Convergence of the Crank-Nicolson-Galerkin finite element method for a class of nonlocal parabolic systems with moving boundaries

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February 3, 2014

Abstract

The aim of this paper is to establish the convergence and error bounds to the fully discrete solution for a class of nonlinear systems of reaction-diffusion nonlocal type with moving boundaries, using a linearized Crank-Nicolson-Galekin finite element method with polynomial approximations of any degree.

A coordinate transformation which fixes the boundaries is used. Some numerical tests to compare our Matlab code with some existence moving finite elements methods are investigated.

keywords: nonlinear parabolic system, nonlocal diffusion term, reaction-diffusion, convergence, numerical simulation, Crank-Nicolson, finite element method.

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1 Introduction

In this work, we study parabolic systems with nonlocal nonlinearity of the following type

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - a_i\left(\displaystyle\int_{\Omega_t} u_1(x,t)dx, \dots, \int_{\Omega_t} u_{ne}(x,t)dx\right) \dfrac{\partial^2 u_i}{\partial x^2} = f_i(x,t), & (x,t) \in Q_t \\ u_i(\alpha(t),t) = u_i(\beta(t),t) = 0, \quad t > 0 \\ u_i(x,0) = u_{i0}(x), \quad x \in \Omega_0 =]\alpha(0),\beta(0)[, \quad i = 1,\dots,ne \end{cases} \tag{1}$$

where Q_t is a bounded non-cylindrical domain defined by

$$Q_t = \left\{(x,t) \in \mathbb{R}^2 : \alpha(t) < x < \beta(t), \text{ for all } 0 < t < T\right\}.$$

Problem (1) is nonlocal in the sense that the diffusion coefficient is determined by a global quantity, that is, a depends on the whole population in the area and it arises in a large class of real models. For example, in biology, where the solution u could describe the density of a population subject to spreading; or in physics, where u could represent the temperature, considering that the measurements are an average in the neighbourhood [8].

This class of problems with nonlocal coefficient in an open bounded cylindrical domain was initially studied by Chipot and Lovat in [9], where they proved the existence and uniqueness of weak solutions. In recent years nonlinear parabolic equations with nonlocal diffusion terms have been extensively studied [10, 1, 13, 7, 11, 12, 14, 25], especially in relation to questions of existence, uniqueness and asymptotic behaviour.

If we want to model interactions then we need to use a system. Raposo et al. [19], in 2008, studied the existence, uniqueness and exponential decay of solutions for reaction-diffusion coupled systems of the form

$$\begin{cases} u_t - a(l(u))\Delta u + f(u-v) = \alpha(u-v) & \text{in} \quad \Omega \times]0,T], \\ v_t - a(l(v))\Delta v - f(u-v) = \alpha(v-u) & \text{in} \quad \Omega \times]0,T], \end{cases}$$

with $a(\cdot) > 0$, l a continuous linear form, f a Lipschitz-continuous function and α a positive parameter. Recently, Duque et al. [16] considered nonlinear systems of parabolic equations with a more general nonlocal diffusion term working on two linear forms l_1 and l_2:

$$\begin{cases} u_t - a_1(l_1(u), l_2(v))\Delta u + \lambda_1|u|^{p-2}u = f_1(x,t) & \text{in} \quad \Omega \times]0,T], \\ v_t - a_2(l_1(u), l_2(v))\Delta v + \lambda_2|v|^{p-2}v = f_2(x,t) & \text{in} \quad \Omega \times]0,T]. \end{cases} \tag{2}$$

They gave important results on polynomial and exponential decay, vanishing of the solutions in finite time and localization properties such as waiting time effect.

Moving boundary problems occur in many physical applications involving diffusion, such as in heat transfer where a phase transition occurs, in moisture

transport such as swelling grains or polymers, and in deformable porous media problems where the solid displacement is governed by diffusion, (see for example, [18, 3, 22, 5]). Cavalcanti et al [6] worked with a time-dependent function $a = a\left(t, \int_{\Omega_t} |\nabla u(x,t)|^2 \, dx\right)$ to establish the solvability and exponential energy decay of the solution for a model given by a hyperbolic-parabolic equation in an open bounded subset of $\mathbb{R}^n$, with moving boundary. Santos et al. [23] established the exponential energy decay of the solutions for nonlinear coupled systems for beam equations with memory in noncylindrical domains. Recently, Robalo et al. [21] proved the existence and uniqueness of weak and strong global in time solutions and gave conditions, on the data, for these solutions to have the exponential decay property. The analysis and numerical simulation of such problems presents other challenges. In [1], Ackleh and Ke propose a finite difference scheme to approximate the solutions and to study their long time behavior. The authors also made numerical simulations, using an implicit finite difference scheme in one dimension [19] and the finite volume discretization in two space dimensions [17]. Bendahmane and Sepulveda [4] in 2009 investigated the propagation of an epidemic disease modeled by a system of three PDE, where the ith equation is of the type

$$(u_i)_t - a_i \left(\int_\Omega u_i dx \right) \Delta u_i = f_i \left(u_1, u_2, u_3 \right),$$

in a physical domain $\Omega \subset \mathbb{R}^n$, $(n = 1, 2, 3)$. They established the existence of solutions to finite volume scheme and its convergence to the weak solution of the PDE. In [15] the authors proved the optimal order of convergence for a linearized Euler-Galerkin finite element method to problem (2) and presented some numerical results. Almeida et al., in [2], established the convergence and error bounds of the fully discrete solutions for a class of nonlinear equations of reaction-diffusion nonlocal type with moving boundaries, using a linearized Crank-Nicolson-Galerkin finite element method with polynomial approximations of any degree. In [20], Robalo et al. proved the existence and uniqueness of a strong regular solution for a certain class of a nonlinear coupled system of reaction-diffusion equations on a bounded domain with moving boundary. The exponential decay of the energy of the solutions, under the same assumptions, was also proved. In addition, they obtained approximate numerical solutions for systems of this type with a Matlab code based on the Moving Finite Element Method (MFEM) with high degree local approximations.

This paper is concerned with the proof of the convergence of a total discrete solution using the Crank-Nicolson-Galekin finite element method. To the best of our knowledge, these results are new for nonlocal reaction-diffusion systems with moving boundaries.

The paper is organized as follows. In Section 2, we formulate the problem and the hypotheses on the data. In Section 3, we define and prove the convergence of the semidiscrete solution. Section 4 is devoted to the proof of the convergence to a fully discrete solution. In Section 5, we obtain approximate numerical solutions for some examples. To finalize this study, in Section 6, we draw some

conclusions.

