epsilon \otimes \omega_\epsilon, f \rangle = \langle \omega \otimes \omega, f \rangle .$$

We can pass to the limit in all expectations written in the statement of the corollary, due to uniform integrability of $|\langle \omega_\epsilon \otimes \omega_\epsilon, f \rangle|$ (Vitali theorem), coming from property (i) of the lemma. The corollary then follows from these limit properties and the lemma. ∎

Remark 7 *In the non symmetric case we simply have*

$$\mathbb{E}\left[|\langle \omega \otimes \omega, f \rangle - \mathbb{E}[\langle \omega \otimes \omega, f \rangle]|^2 \right] = \int\int f^2(x,y)\, dxdy + \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} f(x,y) f(y,x)\, dxdy.$$

Based on the previous key facts we can give a definition of $\langle \omega \otimes \omega, H_\phi \rangle$ when ω is white noise.

Theorem 8 *Let $\omega : \Xi \to C^\infty\left(\mathbb{T}^2\right)'$ be a white noise and $\phi \in C^\infty\left(\mathbb{T}^2\right)$ be given. Assume that $H^n_\phi \in H^{2+}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$ are symmetric and approximate H_ϕ in the following sense:*

$$\lim_{n \to \infty} \int\int \left(H^n_\phi - H_\phi\right)^2 (x,y)\, dxdy = 0$$

$$\lim_{n \to \infty} \int H^n_\phi(x,x)\, dx = 0.$$

Then the sequence of r.v.'s $\left\langle \omega \otimes \omega, H^n_\phi \right\rangle$ is a Cauchy sequence in mean square. We denote by

$$\langle \omega \otimes \omega, H_\phi \rangle$$

its limit. Moreover, the limit is the same if H^n_ϕ is replaced by $\widetilde{H}^n_\phi$ with the same properties and such that $\lim_{n \to \infty} \int\int \left(H^n_\phi - \widetilde{H}^n_\phi\right)^2 (x,y)\, dxdy = 0$.

Proof. Since $\lim_{n\to\infty}\int H_\phi^n(x,x)\,dx = 0$, it is equivalent to show that $\left\langle \omega\otimes\omega, H_\phi^n\right\rangle - \int H_\phi^n(x,x)\,dx$ is a Cauchy sequence in mean square. We have

$$\mathbb{E}\left[\left|\left\langle \omega\otimes\omega, H_\phi^n\right\rangle - \int H_\phi^n(x,x)\,dx - \left\langle \omega\otimes\omega, H_\phi^m\right\rangle + \int H_\phi^m(x,x)\,dx\right|^2\right]$$
$$= \mathbb{E}\left[\left|\left\langle \omega\otimes\omega, \left(H_\phi^n - H_\phi^m\right)\right\rangle - \int \left(H_\phi^n - H_\phi^m\right)(x,x)\,dx\right|^2\right]$$

and now we use properties (ii-iii) of the Corollary

$$= 2\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}\left(H_\phi^n - H_\phi^m\right)^2(x,y)\,dxdy.$$

Due to our assumption, this implies the Cauchy property. Hence $\langle \omega\otimes\omega, H_\phi\rangle$ is well defined. The invariance property is prove in a similar way. ■

Remark 9 *It is easy to construct a sequence $H_\phi^n(x,y)$ with the properties above. Recall that $H_\phi(x,y) := \frac{1}{2}K(x-y)(\nabla\phi(x) - \nabla\phi(y))$, where K smooth for $x\neq y$, $K(y-x) = -K(x-y)$,*

$$|K(x-y)| \leq \frac{C}{|x-y|}$$

for small values of $|x-y|$. We set, for $\epsilon > 0$,

$$K_\epsilon(x) = \begin{cases} K(x)(1-\theta_\epsilon(x)) & \text{for } \; x\neq 0 \\ 0 & \text{for } \; x = 0 \end{cases} \tag{7}$$

where $\theta_\epsilon(x) = \theta\left(\epsilon^{-1}x\right)$, $0\leq\theta\leq 1$, θ is smooth, with support a small ball $B(0,r)$, equal to 1 in $B(0,r/2)$; and, given any sequence $\epsilon_n\to 0$ we set

$$H_\phi^n(x,y) = \frac{1}{2}K_{\epsilon_n}(x-y)(\nabla\phi(x) - \nabla\phi(y)).$$

Then H_ϕ^n is smooth; $H_\phi^n(x,x) = 0$ hence $\int H_\phi^n(x,x)\,dx = 0$; and

$$\lim_{n\to\infty}\int\int\left(H_\phi^n - H_\phi\right)^2(x,y)\,dxdy = \lim_{n\to\infty}\int\int H_\phi^2(x,y)\,\theta_{\epsilon_n}^2(x-y)\,dxdy$$
$$\leq \lim_{n\to\infty}\int\int_{|x-y|\leq\epsilon_n r} H_\phi^2(x,y)\,dxdy = 0$$

(because $H_\phi^2(x,y)$ is bounded above, $\theta_{\epsilon_n}^2 \leq 1$, and $\theta_{\epsilon_n}^2 \neq 0$ only in $B(0,\epsilon_n r)$).

In fact, what we need in Definition 17 below is a definition of $\int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle \, ds$ and for such purpose the previous result is not so strong; it would allow for instance to define such integral as a Bochner integral in the Hilbert space $L^2(\Xi)$. We prefer to have a stronger meaning and for this purpose we refine the previous result.

Theorem 10 *Let $\omega_\cdot : \Xi \times [0,T] \to C^\infty(\mathbb{T}^2)'$ be a measurable map with trajectories of class $C([0,T]; H^{-1-})$. Assume that ω_t is a white noise at every time $t \in [0,T]$. Let H_ϕ^n be an approximation of H_ϕ as above, of class $H^{2+}(\mathbb{T}^2 \times \mathbb{T}^2)$. Then the well defined sequence of real valued process $\left\{ s \mapsto \left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle ; s \in [0,T] \right\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\Xi; L^2(0,T))$.*

Proof. The proof is the same as the one of Theorem 8, but we repeat it, due to the importance of the present result. We have

$$\mathbb{E}\left[\int_0^T \left| \left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle - \int H_\phi^n(x,x)\,dx - \left\langle \omega_s \otimes \omega_s, H_\phi^m \right\rangle + \int H_\phi^m(x,x)\,dx \right|^2 ds \right]$$
$$= \int_0^T \mathbb{E}\left[\int_0^T \left| \left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle - \int H_\phi^n(x,x)\,dx - \left\langle \omega_s \otimes \omega_s, H_\phi^m \right\rangle + \int H_\phi^m(x,x)\,dx \right|^2 \right] ds$$
$$= T \cdot \mathbb{E}\left[\left| \left\langle \omega_0 \otimes \omega_0, \left(H_\phi^n - H_\phi^m \right) \right\rangle - \int \left(H_\phi^n - H_\phi^m \right)(x,x)\,dx \right|^2 \right]$$

and now we use properties (ii-iii) of the Corollary

$$= 2 \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \left(H_\phi^n - H_\phi^m \right)^2 (x,y)\,dxdy.$$

Due to our assumption, this implies the Cauchy property. ■

Definition 11 *Under the assumptions of the previous theorem, we denote by*

$$\{ s \mapsto \langle \omega_s \otimes \omega_s, H_\phi \rangle ; s \in [0,T] \}$$

or more simply by $\langle \omega_\cdot \otimes \omega_\cdot, H_\phi \rangle$ the process of class $L^2(\Xi; L^2(0,T))$, limit of the sequence $\left\{ s \mapsto \left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle ; s \in [0,T] \right\}_{n \in \mathbb{N}}$.

Remark 12 *By the identification $L^2(\Xi; L^2(0,T)) = L^2(0,T; L^2(\Xi))$, we may see $\langle \omega_\cdot \otimes \omega_\cdot, H_\phi \rangle$ as an element of the class $L^2(0,T; L^2(\Xi))$; its value at time s is, for a.e. s, an element of $L^2(\Xi)$; one may check that it is the same element of $L^2(\Xi)$ given by Theorem 8.*

2.5 The nonlinear term for modified white noise vorticity

We may generalize a little bit the previous construction. Assume $\omega : \Xi \to C^\infty(\mathbb{T}^2)'$ is a random distribution with the property that

$$\mathbb{E}[\Phi(\omega)] = \mathbb{E}[\rho(\omega_{WN})\Phi(\omega_{WN})]$$

for every measurable function $\Phi : H^{-1-}(\mathbb{T}^2) \to [0,\infty)$, where $\omega_{WN} : \Xi \to C^\infty(\mathbb{T}^2)'$ is a white noise and $\rho : H^{-1-}(\mathbb{T}^2) \to [0,\infty)$ is a measurable function such that

$$k_q := \mathbb{E}[\rho^q(\omega_{WN})] < \infty$$

for some $q > 1$, and $\int \rho d\mu = 1$. This is equivalent to say that the law of ω is absolutely continuous with respect to μ with density ρ satisfying $\int \rho^q d\mu < \infty$.

Lemma 13 *Under the previous assumptions, if $f \in H^{2+}(\mathbb{T}^2 \times \mathbb{T}^2)$, then:*
i) for every $r \geq 1$ there is a constant $C_r > 0$ such that

$$\mathbb{E}[|\langle \omega \otimes \omega, f\rangle|^r] \leq C_r \|f\|_\infty^r .$$

ii) If f is symmetric, then there exists a constant $C_q > 0$ such that

$$\mathbb{E}\left[\left|\langle \omega \otimes \omega, f\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|\right] \leq C_q \|f\|_{L^2(\mathbb{T}^2\times\mathbb{T}^2)}^{1/p}$$

where p is the number such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. i) We deduce the claim from

$$\begin{aligned}\mathbb{E}[|\langle \omega \otimes \omega, f\rangle|^r] &= \mathbb{E}[\rho(\omega_{WN})|\langle \omega_{WN} \otimes \omega_{WN}, f\rangle|^r] \\ &\leq \mathbb{E}[\rho^q(\omega_{WN})]^{1/q}\,\mathbb{E}[|\langle \omega_{WN} \otimes \omega_{WN}, f\rangle|^{rp}]^{1/p} .\end{aligned}$$

ii) One has

$$\begin{aligned}\mathbb{E}\left[\left|\langle \omega \otimes \omega, f\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|\right] &= \mathbb{E}\left[\rho(\omega_{WN})\left|\langle \omega_{WN} \otimes \omega_{WN}, f\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|\right] \\ &\leq \mathbb{E}[\rho^q(\omega_{WN})]^{1/q}\,\mathbb{E}\left[\left|\langle \omega_{WN} \otimes \omega_{WN}, \phi\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|^p\right]^{1/p} .\end{aligned}$$

Moreover,

$$\begin{aligned}&\mathbb{E}\left[\left|\langle \omega_{WN} \otimes \omega_{WN}, f\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|^p\right] \\ &\leq \mathbb{E}\left[\left|\langle \omega_{WN} \otimes \omega_{WN}, f\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|^2\right]^{1/2} \mathbb{E}\left[\left|\langle \omega_{WN} \otimes \omega_{WN}, f\rangle - \int_{\mathbb{T}^2} f(x,x)\,dx\right|^{2p-2}\right]^{1/2} \\ &= C_q^0 \left(\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} f(x,y)^2\,dxdy\right)^{1/2}\end{aligned}$$

where

$$C_q^0 := 2\mathbb{E}\left[\left|\langle \omega_{WN} \otimes \omega_{WN}, f\rangle - \int_{\mathbb{T}^2} f(x,x)\, dx\right|^{2p-2}\right]^{1/2}$$

is a finite constant, due to property (i) of a previous corollary. We set $C_q = k_q^{1/q}\left(C_q^0\right)^{1/p}$. ■

The next results are the same as those above in the white noise case except that we have a lower order of integrability, nevertheless sufficient for our aims.

