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A Short History of Mathematical Symbolism

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A SHORT HISTORY OF MATHEMATICAL SYMBOLISM

by

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INTRODUCTION

The purpose of this paper is to present a short history of some of the mathematical symbols that are used in high school today. Most of the symbols introduced will be from either arithmetic, algebra, or geometry. The writing will present various symbols that were used in early times. Many of the variations of these symbols are shown and in most cases it is readily understood why some have been discarded, while others are in use today.

It should be pointed out that symbols were invented when the need for them arose. Various men in different countries saw the need and worked on the symbolism of mathematics. In several cases the symbols were different for representing different things. However, in some cases one writer would use a symbol to represent a certain thing while another writer would use the same symbol to represent a different thing. These differences will be discovered in this paper.

When two or more different things were represented by the same symbol, confusion resulted. This led to the need of systematizing the symbols. The less complicated the symbol, the more popularity it received and the more often it was used by the better writers in the field of mathematics.

Today most of the symbols used in arithmetic, algebra, and geometry are well established. An exception might be the symbols of modern mathematics when introduced into these subjects.

This paper, by no means, presents all the symbols with the various derivations of each. There are books written on the history of symbols and notations of mathematics and it would be impossible to include all of them in a paper of this kind.

It is hoped that the most commonly used symbols, with variations and the names of the men responsible for them will stick with the reader in such a way as to enlighten him in the symbolism of mathematics. The paper should reveal the importance of symbols, why they were discovered and used, and an appreciation for the vast amount of work that has been done with symbolism in the field of mathematics.

EARLY NUMBERS AND SYMBOLS

Before the alphabet was invented our savage ancestors used pictures instead of words to represent ideas. If a single lion was to be represented, a picture of a lion was drawn; later only the head was drawn. To represent three lions, three of those pictures were made. Early American Indians used this form of writing, known as picture writing, or hieroglyphics. Later some man conceived the idea of representing three lions by one lion's head with three strokes under it; five lions by the head with five strokes.(1)

Basically the same ideas were used for counting by the early sheep-herders. Instead of using numbers as we do today, these early men placed sticks in a pile to represent the number of sheep they had in their flocks. As the sheep left for pasture in the morning a stick was placed in a pile for each sheep. This is a fine example of our modern idea of a one-to-one correspondence. When the sheep returned in the evening the herder matched a stick with a sheep. If he had any sticks not represented, he knew some of the sheep were missing.

It was somewhat later that number systems started to be developed. Nearly all number-systems, both ancient and modern, are based on the scale of 5, 10, or 20. The reason for this is not difficult to see. When a child learns to count, he makes use of his fingers and perhaps of his toes. In the same way the savages of prehistoric times un-

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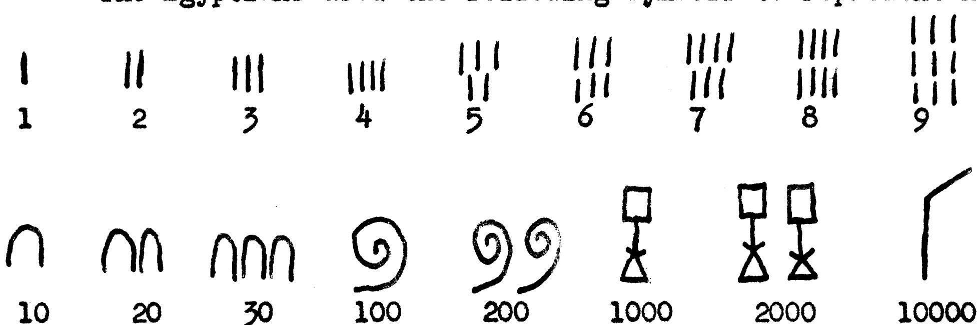
1. Louis C. Karpinski, The History of Arithmetic, p. 1.

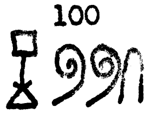
questionably counted on their fingers and in some cases also on their toes.

Such is indeed the practice of the African, the Eskimo, and the South Sea Islander of today. This use of the fingers had often resulted in the development of a more or less pantomine number-system. Evidence of the prevalence of finger symbolism is found among the ancient Egyptians, Babylonians, Greeks, and Romans, as also among the Europeans of the middle ages: even now nearly all Eastern nations use finger symbolisms.(2)

Due to the fact that there are ten fingers on the hands, the decimal number system has prevailed until the present time. However, there has been individuals who believed a number system to the base twelve would be more convenient. Part of their argument is the fact that twelve is divisible by two, three, four, and six, while ten is divisible by two and five. The fractions 1/2, 1/3, 1/4 are used much in business practices and would be more convenient if the number system was to the base twelve.

The Egyptians used the following symbols to represent numbers:



Thus 1,210 was written . They used a system of repeated doubling

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2. Florian Cajori, *A History of Elementary Mathematics*, p. 1.