2   Statement of the problem

In what follows, we study the convergence of a linearized Crank-Nicolson-Galerkin finite element method to the solutions of the one-dimensional Dirichlet problem with two moving boundaries, defined by

$$
\begin{cases}
\dfrac{\partial u_i}{\partial t} - a_i \left(\displaystyle\int_{\Omega_t} u_1(x,t)dx, \ldots, \int_{\Omega_t} u_{ne}(x,t)dx \right) \dfrac{\partial^2 u_i}{\partial x^2} = f_i(x,t), \ (x,t) \in Q_t \\
u_i(\alpha(t), t) = u_i(\beta(t), t) = 0, \quad t > 0 \\
u_i(x, 0) = u_{i0}(x), \quad x \in \Omega_0 =]\alpha(0), \beta(0)[, \quad i = 1, \ldots, ne
\end{cases}
\tag{3}
$$

where

$$
Q_t = \left\{ (x,t) \in \mathbb{R}^2 : \ \alpha(t) < x < \beta(t), \ \text{for all } 0 < t < T \right\}
$$

is a bounded non-cylindrical domain, T is an arbitrary positive real number and a_i denotes a positive real function. The lateral boundary of Q_t is given by $\Sigma_t = \bigcup_{0 \le t < T} (\{\alpha(t), \beta(t)\} \times \{t\})$. Moreover, we assume that $\alpha'(t) < 0$ and $\beta'(t) > 0$, for all $t \in [0, T]$. Note that the hypotheses $\alpha'(t) < 0$ and $\beta'(t) > 0$ imply that Q_t is increasing, in the sense that if $t_2 > t_1$, then the projection of $[\alpha(t_1), \beta(t_1)]$ onto the subspace $t = 0$ is contained in the projection of $[\alpha(t_2), \beta(t_2)]$ onto the same subspace. This also means that the real function $\gamma(t) = \beta(t) - \alpha(t)$ is increasing on $0 \le t < T$.

In [20] Robalo et al. established the existence, uniqueness and asymptotic behaviour of strong regular solutions for these problems using a coordinate transformation which fixes the boundaries. They used the fact that, when (x,t) varies in Q_t, the point (y, t) of $\mathbb{R}^2$, with $y = (x - \alpha(t))/\gamma(t)$, varies in the cylinder $Q =]0, 1[\times]0, T[$. Thus, the function $\tau : Q_t \longrightarrow Q$ given by $\tau(x, t) = (y, t)$, is of class $\mathcal{C}^2$. The inverse τ^{-1} is also of class $\mathcal{C}^2$. The change of variable $v(y, t) = u(x, t)$ and $g(y, t) = f(x, t)$ with $x = \alpha(t) + \gamma(t) y$ transforms problem (3) into problem (4), given by

$$
\begin{cases}
\dfrac{\partial v_i}{\partial t} - a_i(l(v_1), \ldots, l(v_{ne})) \, b_2(t) \dfrac{\partial^2 v_i}{\partial y^2} - b_1(y, t) \dfrac{\partial v_i}{\partial y} = g_i(y, t), \quad (y, t) \in Q \\
v_i(0, t) = v_i(1, t) = 0, \quad t > 0 \\
v_i(y, 0) = v_{i0}(y), \quad y \in \Omega =]0, 1[, \quad i = 1, \ldots, ne
\end{cases}
\tag{4}
$$

where $l(v) = \gamma(t) \int_0^1 v(y, t) \, dy$, $g_i(y, t) = f_i(\alpha + \gamma y, t)$ and $v_{i0}(y) = u_{i0}(\alpha(0) + \gamma(0) y)$. The coefficients $b_1(y, t)$ and $b_2(t)$ are defined by

$$
b_1(y, t) = \frac{\alpha'(t) + \gamma'(t)y}{\gamma(t)} \quad \text{and} \quad b_2(t) = \frac{1}{(\gamma(t))^2}.
$$

Since we need the existence and uniqueness of a strong solution in Q_t, we

consider the same hypotheses as in [20], namely:

$(H1)$ $\alpha,\ \beta \in \mathcal{C}^2\,([0,T])$ and $0 < \gamma_0 < \gamma(t) < \gamma_1 < \infty$, for all $t \in [0,T]$

$(H2)$ $\alpha',\ \beta' \in L_1\,(]0,T[) \cap L_2\,(]0,T[)$

$(H3)$ $u_{i0} \in H_0^1\,(\Omega_0)$, $\Omega_0 =]\alpha(0), \beta(0)[$, $i = 1, \ldots, ne$,

$(H4)$ $f_i \in L_2\,(0,T; L_2\,(\Omega_t)) \cap L_1\,(0,T; L_2\,(\Omega_t))$,
      $\Omega_t =]\alpha(t), \beta(t)[$, $i = 1, \ldots, ne$,

$(H5)$ $a_i : \mathbb{R}^{ne} \longrightarrow \mathbb{R}^+$ is Lipschitz-continuous
      with $0 < m_a \leq a_i(s) \leq M_a$, for all $s \in \mathbb{R}$, $i = 1, \ldots, ne$.

Let $\Omega =]0,1[$. The definition of a weak solution is as follows.

Definition 2.1 (Weak solution). *We say that the function* $\mathbf{v} = (v_1, \ldots, v_{ne})$ *is a weak solution of problem (4) if, for each* $i \in \{1, \ldots, ne\}$,

$$v_i \in L_\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega)), \frac{\partial v_i}{\partial t} \in L_2(0,T; L_2(\Omega)), \tag{5}$$

the following equality in $D'(0,T)$ *is valid for all* $w_i \in H_0^1(\Omega)$, *and* $t \in]0,T[$,

$$\int_0^1 \frac{\partial v_i}{\partial t} w_i\, dy + a_i(l(v_1), \ldots, l(v_{ne})) b_2 \int_0^1 \frac{\partial v_i}{\partial y} \frac{\partial w_i}{\partial y}\, dy - \int_0^1 b_1 \frac{\partial v_i}{\partial y} w_i\, dy = \int_0^1 g_i w_i\, dy \tag{6}$$

and

$$v_i(x,0) = v_{i0}(x), \quad x \in \Omega \tag{7}$$

Henceforth, we assume that $\mathbf{v}$ has the regularity needed to perform all the calculations which follow.

3   Semidiscrete solution

We denote the usual L_2 norm in Ω by $\|.\|$ and the norm in $H^k(\Omega)$ by $\|.\|_{H^k}$. Let $\mathcal{T}_h$ denote a partition of Ω into disjoint intervals T_i, $i = 1, \ldots, nt$ such that $h = \max\{diam(T_i), i = 1, \ldots, nt\}$. Now let S_h^k denote the continuous functions on the closure $\bar{\Omega}$ of Ω which are polynomials of degree k in each interval of $\mathcal{T}_h$ and which vanish on $\partial\Omega$, that is,

$$S_h^k = \{W \in C_0^0(\bar{\Omega}) | W_{|T_i} \text{ is a polynomial of degree } k \text{ for all } T_i \in \mathcal{T}_h\}.$$

If $\{\varphi_j\}_{j=1}^{np}$ is a basis for S_h^k, then we can represent each $W \in S_h^k$ as

$$W = \sum_{j=1}^{np} w_j \varphi_j.$$

Given a smooth function u on Ω, which vanishes on $\partial\Omega$, we may define its interpolant, denoted by $I_h u$, as the function of S_h^k which coincides with u at the points $\{P_j\}_{j=1}^{np}$, that is,

$$I_h u = \sum_{j=1}^{np} u(P_j) \varphi_j.$$

Lemma 3.1 ([24]). *If $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$, then*

$$\|I_h u - u\| + h\|\nabla(I_h u - u)\| \leq Ch^{k+1}\|u\|_{H^{k+1}}.$$

Definition 3.2 ([24] Ritz projection). *A function $\tilde{u} \in S_h^k$ is said to be the Ritz projection of $u \in H_0^1(\Omega)$ onto S_h^k if it satisfies*

$$\int_\Omega \nabla\tilde{u} \cdot \nabla W \, dy = \int_\Omega \nabla u \cdot \nabla W \, dy, \quad \text{for all } W \in S_h^k.$$

Lemma 3.3 ([24]). *If $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$, then*

$$\|\tilde{u} - u\| + h\|\nabla(\tilde{u} - u)\| \leq Ch^{k+1}\|u\|_{H^{k+1}},$$

where C does not depend on h or k.