Theorem 14 *Under the previous assumptions, assume that $H_\phi^n \in H^{2+}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$ are symmetric and approximate H_ϕ as in Theorem 8. Then the sequence of r.v.'s $\left\langle \omega \otimes \omega, H_\phi^n \right\rangle$ is a Cauchy sequence in $L^1(\Xi)$. We denote by $\langle \omega \otimes \omega, H_\phi\rangle$ its limit. It is the same if H_ϕ^n is replaced by $\widetilde{H}_\phi^n$ with the properties described in Theorem 8.*

Proof. Since $\lim_{n\to\infty} \int H_\phi^n(x,x)\, dx = 0$, it is equivalent to show that $\left\langle \omega \otimes \omega, H_\phi^n \right\rangle - \int H_\phi^n(x,x)\, dx$ is a Cauchy sequence in $L^1(\Xi)$. We have

$$\mathbb{E}\left[\left|\langle \omega \otimes \omega, H_\phi^n\rangle - \int H_\phi^n(x,x)\, dx - \langle \omega \otimes \omega, H_\phi^m\rangle + \int H_\phi^m(x,x)\, dx\right|\right]$$
$$= \mathbb{E}\left[\left|\langle \omega \otimes \omega, \left(H_\phi^n - H_\phi^m\right)\rangle - \int \left(H_\phi^n - H_\phi^m\right)(x,x)\, dx\right|\right]$$

and now we use property (ii) of the Corollary

$$\leq C_q \left\|H_\phi^n - H_\phi^m\right\|_{L^2(\mathbb{T}^2\times\mathbb{T}^2)}^{1/p}.$$

Due to our assumptions, this implies the Cauchy property. Hence $\langle \omega \otimes \omega, H_\phi\rangle$ is well defined. The invariance property is prove in a similar way. ■

Theorem 15 *Let $\rho : [0,T] \times H^{-1-}\left(\mathbb{T}^2\right) \to [0,\infty)$ be a function such that $\int \rho_t^q d\mu \leq C$ for some constants $C > 0$, $q > 1$, where μ is the law of white noise; and $\int \rho_t d\mu = 1$ for every $t \in [0,T]$. Let $\omega_\cdot : \Xi \times [0,T] \to C^\infty\left(\mathbb{T}^2\right)'$ be a measurable map with trajectories of class $C\left([0,T]; H^{-1-}\right)$. Assume that the law of ω_t is $\rho_t d\mu$, at every time $t \in [0,T]$. Let H_ϕ^n be an approximation of H_ϕ as above, of class $H^{2+}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$. Then the well defined sequence of real valued process $\left\{s \mapsto \left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle; s \in [0,T]\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1\left(\Xi; L^1(0,T)\right)$.*

Proof. As in previous proofs, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left|\left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle - \int H_\phi^n\left(x,x\right)dx - \left\langle \omega_s \otimes \omega_s, H_\phi^m \right\rangle + \int H_\phi^m\left(x,x\right)dx\right| ds\right] \\
&= \int_0^T \mathbb{E}\left[\left|\left\langle \omega_s \otimes \omega_s, \left(H_\phi^n - H_\phi^m\right)\right\rangle - \int \left(H_\phi^n - H_\phi^m\right)\left(x,x\right)dx\right|\right] ds \\
&\leq C_q T \left\|H_\phi^n - H_\phi^m\right\|_{L^2(\mathbb{T}^2\times\mathbb{T}^2)}^{1/p}
\end{aligned}$$

■

Definition 16 *Under the assumptions of the previous theorem, we denote by* $\langle \omega_\cdot \otimes \omega_\cdot, H_\phi \rangle$ *the process of class* $L^1\left(\Xi; L^1\left(0,T\right)\right)$*, limit of the sequence* $\left\{s \mapsto \left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle; s \in [0,T]\right\}_{n\in\mathbb{N}}$.

2.6 Weak vorticity formulation for white noise vorticity

Definition 17 *We say that a measurable map* $\omega_\cdot : \Xi \times [0,T] \to C^\infty\left(\mathbb{T}^2\right)'$ *with trajectories of class* $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$ *is a white noise solution of Euler equations if* ω_t *is a white noise at every time* $t \in [0,T]$ *and for every* $\phi \in C^\infty\left(\mathbb{T}^2\right)$*, we have the following identity* P*-a.s., uniformly in time,*

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle\, ds.$$

Here $\langle \omega_t, \phi \rangle$ is a.s. a continuous function of time because we assume that trajectories of ω are of class $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$, and $\int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle\, ds$ is the continuous process obtained by integration of the $L^2(0,T)$-process provided by Definition 11.

In the case of the previous definition, in addition, we may require that $\omega_\cdot$ is a time-stationary process. In a sense, the law of white noise is an invariant measure, although we do not have a proper Markov structure allowing us to talk about invariant measures in the classical sense.

Using Definition 16 we may generalize the previous definition to the following case:

Definition 18 *Let* $\rho : [0,T] \times H^{-1-}\left(\mathbb{T}^2\right) \to [0,\infty)$ *satisfy* $\int \rho_t^q d\mu \leq C$ *for some constants* $C > 0$*,* $q > 1$*, where* μ *is the law of white noise; and* $\int \rho_t d\mu = 1$ *for every* $t \in [0,T]$*. Let* $\omega_\cdot :$ $\Xi \times [0,T] \to C^\infty\left(\mathbb{T}^2\right)'$ *be a measurable map with trajectories of class* $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$*, such that* ω_t *has law* $\rho_t d\mu$*, for every* $t \in [0,T]$*. We say that* ω *is a* $\rho-$*white noise solution of Euler equations if for every* $\phi \in C^\infty\left(\mathbb{T}^2\right)$*,* $t \mapsto \langle \omega_t, \phi \rangle$ *is continuous and we have the following identity* P*-a.s., uniformly in time,*

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle\, ds.$$

3 Random point vortex dynamics

Let us introduce some notations. In $\left(\mathbb{T}^2\right)^N$, denote by Δ_N the generalized diagonal

$$\Delta_N = \left\{\left(x^1, ..., x^N\right) \in \left(\mathbb{T}^2\right)^N : x^i = x^j \text{ for some } i \neq j,\ i, j = 1, ..., n\right\}.$$

Then introduce the set of unlabelled and labelled finite sequences of different points

$$F_N\mathbb{T}^2 = \left\{(x_1, ..., x_n) \in \left(\mathbb{T}^2\right)^N : \left(x^1, ..., x^N\right) \in \Delta_N^c\right\}$$

$$\mathcal{L}F_N\mathbb{T}^2 = \left\{\left((\xi_1, x_1), ..., (\xi_N, x_N)\right) \in \left(\mathbb{R} \times \mathbb{T}^2\right)^N : \left(x^1, ..., x^N\right) \in \Delta_N^c\right\}$$

and the unlabelled and labelled configuration space

$$C_N\mathbb{T}^2 = F_N\mathbb{T}^2/\Sigma_N$$

$$\mathcal{L}C_N\mathbb{T}^2 = \mathcal{L}F_N\mathbb{T}^2/\Sigma_N$$

where Σ_N is the group of permutations of coordinates. This set, $\mathcal{L}C_N\mathbb{T}^2$, is in bijection with the set of discrete signed measures with n-point support:

$$\mathcal{M}_N\left(\mathbb{T}^2\right) = \left\{\mu \in \mathcal{M}\left(\mathbb{T}^2\right) : \exists X \in C_N\mathbb{T}^2 : |\mu|\left(X^c\right) = 0, \mu(x) \neq 0 \text{ for every } x \in X\right\}.$$

We do not use extensively these notations but they may help to formalize further the topics we are going to describe.

3.1 Definition for a.e. initial condition

Consider, for every $N \in \mathbb{N}$, the finite dimensional dynamics in $\left(\mathbb{T}^2\right)^N$

$$\frac{dX_t^{i,N}}{dt} = \sum_{j=1}^N \frac{1}{\sqrt{N}}\xi_j K\left(X_t^{i,N} - X_t^{j,N}\right) \qquad i = 1, ..., N \tag{8}$$

with initial condition $\left(X_0^{1,N}, ..., X_0^{N,N}\right) \in \left(\mathbb{T}^2\right)^N \backslash \Delta_N$, where as above K is the Biot-Savart kernel on $\mathbb{T}^2$; we set $K(0) = 0$ so that the self-interaction (namely when $j = i$) in the sum does not count. The intensities $\xi_1, ..., \xi_N$ are (random) numbers of any sign. One can consider (8) as a dynamics on the configuration space $C_N\mathbb{T}^2$. This system corresponds also to the time-evolution of a vorticity distribution concentrated at positions $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$:

$$\omega_t^N = \frac{1}{\sqrt{N}}\sum_{n=1}^N \xi_n \delta_{X_t^n}.$$

There are various ways in which one can relate this finite dimensional dynamics to Euler equations, see [30]; under our assumptions made below we shall clarify one of these connections.

In [30] it is shown an example with $N = 3$ and ξ_1, ξ_2, ξ_3 of different signs such that, starting from different initial positions $X_0^{1,3}, X_0^{2,3}, X_0^{3,3}$, in finite time $X_t^{1,3}, X_t^{2,3}, X_t^{3,3}$ coincide; this vortex collapse corresponds to a blow-up in the finite dimensional dynamics (because $K(x-y)$ diverges as $\frac{1}{|x-y|}$ as $|x-y| \to 0$) and provokes troubles also at the level of a PDE reformulation of the dynamics of ω_t^N (having in mind the weak vorticity formulation above, the measure $\omega_t^N(dx)$ concentrates on the diagonal, where H_ϕ is discontinuous). These difficulties do not happen for constant sign vortices, but they are not interesting for our investigation. Let $\otimes_N Leb_{\mathbb{T}^2}$ be Lebesgue measure on $(\mathbb{T}^2)^N$. The main result we use below, proved in [30] is that, independently of the sign of $\xi_1, ..., \xi_N$, for $\otimes_N Leb_{\mathbb{T}^2}$-a.e. initial condition $\left(X_0^{1,N}, ..., X_0^{N,N}\right) \in (\mathbb{T}^2)^N$ the positions $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ remain different for all times; and in addition the measure $\otimes_N Leb_{\mathbb{T}^2}$ is invariant, in the sense that $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ is distributed as $\otimes_N Leb_{\mathbb{T}^2}$ for all $t \geq 0$. The precise statement is:

Theorem 19 *For every $(\xi_1, ..., \xi_N) \in \mathbb{R}^N$ and for $\otimes_N Leb_{\mathbb{T}^2}$- almost every $\left(X_0^{1,N}, ..., X_0^{N,N}\right) \in \Delta_N^c$, there is a unique solution $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ of system (8), with the property that $\left(X_t^{1,N}, ..., X_t^{N,N}\right) \in \Delta_N^c$ for all $t \geq 0$. Moreover, considering the initial condition as a random variable with distribution $\otimes_N Leb_{\mathbb{T}^2}$, the stochastic process $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ is stationary, with invariant marginal law $\otimes_N Leb_{\mathbb{T}^2}$.*

When this occurs, the measure-valued process $\omega_t^N = \frac{1}{\sqrt{N}} \sum_{n=1}^N \xi_n \delta_{X_t^n}$ satisfies, for every $\phi \in C^\infty(\mathbb{T}^2)$, the identity

$$\begin{aligned}
\frac{d}{dt}\left\langle \omega_t^N, \phi \right\rangle &= \frac{1}{\sqrt{N}} \sum_{n=1}^N \xi_n \frac{d}{dt} \phi(X_t^n) = \frac{1}{\sqrt{N}} \sum_{n=1}^N \xi_n \nabla\phi(X_t^n) \cdot \sum_{j=1}^N \frac{1}{\sqrt{N}} \xi_j K\left(X_t^n - X_t^j\right) \\
&= \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \nabla\phi(x) \cdot K(x-y)\, \omega_t^N(dx)\, \omega_t^N(dy)
\end{aligned}$$

and therefore

$$\left\langle \omega_t^N, \phi \right\rangle = \left\langle \omega_0^N, \phi \right\rangle + \int_0^t \left\langle \omega_s^N \otimes \omega_s^N, H_\phi \right\rangle ds.$$

3.2 Random point vortices, at time $t = 0$, converging to white noise, and their time evolution

On a probability space $(\Xi, \mathcal{F}, \mathbb{P})$, let (ξ_n) be an i.i.d. sequence of $N(0,1)$ r.v.'s and (X_0^n) be an i.i.d. sequence of $\mathbb{T}^2$-valued r.v.'s, independent of (ξ_n) and uniformly distributed.

Denote by

$$\lambda_N^0 := \otimes_N \left(N\left(0,1\right) \otimes Leb_{\mathbb{T}^2}\right)$$

the law of the random vector

$$\left(\left(\xi_1, X_0^1\right), ..., \left(\xi_N, X_0^N\right)\right).$$

For every $N \in \mathbb{N}$, let us consider also the measure-valued vorticity field

$$\omega_0^N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \xi_n \delta_{X_0^n}.$$

Remark 20 *Since product Lebesgue measure does not charge the generalized diagonal Δ_N, the law λ_N^0 can be seen as a probability measure on the set of labelled ordered different points $\mathcal{L}F_N\mathbb{T}^2$ (see the beginning of Section 3). It is an exchangeable measure (namely invariant by permutations) and thus it induces a probability measure on the labelled configuration space $\mathcal{L}C_N\mathbb{T}^2$. It also induces a probability measure on $\mathcal{M}_N\left(\mathbb{T}^2\right)$ or, what we need below, on $H^{-1-}\left(\mathbb{T}^2\right)$. We shall denote this induced measure on discrete measures or on distributions by $\mu_N^0\left(d\omega\right)$. Defined the measurable map $\mathcal{T}_N : \left(\mathbb{R}\times\mathbb{T}^2\right)^N \rightarrow H^{-1-}\left(\mathbb{T}^2\right)$ as*

$$\left(\left(\xi_1, X_0^1\right), ..., \left(\xi_N, X_0^N\right)\right) \overset{\mathcal{T}_N}{\mapsto} \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \xi_n \delta_{X_0^n}$$

we have (with the push-forward notation)

$$\mu_N^0 = \left(\mathcal{T}_N\right)_* \lambda_N^0.$$

The random distribution ω_0^N is centered, becuase

$$\mathbb{E}\left[\xi_n \left\langle \delta_{X_0^n}, \varphi \right\rangle\right] = 0$$

(true since ξ_n and $\left\langle \delta_{X_0^n}, \varphi \right\rangle$ are independent and ξ_n is centered). Let us denote by Q_N the covariance operator of ω_0^N, defined as

$$\left\langle Q_N \varphi, \psi \right\rangle = \mathbb{E}\left[\left\langle \omega_0^N, \varphi \right\rangle \left\langle \omega_0^N, \psi \right\rangle\right]$$

for all $\varphi, \psi \in C^\infty\left(\mathbb{T}^2\right)$. We have

$$\begin{aligned}
\left\langle Q_N \varphi, \psi \right\rangle &= \frac{1}{N} \sum_{n,m=1}^{N} \mathbb{E}\left[\xi_n \xi_m \left\langle \delta_{X_0^n}, \varphi \right\rangle \left\langle \delta_{X_0^m}, \psi \right\rangle\right] \\
&= \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}\left[\xi_n^2\right] \mathbb{E}\left[\left\langle \delta_{X_0^n}, \varphi \right\rangle \left\langle \delta_{X_0^n}, \psi \right\rangle\right] \\
&= \mathbb{E}\left[\xi_1^2\right] \mathbb{E}\left[\varphi\left(X_0^1\right) \psi\left(X_0^1\right)\right] \\
&= \int_{\mathbb{T}^2} \varphi\left(x\right) \psi\left(x\right) dx
\end{aligned}$$

hence ω_0^N has the same covariance as white noise, but obviously it is not Gaussian. However, a Hilbert-valued version of the Central Limit Theorem gives us

Proposition 21 *If ω_{WN} denotes white noise, then*

$$\omega_0^N \overset{Law}{\rightharpoonup} \omega_{WN}$$

where convergence takes place in $H^{-1-\delta}$ for every $\delta > 0$.