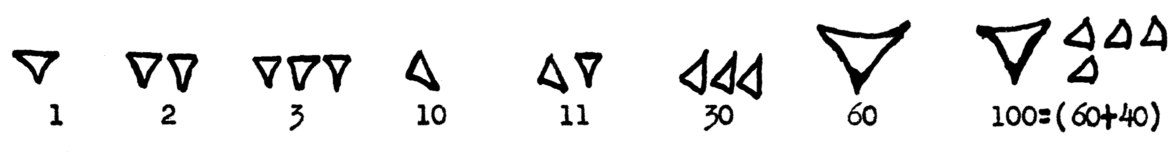
to multiply. Thus to multiply 37 by 11 the Egyptian wrote with his symbols:

37	1
74	2
148	4
296	8
407	

In this case, he used 1 X 37 + 2 X 37 + 8 X 37 which is the same as (1 + 2 + 8) 37 or 407. Division was done by reversing the above process.[3]

The Egyptians wrote their fractions with the numerator unity, together with 2/3. Their notation was simple. The symbol [symbol] above a number meant one divided by the number. This served for all possible unit fractions and the symbol [symbol] stood for 2/3. Thus 3/4 would be written as 1/2 1/4 or as 2/3 1/12; 5/8 would be written 1/2 1/8. They would multiply using the same method as above, but use the fraction as a second multiplier and then add the results of the two multiplications. To multiply 37 by 11 5/8, find the product of 37 times 11 and 37 times 1/2 1/8 which was 407 + 1/2 (37) + 1/4 1/2(37). The result was 407 + 18 1/2 + 4 1/2 1/8 = 430 1/8.[4]

The Babylonians used cuneiform writing, a combination of wedge-shaped symbols to represent their numbers.



Probably for a long time the Babylonian system did not go beyond 60.

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3. Karpinski, op. cit., pp. 3-5.

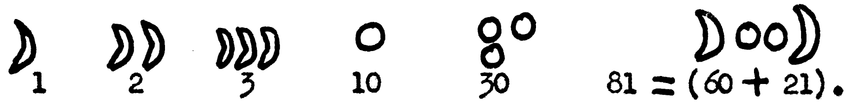
4. Vera Sanford, A Short History of Mathematics, p. 102.

From the use of the sixty or sexagesimal system we get our minutes and seconds, both in measurement of time and angles.[5]

The Babylonian multiplication consisted of a table of multipliers from 1 to 60. The table of 18 begins 18 X 1, 18 X 2, and so on to 18 X 19, 18 X 20; then the tables give 18 X 30, 18 X 40, and 18 X 50. Eventually 18 X 57 would have been obtained as 18 X 50 added to 18 X 7. The tables were not convenient due to their length and weight.

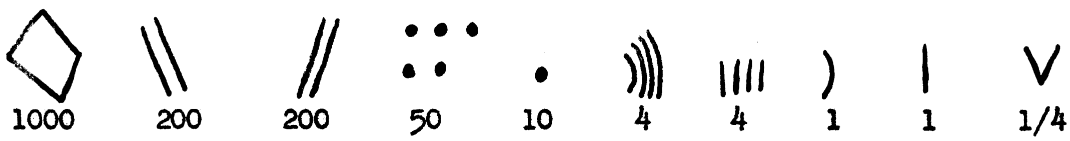
There were curvilinear numerals used by the Babylonians also. These numbers were used primarily for the number of the year, for the age of an animal, and in stating that a second or third payment had been made. It was used also concerning the allotment of food.

The symbols were written as follows:



[6]

The Greeks used different symbols to represent their numbers. These symbols did not come from Greece but from the island of Crete.



In the time of Thales (624-547 B.C.), the initial letters of words were used to represent numbers.

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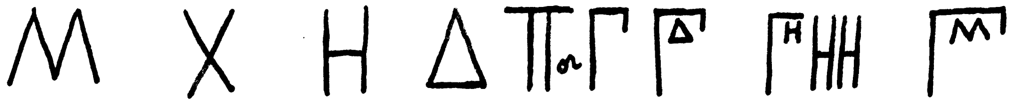
5. Carl Fink, A Brief History of Mathematics, pp. 9-10.

6. Karpinski, op. cit., pp. 7-11.

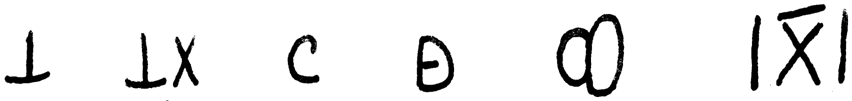


10,000 1000 100 10 5 50 700 50,000

The Hebrews used this same system with Hebrew letters, and the Arabs continued its use until 800 or 900 A.D. .[7]

The Romans came along with a system of symbols called the Roman numerals. Some of the symbols were as follows:

I	II	III	IIII	V	VI	VII	VIII	VIIII	X	XXXX
1	2	3	4	5	6	7	8	9	10	40



50 60 100 500 1000 1,000,000. Later four was written IV and nine was changed to IX. Their numbers were written using the combinations of the symbols. One hundred twenty-eight was written CXXVIII. [8]

Other systems used were finger reckoning with the fingers used as digits and also the Mayas system. In finger reckoning the position of the hand and fingers represented certain numbers. This method is still used by savage races in Africa, by Arabs, and by Persians, while the Mayas used a highly developed twenty system. They used words for 20; 400 or 20 times 20, and for 8000 or 20 times 20 times 20. The following are some of their symbols:

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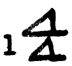
7. Ibid., pp. 11-13.