The semidiscrete problem, based on Definition 2.1, consists in finding $\mathbf{V} = (V_1, \ldots, V_{ne})$ belonging to $(S_h^k)^{ne}$, for $t \geq 0$, such that for all $\mathbf{W} = (W_1, \ldots, W_{ne}) \in (S_h^k)^{ne}$ and $t \in]0, T[$:

$$\begin{cases} \displaystyle\int_0^1 \frac{\partial V_i}{\partial t} W_i dy + a(l(V_1), \ldots, l(V_{ne})) b_2 \int_0^1 \frac{\partial V_i}{\partial y}\frac{\partial W_i}{\partial y} \, dy - \int_0^1 b_1 \frac{\partial V_i}{\partial y} W_i \, dy \\ \\ \hspace{6cm} = \displaystyle\int_0^1 g_i W_i \, dy \\ V_i(y, 0) = I_h v_{i0}, \quad i = 1, \ldots, ne \end{cases} .$$

(8)

Theorem 3.4. *If $\mathbf{v}$ is the solution of problem (4) and $\mathbf{V}$ is the solution of (8), then*

$$\|V_i - v_i\| \leq Ch^{k+1}, \quad t \in]0, T], \quad i = 1, \ldots, ne$$

where C does not depend on h, k or i.

Proof. Let $e_i = V_i - v_i$ be written as

$$e_i(y, t) = (V_i(y, t) - \tilde{V}_i(y, t)) + (\tilde{V}_i(y, t) - v_i(y, t)) = \theta_i(y, t) + \rho_i(y, t),$$

with $\tilde{V}_i^{(h)}(y, t) \in S_h^k$ being the Ritz projection of v_i. Then

$$\|e_i(y, t)\| \leq \|\theta_i(y, t)\| + \|\rho_i(y, t)\|$$

and, by lemma 3.3, it follows that

$$\|\rho_i(y, t)\| \leq Ch^{k+1}\|v_i\|_{H^{k+1}}$$

Next, we determine an upper limit for $\|\theta_i(y, t)\|$. Let

$$a_i^{(h)} = a_i(l(V_1), \ldots, l(V_{ne})).$$

Then, for every $i \in \{1, \ldots, ne\}$, we have that

$$\int_0^1 \frac{\partial \theta_i}{\partial t} W_i dy + a_i^{(h)} b_2 \int_0^1 \frac{\partial \theta_i}{\partial y} \frac{\partial W_i}{\partial y} dy - \int_0^1 b_1 \frac{\partial \theta_i}{\partial y} W_i dy$$

$$= \int_0^1 \frac{\partial V_i}{\partial t} W_i dy + a_i^{(h)} b_2 \int_0^1 \frac{\partial V_i}{\partial y} \frac{\partial W_i}{\partial y} dy - \int_0^1 b_1 \frac{\partial V_i}{\partial y} W_i dy$$

$$- \int_0^1 \frac{\partial \tilde{V}_i}{\partial t} W_i dy - a_i^{(h)} b_2 \int_0^1 \frac{\partial \tilde{V}_i}{\partial y} \frac{\partial W_i}{\partial y} dy + \int_0^1 b_1 \frac{\partial \tilde{V}_i}{\partial y} W_i dy$$

$$= \int_0^1 g_i W_i dy - \int_0^1 \frac{\partial v_i}{\partial t} W_i dy - a_i b_2 \int_0^1 \frac{\partial \tilde{V}_i}{\partial y} \frac{\partial W_i}{\partial y} dy + \int_0^1 b_1 \frac{\partial v_i}{\partial y} W_i dy$$

$$+ (a_i - a_i^{(h)}) b_2 \int_0^1 \frac{\partial \tilde{V}_i}{\partial y} \frac{\partial W_i}{\partial y} dy + \int_0^1 b_1 (\frac{\partial \tilde{V}_i}{\partial y} - \frac{\partial v_i}{\partial y}) W_i dy$$

$$+ \int_0^1 (\frac{\partial v_i}{\partial t} - \frac{\partial \tilde{V}_i}{\partial t}) W_i dy$$

$$= (a_i - a_i^{(h)}) b_2 \int_0^1 \frac{\partial \tilde{V}_i}{\partial y} \frac{\partial W_i}{\partial y} dy + \int_0^1 b_1 (\frac{\partial \tilde{V}_i}{\partial y} - \frac{\partial v_i}{\partial y}) W_i dy$$

$$+ \int_0^1 (\frac{\partial v_i}{\partial t} - \frac{\partial \tilde{V}_i}{\partial t}) W_i dy.$$

If we consider $W_i = \theta_i$, then

$$\int_0^1 \frac{\partial \theta_i}{\partial t} \theta_i dy + a_i^{(h)} b_2 \int_0^1 \left(\frac{\partial \theta_i}{\partial y} \right)^2 dy - \int_0^1 b_1 \frac{\partial \theta_i}{\partial y} \theta_i dy$$

$$= (a_i - a_i^{(h)}) b_2 \int_0^1 \frac{\partial \tilde{V}_i}{\partial y} \frac{\partial \theta_i}{\partial y} dy + \int_0^1 b_1 \frac{\partial \rho_i}{\partial y} \theta_i dy - \int_0^1 \frac{\partial \rho_i}{\partial t} \theta_i dy.$$

Integrating by parts the third term on the left side and the second term on the right side of the above equation, we obtain

$$\int_0^1 \frac{1}{2} \frac{d}{dt} \theta_i^2 dy + a_i^{(h)} b_2 \int_0^1 \left(\frac{\partial \theta_i}{\partial y} \right)^2 dy + \frac{\gamma'(t)}{2\gamma(t)} \int_0^1 \theta_i^2 dy$$

$$= (a_i - a_i^{(h)}) b_2 \int_0^1 \frac{\partial \tilde{V}_i}{\partial y} \frac{\partial \theta_i}{\partial y} dy - \int_0^1 \frac{\partial \rho_i}{\partial t} \theta_i dy - \frac{\gamma'(t)}{\gamma(t)} \int_0^1 \rho_i \theta_i dy - \int_0^1 b_1 \rho_i \frac{\partial \theta_i}{\partial y} dy.$$