Proof. The condition for the validity of this claim, a part from the computation above on the covariance, is that the space is Hilbert and the second moment is finite:

$$\mathbb{E}\left[\left\|\xi_n \delta_{X_0^n}\right\|_{H^{-1-\delta}}^2\right] < \infty \tag{9}$$

(see [26]). Condition (9) is true because $\mathbb{E}\left[\left\|\xi_n \delta_{X_0^n}\right\|_{H^{-1-\delta}}^2\right] = \mathbb{E}\left[\left\|\delta_{X_0^n}\right\|_{H^{-1-\delta}}^2\right]$ and

$$\begin{aligned}\left\|\delta_{X_0^n}\right\|_{H^{-1-\delta}} &= \sup_{\|\phi\|_{H^{1+\delta}} \leq 1} \left\langle \delta_{X_0^n}, \phi \right\rangle = \sup_{\|\phi\|_{H^{1+\delta}} \leq 1} \phi\left(X_0^n\right) \\ &\leq \sup_{\|\phi\|_{H^{1+\delta}} \leq 1} \|\phi\|_\infty \leq C \sup_{\|\phi\|_{H^{1+\delta}} \leq 1} \|\phi\|_{H^{1+\delta}} = C\end{aligned}$$

where we have used Sobolev embedding theorem $H^{1+\delta}\left(\mathbb{T}^2\right) \subset C\left(\mathbb{T}^2\right)$. ■

Obviously, using proper versions of the Central Limit Theorem, one can provide much more general random point vortices that converge in law to ω_{WN}; our aim here is not the generality but the construction of an approximation scheme for our main existence theorem.

As a consequence of Theorem 19 we have:

Proposition 22 *Consider the vortex dynamics with random intensities $(\xi_1, ..., \xi_N)$ and random initial positions $\left(X_0^1, ..., X_0^N\right)$ distributed as λ_N^0. For a.e. value of $\left(\left(\xi_1, X_0^1\right), ..., \left(\xi_N, X_0^N\right)\right)$ the dynamics $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ is well defined in Δ_N^c for all $t \geq 0$, and the associated measure-valued vorticity ω_t^N satisfies the weak vorticity formulation. The stochastic process ω_t^N is stationary in time and space-homogeneous; in particular the law of $\left(\left(\xi_1, X_t^1\right), ..., \left(\xi_N, X_t^N\right)\right)$ is λ_N^0 at any time $t \geq 0$.*

Proof. The first claims are obvious consequences of Theorem 19. Given $(\xi_1, ..., \xi_N)$, the process $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ is stationary. Hence, denoted $(\xi_1, ..., \xi_N)$ by ξ and $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ by X_t, for every $0 \leq t_1 \leq ... \leq t_n$ and bounded measurable F, the random variable (conditional expectation given the σ-field generated by ξ)

$$\mathbb{E}\left[F\left(\left(\xi, X_{t_1+h}\right), ..., \left(\xi, X_{t_n+h}\right)\right) | \xi\right]$$

is independent of h (in the equivalence class of conditional expectation). Therefore its expectation, namely $\mathbb{E}\left[F\left(\left(\xi, X_{t_1+h}\right), ..., \left(\xi, X_{t_n+h}\right)\right)\right]$, is independent of h, which implies that (ξ, X_t) (and therefore ω_t^N) is a stationary process. Space homogeneity is not used below and thus we do not prove it, but it is not difficult due to the symmetries of the system. ■

3.3 Integrability properties of the random point vortices

Let ω_t^N be given by Proposition 22. It satisfies estimates similar to those of white noise.

Lemma 23 *Assume $f : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}$ is symmetric, bounded and measurable. Then, for every $p \geq 1$ and $\delta > 0$ there are constants $C_p, C_{p,\delta} > 0$ such that*

$$\mathbb{E}\left[\left\langle \omega_t^N \otimes \omega_t^N, f \right\rangle^p\right] \leq C_p \left\|f\right\|_\infty^p$$

$$\mathbb{E}\left[\left\|\omega_t^N\right\|_{H^{-1-\delta}}^p\right] \leq C_{p,\delta}$$

and moreover

$$\mathbb{E}\left[\left\langle \omega_t^N \otimes \omega_t^N, f \right\rangle^2\right] = \frac{3}{N}\int f^2(x,x)\,dx + \left(\int f(x,x)\,dx\right)^2 + 2\int\int f^2(x,y)\,dxdy.$$

Proof. Step 1. It is sufficient to consider integer values of p. One has

$$\begin{aligned}
\mathbb{E}\left[\left\langle \omega_t^N \otimes \omega_t^N, f \right\rangle^p\right] &= \mathbb{E}\left(\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} f(x,y)\,\omega_t^N(dx)\,\omega_t^N(dy)\right)^p \\
&= \int_{(\mathbb{T}^2)^{2p}} \mathbb{E}\left[\prod_{i=1}^p f(x_i,y_i) \prod_{i=1}^p \left(\omega_t^N(dx_i)\,\omega_t^N(dy_i)\right)\right] \\
&= \frac{1}{N^p}\sum_{k_1,h_1,...,k_p,h_p=1}^{N} \mathbb{E}\left[\prod_{i=1}^p \xi_{k_i}\xi_{h_i}\right]\mathbb{E}\left[\prod_{i=1}^p f\left(X_t^{k_i}, X_t^{h_i}\right)\right].
\end{aligned}$$

We replace here Isserlis-Wick theorem by a combinatorial argument based on the independence of the r.v.'s ξ_i. Denote by $\mathcal{P}_p$ the family of all $(2p)$-ples $(k_1, h_1, ..., k_p, h_p)$ that are "paired", namely such that we may split $(k_1, h_1, ..., k_p, h_p)$ in p pairs such that in each pair the two elements have the same value; an example is when $h_1 = k_1, \ldots, h_p = k_p$. Notice that we do not require that the values in different pairs are different. One has $\mathbb{E}\left[\prod_{i=1}^p \xi_{k_i}\xi_{h_i}\right] = 0$ if $(k_1, h_1, ..., k_p, h_p) \notin \mathcal{P}_p$, hence

$$\begin{aligned}
\mathbb{E}\left[\left\langle \omega_t^N \otimes \omega_t^N, f \right\rangle^p\right] &= \frac{1}{N^p}\sum_{(k_1,h_1,...,k_p,h_p)\in\mathcal{P}_p} \mathbb{E}\left[\prod_{i=1}^p \xi_{k_i}\xi_{h_i}\right]\mathbb{E}\left[\prod_{i=1}^p f\left(X_t^{k_i}, X_t^{h_i}\right)\right] \\
&\leq \|f\|_\infty^p \frac{C_p'}{N^p} Card\left(\mathcal{P}_p\right)
\end{aligned}$$

where C_p' is a constant that bounds from above $\mathbb{E}\left[\prod_{i=1}^{p}\xi_{k_i}\xi_{h_i}\right]$ independently of the index. The cardinality of $\mathcal{P}_p$ is bounded above by $C_p''N^p$ for another constant $C_p''>0$ (the idea is that given any one of the N values of k_1, either h_1 or k_2 or one of the next indexes is equal to k_1, and this constraints the variability of that index to one value; then repeat p times this argument). Therefore $\mathbb{E}\left[\left\langle\omega_t^N\otimes\omega_t^N,f\right\rangle^p\right]\le\|f\|_\infty^p C_p'C_p''$. This proves the first claim of the lemma, with $C_p=C_p'C_p''$.

Step 2. Similarly,

$$\begin{aligned}
\mathbb{E}\left[\left\|\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\xi_n\delta_{X_t^n}\right\|_{H^{-1-\delta/2}}^{2p}\right] &= \mathbb{E}\left[\left(\left\langle\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\xi_n\delta_{X_t^n},\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\xi_n\delta_{X_t^n}\right\rangle_{H^{-1-\delta/2}}\right)^p\right]\\
&=\frac{1}{N^p}\mathbb{E}\left[\left(\sum_{n,m=1}^{N}\xi_n\xi_M\left\langle\delta_{X_t^n},\delta_{X_t^m}\right\rangle_{H^{-1-\delta/2}}\right)^p\right]\\
&=\frac{1}{N^p}\sum_{k_1,h_1,\ldots,k_p,h_p=1}^{N}\mathbb{E}\left[\prod_{i=1}^{p}\xi_{k_i}\xi_{h_i}\right]\mathbb{E}\left[\prod_{i=1}^{p}\left\langle\delta_{X_t^{k_i}},\delta_{X_t^{h_i}}\right\rangle_{H^{-1-\delta/2}}\right]\\
&=\frac{1}{N^p}\sum_{(k_1,h_1,\ldots,k_p,h_p)\in\mathcal{P}_p}\mathbb{E}\left[\prod_{i=1}^{p}\xi_{k_i}\xi_{h_i}\right]\mathbb{E}\left[\prod_{i=1}^{p}\left\langle\delta_{X_0^{k_i}},\delta_{X_0^{h_i}}\right\rangle_{H^{-1-\delta/2}}\right]\\
&\le C_{p,\delta}
\end{aligned}$$

because we use the same bounds above for $\mathbb{E}\left[\prod_{i=1}^{p}\xi_{k_i}\xi_{h_i}\right]$ and $Card\left(\mathcal{P}_p\right)$ and a trivial uniform bound on $\mathbb{E}\left[\prod_{i=1}^{p}\left\langle\delta_{X_0^{k_i}},\delta_{X_0^{h_i}}\right\rangle_{H^{-1-\delta/2}}\right]$ due to the property $\left\|\delta_{X_0^i}\right\|_{H^{-1-\delta/2}}\le C$ showed in the proof of Proposition 21.

Step 3.

$$\begin{aligned}
\mathbb{E}\left[\left\langle\omega_t^N\otimes\omega_t^N,f\right\rangle^2\right] &= \mathbb{E}\left(\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}f(x,y)\,\omega_t^N(dx)\,\omega_t^N(dy)\right)^2\\
&=\mathbb{E}\int_{(\mathbb{T}^2)^4}f(x,y)\,f\left(x',y'\right)\omega_t^N(dx)\,\omega_t^N(dy)\,\omega_t^N\left(dx'\right)\omega_t^N\left(dy'\right)\\
&=\frac{1}{N^2}\sum_{ijkh=1}^{N}\mathbb{E}\left[f\left(X_t^i,X_t^j\right)f\left(X_t^k,X_t^h\right)\right]\mathbb{E}\left[\xi_i\xi_j\xi_k\xi_h\right].
\end{aligned}$$

In this sum there are various terms. The term with $i=j=k=h$ is

$$\frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\left[f\left(X_t^i,X_t^i\right)f\left(X_t^i,X_t^i\right)\right]\mathbb{E}\left[\xi_i^4\right]=\frac{\mathbb{E}\left[\xi^4\right]}{N}\int f^2(x,x)\,dx.$$

Then there are terms with $j = i$, $h = k$:

$$\frac{1}{N^2} \sum_{i \neq k=1}^{N} E\left[\xi_i^2\right] E\left[\xi_k^2\right] \mathbb{E}\left[f\left(X_t^i, X_t^i\right) f\left(X_t^k, X_t^k\right)\right]$$

$$= \frac{E\left[\xi^2\right]^2}{N^2} \sum_{i \neq k=1}^{N} \mathbb{E}\left[f\left(X_t^i, X_t^i\right)\right] \mathbb{E}\left[f\left(X_t^k, X_t^k\right)\right]$$

$$\leq E\left[\xi^2\right]^2 \left(\int f(x,x)\,dx\right)^2.$$

Then there are terms with $k = i$, $h = j$:

$$\frac{E\left[\xi^2\right]^2}{N^2} \sum_{i \neq j=1}^{N} \mathbb{E}\left[f\left(X_t^i, X_t^j\right) f\left(X_t^i, X_t^j\right)\right] \leq E\left[\xi^2\right]^2 \int\int f^2(x,y)\,dxdy.$$

Finally, then there are terms with $k = j$, $h = i$: (here we use symmetry)

$$\frac{E\left[\xi^2\right]^2}{N^2} \sum_{i \neq j=1}^{N} \mathbb{E}\left[f\left(X_t^i, X_t^j\right) f\left(X_t^j, X_t^i\right)\right] \leq E\left[\xi^2\right]^2 \int\int f^2(x,y)\,dxdy.$$

■

4 Main results

Denote by μ the law of White Noise. We first formulate our version of Albeverio-Cruzeiro result [1].