8. Alfred Hooper, Makers of Mathematics, p. 89.

1 2 3 4 5 6 7 10 11 15 18 20.[9]

The Hindus and Greeks were among the first to work with writing large numbers. The writing was from right to left. If a digit was to be left out the Greeks replaced the missing digit with the word vacant. Writing the number 3,080,046, the Greeks would use their symbols and write it in this manner. Six, four, vacant, vacant, eight, vacant, three. It was this type of number writing that led to the development of the zero.

The Babylonian used the symbol  for zero. The Hindus were the first to develop the idea of place value and for a missing digit the symbol zero was inserted. The symbol for zero was dot or a small circle. The Arabs used the word sifr, meaning vacant.[10] The Arabs represented the presence of a zero by an (X).[11]

The symbol (O) has been used to represent different ideas. At one time the symbol meant unity. Bhaskara used a small circle above a number to indicate subtraction, and in the Tartar writing a redundant word is removed by drawing an oval around it. O'Creat used the symbol  for zero, being an abbreviation for the word teca which was one of the names used for zero.

Although the dot and the small circle were both used to represent

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9. Sanford, op. cit., p. 84.

10. Karpinski, op. cit., pp. 39-41.

11. Louis Karpinski & David Smith, The Hindu-Arabic Numerals, p. 53.

zero, the small circle continues to be in use today. The Arabs did not adopt the small circle to represent zero because it looked so much like the symbol they used for five. Today the Arabs use the 0 only when, under European influence, they adopt the ordinary system.(12)

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12. Ibid., pp. 54-56.

DEVELOPMENT OF FRACTIONS AND OPERATION SYMBOLS

The writing of common fractions was written differently by different people. The Greeks used two sets of fractions. One of these was for scientific work and the other for ordinary computation. Those used for scientific work were fractions with the denominators being limited to powers of 60. The others known as "usual small fractions" were called English "vulgar fractions" and then, by the substitution of a synonym, the American "common fraction".

The Ionic system of numerals included symbols for 1/2 and 2/3. Unit fractions were represented by writing the denominator followed by an accent ('). Thus one-fifth was 5'. An early approach to our modern notation was in the work of Diophantus. He avoided the ambiguities of the Ionic system by writing the denominator above the numerator. This is equivalent to having 4/3 stand for 3/4.

This same scheme was used by Brahmagupta in the seventh century. The Arabs improved this by inserting a bar between the two numbers. Fibonacci wrote his fractions in this way, but he habitually placed the fractional part of a mixed number to the left of the number itself. Kobel tried to combine Roman numerals with the Hindu-Arabic notation writing 1/4 as I/IIII and 6/8 as VI/VIII. This never became popular. Leibniz advocated the use of a colon between numerator and denominator, a scheme that is in some ways preferable to the commonly used / today.

Decimal fractions were also written with a variety of symbols, Adam Reise gave the square root of two as 1 414/1000. Rudolff used an upright bar in an interest table to mark off fractions whose denominators were powers of ten. In 1492, Pellos used a period to cut off one, two, or three places in the dividend when his divisor was a multiple of 10, 100, or 1000. (13)

Simon Stevin (1548-1620), was the first writer to give a systematic treatment of decimal fractions. In 1585 he published seven pages in which decimal fractions were explained. He recognized full importance of decimal fractions, and applied them to all the operations of ordinary arithmetic. His notation was not too good. In place of our decimal point, he used a cipher; to each place in the fraction was attached the corresponding index. Thus the number 5.912 would be written $\overset{0}{5}\overset{1}{9}\overset{2}{1}\overset{3}{2}$; $5_{⓪}\ 9_{①}\ 1_{②}\ 2_{③}$; or $5912_{③}$ according to the exigencies of the case. Later this system led Stevin to an important notation-- the exponential notation.(14)

The development of symbolism for decimal fractions was slow. In fact, it has not been settled today. The comma is used as a decimal point on the continent and the form 3•14 is the accepted one in England. The following symbols are some of those used after the invention of decimals: Burgi (1592) 3.14 and 3,14; Napier (1617) 3,14 and 3,1'4"; Kepler (1616) 3,14 and 3(14); Oughtred (1631) 3|<u>14</u>; and van Schooten (1657) 314. (15)

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13. Sanford, op. cit., pp. 104-111.

14. Cajori, op. cit., pp. 151-153.

15. Sanford, op. cit., pp. 113-114.

The early Babylonians had an ideogram, which is translated LAL, to signify "minus". In the hieratic papyrus of Ahmes and, more clearly in the hieroglyphic translation of it, a pair of legs walking forward is the sign of addition; away, the sign of subtraction. In another Egyptian papyrus kept in the Museum of Fine Arts in Moscow, a pair of legs walking forward has a different significance; there it means to square a number.  are the symbols for addition and subtraction.