Taking the absolute value of the right side of this equation, ignoring the third term on the left side and considering the lower limits of a and b_i, it follows that

$$\frac{1}{2} \frac{d}{dt} \|\theta_i\|^2 + \frac{m_a}{\gamma_1^2} \left\| \frac{\partial \theta_i}{\partial y} \right\|^2$$

$$\leq \left| a_i - a_i^{(h)} \right| \frac{1}{\gamma_0^2} \int_0^1 \left| \frac{\partial \tilde{V}_i}{\partial y} \right| \left| \frac{\partial \theta_i}{\partial y} \right| dy + \int_0^1 \left| \frac{\partial \rho_i}{\partial t} \right| |\theta_i| \, dy + \frac{\gamma'_{\max}}{\gamma_0} \int_0^1 |\rho_i| \, |\theta_i| \, dy$$

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$$+\frac{\alpha'_{\max}+\gamma'_{\max}}{\gamma_0}\int_0^1|\rho_i|\left|\frac{\partial\theta_i}{\partial y}\right|dy$$

$$\leq C_1\left|a_i-a_i^{(h)}\right|^2+\frac{m_a}{2\gamma_1^2}\int_0^1\left|\frac{\partial\theta_i}{\partial y}\right|^2dy+\frac{1}{2}\int_0^1\left|\frac{\partial\rho_i}{\partial t}\right|^2dy+\frac{1}{2}\int_0^1|\theta_i|^2\,dy$$

$$+\frac{\gamma'_{\max}}{2\gamma_0}\int_0^1|\rho_i|^2\,dy+\frac{\gamma'_{\max}}{2\gamma_0}\int_0^1|\theta_i|^2\,dy+C_2\int_0^1|\rho_i|^2\,dy+\frac{m_a}{2\gamma_1^2}\int_0^1\left|\frac{\partial\theta_i}{\partial y}\right|^2dy.$$

Then, by (H5) we have that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\theta_i\|^2 &\leq C_3\sum_{j=1}^{ne}\|\rho_j\|^2+C_4\sum_{j=1}^{ne}\|\theta_j\|^2+\frac{1}{2}\left\|\frac{\partial\rho_i}{\partial t}\right\|^2+\frac{1}{2}\|\theta_i\|^2\\
&\quad+\frac{\gamma'_{\max}}{2\gamma_0}\|\rho_i\|^2+\frac{\gamma'_{\max}}{2\gamma_0}\|\theta_i\|^2+C_2\|\rho_i\|^2\\
&\leq C\sum_{j=1}^{ne}\|\theta_j\|^2+C\sum_{j=1}^{ne}\|\rho_j\|^2+\frac{1}{2}\left\|\frac{\partial\rho_i}{\partial t}\right\|^2.
\end{aligned}
$$

and hence, we obtain

$$\frac{d}{dt}\left(\sum_{i=1}^{ne}\|\theta_i\|^2\right)\leq C\sum_{i=1}^{ne}\|\theta_i\|^2+C\sum_{i=1}^{ne}\|\rho_i\|^2+\sum_{i=1}^{ne}\left\|\frac{\partial\rho_i}{\partial t}\right\|^2.$$

Applying Gronwall's Theorem, we arrive at the inequality

$$\sum_{i=1}^{ne}\|\theta_i\|^2\leq C\sum_{i=1}^{ne}\|\theta_i(y,0)\|^2+C\sum_{i=1}^{ne}\int_0^T\|\rho_i\|^2+\left\|\frac{\partial\rho_i}{\partial t}\right\|^2dt.$$

By the hypothesis of the theorem, we have, for every $i\in 1,\ldots,ne,$

$$
\begin{aligned}
\|\theta_i(y,0)\|^2 &\leq \|e_i(y,0)\|^2=\|V_i(y,0)-v_{i0}\|^2\leq Ch^{2(k+1)}\|v_{i0}\|_{H^{k+1}}^2,\\
\int_0^T\|\rho_i\|^2\,dt &\leq CTh^{2(k+1)}\|v_i\|_{H^{k+1}}^2,\\
\int_0^T\left\|\frac{\partial\rho_i}{\partial t}\right\|^2dt &\leq CTh^{2(k+1)}\left\|\frac{\partial v_i}{\partial t}\right\|_{H^{k+1}}^2
\end{aligned}
$$

and so

$$\sum_{i=1}^{ne}\|\theta_i\|^2\leq C\left(\sum_{i=1}^{ne}\|v_{i0}\|_{H^{k+1}}^2+\sum_{i=1}^{ne}\|v_i\|_{H^{k+1}}^2+\sum_{i=1}^{ne}\left\|\frac{\partial v_i}{\partial t}\right\|_{H^{k+1}}^2\right)h^{2(k+1)}.$$

Hence

$$\|\theta_i\|\leq Ch^{k+1},\quad i=1,\ldots ne$$

and adding the estimate of ρ_i, we obtain the desired result. $\qquad\square$

4 Discrete problem

Let $\delta > 0$ and consider the partition $]0, T] = \overset{n_i-1}{\underset{j=1}{\cup}}]t_{j-1}, t_j] = \overset{n_i-1}{\underset{j=1}{\cup}} I_j$, $\delta = t_j - t_{j-1}$ and $int(I_j) \cap int(I_i) = \emptyset$. The time discretization is made utilizing the Crank-Nicolson method. Let $\mathbf{V}^{(n)}(y)$ be the approximation of $\mathbf{v}(y, t_n)$, in the space $(S_h^k)^{ne}$. This method evaluates the equation at the points $t_{n-1/2} = \frac{t_n + t_{n-1}}{2}$, $n = 1, \ldots, ni$, and uses the approximations

$$\mathbf{V}(y, t_{n-1/2}) \approx \frac{\mathbf{V}^{(n)}(y) + \mathbf{V}^{(n-1)}(y)}{2} = \hat{\mathbf{V}}^{(n)}(y)$$

and

$$\frac{\partial \mathbf{V}}{\partial t}(y, t_{n-1/2}) \approx \frac{\mathbf{V}^{(n)}(y) - \mathbf{V}^{(n-1)}(y)}{\delta} = \overline{\partial}\mathbf{V}^{(n)}(y).$$

Then we have the problem of finding $\mathbf{V}^{(n)} \in (S_h^k)^{ne}$ such that it is zero on the boundary of Ω, satisfies $V_i^{(0)} = I_h(v_{i0})$, $i = 1, \ldots, ne$, and

$$\int_0^1 \overline{\partial}V_i^{(n)} W_i \ dy + a_i(l(\hat{V}_1^{(n)}), \ldots, l(\hat{V}_{ne}^{(n)})) b_2^{(n-1/2)} \int_0^1 \frac{\partial \hat{V}_i^{(n)}}{\partial y} \frac{\partial W_i}{\partial y} \ dy$$

$$- \int_0^1 b_1^{(n-1/2)} \frac{\partial \hat{V}_i^{(n)}}{\partial y} W_i \ dy = \int_0^1 g_i^{(n-1/2)} W_i \ dy, \qquad (9)$$

with $f^{(n-1/2)} = f(y, t_{n-1/2})$.