Theorem 24 *There exists a probability space $(\Xi, \mathcal{F}, P)$ with the following properties.*

i) There exists a measurable map $\omega_\cdot : \Xi \times [0,T] \to C^\infty\left(\mathbb{T}^2\right)'$ such that $\omega_\cdot$ is a time-stationary white noise solution of Euler equations, in the sense of Definition 17.

ii) On $(\Xi, \mathcal{F}, P)$ one can define the random point vortex system described in Section 3.2; it has a subsequence which converges P-a.s. to the solution of point (i) in $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$.

We prove also a generalization to $\rho-$white noise solutions; the assumption on ρ_0 is presumably too restrictive but further investigation is needed for more generality.

Theorem 25 *Given $\rho_0 \in C_b\left(H^{-1-}\left(\mathbb{T}^2\right)\right)$ such that $\rho_0 \geq 0$ and $\int \rho_0 d\mu = 1$, there exist a probability space $(\Xi, \mathcal{F}, P)$, a bounded measurable function $\rho : [0,T] \times H^{-1-}\left(\mathbb{T}^2\right) \to [0, \|\rho_0\|_\infty]$ and a measurable map $\omega_\cdot : \Xi \times [0,T] \to C^\infty\left(\mathbb{T}^2\right)'$ such that $\omega_\cdot$ is a $\rho-$white noise solution of Euler equations, in the sense of Definition 18. It is also the limit P-a.s. in $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$ of a suitable sequence of random point vortices.*

4.1 Remarks on disintegration, uniqueness an Gaussianity

In this section we discuss several limits of the previous results and open problems arising from them.

Consider the law Q, on path space $C\left([0,T];H^{-1-}\left(\mathbb{T}^2\right)\right)$, of a solutions provided by Theorem 24 (similarly for Theorem 25). If we disintegrate Q with respect to the marginal law at time $t=0$ (namely the white noise law μ for Theorem 24 or law $\rho_0 d\mu$ for Theorem 25), we find a probability kernel $Q(\cdot,\omega_0)$, indexed by $\omega_0 \in H^{-1-}\left(\mathbb{T}^2\right)$, such that for μ-a.e. $\omega_0 \in H^{-1-}\left(\mathbb{T}^2\right)$ the probability measure $Q(\cdot,\omega_0)$ is concentrated on solutions of Euler equations (in the sense described above). But $Q(\cdot,\omega_0)$ is not of the form $\delta_{\omega_\cdot^{\omega_0}}$, namely it is not concentrated on a single solution $\omega_t^{\omega_0}$ with initial condition ω_0; or at least we do not know this information. In the language of [6], we have a superposition solution that we do not know to be a graph. For μ-a.e. $\omega_0 \in H^{-1-}\left(\mathbb{T}^2\right)$, we have at least one solution ω of Euler equations, but we could have many; also in the sense of the Lagrangian flows described in [6], see below.

In the case of Theorem 25 on $\rho-$white noise solutions, we are certainly far away from any uniqueness claim, even in law. Presumably one should try first to investigate uniqueness of ρ_t, maybe with tools related to those of [7], [8], [18], [20], which already looks a formidable task.

In the case however of Theorem 24, due to fact that the law at any time t is uniquely determined, it could seem that a statement of uniqueness in law is not far (notice that uniqueness in law would also imply that the full sequence of point vortices converges to it, in law). And perhaps a statement of uniqueness of Lagrangian flows. These are however open problems, potentially of very difficult solution. Let us mention where two approaches, both based on uniqueness of the 1-dimensional marginals, meet essential difficulties.

One approach is by the criteria of uniqueness for martingale solutions of stochastic equations (applicable in principle to deterministic equations with random solutions). Take as an example Theorem 6.2.3 of [40]. It does not apply here, at the present stage of our understanding, since we do not have any information of uniqueness of 1-dimensional marginals starting from generic deterministic initial conditions. As remarked above, by disintegration we may construct solutions $Q(\cdot,\omega_0)$ (in the sense of the martingale problem; we do not develop the details) for μ-a.e. $\omega_0 \in H^{-1-}\left(\mathbb{T}^2\right)$, but we do not know the uniqueness of their 1-point marginals.

A second approach is described in [6], see Theorem 16. It requires the validity of comparison principle, a variant of 1-point marginal uniqueness, for the associated continuity equation. The comparison principle should hold in a convex class of solutions (denoted by $\mathcal{L}_b$ in [6]); if only this, one could take the class defined by the rule that it is white noise at every time. However, the class $\mathcal{L}_b$ in [6] has to satisfy also a monotonicity property (see (14) in [6], used in essential way in Theorem 18), which is not satisfied by the trivial class defined by being white noise at every time. If we enlarge the class to have the monotonicity property, we are faced with a very difficult question of uniqueness - or comparison principle

- for weak solutions of the continuity equation associated to Euler equations, which is an open problem.

The k-dimensional time marginals are not easily identified by the Euler equations or by the random point vortex dynamics. The question is, given $0 \leq t_1 < \cdots < t_k \leq T$, to understand the limit as $N \to \infty$ of the marginal $\left(\omega_{t_1}^N, ..., \omega_{t_k}^N\right)$, given by

$$\left(\omega_{t_1}^N, ..., \omega_{t_k}^N\right) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \xi_n \left(\delta_{X_{t_1}^n}, ..., \delta_{X_{t_k}^n}\right).$$

This is an open problem.

For Burgers equations with white noise initial conditions, thanks to special representation formulae, it was possible to compute the two-point distribution, see [21]. Here we do not see yet a method. But, also due to the comparison with [21], one should be aware that there is no reason why k-dimensional time marginals are Gaussian! Nonlinearity, still preserving a Gaussian initial condition, should distroy Gaussianity at the level of the process.

Another example of nonlinear equation with stationary solutions having Gaussian 1-dimensional marginals is KPZ equation or the stochastic Burgers equations, see [25], [23], [24].

4.2 Proof of Theorem 24

Consider the Polish space $\mathcal{X} = C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$ with the metric $d_{\mathcal{X}}\left(\omega_\cdot, \omega_\cdot'\right)$ defined in Section 1.1. J. Simon [38], in Corollary 8, gives a useful class of compact sets in this space, generalizing the more classical Aubin-Lions compactness lemma (and Ascoli-Arzelà criterion). Let us explain the result of Simon in our context. Take $\delta \in (0,1)$, $\gamma > 3$ (this special choice of γ is due to the estimates below) and consider the spaces

$$X = H^{-1-\delta/2}\left(\mathbb{T}^2\right), \qquad B = H^{-1-\delta}\left(\mathbb{T}^2\right), \qquad Y = H^{-\gamma}\left(\mathbb{T}^2\right).$$

We have

$$X \subset B \subset Y$$

with compact dense embeddings and we also have, for a suitable constant $C > 0$ and for

$$\theta = \frac{\delta/2}{\gamma - 1 - \delta/2}$$

the interpolation inequality

$$\|\omega\|_B \leq C \|\omega\|_X^{1-\theta} \|\omega\|_Y^{\theta}$$

for all $\omega \in X$. These are preliminary assumptions of Corollary 8 of [38]. Then such Corollary, in the second part, in the particular case $r_1 = 2$, states that a bounded family F in

$$L^{p_0}(0,T;X) \cap W^{1,2}(0,T;Y)$$

is relatively compact in

$$C([0,T];B)$$

if

$$\frac{\theta}{2} > \frac{1-\theta}{p_0}.$$

Here p_0 is any number in $[1,\infty]$. We apply this result to our spaces X, B, Y, taking p_0 large enough to have the previous inequality. More precisely, we use the following statement (notice that $\frac{1-\theta}{\theta} = \frac{\gamma-1-\delta}{\delta/2}$):

Lemma 26 *Let $\delta > 0$, $\gamma > 3$ be given. If*

$$p_0 > \frac{\gamma - 1 - \delta}{\delta/2}$$

then

$$L^{p_0}\left(0,T;H^{-1-\delta/2}\left(\mathbb{T}^2\right)\right) \cap W^{1,2}\left(0,T;H^{-\gamma}\left(\mathbb{T}^2\right)\right)$$

is compactly embedded into

$$C\left([0,T];H^{-1-\delta}\left(\mathbb{T}^2\right)\right).$$

In fact we need compactness in $\mathcal{X}$. Denote by $L^{\infty-}\left(0,T;H^{-1-}\left(\mathbb{T}^2\right)\right)$ the space of all functions of class $L^{p_0}\left(0,T;H^{-1-\delta}\left(\mathbb{T}^2\right)\right)$ for any $p_0 > 0$ and $\delta > 0$, endowed with the metric

$$d_{L_t^{\infty-}(H^{-1-})}\left(\omega_\cdot,\omega'_\cdot\right) = \sum_{n=1}^{\infty} 2^{-n}\left(\left(\int_0^T \left\|\omega_t - \omega'_t\right\|^n_{H^{-1-\frac{1}{n}}}\right)^{1/n} \wedge 1\right).$$

It is a simple exercise to check that:

Corollary 27 *Let $\gamma > 3$ be given. Then*

$$\mathcal{Y} := L^{\infty-}\left(0,T;H^{-1-}\left(\mathbb{T}^2\right)\right) \cap W^{1,2}\left(0,T;H^{-\gamma}\left(\mathbb{T}^2\right)\right)$$

is compactly embedded into $\mathcal{X}$.

Let Q^N be the law of ω^N on Borel subsets of $\mathcal{X}$. We want to prove that the family $\left\{Q^N\right\}_{N\in\mathbb{N}}$ is tight in this space. In order to prove this, it is sufficient to prove that the family $\left\{Q^N\right\}_{N\in\mathbb{N}}$ is bounded in probability in the space $\mathcal{Y}$ given by the previous corollary. For this purpose, it is sufficient to prove that $\left\{Q^N\right\}_{N\in\mathbb{N}}$ is bounded in probability in $W^{1,2}\left(0,T;H^{-\gamma}\left(\mathbb{T}^2\right)\right)$ and in each $L^{p_0}\left(0,T;H^{-1-\delta}\left(\mathbb{T}^2\right)\right)$, for any $p_0 > 0$ and $\delta > 0$. Let us prove these conditions.

The family $\{Q^N\}_{N\in\mathbb{N}}$ is bounded in probability in $L^{p_0}\left(0,T;H^{-1-\delta}\left(\mathbb{T}^2\right)\right)$ (by Chebyshev inequality) because

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\int_0^T \left\|\omega_t^N\right\|_{H^{-1-\delta}}^{p_0} dt\right] < \infty.$$

This inequality (that we could conceptually summarize as the "compactness in space") comes from stationarity of ω_t^N:

$$\mathbb{E}\left[\int_0^T \left\|\omega_t^N\right\|_{H^{-1-\delta}}^{p_0} dt\right] = \int_0^T \mathbb{E}\left[\left\|\omega_t^N\right\|_{H^{-1-\delta}}^{p_0}\right] dt \le C_{p_0,\delta} T$$

by Lemma 23.

To prove "compactness in time", namely the property that the family $\{Q^N\}_{N\in\mathbb{N}}$ is bounded in probability in $W^{1,2}\left(0,T;H^{-\gamma}\left(\mathbb{T}^2\right)\right)$, we use the equation, in its weak vorticity formulation. We have, for all $\phi \in C^\infty\left(\mathbb{T}^2\right)$,

$$\left\langle \omega_t^N, \phi\right\rangle = \left\langle \omega_0^N, \phi\right\rangle + \int_0^t \left\langle \omega_s^N \otimes \omega_s^N, H_\phi\right\rangle ds$$

where P-a.s. the function $s \mapsto \left\langle \omega_s^N \otimes \omega_s^N, H_\phi\right\rangle$ is continuous (the trajectories of point vortices are continuous and never touch the diagonal), hence, P-a.s., the function $t \mapsto \left\langle \omega_t^N, \phi\right\rangle$ is continuously differentiable and $\partial_t \left\langle \omega_t^N, \phi\right\rangle = \left\langle \omega_t^N \otimes \omega_t^N, H_\phi\right\rangle$. Thus

$$\begin{aligned}\mathbb{E}\left[\left|\partial_t \left\langle \omega_t^N, \phi\right\rangle\right|^2\right] &= \mathbb{E}\left[\left|\left\langle \omega_t^N \otimes \omega_t^N, H_\phi\right\rangle\right|^2\right] \\ &\le C \left\|H_\phi\right\|_\infty^2 \le C \left\|D^2\phi\right\|_\infty^2\end{aligned}$$

by Lemma 23. Then we apply this inequality to $\phi = e_k$ and get

$$\mathbb{E}\left[\left|\partial_t \left\langle \omega_t^N, e_k\right\rangle\right|^2\right] \le C \left|k\right|^4.$$

Therefore

$$\begin{aligned}\mathbb{E}\left[\int_0^T \left\|\partial_t \omega_t^N\right\|_{H^{-\gamma}}^2 dt\right] &= \mathbb{E}\left[\int_0^T \sum_k \left(1+|k|^2\right)^{-\gamma} \left|\left\langle \partial_t \omega_t^N, e_k\right\rangle\right|^2 dt\right] \\ &\le C\mathbb{E}\left[\int_0^T \sum_k \left(1+|k|^2\right)^{-\gamma} |k|^4 \, dt\right] < \infty\end{aligned}$$

for $2\gamma - 4 > 2$, hence $\gamma > 3$. The estimate for $\mathbb{E}\left[\int_0^T \left\|\omega_t^N\right\|_{H^{-\gamma}}^2 dt\right]$ is similar to the one for "compactness in space" above. By Chebyshev inequality, $\{Q^N\}_{N\in\mathbb{N}}$ is bounded in probability in $W^{1,2}\left(0,T;H^{-\gamma}\left(\mathbb{T}^2\right)\right)$.