Diophantus used a slanting line / for addition, also a semi-elliptical curve) for subtraction, and a combination of the two  for the total result. This has been detected in Greek papyri. He also used the sign  for subtraction. The Hindus had no mark for addition except that, in the Bakhshali Arithmetic, yu is used for the purpose. The Hindus distinguished negative quantities by a dot. The Frenchman Chuquet (1484); the Italian Pacioli (1494), and the sixteenth-century mathematicians in Italy used p̄ or p: for plus and m̄ or m: for "minus". [16]

In the Ahmes Papyrus (c. 1650 B.C.), addition is indicated by the sign  and subtraction by . The first appearance of the signs + and − in print was in a book by Widman published in Leipzig in 1489. The first problem in which these signs are used asked the value of 13 barrels of figs at 4 1/8 florins a hundred weight, the weight of the barrels being 4ct.+5 lb., 4ct. − 17 lb., 3ct.+36 lb.,

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16. Florian Cajori, A History of Mathematical Notations, Vol. I, pp. 229-230.

etc.. These signs occurred in connection with problems worked by "false position". [17]

Leonardo of Pisa used the words "minus" and "more", or "plus" in connection with the method of false position in the sense of "positive error" and "negative error". He used "minus" to indicate an operation (of subtraction), he does not use the word "plus". The word "plus" signifying the operation of addition, was first found by Enestrom in an Italian algebra of the fourteenth century.[18]

The plus sign as found in print has had three principal varieties of form: (1) the Greek cross +, as it is used by Widman (1489); (2) the Latin cross, † more frequently placed horizontally, —+ or +— ; and (3) the form ✠ , or a cross having four rounded vases with tendrils drooping from their edges.

The minus sign has been generally written in three symbols. The most common and still used today is the — sign. Others that have been used are the symbols ∸ and ÷ . The ∸ sign occurred in a book written in 1535.[19]

Other signs of operation are those of multiplication and division. In the early Babylonian tablets there is an ideogram A-DU signifying "times" or multiplication. Diophantus used no symbol for multiplication. In some cases a dot placed between two factors meant to multiply. Stifel in 1545 used the capital letter M to

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17. Sanford, op. cit., p. 149.

18. Cajori, A History of Elementary Mathematics, pp. 236-241.

19. Cajori, A History of Mathematical Notation, (Vol. I), pp. 236-242.

designate multiplication, and D to designate division. Vieta indicated the product of A and B by writing "A in B".[20]

The use of the sign X for multiplication is probably due to Oughtred. Leibniz objected to the notation with the plea that the sign X was easily confused with the unknown quantity x. Leibniz experimented with six signs for multiplication but the dot and the X are the only ones used extensively today. Others he experimented with were (⌒), (,), (;), and (*).

Division like multiplication was represented by various symbols. The horizontal bar between the numerator and denominator of a fraction was used to show division. Fibonacci omitted the bar and merely wrote one number above the other. A manuscript dating from about 1460, now in Munich, for example gives the fraction $\frac{12x+45}{x^2+3x}$ in the form $\frac{\text{12 res et 45}}{\text{1 census et 3 res.}}$. Leibniz preferred the colon (:) to the fractional form of indicating division but he also used the symbol ◡ to represent division. The sign ÷, once used to indicate subtraction, is of comparatively late use, dating from the latter half of the seventeenth century.[21]

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20. Ibid., p. 217.

21. Sanford, op. cit., p. 152.

SYMBOLS THAT SHOW RELATIONSHIPS

Some of the same symbols used for division were also used to show proportion. Oughtred used the symbol $\frac{\cdot\cdot}{\cdot\cdot}$ to indicate that the numbers following were in continued geometrical proportion. Thus, $\frac{\cdot\cdot}{\cdot\cdot}$ 2, 6, 18, 54, 162 are in continued geometric proportion. Several writers used the same symbol.

Symbols for arithmetic progression were less common than for geometric progression. Oughtred had no symbol. The sign $\div$ was used by some writers. Stone and Wilson use six dots ⁚⁚⁚ . Kirkby and Emerson used $\frac{\cdot\cdot}{\cdot}$. Clark used $\frac{\cdot\cdot\cdot}{\cdot\cdot\cdot}$. Blassiere used $\frac{\cdot}{\cdot\cdot}$. Some French writers used $\div$ for arithmetic progression and $\frac{\cdot\cdot}{\cdot\cdot}$ for geometric progression. Regiomontanus in a letter wrote our modern a:b:c in the form a.b.c, the dots being simply signs of separation.

As the symbolism of algebra was being developed, the need for more precise symbolism became apparent. In 1631, Oughtred introduced the notation 5.10 :: 6.12 to represent proportion. He could not employ two dots (:) for ratio, because the two dots were already pre-empted by him for the designation of aggression, : A+B: signifying (A+B). His work would have been far superior if an (=) sign had been used in place of the four dots between the ratios.

Oughtred's use of the dot was not universally accepted in England and as late as 1651, Wing used the colon (:) as the symbol for ratio.

The use of these two symbols for ratio caused a struggle in

England. During the second half of the seventeenth century there was much competition with the dot maintaining its ascendancy. The dot was used by both Wallis and Barrow, but Newton in his writings used the colon to designate ratio.