System (9) is a non linear algebraic system due to the presence of $a_i(l(\hat{V}_1^{(n)}), \ldots, l(\hat{V}_{ne}^{(n)}))$. Obtaining the solution of (9) implies the use of an iterative method in each time step. We could apply Newton's method, the fixed point method or some secant method, but it would be very time consuming. In order to avoid this, we choose a linearization method and, as suggested in [24], we substitute $\hat{V}_i^{(n)}$ with $\overline{V}_i^{(n)} = \frac{3}{2}V_i^{(n-1)} - \frac{1}{2}V_i^{(n-2)}$ in the diffusion coefficient. So the totally discrete problem, in this case, will be to calculate the functions $\mathbf{V}^{(n)}$, $n \geq 2$, belonging to $(S_h^k)^{ne}$, which are zero on the boundary of Ω and satisfy

$$\int_0^1 \overline{\partial}V_i^{(n)} W_i \ dy + a_i(l(\overline{V}_1^{(n)}), \ldots, l(\overline{V}_{ne}^{(n)})) b_2^{(n-1/2)} \int_0^1 \frac{\partial \hat{V}_i^{(n)}}{\partial y} \frac{\partial W_i}{\partial y} \ dy$$

$$- \int_0^1 b_1^{(n-1/2)} \frac{\partial \hat{V}_i^{(n)}}{\partial y} W_i \ dy = \int_0^1 g_i^{(n-1/2)} W_i \ dy, \quad n \geq 2, \quad i = 1, \ldots, ne. \ (10)$$

In this way, we have a linear multistep method which requires two initial estimates $\mathbf{V}^{(0)}$ and $\mathbf{V}^{(1)}$. The estimate $\mathbf{V}^{(0)}$ is obtained by the initial condition as $V_i^{(0)} = I_h(v_{i0})$. In order to calculate $\mathbf{V}^{(1)}$ with the same accuracy, we follow [24] and we use the following predictor-corrector scheme.

$$\int_0^1 \frac{V_i^{(1,0)} - V_i^{(0)}}{\delta} W_i \ dy + a_i(l(V_1^{(0)}), \ldots, l(V_{ne}^{(0)})) b_2^{(1/2)}$$

$$\times \int_0^1 \frac{\partial}{\partial y}\left(\frac{V_i^{(1,0)} + V_i^{(0)}}{2}\right)\frac{\partial W_i}{\partial y}\ dy - \int_0^1 b_1^{(1/2)}\frac{\partial}{\partial y}\left(\frac{V_i^{(1,0)} + V_i^{(0)}}{2}\right)W_i\ dy$$

$$= \int_0^1 g_i^{(1/2)}W_i\ dy, \quad i = 1,\ldots,ne. \tag{11}$$

$$\int_0^1 \overline{\partial}V_i^{(1)}W_i\ dy + a_i\left(l\left(\frac{V_1^{(1,0)} + V_1^{(0)}}{2}\right),\ldots,l\left(\frac{V_{ne}^{(1,0)} + V_{ne}^{(0)}}{2}\right)\right)b_2^{(1/2)}$$

$$\times \int_0^1 \frac{\partial \hat{V}_i^{(1)}}{\partial y}\frac{\partial W_i}{\partial y}\ dy - \int_0^1 b_1^{(1/2)}\frac{\partial \hat{V}_i^{(1)}}{\partial y}W_i\ dy = \int_0^1 g_i^{(1/2)}W_i\ dy, \quad i = 1,\ldots,ne. \tag{12}$$

Theorem 4.1. *If* $\mathbf{v}$ *is the solution of equation (4) and* $\mathbf{V}^{(n)}$ *is the solution of (10)-(12), then*

$$\|V_i^{(n)}(y) - v_i(y,t_n)\| \le C(h^{k+1} + \delta^2), \quad n = 1,\ldots,nt, \quad i = 1,\ldots,ne,$$

where C *does not depend on* h, k *or* δ.

Proof. First we will determine the estimate for $n = 1$. Let $\theta_i^{(1,0)} = V_i^{(1,0)} - \tilde{v}_i^{(1)}$, $\hat{\theta}_i^{(1,0)} = \frac{\theta_i^{(1,0)} + \theta_i^{(0)}}{2}$ and $\overline{\partial}\theta_i^{(1,0)} = \frac{\theta_i^{(1,0)} - \theta_i^{(0)}}{\delta}$. We have that

$$\int_0^1 \overline{\partial}\theta_i^{(1,0)}W_i\ dy + a_i(l(V_1^{(0)}),\ldots,l(V_{ne}^{(0)}))b_2^{(1/2)}\int_0^1 \frac{\partial \hat{\theta}_i^{(1,0)}}{\partial y}\frac{\partial W_i}{\partial y}\ dy$$

$$- \int_0^1 b_1^{(1/2)}\frac{\partial \hat{\theta}_i^{(1,0)}}{\partial y}W_i\ dy$$

$$= \int_0^1 \overline{\partial}V_i^{(1,0)}W_i\ dy + a_i(l(V_1^{(0)}),\ldots,l(V_{ne}^{(0)}))b_2^{(1/2)}\int_0^1 \frac{\partial \hat{V}_i^{(1,0)}}{\partial y}\frac{\partial W_i}{\partial y}\ dy$$

$$- \int_0^1 b_1^{(1/2)}\frac{\partial \hat{V}_i^{(1,0)}}{\partial y}W_i\ dy - \int_0^1 \overline{\partial}\tilde{v}_i^{(1)}W_i\ dy$$

$$- a_i(l(V_1^{(0)}),\ldots,l(V_{ne}^{(0)}))b_2^{(1/2)}\int_0^1 \frac{\partial \hat{\tilde{v}}_i^{(1)}}{\partial y}\frac{\partial W_i}{\partial y}\ dy + \int_0^1 b_1^{(1/2)}\frac{\partial \hat{\tilde{v}}_i^{(1)}}{\partial y}W_i\ dy$$

$$= \int_0^1 g_i^{(1/2)}W_i\ dy - \int_0^1 \left(\frac{\partial v_i}{\partial t}\right)^{(1/2)}W_i\ dy - a_i(l(v_1^{(1/2)}),\ldots,l(v_{ne}^{(1/2)}))b_2^{(1/2)}$$

$$\times \int_0^1 \frac{\partial v_i^{(1/2)}}{\partial y}\frac{\partial W_i}{\partial y}\ dy + \int_0^1 b_1^{(1/2)}\frac{\partial v_i^{(1/2)}}{\partial y}W_i\ dy + \int_0^1 \left(\left(\frac{\partial v_i}{\partial t}\right)^{(1/2)} - \overline{\partial}\tilde{v}_i^{(1)}\right)W_i\ dy$$

$$+ b_2^{(1/2)}\int_0^1 \left(a_i(l(v_1^{(1/2)}),\ldots,l(v_{ne}^{(1/2)}))\frac{\partial v_1^{(1/2)}}{\partial y}\right.$$

$$\left. - a_i(l(V_1^{(0)}),\ldots,l(V_{ne}^{(0)}))\frac{\partial \hat{v}_i^{(1)}}{\partial y}\right)\frac{\partial W_i}{\partial y}\ dy + \int_0^1 b_1^{(1/2)}\left(\frac{\partial \hat{\tilde{v}}_i^{(1)}}{\partial y} - \frac{\partial v_i^{(1/2)}}{\partial y}\right)W_i\ dy$$