We have proved that the family $\{Q^N\}_{N\in\mathbb{N}}$ is bounded in probability in $\mathcal{Y}$ and thus it is tight in $\mathcal{X}$. From Prohorov theorem, it is relatively compact in $\mathcal{X}$. Let $\{Q^{N_k}\}_{k\in\mathbb{N}}$ be a subsequence which converges weakly, in $\mathcal{X}$, to a Borel probability measure Q. First, convergence in $\mathcal{X}$ implies that Q is invariant by time-shift (because Q^N is; by shift we mean shift of finite dimensional distributions such that all involved time points are in $[0,T]$) and the marginal at any time is the law of white noise, by Proposition 21 (recall that ω_t^N is stationary, hence this proposition applies at every time).

By Skorokhod representation theorem, there exist a new probability space $\left(\widehat{\Xi},\widehat{\mathcal{F}},\widehat{P}\right)$ and r.v.'s $\widehat{\omega}^{N_k}$, $\widehat{\omega}$ with values in $\mathcal{X}$, such that the laws of $\widehat{\omega}^{N_k}$ and $\widehat{\omega}$ are Q^{N_k} and Q respectively, and $\widehat{\omega}^{N_k}$ converges P-a.s. to $\widehat{\omega}$ in the topology of $\mathcal{X}$; since $\mathcal{X}$ is made of functions of time, we may see $\widehat{\omega}^{N_k}$ and $\widehat{\omega}$ as stochastic processes, $\widehat{\omega}_t^{N_k}$ and $\widehat{\omega}_t$ being the result of application of the projection at time t. We are going to check that $\widehat{\omega}$, or more precisely another process closely defined, is the solution claimed by the theorem. We already know it has trajectories of class $C\left([0,T];H^{-1-}\left(\mathbb{T}^2\right)\right)$, it is time stationary and with marginal being a white noise. We have to show that it satisfies the equation, in the sense specified by the definitions.

We have to enlarge the probability space $\left(\widehat{\Xi},\widehat{\mathcal{F}},\widehat{P}\right)$ to be sure it contains certain independent r.v.'s we need in the construction. Denote by $\left(\widetilde{\Xi},\widetilde{\mathcal{F}},\widetilde{P}\right)$ a probability space where, for every N, it is defined a random permutation $\widetilde{s}_N:\widetilde{\Xi}\to\Sigma_N$, uniformly distributed. Define the new probability space

$$(\Xi,\mathcal{F},P):=\left(\widehat{\Xi}\times\widetilde{\Xi},\widehat{\mathcal{F}}\otimes\widetilde{\mathcal{F}},\widehat{P}\otimes\widetilde{P}\right)$$

and the new processes

$$\omega^{N_k}=\widehat{\omega}^{N_k}\circ\pi_1,\qquad \omega=\widehat{\omega}\circ\pi_1,\qquad s_N=\widetilde{s}_N\circ\pi_2$$

where π_1 and π_2 are the projections on $\widehat{\Xi}\times\widetilde{\Xi}$. We adopt a little abuse of notation here, because we indicate the final spaces and processes like the original ones, but we shall try to clarify everywhere which ones we are investigating. Notice that the properties of convergence and of the laws of the processes ω^{N_k} and ω are the same as those of $\widehat{\omega}^{N_k}$ and $\widehat{\omega}$.

Lemma 28 *The process $\omega_t^{N_k}$ (the one on the new probability space) can be represented in the form $\frac{1}{\sqrt{N_k}}\sum_{i=1}^{N_k}\xi_i\delta_{X_t^{i,N_k}}$, where*

$$\left(\left(\xi_1,X_0^{1,N_k}\right),...,\left(\xi_{N_k},X_0^{N_k,N_k}\right)\right)\tag{10}$$

is a random vector with law λ_N^0 and $\left(X_t^{1,N_k},...,X_t^{N_k,N_k}\right)$ solves system (8) with initial condition $\left(X_0^{1,N_k},...,X_0^{N_k,N_k}\right)$.

Proof. Step 1. Let us list a few preliminary facts; we omit some detail in the proofs; we extensively use the notations at the beginning of Section 3.

Identify for a second $\mathbb{T}^2$ with $[0,1)^2$. On $[0,1)^2$, consider the lexicographic order: $x = (a,b)$ is smaller than $y=(c,d)$ either if $a<c$ or if $a=c$ but $b<d$. It is a total order. We write $<_L$ for the strict lexicographic order just defined. Let us denote by $\mathcal{L}\Lambda_N^1 \subset \mathcal{L}F_N\mathbb{T}^2$ the set of strings $((\xi_1,x_1),...,(\xi_N,x_N))$ such that $x_1 <_L ... <_L x_N$, with x_i seen as elements of $[0,1)^2$. The set $\mathcal{L}F_N\mathbb{T}^2$ is partitioned in $N!$ subsets $\mathcal{L}\Lambda_N^1,...,\mathcal{L}\Lambda_N^{N!}$ obtained applying to $\mathcal{L}\Lambda_N^1$ each one of the $N!$ permutations of indexes.

Given $\omega \in \mathcal{M}_N(\mathbb{T}^2)$, there is a unique element $\{(\xi_i,x_i), i=1,..,N\} \in \mathcal{L}C_N\mathbb{T}^2 = \mathcal{L}F_N\mathbb{T}^2/\Sigma_N$ such that $\omega = \frac{1}{\sqrt{N}}\sum_{i=1}^N \xi_i\delta_{x_i}$. Notice that the indexing $i=1,..,N$ here, a priori, is not canonical. However, we may use the lexicographic order, and the fact that point are disjoint, to attribute the indexes $i=1,..,N$ to the elements of the set $\{(\xi_i,x_i), i=1,..,N\}$, in such a way that $((\xi_1,x_1),...,(\xi_N,x_N)) \in \mathcal{L}\Lambda_N^1$. This way, we have uniquely defined maps $\omega \overset{h_1}{\mapsto} (\xi_1,x_1)$, ..., $\omega \overset{h_N}{\mapsto} (\xi_N,x_N)$, from $\mathcal{M}_N(\mathbb{T}^2)$ to $\mathbb{R}\times\mathbb{T}^2$.

On $\mathcal{M}_N(\mathbb{T}^2) \subset H^{-1-}(\mathbb{T}^2)$ let us put the topology induced by $d_{H^{-1-}}$ and consider the functions of class $C([0,T];\mathcal{M}_N(\mathbb{T}^2))$. The set $\mathcal{M}_N(\mathbb{T}^2)$ is measurable in $H^{-1-}(\mathbb{T}^2)$, and the set $C([0,T];\mathcal{M}_N(\mathbb{T}^2))$ is measurable in $C([0,T];H^{-1-}(\mathbb{T}^2))$ (the proof is not difficult arguing on suitable close subfamilies of $\mathcal{M}_N(\mathbb{T}^2)$, constrained by the minimal distance between elements in the support).

If $\omega_t^N = \frac{1}{\sqrt{N}}\sum_{i=1}^N \xi_i\delta_{X_t^{i,N}}$ comes from the vortex point dynamics with an initial condition such that coalescence does not occur, then $\omega_\cdot^N \in C([0,T];\mathcal{M}_N(\mathbb{T}^2))$: to prove this, one has to use the embedding of $H^{-1-}(\mathbb{T}^2)$ into Hölder continuous functions, in evaluating

$$\sup_{\|\phi\|_{H^{-1-\delta}}\leq 1}\left|\sum_{i=1}^N \xi_i\left(\phi\left(X_t^{i,N}\right)-\phi\left(X_s^{i,N}\right)\right)\right|.$$

Conversely, if $\omega_\cdot^N \in C([0,T];\mathcal{M}_N(\mathbb{T}^2))$, then there exist functions $x_\cdot^{i,N} \in C([0,T];\mathbb{T}^2)$ and numbers ξ_i, $i=1,...,N$, such that $\omega_t^N = \frac{1}{\sqrt{N}}\sum_{i=1}^N \xi_i\delta_{x_t^{i,N}}$; the lengthy proof requires identification of these functions locally in time by means of very concentrated test functions. The indexing $i=1,...,N$ of this functions however cannot correspond to lexicographic order: to have lexicographic order at every time we should accept jumps in time (these jumps occur every time the first coordinates of two points exchange their order, also due to the difference between $\mathbb{T}^2$ and $[0,1)^2$). Let us impose lexicographic order only at time $t=0$ (in doing so there is no problem to identify $\mathbb{T}^2$ with $[0,1)^2$) and then accept that particles exchange lexicographic order later in time, with the advantage that $x_\cdot^{i,N} \in C([0,T];\mathbb{T}^2)$. Thus we have uniquely defined the maps $\omega_\cdot^N \overset{\widetilde{h}_1}{\mapsto} \left(\xi_1, x_\cdot^{1,N}\right)$, ..., $\omega_\cdot^N \overset{\widetilde{h}_N}{\mapsto} \left(\xi_N, x_\cdot^{N,N}\right)$ from $C([0,T];\mathcal{M}_N(\mathbb{T}^2))$ to $\mathbb{R}\times C([0,T];\mathbb{T}^2)$: at time zero we impose $x_0^{1,N} <_L ... <_L x_0^{N,N}$ (at later times this may be not true anymore). These maps are measurable.

Finally let us discuss the last preliminary fact we need below. Given a probability measure ρ on $\mathcal{L}F_N\mathbb{T}^2$, assume it is exchangeable, namely its law is invariant by permutation of the indexes; it is thus uniquely determined by its restriction to $\mathcal{L}\Lambda_N^1$. Consider ρ restricted to $\mathcal{L}\Lambda_N^1$, remormalized by $N!$ so to be a probability measure; call $\widehat{\rho}$ such measure. We have a one-to-one correspondence between ρ and $\widehat{\rho}$, measures on $\mathcal{L}F_N\mathbb{T}^2$ and $\mathcal{L}\Lambda_N^1$ respectively.

In particular, given a measure $\widehat{\rho}$ on $\mathcal{L}\Lambda_N^1$, we may reconstruct an exchangeable measure on $\mathcal{L}F_N\mathbb{T}^2$, the unique one that restricted to $\mathcal{L}\Lambda_N^1$ gives values proportional to $\widehat{\rho}$ up to $N!$. Assume more, namely that $\widehat{\rho}$ on $\mathcal{L}\Lambda_N^1$ is the law of a vector $\left(\left(\widehat{\xi}_1, \widehat{X}_1\right), ..., \left(\widehat{\xi}_N, \widehat{X}_N\right)\right)$, defined on a probability space $\left(\widehat{\Xi}, \widehat{\mathcal{F}}, \widehat{P}\right)$. Enlarge the probability space as described before the lemma, incorporating independent permutations $\widetilde{s}_N : \widetilde{\Xi} \to \Sigma_N$. On the product space $(\Xi, \mathcal{F}, P)$, with the notations above plus $(\xi_i, X_i) = \left(\widehat{\xi}_i, \widehat{X}_i\right) \circ \pi_1$, consider the new vector

$$\left(\left(\xi_1^*, X_1^*\right), ..., \left(\xi_N^*, X_N^*\right)\right) := \left(\left(\xi_{\widetilde{s}_N(1)}, X_{\widetilde{s}_N(1)}\right), ..., \left(\xi_{\widetilde{s}_N(N)}, X_{\widetilde{s}_N(N)}\right)\right).$$

This vector takes values in $\mathcal{L}F_N\mathbb{T}^2$, not in $\mathcal{L}\Lambda_N^1$ as the previous one $\left(\left(\widehat{\xi}_1, \widehat{X}_1\right), ..., \left(\widehat{\xi}_N, \widehat{X}_N\right)\right)$. We claim its law is ρ, in the correspondence $\rho \leftrightarrow \widehat{\rho}$ described above. Indeed, $\left(\left(\xi_1^*, X_1^*\right), ..., \left(\xi_N^*, X_N^*\right)\right)$ is exchangeable, because given a single deterministic permutation s, $\widetilde{s}_N \circ s$ is uniformly distributed. And conditioning to have $X_{\widetilde{s}_N(1)} <_L ... <_L X_{\widetilde{s}_N(N)}$ is like conditioning to have $\widetilde{s}_N = id$, which gives $\widehat{\rho}$. Let us call *shuffling* the procedure illustrated here of composition with independent permutations, to get the exchangeable distribution from a distribution on $\mathcal{L}\Lambda_N^1$.