Another notation for proportion was a symbolism consisting of vertical lines. Descartes (1619-21) used this notation. He wrote a|b||c|d. Also, Slusius used two vertical strokes (||) to signify equality. La Hire wrote "aa||xx||ab" for $a^2: x^2 = x^2:$ ab. Herigone used 2|2 for equality and π for ratio.

In 1639, Stampioen used the designation A,,B=C,,D. Nearly a century later, a French writer used the two commas to represent ratio. Stampioen used Recorde' s equal sign to show equality in writing proportions. In 1668, Gregory used the same notation, but had few followers in Great Britain.

In 1708, Leibniz's notation, a:b=c:d was used in a writing. In the twentieth century this notation came to be generally adopted in the United States. It replaced the : :: : notation for ratio and proportion.[22]

Ahmes (c. 1550 B.C.) defined an unknown quantity as a ("mass", "quantity", or "heap"). Diophantus called it "an undefined number of units". The Arabs called it sha meaning "thing" or "anything".[23] In modern mathematics usually a letter is chosen to represent the

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22. Cajori, History, Vol. I, pp. 278-296.

23. David Smith, History of Mathematics, Vol. II, p. 393.

unknown. Some books suggest that letters near the end of the alphabet be chosen to represent unknowns.

There were several symbols used to represent equality and inequality. Robert Recorde in 1557 used the symbol $=$ to represent equal. At the time it was not widely accepted but due to its simplicity it has been used until today.

In 1613, the Italian writer Giovanni Camillo Glorioso used two vertical lines for equality. An example was $1 \mid 2 \parallel 4 \mid 8 \parallel 8 \mid 16$. Though used by occasional writers for more than a century, this mark $\parallel$ never gave promise of becoming a universal symbol for equality. A single vertical line was used for equality by S. Reyher in 1698. With him, "A | B" meant $A = B$. In England it was Leonard and Thomas Diggers, father and son, who introduced new symbols, including a line complex  for equality.

The greatest oddity was produced by Herigone in 1634. It was the symbol "2 | 2" for two equal two. Based on the same idea is his "3 | 2" for "greater than", and his "2 | 3" for "less than". In some cases he used $\sqcup$ to express equality. Leibniz, on one occasion used the Cartesian  for identity.

Different yet was the equality sign 3 used by J. V. Andrea in 1614. Descartes used ae for equal because it was the first two letters of the word aequalis meaning "equal". Jean Prestet, about 1691, used $\sim$ to represent equal. Fermat, in 1679, used "DA $\{$ BE" in an example, where $DA = BE$.

Bolyai in 1832 used $\doteq$ to signify absolute equality; $\overline{\underline{\;\cdot\;}}$ equality in content; A(=B or B=)A, to signify that each value of A is equal to some value B; A(=)B, that each of the values of A is equal to some value of B, and vice versa. To mark the equality of vectors, Bellavitis used in 1832 and later the sign $\bumpeq$.

Gustave du Pasquier in discussing general complex numbers employs the sign of double equality $\equiv$ to signify "equal by definition". Greenhill denotes approximate equality by $\approx$. An early suggestion due to Fischer was the sign $\asymp$ for "approximately equal to". (24)

In 1631, Oughtred used the symbols $\sqsubset$ and $\sqsupset$ with the meaning "is greater than" and "is less than". These symbols lasted for at least a century. Thomas Harriot used the symbols $>$ and $<$ for "is greater than" and "is less than".

The symbols $\neq$ for is not equal to, $\not>$ is not greater than, and $\not<$ for is not less than, are now rarely used outside Great Britain; they were employed in 1734. The vinculum was introduced by Vieta in 1591; and brackets were first used by Girard in 1629. (25)

Al-Khowarizm spoke of the first power of the unknown as a "root" and the second power as a "square". Fibonacci translated the Arabic names into radix and quadratus and used cubus numerus for the third power. Later Italian writers sometimes used res or cosa ("thing"), censo, and cubus while German mathematicians used Coss, Zenso, and

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24. Cajori, <u>A History of Mathematical Notations</u>, Vol. I, pp. 297-309.

25. Ball, <u>op</u>. <u>cit</u>., p. 242.

Cubus for x, x^2, and x^3. These words were soon abbreviated. Diophantus used symbols made of the initial letters of the words for number, power, and cube.

Pictorial representations was attempted by several writers. Ghaligai (1521) used C^o (from cosa) for x, and □ for x. Buteo (1559) included the third power using ρ, ◇, and ⬢. These symbols were clumsy and could not be adopted to the representation of higher powers. [26]

The difficulty of not being able to write powers of unknowns in a successful form led Bombelli, in 1572, to representing the unknown quantity by $\underset{\smile}{1}$, its square by $\underset{\smile}{2}$, its cube by $\underset{\smile}{3}$, etc.. In 1591, Vieta improved on this by denoting the different powers of A by A, A quad., A cub., etc., so that he could indicate the powers of different magnitudes; Harriot in 1631 improved on Vieta's notation by writing aa for a^2, aaa for a^3, etc.. This remained in use for fifty years. In 1634, Herigonus wrote a, a2, a3,..., for a, a^2, a^3... .