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$$= \int_0^1 \left(\left(\frac{\partial v_i}{\partial t} \right)^{(1/2)} - \overline{\partial} \tilde{v}_i^{(1)} \right) W_i \ dy + a_i(l(v_1^{(1/2)}), \ldots, l(v_{ne}^{(1/2)})) b_2^{(1/2)}$$

$$\times \int_0^1 \left(\frac{\partial v_i^{(1/2)}}{\partial y} - \frac{\partial \hat{v}_i^{(1)}}{\partial y} \right) \frac{\partial W_i}{\partial y} \ dy + \left(a_i(l(v_1^{(1/2)}), \ldots, l(v_{ne}^{(1/2)})) \right.$$

$$\left. - a_i(l(V_1^{(0)}), \ldots, l(V_{ne}^{(0)})) \right) b_2^{(1/2)} \int_0^1 \frac{\partial \hat{v}_i^{(1)}}{\partial y} \frac{\partial W_i}{\partial y} \ dy$$

$$+ \int_0^1 b_1^{(1/2)} \left(\frac{\partial \hat{\tilde{v}}_i^{(1)}}{\partial y} - \frac{\partial v_i^{(1/2)}}{\partial y} \right) W_i \ dy.$$

Setting $W_i = \hat{\theta}_i^{(1,0)}$, we arrive at

$$\int_0^1 \overline{\partial} \theta_i^{(1,0)} \hat{\theta}_i^{(1,0)} \ dy + a_i(l(V_1^{(0)}), \ldots, l(V_{ne}^{(0)})) b_2^{(1/2)} \int_0^1 \left(\frac{\partial \hat{\theta}_i^{(1,0)}}{\partial y} \right)^2 \ dy$$

$$- \int_0^1 b_1^{(1/2)} \frac{\partial \hat{\theta}_i^{(1,0)}}{\partial y} \hat{\theta}_i^{(1,0)} \ dy$$

$$= \int_0^1 \left(\left(\frac{\partial v_i}{\partial t} \right)^{(1/2)} - \overline{\partial} \tilde{v}_i^{(1)} \right) \hat{\theta}_i^{(1,0)} \ dy$$

$$+ a_i(l(v_1^{(1/2)}), \ldots, l(v_{ne}^{(1/2)})) b_2^{(1/2)} \int_0^1 \left(\frac{\partial v_i^{(1/2)}}{\partial y} - \frac{\partial \hat{v}_i^{(1)}}{\partial y} \right) \frac{\partial \hat{\theta}_i^{(1,0)}}{\partial y} \ dy$$

$$+ \left(a_i(l(v_1^{(1/2)}), \ldots, l(v_{ne}^{(1/2)})) - a_i(l(V_1^{(0)}), \ldots, l(V_{ne}^{(0)})) \right) b_2^{(1/2)}$$

$$\times \int_0^1 \frac{\partial \hat{v}_i^{(1)}}{\partial y} \frac{\partial \hat{\theta}_i^{(1,0)}}{\partial y} \ dy + \int_0^1 b_1^{(1/2)} \left(\frac{\partial \hat{\tilde{v}}_i^{(1)}}{\partial y} - \frac{\partial v_i^{(1/2)}}{\partial y} \right) \hat{\theta}_i^{(1,0)} \ dy.$$

Applying integration by parts and hypothesis H_1 and H_2, it follows that

$$- \int_0^1 b_1^{(1/2)} \frac{\partial \hat{\theta}_i^{(1,0)}}{\partial y} \hat{\theta}_i^{(1,0)} \ dy = - \int_0^1 b_1^{(1/2)} \frac{1}{2} \frac{\partial (\hat{\theta}_i^{(1,0)})^2}{\partial y} \ dy$$

$$= \frac{(\gamma')^{(1/2)}}{2\gamma^{(1/2)}} \int_0^1 (\hat{\theta}_i^{(1,0)})^2 \ dy \geq 0$$

and

$$\int_0^1 b_1^{(1/2)} \left(\frac{\partial \hat{\tilde{v}}_i^{(1)}}{\partial y} - \frac{\partial v_i^{(1/2)}}{\partial y} \right) \hat{\theta}_i^{(1,0)} \ dy$$

$$= -\frac{(\gamma')^{(1/2)}}{\gamma^{(1/2)}} \int_0^1 \left(\hat{\tilde{v}}_i^{(1)} - v_i^{(1/2)} \right) \hat{\theta}_i^{(1,0)} \ dy - \int_0^1 b_1^{(1/2)} (\hat{\tilde{v}}_i^{(1)} - v_i^{(1/2)}) \frac{\partial \hat{\theta}_i^{(1,0)}}{\partial y} \ dy.$$

Then, by the Poincaré and Hölder inequalities, we can conclude that

$$\frac{1}{2}\overline{\partial}\|\theta_i^{(1,0)}\|^2 + \frac{m_a}{\gamma_0^2}\left\|\frac{\partial\hat{\theta}_i^{(1,0)}}{\partial y}\right\|^2 \le$$

$$\le C\left(\left\|\left(\frac{\partial v_i}{\partial t}\right)^{(1/2)} - \overline{\partial}\tilde{v}_i^{(1)}\right\| + \left\|\frac{\partial v_i^{(1/2)}}{\partial y} - \frac{\partial\hat{v}_i^{(1)}}{\partial y}\right\| + \sum_{j=1}^{ne}\|v_j^{(1/2)} - V_j^{(0)}\|\right.$$

$$\left. + \|\hat{\tilde{v}}_i^{(1)} - v_i^{(1/2)}\|\right)\left\|\frac{\partial\hat{\theta}_i^{(1,0)}}{\partial y}\right\|.$$

Using Cauchy's inequality, we have that

$$\overline{\partial}\|\theta_i^{(1,0)}\|^2 \le C\left(\left\|\left(\frac{\partial v_i}{\partial t}\right)^{(1/2)} - \overline{\partial}\tilde{v}_i^{(1)}\right\| + \left\|\frac{\partial v_i^{(1/2)}}{\partial y} - \frac{\partial\hat{v}_i^{(1)}}{\partial y}\right\|\right.$$

$$\left. + \sum_{j=1}^{ne}\|v_j^{(1/2)} - V_j^{(0)}\| + \|\hat{\tilde{v}}_i^{(1)} - v_i^{(1/2)}\|\right).$$