Step 2. Now let us prove the lemma. The law of $\widehat{\omega}_\cdot^{N_k}$, being the same as the law of the original process, is concentrated on $C\left([0,T]; \mathcal{M}_{N_k}\left(\mathbb{T}^2\right)\right)$. Hence, by the measurable maps $\widetilde{h}_i$ described above, it defines random elements $\left(\widehat{\xi}_1, \widehat{X}_\cdot^{1,N_k}\right)$, ..., $\left(\widehat{\xi}_{N_k}, \widehat{X}_\cdot^{N_k,N_k}\right)$ in $\mathbb{R} \times C\left([0,T]; \mathbb{T}^2\right)$. One has $\widehat{\omega}_t^{N_k} = \frac{1}{\sqrt{N_k}} \sum_{i=1}^{N_k} \widehat{\xi}_i \delta_{\widehat{X}_t^{i,N_k}}$; therefore we have proved a first claim of the lemma (in fact we shall redefine the random vector but the redefinition will not change this statement). We still have to prove that λ_N^0 is the law of (10) (in fact we still have to define properly (10)) and $\left(\widehat{X}_t^{1,N_k}, ..., \widehat{X}_t^{1,N_k}\right)$ solves system (8).

Since the original process $\omega_\cdot^{N_k}$ had the property that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left| \left\langle \omega_t^{N_k}, \phi \right\rangle - \left\langle \omega_0^{N_k}, \phi \right\rangle - \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \nabla\phi(x) \cdot K(x-y)\, \omega_s^{N_k}(dx)\, \omega_s^{N_k}(dy)\, ds \right| \wedge 1 \right] = 0$$

for every $\phi \in C^\infty\left(\mathbb{T}^2\right)$, the same property holds for the new process $\widehat{\omega}_t^{N_k}$ (because they have the same law), hence $\widehat{P}$-a.s. it holds

$$\sup_{t\in[0,T]} \left| \left\langle \widehat{\omega}_t^{N_k}, \phi \right\rangle - \left\langle \widehat{\omega}_0^{N_k}, \phi \right\rangle - \int_0^t \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \nabla\phi(x) \cdot K(x-y)\, \widehat{\omega}_s^{N_k}(dx)\, \widehat{\omega}_s^{N_k}(dy)\, ds \right| = 0$$

on a dense countable set of $\phi \in C^{\infty}\left(\mathbb{T}^2\right)$, which implies (using the structure $\widehat{\omega}_t^{N_k} = \frac{1}{\sqrt{N_k}}\sum_{i=1}^{N_k}\widehat{\xi}_i\delta_{\widehat{X}_t^{i,N_k}}$) that $\left(\widehat{X}_t^{1,N_k},...,\widehat{X}_t^{N_k,N_k}\right)$ satisfies (8). Below we shall redefine this process but the redefinition will not change this property.

It remains to understand the law of (10). We have constructed the random vector $\left(\left(\widehat{\xi}_1,\widehat{X}_0^{1,N_k}\right),...,\left(\widehat{\xi}_{N_k},\widehat{X}_0^{N_k,N_k}\right)\right)$, with $\widehat{X}_0^{1,N} <_L ... <_L \widehat{X}_0^{N,N}$. We apply the shuffling procedure described at the end of Step 1, hence redefining all r.v.'s and processes by composition with random permutations. The result is an initial random vector of the form (10) and the associated process $\left(X_t^{1,N_k},...,X_t^{N_k,N_k}\right)$. The modifications introduced by shuffling do not change the representation $\omega_t^{N_k} = \frac{1}{\sqrt{N_k}}\sum_{i=1}^{N_k}\xi_i\delta_{X_t^{i,N_k}}$ (now $\omega_t^{N_k}$ is the process defined before the lemma) and the fact that $\left(X_t^{1,N_k},...,X_t^{N_k,N_k}\right)$ solves system (8). We claim that the new initial random vector (10) has law λ_N^0. By construction the vector (10) is exchangeable and its law is the unique exchangeable law on $\mathcal{L}F_N\mathbb{T}^2$ corresponding to a certain probability measure $\widehat{\rho}$ on $\mathcal{L}\Lambda_N^1$ that we now describe. Since λ_N^0 has this property, we deduce that λ_N^0 is the law of (10). Let us describe $\widehat{\rho}$. It is the law of $\left(\left(\widehat{\xi}_1,\widehat{X}_0^{1,N_k}\right),...,\left(\widehat{\xi}_{N_k},\widehat{X}_0^{N_k,N_k}\right)\right)$, random vector constructed through the unique maps h_i, hence $\widehat{\rho}$ is the push forward under $(h_1,...,h_N)$ of the law of $\omega_0^{N_k}$; call it $\pi_{t=0}Q^{N_k}$. These correspondences are bijections and, as already said, if we start by λ_N^0 and push it forward (in opposite direction) to a law on $\omega_0^{N_k}$ we find $\pi_{t=0}Q^{N_k}$. Thus we have the identification. ■

Given $\phi \in C^{\infty}\left(\mathbb{T}^2\right)$ and $t \in [0,T]$, we are going to prove that

$$E\left[\left|\langle\omega_t,\phi\rangle - \langle\omega_0,\phi\rangle - \int_0^t \langle H_\phi,\omega_s\otimes\omega_s\rangle\, ds\right| \wedge 1\right] = 0.$$

This implies that $\langle\omega_t,\phi\rangle = \langle\omega_0,\phi\rangle - \int_0^t \langle H_\phi,\omega_s\otimes\omega_s\rangle\, ds$ with P-probability one, at time t. Since the processes involved are continuous, this implies that the identity holds uniformly in time, with P-probability one.

Based on the identity

$$\left\langle\omega_t^{N_k},\phi\right\rangle - \left\langle\omega_0^{N_k},\phi\right\rangle - \int_0^t \left\langle H_\phi,\omega_s^{N_k}\otimes\omega_s^{N_k}\right\rangle ds = 0$$

and the general fact that $|x+y|\wedge 1 \le (|x|\wedge 1) + (|y|\wedge 1)$, one has the inequality

$$\begin{aligned}
&E\left[\left|\langle\omega_t,\phi\rangle - \langle\omega_0,\phi\rangle - \int_0^t \langle H_\phi,\omega_s\otimes\omega_s\rangle\, ds\right| \wedge 1\right] \\
&\le E\left[\left(\left|\langle\omega_t,\phi\rangle - \left\langle\omega_t^{N_k},\phi\right\rangle\right|\right)\wedge 1\right] + E\left[\left(\left|\langle\omega_0,\phi\rangle - \left\langle\omega_0^{N_k},\phi\right\rangle\right|\right)\wedge 1\right] \\
&+ E\left[\left(\left|\int_0^t \left\langle H_\phi,\omega_s^{N_k}\otimes\omega_s^{N_k}\right\rangle ds - \int_0^t \langle H_\phi,\omega_s\otimes\omega_s\rangle\, ds\right|\right)\wedge 1\right].
\end{aligned}$$

We have, for $\phi \in C^{\infty}\left(\mathbb{T}^2\right)$ and $t \in [0,T]$,

$$\lim_{k\to\infty} E\left[\left(\left|\langle \omega_t, \phi\rangle - \left\langle \omega_t^{N_k}, \phi\right\rangle\right|\right) \wedge 1\right] = 0$$

simply because we have a.s. convergence in $C\left([0,T]\,; H^{-1-\delta}\left(\mathbb{T}^2\right)\right)$. Hence it remains to prove

$$\lim_{k\to\infty} E\left[\left(\left|\int_0^t \left\langle H_\phi, \omega_s^{N_k} \otimes \omega_s^{N_k}\right\rangle ds - \int_0^t \langle H_\phi, \omega_s \otimes \omega_s\rangle\, ds\right|\right) \wedge 1\right] = 0$$

which is the most demanding part of the passage to the limit. Let us consider a smooth (of class H^{2+} is sufficient) approximation H_ϕ^δ of H_ϕ, $\delta > 0$, with the property $H_\phi^\delta(x,x) = 0$ (see Remark 9). We have

$$\lim_{n\to\infty} E\left[\left(\left|\int_0^t \left\langle H_\phi^\delta, \omega_s^{N_k} \otimes \omega_s^{N_k}\right\rangle ds - \int_0^t \left\langle H_\phi^\delta, \omega_s \otimes \omega_s\right\rangle ds\right|\right) \wedge 1\right] = 0$$

again because of a.s. convergence of ω^{N_k} to ω in $C\left([0,T]\,; H^{-1-}\left(\mathbb{T}^2\right)\right)$ and thus of $\omega^{N_k} \otimes \omega^{N_k}$ to $\omega \otimes \omega$ in $C\left([0,T]\,; H^{-2-}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)\right)$. Therefore

$$\begin{aligned}
&\limsup_{k\to\infty} E\left[\left(\left|\int_0^t \left\langle H_\phi, \omega_s^{N_k} \otimes \omega_s^{N_k}\right\rangle ds - \int_0^t \langle H_\phi, \omega_s \otimes \omega_s\rangle\, ds\right|\right) \wedge 1\right] \\
&\leq E\left[\left(\left|\int_0^t \left\langle H_\phi - H_\phi^\delta, \omega_s \otimes \omega_s\right\rangle ds\right|\right) \wedge 1\right] + \sup_{k\in\mathbb{N}} E\left[\left(\left|\int_0^t \left\langle H_\phi - H_\phi^\delta, \omega_s^{N_k} \otimes \omega_s^{N_k}\right\rangle ds\right|\right) \wedge 1\right].
\end{aligned}$$

We know that

$$\begin{aligned}
&E\left[\left(\left|\int_0^t \left\langle H_\phi - H_\phi^\delta, \omega_s \otimes \omega_s\right\rangle ds\right|\right) \wedge 1\right] \leq \int_0^t E\left[\left|\left\langle H_\phi - H_\phi^\delta, \omega_s \otimes \omega_s\right\rangle\right|\right] ds \\
&\leq C \int_0^t E\left[\left|\left\langle H_\phi - H_\phi^\delta, \omega_s \otimes \omega_s\right\rangle\right|^2\right]^{1/2} ds
\end{aligned}$$

and the last term is arbitrarily small with δ, due to Corollary 6 (a little argument is needed because $H_\phi - H_\phi^\delta$ is not smooth but the computation is similar to the Cauchy property of Theorem 8). It remain to show that

$$E\left[\left(\left|\int_0^t \left\langle H_\phi - H_\phi^\delta, \omega_s^{N_k} \otimes \omega_s^{N_k}\right\rangle ds\right|\right) \wedge 1\right]$$

is small for small δ, uniformly in k. But this case is similar to the previous one, using now Lemma 23. The proof is complete.

5 Proof of Theorem 25

Recall the definitions of $\lambda_N^0(d\theta)$, $\mathcal{T}_N$, $\mu_N^0(d\omega)$ from Remark 20.

Lemma 29 *Given a measurable function $\rho : H^{-1-\delta}\left(\mathbb{T}^2\right) \to [0,\infty)$ such that $\int_{H^{-1-\delta}(\mathbb{T}^2)} \rho(\omega)\,\mu_N^0(d\omega) < \infty$, the measure $\lambda_N^\rho(d\theta) := \rho(\mathcal{T}_N(\theta))\,\lambda_N^0(d\theta)$ on $\left(\mathbb{R}\times\mathbb{T}^2\right)^N$ has the property that its image measure $\mu_N^\rho(d\omega)$ on $H^{-1-\delta}\left(\mathbb{T}^2\right)$ under the map $\mathcal{T}_N$ is $\rho(\omega)\,\mu_N^0(d\omega)$.*

Proof. By definition of $\mu_N^\rho(d\omega)$ and $\lambda_N^\rho(d\theta)$, for every non-negative measurable function F we have

$$\begin{aligned}
\int_{H^{-1-\delta}(\mathbb{T}^2)} F(\omega)\,\mu_N^\rho(d\omega) &= \int_{\mathbb{R}^N\times\mathbb{R}^{2N}} F(\mathcal{T}_N(\theta))\,\lambda_N^\rho(d\theta) \\
&= \int_{\mathbb{R}^N\times\mathbb{R}^{2N}} F(\mathcal{T}_N(\theta))\,\rho(\mathcal{T}_N(\theta))\,\lambda_N^0(d\theta) \\
&= \int_{H^{-1-\delta}(\mathbb{T}^2)} F(\omega)\,\rho(\omega)\,\mu_N^0(d\omega).
\end{aligned}$$

■

We may now prove Theorem 25. Given $\rho_0 \in C_b\left(H^{-1-}\left(\mathbb{T}^2\right)\right)$, $\rho_0 \geq 0$, $\int \rho_0 d\mu = 1$ (μ here is the white noise Gaussian law on $H^{-1-}\left(\mathbb{T}^2\right)$), there is a constant $C_N > 0$ such that $C_N \int_{H^{-1-\delta}(\mathbb{T}^2)} \rho_0(\omega)\,\mu_N^0(d\omega) = 1$, for any $\delta > 0$. Since μ_N^0 converges weakly to μ on $H^{-1-\delta}\left(\mathbb{T}^2\right)$ and ρ_0 is continuous and bounded on $H^{-1-\delta}\left(\mathbb{T}^2\right)$, we deduce $\lim_{N\to\infty} C_N = 1$. Let us consider, on Borel sets of $\left(\mathbb{R}\times\mathbb{T}^2\right)^N$, the finite positive measure $C_N\rho_0(\mathcal{T}_N(\theta))\,\lambda_N^0(d\theta)$. By the lemma, its image measure on $H^{-1-\delta}\left(\mathbb{T}^2\right)$ under the map $\mathcal{T}_N$ is $C_N\rho_0(\omega)\,\mu_N^0(d\omega)$ (we apply the lemma to $\rho(\omega) := C_N\rho_0(\omega)$). The point vortex dynamics is well defined for a.e. $\left(\left(\xi_1, X_0^1\right), ..., \left(\xi_N, X_0^N\right)\right) \in \left(\mathbb{R}\times\mathbb{T}^2\right)^N$ with respect to $C_N\rho_0(\mathcal{T}_N(\theta))\,\lambda_N^0(d\theta)$, because this fact holds for $\lambda_N^0(d\theta)$. Denote by ω_t^N the vorticity of this point vortex dynamics; the law of ω_0^N is $C_N\rho_0(\omega)\,\mu_N^0(d\omega)$.