The idea of exponents to mark the power was introduced by Descartes in 1637. He used only positive integral indices a^1, a^2, a^3, ... Wallis in 1659 explained the meaning of negative and fractional indices in expressions such as a^{-1}, $ax^{1/2}$, etc.. The symbol ∞ for infinity was first employed by Wallis in 1655. [27]

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26. Sanford, op. cit., p. 155.

27. Rouse Ball, A Short Account of the History of Mathematics, pp. 242-244.

The symbol √ to denote the square root was introduced by Rudolff in 1526. √√√ indicated the cube root, and √√ the fourth root. Similar notations had been used a short time before by others.[28]

The sign ⌈⌉ for square root occurs in two Egyptian papyri, both found at Kahun. Scheubel used the abbreviation ra for the sign √. He indicated cube root by ra. cu. or by √√√, fourth root by ra. ra. or by √√. Perez de Moya, in 1784, used rrr as signifying cube root, and rr as fourth root.

John Napier used the notation derived from this figure

1	2	3
4	5	6
7	8	9

in the following way: ⊔ prefixed to a number means its square root, ⊐ its fourth root, □ its fifth root, ⌈ its ninth root, and so on, with extensions of obvious kinds for higher roots.

Dibuadius in 1605 gives three designations of square root, √, √Q, √Z; also three designations of cube root, √C, √c, √ce, and three designations of the fourth root, √√, √QQ, √ZZ.

It was G. Peano, who indicated by $\sqrt[m]{a}$ all the m values of the radical, reserving $\sqrt[m]{a}$ for the designation of its "principal value".[29] There were many others who introduced various symbols but they were too complicated for notational purposes and as a result were not widely accepted.

The square root of a negative number was avoided by many writers. Cardan found a solution to a similar problem but called the roots fictitious and did nothing to develop them further. Bombelli and Stevin

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28. J. W. Sullivan, The History of Mathematics in Europe, p. 36.

29. Cajori, History, I, pp. 360-379.

spoke of imaginary numbers but did not explain them. Other men worked with the problem and had ideas but Leibniz (1702) characterized imaginaries as "that wonderful creature of an ideal world, almost an amphibian between things that are and things that are not".

The symbolism and the names of these quantities were standardized soon after the numbers were accepted as being worthy of notice. Descartes (1637) was the originator of the words vrage (real) and imaginaire (imaginary) in this connection. Euler (1748) used the letter i for $\sqrt{-1}$. Chauchy (1821) used the word conjugate for the pair of values $a+bi$ and $a-bi$, and the word modulus for the distance from the origin to a point in the complex plane.

The problem of trying to find the ratio of the circumference of a circle to its diameter troubled mathematicians for many years. It was not until the nineteenth century that it was actually proved that the value of π is transcendental. This was done by Lindemann in 1882.

The symbol π was first used to mean the circumference or periphery of a circle and appeared in Oughtred's book. Others copied this idea but in 1707 William Jones gave the symbol its present meaning of the ratio of the circumference to the diameter. Euler's adoption of the symbol in 1737 brought it into general use.(30)

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30. Sanford, <u>op. cit.</u>, pp. 186-189.

SYMBOLS USED IN GEOMETRY AND PROBLEM SOLVING

The symbols usually used in geometry may be grouped roughly under three heads: (1) pictographs or pictures representing geometrical concepts, as △ representing a triangle; (2) ideographs designed especially for geometry, as ∼ for "similar"; (3) symbols of elementary algebra, like + and −.

The use of geometrical drawings goes back at least to the time of Ahmes, but the employment of pictographs in the place of words is first found in Heron's Dioptra. In 150 A.D. Heron wrote ▷ for triangle, $\underline{\underline{ov}}$ for parallel and parallelogram, also $\underline{\underline{p'}}$ for parallelogram, ▭' for rectangle, ő for circle. Pappus (fourth century A.D.) wrote ○ and ⊙ for circle, ▽ and △ for triangle, ∟ for right angle, $\underline{\underline{ov}}$ or = for parallel, □ for square. These symbols were not adopted well by the authors of that time.(31)

In 1555 the Italian Fr. Maurolycus employed △ , □ , also * for hexagon and ⁙ for pentagon, while in 1575 he also used ▭. In 1623, Metius in the Netherlands adopted not only △ , □ , but a circle with a horizontal diameter and small drawings representing a sphere, a cube, a tetrahedron, and an octohedran. The latter four were never adopted due to difficulty in drawing.

Oughtred introduced many symbols and over forty of them appeared in the tenth book of Euclid's Elements printed in 1648; of these symbols only three were pictographs, namely, □ for rectangle,

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31. Cajori, History, I, p. 401.

□ for square, and △ for triangle. Kambly used # to signify rectangle. Other symbols are those for parallelogram and trapezoid. Halsted denoted a parallelogram by || g' m. In 1634, Herigone used the symbol ◊ for parallelogram. The ▱ symbol for parallelogram is of rare occurance in geometries preceding the last quarter of the nineteenth century. Reyher used ⏢ symbol to designate trapezoid.

The symbols for triangle, square, rectangle, and parallelogram are among the most widely used pictographs. Some authors used only two or three of the symbols. A rather curious occurrence is the Hebrew letter "mem", מ, to represent a rectangle.