The following estimates are true for every $i \in \{1,\dots,ne\}$,

$$\left\|\left(\frac{\partial v_i}{\partial t}\right)^{(1/2)} - \overline{\partial}\tilde{v}_i^{(1)}\right\| \le \left\|\left(\frac{\partial v_i}{\partial t}\right)^{(1/2)} - \overline{\partial}v_i^{(1)}\right\| + \|\overline{\partial}v_i^{(1)} - \overline{\partial}\tilde{v}_i^{(1)}\|$$

$$\le C\delta^2 + Ch^{k+1},$$

$$\left\|\frac{\partial v_i^{(1/2)}}{\partial y} - \frac{\partial\hat{v}_i^{(1)}}{\partial y}\right\| \le C\delta\int_{t_0}^{t_1}\left\|\frac{\partial^3 v_i}{\partial y\partial t^2}\right\|\,dt \le C\delta^2,$$

$$\|v_i^{(1/2)} - V_i^{(0)}\| \le \|v_i^{(1/2)} - v_i^{(0)}\| + \|v_i^{(0)} - V_i^{(0)}\| \le C\delta + Ch^{k+1},$$

and

$$\|\hat{\tilde{v}}_i^{(1)} - v_i^{(1/2)}\| \le \|\hat{\tilde{v}}_i^{(1)} - \hat{\tilde{v}}_i^{(1/2)}\| + \|\hat{\tilde{v}}_i^{(1/2)} - v_i^{(1/2)}\| \le C\delta^2 + Ch^{k+1}.$$

Hence

$$\overline{\partial}\|\theta_i^{(1,0)}\|^2 \le C(h^{k+1} + \delta)^2,$$

and, we have the estimate

$$\|\theta_i^{(1,0)}\|^2 \le \|\theta_i^{(0)}\|^2 + C\delta(h^{k+1} + \delta)^2 \le C(h^{2(k+1)} + \delta^3), \quad i = 1,\dots,ne.$$

Repeating this process for equation (12), we arrive at

$$\frac{1}{2}\overline{\partial}\|\theta_i^{(1)}\|^2 + \frac{m_a}{\gamma_0^2}\left\|\frac{\partial\hat{\theta}_i^{(1)}}{\partial y}\right\|^2$$

$$\leq C\left(\left\|\left(\frac{\partial v_i}{\partial t}\right)^{(1/2)} - \overline{\partial}\tilde{v}_i^{(1)}\right\| + \left\|\frac{\partial v_i^{(1/2)}}{\partial y} - \frac{\partial\hat{v}_i^{(1)}}{\partial y}\right\|\right.$$

$$\left. + \sum_{j=1}^{ne}\left\|v_j^{(1/2)} - \frac{V_j^{(1,0)} - V_j^{(0)}}{2}\right\| + \|\hat{v}_i^{(1)} - v_i^{(1/2)}\|\right)\left\|\frac{\partial\hat{\theta}_i^{(1)}}{\partial y}\right\|.$$

In this case, we use the estimate

$$\left\|v_i^{(1/2)} - \frac{V_i^{(1,0)} - V_i^{(0)}}{2}\right\| \leq \|v_i^{(1/2)} - \hat{v}_i^{(1)}\| + \|\hat{v}_i^{(1)} - \frac{V_i^{(1,0)} - V_i^{(0)}}{2}\|$$

$$\leq \|v_i^{(1/2)} - \hat{v}_i^{(1)}\| + \frac{1}{2}\|\theta_i^{(1,0)}\| + \frac{1}{2}\|\theta_i^{(0)}\|$$

$$\leq C(h^{k+1} + \delta^2) + Ch^{k+1} + C(h^{k+1} + \delta^{\frac{3}{2}})$$

$$\leq C(h^{k+1} + \delta^{\frac{3}{2}}),$$

and then, by Cauchy's inequality, we conclude that

$$\overline{\partial}\|\theta_i^{(1)}\|^2 \leq C(h^{2(k+1)} + \delta^3),$$

whence

$$\|\theta_i^{(1)}\|^2 \leq \|\theta_i^{(0)}\|^2 + C\delta(h^{2(k+1)} + \delta^3) \leq C(h^{2(k+1)} + \delta^4).$$

To conclude the proof, we obtain the result for $n \geq 2$, applying the same process to equation (10). In this way, we obtain

$$\frac{1}{2}\overline{\partial}\|\theta_i^{(n)}\|^2 + \frac{m_a}{\gamma_0^2}\left\|\frac{\partial\hat{\theta}_i^{(n)}}{\partial y}\right\|^2$$

$$\leq C\left(\left\|\left(\frac{\partial v_i}{\partial t}\right)^{(n-1/2)} - \overline{\partial}\tilde{v}_i^{(n)}\right\| + \left\|\frac{\partial v_i^{(n-1/2)}}{\partial y} - \frac{\partial\hat{v}_i^{(n)}}{\partial y}\right\|\right.$$

$$\left. + \sum_{j=1}^{ne}\left\|v_j^{(n-1/2)} - \bar{V}_j^{(n)}\right\| + \|\hat{v}_i^{(n)} - v_i^{(n-1/2)}\|\right)\left\|\frac{\partial\hat{\theta}_i^{(n)}}{\partial y}\right\|.$$

Now, we need the estimate

$$\left\|v_i^{(n-1/2)} - \bar{V}_i^{(n)}\right\| \leq \|v_i^{(n-1/2)} - \bar{v}_i^{(n)}\| + \|\bar{v}_i^{(n)} - \bar{V}_i^{(n)}\|$$

$$\leq \|v_i^{(n-1/2)} - \bar{v}_i^{(n)}\| + \|\bar{\rho}_i^{(n)}\| + \|\bar{\theta}_i^{(n)}\|$$

$$\leq C\delta^2 + Ch^{k+1} + C(\|\theta_{n-1}\| + \|\theta_{n-2}\|)$$

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to prove that

$$\overline{\partial}\|\theta_i^{(n)}\|^2 \le C\sum_{j=1}^{ne}\|\theta_j^{(n-1)}\|^2 + C\sum_{j=1}^{ne}\|\theta_j^{(n-2)}\|^2 + C(h^{(k+1)}+\delta^2)^2, \quad i=1,\dots,ne.$$

Summing up for all i, it follows that

$$\overline{\partial}\sum_{i=1}^{ne}\|\theta_i^{(n)}\|^2 \le C\sum_{j=1}^{ne}\|\theta_j^{(n-1)}\|^2 + C\sum_{j=1}^{ne}\|\theta_j^{(n-2)}\|^2 + C(h^{(k+1)}+\delta^2)^2.$$

Iterating, we obtain

$$\sum_{i=1}^{ne}\|\theta_i^{(n)}\|^2 \le (1+C\delta)\sum_{i=1}^{ne}\|\theta_i^{(n-1)}\|^2 + C\delta\sum_{i=1}^{ne}\|\theta_i^{(n-2)}\|^2 + C\delta(h^{k+1}+\delta^2)^2$$

$$\le C\sum_{i=1}^{ne}\|\theta_i^{(1)}\|^2 + C\sum_{i=1}^{ne}\delta\|\theta_i^{(0)}\|^2 + C\delta(h^{k+1}+\delta^2)^2$$

and recalling the estimates for $\|\theta_i^{(0)}\|$, $\|\theta_i^{(1)}\|$ and $\|\rho_i^{(n)}\|$, the proof is complete. $\qquad\square$

5   Examples

The final step is to implement this method using a programming language. To perform this task, we choose the Matlab environment.