Denote by Φ_t^N the map in $H^{-1-}\left(\mathbb{T}^2\right)$, defined a.s. with respect to μ_N^0, which gives $\omega_t^N = \Phi_t^N\omega_0^N$. The law of ω_t^N has the form

$$C_N\rho_0\left(\left(\Phi_t^N\right)^{-1}(\omega)\right)\mu_N^0(d\omega)$$

where $\left(\Phi_t^N\right)^{-1}$ is the inverse map of Φ_t^N and it is defined for μ_N^0-a.e. $\omega \in H^{-1-}\left(\mathbb{T}^2\right)$. Indeed, for every non-negative measurable function F we have

$$\begin{aligned}
\mathbb{E}\left[F\left(\omega_t^N\right)\right] = \mathbb{E}\left[F\left(\Phi_t^N\omega_0^N\right)\right] &= \int_{H^{-1-\delta}(\mathbb{T}^2)} F\left(\Phi_t^N\omega\right) C_N\rho_0(\omega)\,\mu_N^0(d\omega) \\
&= \int_{H^{-1-\delta}(\mathbb{T}^2)} F(\omega)\,C_N\rho_0\left(\left(\Phi_t^N\right)^{-1}(\omega)\right)\left(\Phi_t^N\right)_*\mu_N^0(d\omega)
\end{aligned}$$

but $\left(\Phi_t^N\right)_* \mu_N^0 = \mu_N^0$, see Proposition 22.

Therefore, for every non-negative measurable function F on $H^{-1-}\left(\mathbb{T}^2\right)$, one has

$$\mathbb{E}\left[F\left(\omega_t^N\right)\right] = \mathbb{E}\left[C_N \rho_0 \left(\left(\Phi_t^N\right)^{-1}\left(\omega_{WN}^N\right)\right) F\left(\omega_{WN}^N\right)\right]$$

where ω_{WN}^N denotes the random point vortices initial condition with law μ_N^0.

Let Q^N be the law of ω^N on Borel subsets of the space $\mathcal{X}$, as in the previous section. We want to prove that the family $\left\{Q^N\right\}_{N\in\mathbb{N}}$ is tight in $\mathcal{X}$, by proving that it is bounded in probability in $\mathcal{Y}$ (see previous section). The family $\left\{Q^N\right\}_{N\in\mathbb{N}}$ is bounded in probability in $L^{p_0}\left(0,T;H^{-1-\delta}\left(\mathbb{T}^2\right)\right)$, because

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \left\|\omega_t^N\right\|_{H^{-1-\delta}}^{p_0} dt\right] &= \int_0^T \mathbb{E}\left[\left\|\omega_t^N\right\|_{H^{-1-\delta}}^{p_0}\right] dt \\
&= \int_0^T \mathbb{E}\left[C_N \rho_0 \left(\left(\Phi_t^N\right)^{-1}\left(\omega_{WN}^N\right)\right) \left\|\omega_{WN}^N\right\|_{H^{-1-\delta}}^{p_0}\right] dt \\
&\leq C_N \left\|\rho_0\right\|_\infty T \mathbb{E}\left[\left\|\omega_{WN}^N\right\|_{H^{-1-\delta}}^{p_0}\right] \leq C_{p_0,\delta} C_N \left\|\rho_0\right\|_\infty T
\end{aligned}$$

(see the estimate of the previous section). It is bounded in probability in $W^{1,2}\left(0,T;H^{-\gamma}\left(\mathbb{T}^2\right)\right)$, by the same arguments given in the previous section, because

$$\begin{aligned}
&\mathbb{E}\left[\left|\left\langle \omega_t^N \otimes \omega_t^N, H_\phi \right\rangle\right|^2\right] \\
&= \mathbb{E}\left[C_N \rho_0 \left(\left(\Phi_t^N\right)^{-1}\left(\omega_{WN}^N\right)\right) \left|\left\langle \omega_{WN}^N \otimes \omega_{WN}^N, H_\phi \right\rangle\right|^2\right] \\
&\leq C_N \left\|\rho_0\right\|_\infty \mathbb{E}\left[\left|\left\langle \omega_{WN}^N \otimes \omega_{WN}^N, H_\phi \right\rangle\right|^2\right] \\
&\leq C_N \left\|\rho_0\right\|_\infty C \left\|H_\phi\right\|_\infty^2 \leq C_N \left\|\rho_0\right\|_\infty C \left\|D^2\phi\right\|_\infty^2
\end{aligned}$$

(all the other steps of the proof are the same). This proves tightness in $\mathcal{X}$.

Repeating the arguments of the previous section (we use Prohorov and Skorokhod theorems) we extract a subsequence N_k, construct a new probability space, denoted by $(\Xi, \mathcal{F}, P)$ and processes $\omega_t^{N_k}$, ω_t with trajectories in $\mathcal{X}$, such that the laws of ω^{N_k} and ω are Q^{N_k} and Q respectively, and ω^{N_k} converges to ω in the topology of $\mathcal{X}$, P-a.s.; and the structure of ω^{N_k} as sum of delta Dirac is identified, namely Lemma 28 is still true in the case treated here (the proof does not require modifications). The only difference is that here the law of (10) is $C_N \rho_0(\omega) \mu_N^0(d\omega)$. Let us first prove that the law of ω_t on $H^{-1-}\left(\mathbb{T}^2\right)$, called herewith μ_t, is absolutely continuous with respect to μ (the law of white noise) with bounded density. For every $F \in C_b\left(H^{-1-}\left(\mathbb{T}^2\right)\right)$, we have

$$\begin{aligned}
\int F(\omega) \mu_t(d\omega) &= \lim_{N\to\infty} \mathbb{E}\left[F\left(\omega_t^N\right)\right] = \lim_{N\to\infty} \mathbb{E}\left[C_N \rho_0 \left(\left(\Phi_t^N\right)^{-1}\left(\omega_{WN}^N\right)\right) F\left(\omega_{WN}^N\right)\right] \\
&\leq \left\|\rho_0\right\|_\infty \lim_{N\to\infty} \mathbb{E}\left[F\left(\omega_{WN}^N\right)\right] = \left\|\rho_0\right\|_\infty \int F(\omega) \mu(d\omega).
\end{aligned}$$

This implies $\mu_t << \mu$ with bounded density, denoted in the sequel by ρ_t.

We can pass to the limit as in the previous section. Inspection in that proof reveals that we have only to explain why $E\left[\left(\left|\int_0^t \left\langle H_\phi - H_\phi^\delta, \omega_s \otimes \omega_s \right\rangle ds\right|\right) \wedge 1\right]$ and

$$E\left[\left(\left|\int_0^t \left\langle H_\phi - H_\phi^\delta, \omega_s^{N_k} \otimes \omega_s^{N_k} \right\rangle ds\right|\right) \wedge 1\right] \tag{11}$$

are small for small δ, uniformly in k for the second term. We have

$$E\left[\left(\left|\int_0^t \left\langle H_\phi - H_\phi^\delta, \omega_s \otimes \omega_s \right\rangle ds\right|\right) \wedge 1\right] \leq C \int_0^t E\left[\left|\left\langle H_\phi - H_\phi^\delta, \omega_s \otimes \omega_s \right\rangle\right|^2\right]^{1/2} ds$$

$$= C \int_0^t E\left[\rho_s\left(\omega_{WN}\right)\left|\left\langle H_\phi - H_\phi^\delta, \omega_{WN} \otimes \omega_{WN} \right\rangle\right|^2\right]^{1/2} ds$$

$$\leq C \int_0^t E\left[\left|\left\langle H_\phi - H_\phi^\delta, \omega_{WN} \otimes \omega_{WN} \right\rangle\right|^2\right]^{1/2} ds$$

that is arbitrarily small with δ, due to Corollary 6. The proof for (11) is similar.

6 Proof of Theorem 1

We have proved, see Theorem 24 part (i), that there exist a probability space $(\Xi, \mathcal{F}, P)$ and a measurable map $\omega_\cdot : \Xi \times [0,T] \to C^\infty\left(\mathbb{T}^2\right)'$ such that $\omega_\cdot$ is a time-stationary white noise solution of Euler equations, in the sense of Definition 17, and the random point vortex system, defined on $(\Xi, \mathcal{F}, P)$, has a subsequence which converges in law to this solution, in $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$.

This means that:

- ω_0 is distributed as a white noise, hence it takes values in $H^{-1-}\left(\mathbb{T}^2\right) \setminus \left(H^{-1}\left(\mathbb{T}^2\right) \cup \mathcal{M}\left(\mathbb{T}^2\right)\right)$ and it is a full μ-measure set, where μ is the enstrophy Gaussian measure;
- there exists a set $\Xi_1 \in \mathcal{F}$ with $P(\Xi_1) = 1$ such that for all $\theta \in \Xi_1$ one has $\omega_\cdot(\theta) \in C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$.

Moreover, for every $\phi \in C^\infty\left(\mathbb{T}^2\right)$, the following two claims hold true:

- for P-a.e. $\theta \in \Xi$, $s \mapsto \langle \omega_s \otimes \omega_s, H_\phi \rangle(\theta)$ is well defined as $L^2(0,T)$-limit of a subsequence of $s \mapsto \left\langle \omega_s(\theta) \otimes \omega_s(\theta), H_\phi^n \right\rangle$ (Definition 11 identifies $s \mapsto \langle \omega_s \otimes \omega_s, H_\phi \rangle$ by an $L^2(\Xi)$-limit, from which we can extract a subsequence which converges P-almost surely)

- for P-a.e. $\theta \in \Xi$, we have the identity uniformly in time:

$$\langle \omega_t (\theta), \phi \rangle = \langle \omega_0 (\theta), \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_\phi \rangle (\theta) \, ds.$$

Therefore, if $\mathcal{D}$ is a countable set in $C^\infty (\mathbb{T}^2)$, applying a diagonal procedure to extract a single subsequence with P-a.s. convergence of $\left\langle \omega_s \otimes \omega_s, H_\phi^n \right\rangle$, we can find a set $\Xi_2 \in \mathcal{F}$ with $P(\Xi_2) = 1$ such that for all $\theta \in \Xi_2$:

- for every $\phi \in \mathcal{D}$, $s \mapsto \langle \omega_s \otimes \omega_s, H_\phi \rangle (\theta)$ is well defined as $L^2(0,T)$-limit of a subsequence of $s \mapsto \left\langle \omega_s (\theta) \otimes \omega_s (\theta), H_\phi^n \right\rangle$
- for every $\phi \in \mathcal{D}$, we have the identity above uniformly in time.

Putting together $\Xi_{1,2} := \Xi_1 \cap \Xi_2$, for all $\theta \in \Xi_{1,2}$ the function $\omega_\cdot (\theta)$ satisfies the conditions of Theorem 1, part (i), for all $\phi \in \mathcal{D}$. We have thus proved such claim, limited to $\phi \in \mathcal{D}$.

Assume $\mathcal{D}$ is also dense in $C^\infty (\mathbb{T}^2)$; precisely we shall use density in $H^{-\gamma} (\mathbb{T}^2)$ for some $\gamma > 3$. Given $\phi \in H^{-\gamma} (\mathbb{T}^2)$, take $\phi_k \to \phi$ in $H^{-\gamma} (\mathbb{T}^2)$, $\phi_k \in \mathcal{D}$. We have

$$\begin{aligned} &\int_0^T \left| \left\langle \omega_s (\theta) \otimes \omega_s (\theta), H_\phi^n - H_\phi^m \right\rangle \right|^2 ds \\ &\leq 2 \int_0^T \left| \left\langle \omega_s (\theta) \otimes \omega_s (\theta), H_{\phi_k}^n - H_{\phi_k}^m \right\rangle \right|^2 ds + 2 \int_0^T \left| \left\langle \omega_s (\theta) \otimes \omega_s (\theta), H_{\phi_k - \phi}^n - H_{\phi_k - \phi}^m \right\rangle \right|^2 ds \end{aligned}$$

hence, to get that $s \mapsto \left\langle \omega_s (\theta) \otimes \omega_s (\theta), H_\phi^n \right\rangle$ is Cauchy in $L^2(0,T)$ it is sufficient to prove that

$$\int_0^T \left| \left\langle \omega_s (\theta) \otimes \omega_s (\theta), H_{\phi_k - \phi}^n \right\rangle \right|^2 ds$$

is small uniformly in n, if k is large enough. Let us prove that this property is true in a set $\Xi_3 \in \mathcal{F}$ with $P(\Xi_3) = 1$. Then the proof of Theorem 1, part (i), will be complete, considering $\theta \in \Xi_{1,2,3} := \Xi_1 \cap \Xi_2 \cap \Xi_3$.