Although a small image of a circle to take the place of the word was used in Greek time by Heron and Pappus, the introduction of the symbol was slow. Herigone used ⊙, but Oughtred did not. There were other authors who used this symbol, however, some authors used both the small circle and the small circle with the dot in the middle. In some cases the first meant circle and the second, the area of the circle.

In Plato of Tivoli's translation (middle of twelfth century) one finds repeatedly the designations $\overset{\frown}{abc}$ and $\overset{\frown}{ab}$ for arcs of circles. W. R. Hamilton designated by ⌒LF the arc "from F to L". In 1634, Herigone used ⌒ for arc of circle. The symbol was used to designate arc in different countries. In 1755 John Landen used the sign (PQR) for the circular arc which measures the angle PQR, the radius being unity. A report given by The National Committee on Mathematical Requirements in 1923 stated the opinion that "the value of the symbol $\frown$ in place of the short word arc is doubtful".(32)

In 1634, Herigone used $<$ symbol for angle. Unfortunately, in 1631, Harriot had used the same symbol for "less than". In 1648, Oughtred used $\angle$ for an angle. There were many other symbols used. John Ward inverted the sign $\urcorner$. The $\wedge$ symbol appeared in the Ladies Diary and other writings. In 1803, Carnot denoted angle ABC by $\widehat{ABC}$ in his geometry. $A\hat{\ }B$ designated angle AB. Binet, Mobius, and Favaro used $\widehat{ab}$ as the angle formed by two straight lines a and b. Nixon adopted $\hat{A}$ for angle A.

Some authors, especially German, adopted the sign $\sphericalangle$ to represent angle. Byrne used a slight modification $\triangleleft$ for angle. Various other representations were used as follows: the capital letter L, the capital letter V, or the inverted capital letter $\forall$, the perpendicular lines $\lrcorner$ or $\llcorner$, $\overline{pq}$ the angle made by the lines p and q, (ab) the angle between the rays, a and b, $\overset{\frown}{ab}$ the angle between the lines a and b, or (u,v) the angles formed by u and v.

It should be pointed out, that Herigone's symbol $<$ persisted in the seventeenth and eighteenth centuries. However, the symbol $\angle$ used by Oughtred enjoyed wide popularity in different countries and is still used today. Caswell designated the plural of angle by the symbol $\angle\angle$. Oughtred used $\angle\!\!\angle$. Many writers have adopted the

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32. Ibid., pp. 402-405.

symbol $\angle s$. Caswell also wrote Z$\angle\angle$ for the "sum of two angles".

The symbol $\underline{\angle}$ designated an obtuse angle. In 1634, Herigone used the symbol $\lrcorner$ for a right angle. Reyher used two perpendicular lines $\llcorner$ to designate a right angle. He let $\wedge$ A $|\llcorner$ stand for "angle A is a right angle". The vertical bar stands for equality. The same idea is involved in the signs a$\uparrow$b, i.e. "angle a is equal to angle b". Many writers used the $\llcorner$ to designate a right angle. Kersey used the sign $\perp$. Bryan used $\triangle$. Mach used $\dot{\perp}$ for right angles. Dupius used $\urcorner$ for right angle. More common among recent writers is "rt.$\angle$"for right angle.

There were symbols designed to show equality. D'Alembert, in 1754, used $\overset{\vee}{=}$ to signify equality of the angles and $\overset{|}{=}$ to signify the equality of the sides of a figure. Stone defines $\overset{\vee}{=}$ as signifying "equiangular or similar". Palmer and Taylor's Geometry used the sign $\overset{\circ}{=}$ to signify "equal number of degrees". John Caswell used the sign $\geqq$ to express "equiangular".

Herigone used $\perp$ for perpendicular lines. Dulaurens used the same symbol in 1667. In 1673 Kersey in England employed it. Several writers including Caswell used the inverted capital letter $\perp$. Emerson had the vertical bar extremely short, $\perp$. Baker adopted the symbol $\neg\!\!\perp$ for perpendicular. During the nineteenth century the symbol $\perp$ was adopted by most writers. Some used $\perp s$ for perpendiculars.

There were several symbols used to show parallel lines. Herigone used the Heronic symbol $=$, for parallel. Heron and Pappus used similar symbols. After Recorde's equal sign won its way upon the continent, vertical lines came to be used for parallelism. Many writers adopted || as the symbol for parallel lines but there were other symbols used also.

Hall and Stevens used "par or || " for parallel. Kambly mentioned the symbols ⧺ and ǂ for parallel. Bolyai used ||| . Karsten used ⋕ . Baker employed the sign ⥊. Haseler employed ⋕ as "the sign of parallelism of two lines or surfaces.(33)

There were other symbols used to represent pentagons, hexagons, segments, straight lines, and solids. Herigone used 5< for pentagon and 6< for hexagon. He also used ⌓ for segment and — for a straight line. Pictographs for solids are very rare. Usually those symbols were completely shaded. ■ was used for cube, ▲ was used for pyramid, ▰ for parallelopiped, and ▬ for a rectangular parallelopiped. These symbols were difficult to draw and as a result not widely used.

There were many symbols and many uses for them in geometry but there were some symbols used in astrology to indicate roughly the relative positions of two heavenly bodies with respect to an observer. Thus ☌ , ☍ , □ , △ , * designated, respectively, conjunction, opposition, at right angles, at 120°, and at 60°.

Other symbols used were those for similarity and congruence.

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33. Ibid., pp. 407-413.

There was much confusion as to who actually used what. Early, Leibniz used $\backsim$, $\simeq$ and $|\simeq|$ for congruent. Later he used $\sim$ for similar and $\simeq$ for congruent. Wolf was the first to use explicitly $\backsim$ and $=$ for congruence, but he did not combine the two into one symbolism. In Klugel's book one reads; "$\backsim$ with English and French authors means difference", "with German authors $\backsim$ is the sign of similarity". The earliest uses of $\sim$ and $\cong$ for "similar" and "congruent" in the United States was by G. A. Hill and Halsted. Alan Sanders used the $\sim$ for "is (or are) measured by". Hudson and Lipka used the sign $\approxeq$ for "equals approximately".

Occasionally the sign $\equiv$ was used for congruent. It was first introduced by Riemann, and later used by many writers. Jordan employed the same symbol to express equivalence. Hopkin's Geometry used $\equiv$ or $\asymp$ to express "equivalent to". The symbol $\not\backsim$ was adopted at one time by a member of the University of Cambridge, to express, "is similar to" in an edition of Euclid.

There were symbols devised for the lettering of triangles. Carnot marked points with capital letters A,B,C,..., $\overline{AB}$ for segment AB, $\overset{\frown}{ABCD}$ marked four points on a circular arc, $=\!|\!=$ for identity of two objects, $\triangle$ ABC for the triangle having vertices A,B,C, ⊿ ABC for a right triangle, and $\overline{\overline{ABC}}$ for the area of the triangle ABC.

In 1866, Reye proposed the use of capital letters, A,B,C,..., for points; the small letters a,b,c,..., for lines; α, β, γ,..., for planes. This notation has been adopted by Favaro and others.

Hart and Feldman's Geometry used the sign $\underline{\alpha}$ for "is measured by". Shutt used the sign I for the same meaning. Veronese employed $\equiv|\equiv$ to mark "not equal" line segments. Kambly used a horizontal line drawn underneath an equation for "therefore". An example is as follows:

$$\frac{\angle r+q = s+q}{\angle r = s}$$

The use of Algebraic symbols in the solution of geometric problems began at the very time when the symbols themselves were introduced. The solving of geometric problems created a need for algebraic symbols. The use of algebraic symbolism in applied geometry is seen in the writings of Pacioli, Tartaglia, Cardan, Bombelli, Widman, Stevin, and many others since the sixteenth century.

Usually printed works which contained pictographs had also algebraic symbols. However, Barrow's Euclid contained algebraic symbols in superabundance, but no pictographs.

The case was different in works containing a systematic development of geometric theory. The geometric works of Euclid, Archimedes, and others did not employ algebraic symbolism. Not until the seventeenth century, in the writings of Herigone in France, and Oughtred, Wallis, and Barrow in England, was there a formal translation of the geometric classics of antiquity into the language of symbolic algebra.(34)

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34. Ibid., pp. 415-425.

CONCLUSION

There are many symbols written in this paper. The purpose of so many is to show the different representations before a definite one was settled upon. In a few instances, this has not been done yet today, but there are only a few exceptions.

The symbols used in mathematics are a language. They are a language in the sense that they are capable of relaying ideas. They are words written in a different form and have a definite meaning. A trained person in mathematics upon seeing an equal sign immediately associates two quantities being equal. The same is true of all the symbols used in mathematics.

The need for a standard form of symbols was seen by the early writers. The number systems were introduced to replace pictographs. The need for simple symbols is obvious when trying to divide or multiply large numbers. A system has to be used with the digits in a certain place meaning a certain thing.

The development of zero, for example, enabled the Hindus to write large numbers with little difficulty. For the Greeks, the symbol replaced the word vacant. Few people recognize the importance of this symbol. In my opinion, many people take its use for granted and do not know by whom or how it was invented. I think the same could be said of many of the other symbols that many of us use every day.

It is hoped that this paper has caused one to appreciate the symbolism used in mathematics. It should have brought out, at least intuitively, some of the struggles that must have existed before an agreement was made upon a definite meaning for a symbol.

APPENDIX

The following is a list of symbols that are used in Mallory's Plane Geometry book. These same symbols are also used in other books.

SYMBOL	MEANING
$=$	is equal to; equals
$\neq$	is not equal to
$\sim$	is similar to
$\cong$	is congruent to; congruent
$>$	is greater than
$<$	is less than
$\parallel$	is parallel to; parallel
$\perp$	is perpendicular to
$\therefore$	therefore
$\ldots$	and so on
$\angle$	angle
$\angle$s	angles
$\triangle$	triangle
$\triangle$s	triangles
▱	parallelogram
▱s	parallelograms
▭	rectangle
▭s	rectangles

$\odot$	circle
$\circledS$	circles
$\frown$	arc

This list includes most of the symbols used in todays geometry books.

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