Consider the distribution $g_s^n (\theta)$ defined as

$$\langle g_s^n (\theta), \phi \rangle := \left\langle \omega_s (\theta) \otimes \omega_s (\theta), H_\phi^n \right\rangle.$$

We have

$$\begin{aligned} \| g_s^n (\theta) \|_{H^{-\gamma}}^2 &= \sum_k \left(1 + |k|^2 \right)^{-\gamma} \left| \langle g_s^n (\theta), e_k \rangle \right|^2 \\ &= \sum_k \left(1 + |k|^2 \right)^{-\gamma} \left| \left\langle \omega_s (\theta) \otimes \omega_s (\theta), H_{e_k}^n \right\rangle \right|^2 \end{aligned}$$

$$\mathbb{E}\left[\int_0^T \|g_s^n\|_{H^{-\gamma}}^2 \, ds\right] = \sum_k \left(1+|k|^2\right)^{-\gamma} \mathbb{E}\left[\int_0^T \left|\left\langle \omega_s \otimes \omega_s, H_{e_k}^n \right\rangle\right|^2 ds\right]$$
$$\leq CT \sum_k \left(1+|k|^2\right)^{-\gamma} \|e_k\|_{C^2}^2 \leq CT \sum_k \left(1+|k|^2\right)^{-\gamma} |k|^4$$

and this is finite when $\gamma > 3$. Hence there is a set $\Xi_3 \in \mathcal{F}$ with $P(\Xi_3) = 1$, such that $\int_0^T \|g_s^n(\theta)\|_{H^{-\gamma}}^2 \, ds < \infty$ for all $\theta \in \Xi_3$. For such θ we have

$$\int_0^T \left|\left\langle \omega_s(\theta) \otimes \omega_s(\theta), H_{\phi_k - \phi}^n \right\rangle\right|^2 ds = \int_0^T |\langle g_s^n(\theta), \phi_k - \phi \rangle|^2 \, ds \leq C(\theta) \|\phi_k - \phi\|_{H^\gamma}^2$$

where $C(\theta) := \int_0^T \|g_s^n(\theta)\|_{H^{-\gamma}}^2 \, ds < \infty$. Hence we have the required property.

As to claim (ii) of Theorem 1, we invoke the result of [29]. First, let us recall Theorem 24 part (ii): the solution (not unique) provided by part (i) is the P-a.s. limit in $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$ of a subsequence of the random point vortex system (8), defined also on $(\Xi, \mathcal{F}, P)$. This means that there is $(N_k)_{k\in\mathbb{N}}$ and a subset $\Xi_4 \in \mathcal{F}$ of $\Xi_{1,2,3}$, still with $P(\Xi_4) = 1$ such that for all $\theta \in \Xi_4$ the function $\omega_\cdot(\theta)$ is the $C\left([0,T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$-limit of the sequence $\frac{1}{\sqrt{N_k}} \sum_{n=1}^{N_k} \xi_n(\theta) \delta_{X_t^n(\theta)}$; with the understanding that Ξ_4 is such that for all $\theta \in \Xi_4$ the corresponding point vortex dynamics is well defined for all times, without coalescence of points.

Taken $\theta \in \Xi_4$, the function $\omega_\cdot(\theta)$ satisfies the conditions of Theorem 1, part (i). In addition, given any $\epsilon > 0$, there is $k_\epsilon \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} d_{H^{-1-}}\left(\omega_t(\theta), \frac{1}{\sqrt{N_{k_\epsilon}}} \sum_{i=1}^{N_{k_\epsilon}} \xi_i(\theta) \delta_{X_t^i(\theta)}\right) < \epsilon/2.$$

Hence, for every $\phi \in C^\infty\left(\mathbb{T}^2\right)$ one has

$$\sup_{t\in[0,T]} \left| \langle \omega_t(\theta), \phi \rangle - \left\langle \frac{1}{\sqrt{N_{k_\epsilon}}} \sum_{i=1}^{N_{k_\epsilon}} \xi_i(\theta) \delta_{X_t^i(\theta)}, \phi \right\rangle \right| < \epsilon/2.$$

We now apply Theorem 2.1 of [29] to $\frac{1}{\sqrt{N_{k_\epsilon}}} \sum_{i=1}^{N_{k_\epsilon}} \xi_i(\theta) \delta_{X_t^i(\theta)}$, applicable because this solution of the point vortex dynamics is global (namely without coalescence). It claims that there exists a sequence $\omega_\cdot^{(n)}$ of solutions of class $L^\infty\left([0,T] \times \mathbb{T}^2\right) \cap C\left([0,T]; L^p\left(\mathbb{T}^2\right)\right)$ for every $p \in [1,\infty)$, such that for every $\phi \in C\left(\mathbb{T}^2\right)$ one has

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \left\langle \omega_t^{(n)}, \phi \right\rangle - \left\langle \frac{1}{\sqrt{N_{k_\epsilon}}} \sum_{i=1}^{N_{k_\epsilon}} \xi_i(\theta) \delta_{X_t^i(\theta)}, \phi \right\rangle \right| = 0.$$

Hence, given the value of ϵ above, there is n_0 such that for all $n > n_0$

$$\sup_{t\in[0,T]}\left|\left\langle \omega_t^{(n)},\phi\right\rangle - \left\langle \frac{1}{\sqrt{N_{k_\epsilon}}}\sum_{i=1}^{N_{k_\epsilon}} \xi_i(\theta)\,\delta_{X_t^i(\theta)},\phi\right\rangle\right| < \epsilon/2.$$

We deduce $\sup_{t\in[0,T]}\left|\langle \omega_t(\theta),\phi\rangle - \left\langle \omega_t^{(n)},\phi\right\rangle\right| < \epsilon$, concluding the proof of Theorem 1, part (ii).

7 Remarks on ρ-white noise solutions

7.1 The continuity equation

Let μ be the law of white noise. Following [16], [18] and related literature, let us denote by $\mathcal{FC}_{b,T}^1$ the set of all functionals $F : [0,T]\times C^\infty(\mathbb{T}^2)' \to \mathbb{R}$ of the form $F(t,\omega) = \sum_{i=1}^m \widetilde{f}_i(\langle\omega,\phi_1\rangle,...,\langle\omega,\phi_n\rangle)\, g_i(t)$, with $\phi_1,...,\phi_n \in C^\infty(\mathbb{T}^2)$, $\widetilde{f}_i \in C_b^1(\mathbb{R}^n)$, $g_i \in C^1([0,T])$ with $g_i(T) = 0$. Given $F \in \mathcal{FC}_{b,T}^1$, denote by $D_\omega F(t,\omega)$ the function

$$\sum_{i=1}^m\sum_{j=1}^n \partial_j \widetilde{f}_i(\langle\omega,\phi_1\rangle,...,\langle\omega,\phi_n\rangle)\, g_i(t)\,\phi_j.$$

Definition 30 *Given $F \in \mathcal{FC}_{b,T}^1$, we set*

$$\langle D_\omega F(t,\omega), b(\omega)\rangle := \sum_{i=1}^m\sum_{j=1}^n \partial_j \widetilde{f}_i(\langle\omega,\phi_1\rangle,...,\langle\omega,\phi_n\rangle)\, g_i(t)\left\langle \omega\otimes\omega, H_{\phi_j}\right\rangle$$

where $\left\langle \omega\otimes\omega, H_{\phi_j}\right\rangle$, $j = 1,...,n$, are the elements of $L^2(\Xi)$ given by Theorem 8. Hence $\langle D_\omega F(t,\omega), b(\omega)\rangle$ is an element of $C\left([0,T]; L^2(\Xi)\right)$.

Definition 31 *We say that a bounded measurable function $\rho : [0,T]\times H^{-1-}(\mathbb{T}^2) \to [0,\infty)$ is a bounded weak solution of the continuity equation*

$$\partial_t\rho_t + \mathrm{div}_\mu(\rho_t b) = 0 \tag{12}$$

with initial condition ρ_0, if

$$\int_0^T\int_{H^{-1-\delta/2}} \left(\partial_t F(t,\omega) + \langle D_\omega F(t,\omega), b(\omega)\rangle\right)\rho_t(\omega)\,\mu(d\omega)\,dt = -\int_{H^{-1-\delta/2}} F(0,\omega)\,\rho_0(\omega)\,\mu(d\omega)$$

for all $F \in \mathcal{FC}_{b,T}^1$.

Proposition 32 *Any function ρ given by Theorem 25 is a bounded weak solution of the continuity equation (12).*

Proof. Let ω be a solution of Euler equations given by Theorem 25, with the associated density function ρ. Given $F \in \mathcal{FC}^1_{b,T}$ of the form $F(t,\omega) = \sum_{i=1}^m \widetilde{f}_i(\langle \omega, \phi_1 \rangle, ..., \langle \omega, \phi_n \rangle) g_i(t)$, we know that

$$\langle \omega_t, \phi_j \rangle = \langle \omega_0, \phi_j \rangle + \int_0^t \left\langle \omega_s \otimes \omega_s, H_{\phi_j} \right\rangle ds$$

for every $j = 1, ..., n$. Here P-a.s. the function $s \mapsto \left\langle \omega_s \otimes \omega_s, H_{\phi_j} \right\rangle$ is of class $L^2(0,T)$. Hence $\langle \omega_t, \phi_j \rangle$ is differentiable a.s. in time. We have, P-a.s., a.s. in time,

$$\begin{aligned}
&\partial_t (F(t,\omega_t)) \\
&= \sum_{i=1}^m \sum_{j=1}^n \partial_j \widetilde{f}_i(\langle \omega_t, \phi_1 \rangle, ..., \langle \omega_t, \phi_n \rangle) g_i(t) \partial_t \langle \omega_t, \phi_j \rangle + \sum_{i=1}^m \widetilde{f}_i(\langle \omega_t, \phi_1 \rangle, ..., \langle \omega_t, \phi_n \rangle) g_i'(t) \\
&= \langle D_\omega F(t,\omega_t), b(\omega_t) \rangle + \partial_t F(t,\omega)|_{\omega=\omega_t}
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_0^T \int_{H^{-1-\delta/2}} (\partial_t F(t,\omega) + \langle D_\omega F(t,\omega), b(\omega) \rangle) \rho_t(\omega) \mu(d\omega) dt \\
&= \int_0^T \mathbb{E}\left[\partial_t F(t,\omega)|_{\omega=\omega_t} + \langle D_\omega F(t,\omega_t), b(\omega_t) \rangle\right] dt \\
&= \int_0^T \mathbb{E}\left[\partial_t (F(t,\omega_t))\right] dt = \int_0^T \partial_t \mathbb{E}\left[F(t,\omega_t)\right] dt \\
&= \mathbb{E}\left[F(T,\omega_T)\right] - \mathbb{E}\left[F(0,\omega_0)\right] \\
&= -\int_{H^{-1-\delta/2}} F(0,\omega) \rho_0(\omega) \mu(d\omega)
\end{aligned}$$

where the exchange of time-derivative and expectation is possible due to the boundedness of terms in F; and we have used $g_i(T) = 0$. ■

The analysis of this continuity equation deserves more attention; we have just mentioned here as a starting point of future investigations.

7.2 An open problem

We have treated above the problem of approximating Albeverio-Cruzeiro solution by smoother solutions of the Euler equations. Let us mention a sort of dual problem, that can be formulated thanks to Theorem 25.

Given $\overline{\omega}_0 \in L^\infty(\mathbb{T}^2)$, there exists a unique solution $\overline{\omega}_t$ in $L^\infty(\mathbb{T}^2)$ of the Euler equations (point 1 of the Introduction). For every $\epsilon > 0$, consider the density

$$\rho_0^{(\epsilon)}(\omega) = \frac{1}{Z_R} \exp\left(-\frac{d_{H^{-1-}}(\omega, \overline{\omega}_0)^2}{2\epsilon} \right)$$

defined on $H^{-1-}(\mathbb{T}^2)$, where

$$Z_R = \int_{H^{-1-\delta}(\mathbb{T}^2)} \exp\left(-\frac{d_{H^{-1-}}(\omega, \overline{\omega}_0)^2}{2\epsilon}\right)\mu(d\omega).$$

Let $\omega_t^{(\epsilon)}$ be a ρ-white noise solution, provided by Theorem 25, corresponding to this initial density $\rho_0^{(\epsilon)}$. Can we prove that $\omega_t^{(\epsilon)}$ converges, in a suitable sense, to $\overline{\omega}_t$?

We do not know the solution of this problem. Let us only remark that it looks similar to the question of vortex point approximation of solutions of Euler equations, solved in a smoothed Biot-Savart kernel scheme by [30] and in great generality by [36]. Also, very roughly, reminds large deviation approximations of smooth paths by diffusion processes.

Theorems 24 and 25 give some intuition into Albeverio-Cruzeiro solution and its variants, as a limit of random point vortices. A positive solution of the previous problem would add more.

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