In this section, we present some examples to illustrate the applicability and robustness of this method, comparing the results with the theoretical results proved and with the results presented in [20].

5.1   Example 1

As a first example we simulate a problem with a known exact solution, which will permit us to calculate the error and confirm numerically the theoretical convergence rates. Let us consider problem (3) with two equations in Q_t and $T=3$. The diffusion coeficientes are

$$a_1(r,s) = 2 - \frac{1}{1+r^2} + \frac{1}{1+s^2}, \quad a_2(r,s) = 3 + \frac{2}{1+r^2} - \frac{1}{1+s^2},$$

the movement of the boundaries is given by the functions

$$\alpha(t) = -\frac{t}{1+t}, \quad \beta(t) = 1 + \frac{2t}{1+t},$$

the functions $f_1(x,t)$, $f_2(x,t)$, $u_{10}(x,t)$ and $u_{20}(x,t)$ are chosen such that

$$u_1(x,t) = \frac{1}{t+1}\left(\frac{611}{70}z - \frac{10513}{210}z^2 + \frac{646}{7}z^3 - \frac{1070}{21}z^4\right)$$

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and
$$u_2(x,t) = e^{-t}\left(\frac{2047}{140}z - \frac{27701}{420}z^2 + \frac{691}{7}z^3 - \frac{995}{21}z^4\right)$$

with
$$z = \frac{(2t+1)(x+tx+t)}{5t^2+5t+1}$$
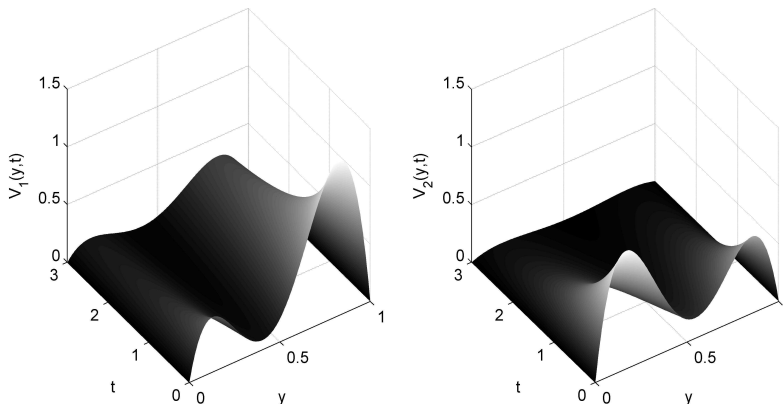
are the exact solutions.

Figure 1: Evolution in time of the approximated solution in the fixed boundary problem for v_1 (left) and v_2 (right).

The picture on the left in Figure 1 illustrates the evolution in time of the obtained solution for v_1 in the fixed boundary problem, and the picture on the right illustrates the evolution in time of the obtained solution for v_2. This solution was calculated with approximations of degree two and $h = \delta = 10^{-2}$.

The pictures in Figure 2 represent the obtained solutions in the moving boundary domain, after applying the inverse transformation $\tau^{-1}(y,t)$. In this case u and v could represent the density of two populations of bacteria. We observe that, initially, each population is concentrated mainly in two regions and, as time increases, the two populations decrease and spread out in the domain, as expected.

In order to analyze the convergence rates, this problem was simulated with different combinations of k, h and δ and the error results are represented in Figure 3.

The error was calculated in $t = T$ and using the $L_2(\alpha(T), \beta(T))$-norm in the space variable. In the picture on the left the logarithms of the errors versus the logarithm of h for the simulations done with $\delta = 10^{-4}$ and approximations of degree 2, are represented. The errors versus the logarithm of h for the simulations done with $\delta = 10^{-4}$ and approximations of degree 3 are represented in the picture in the center. The logarithms of the errors versus the logarithm of δ for
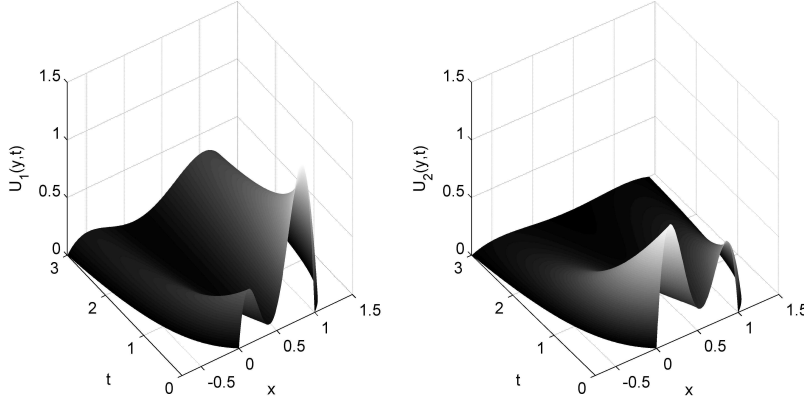
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Figure 2: Evolution in time of the approximated solution in the moving boundary problem for u_1 (left) and u_2 (right).

the simulations done with $h = 10^{-3}$ and approximations of degree 2, are represented in the picture on the right. As expected, the pictures are in accordance with the orders of convergence for h and δ, as was proved in Theorem 4.1. In Table 1 we compare the error of the present method with the error of the moving finite element method presented in [20]. Both simulations were done with approximations of degree five and four finite elements. We used $\delta = 10^{-4}$ for the present method and 10^{-10} for the integrator's error tolerance in the moving finite element method.

t_i	$\max\limits_{j=1,\ldots,np} \{|u_1(P_j, t_i) - U_1^{(i)}(P_j)|\}$		$\max\limits_{j=1,\ldots,np} \{|u_2(P_j, t_i) - U_2^{(i)}(P_j)|\}$	
	MFEM[20]	present	MFEM[20]	present
0.001	7.3025e-08	2.6502e-10	4.2464e-08	6.3606e-10
0.005	8.9490e-08	1.0338e-09	5.2037e-08	1.4457e-09
0.01	2.7945e-08	1.4645e-09	1.6249e-08	1.8010e-09
0.02	1.3320e-08	1.8424e-09	7.7437e-09	2.0228e-09
0.05	7.2664e-08	2.0530e-09	4.2203e-08	2.1597e-09
0.5	1.9044e-08	1.0564e-09	1.0743e-08	1.0907e-09
1	2.1230e-08	5.0614e-10	9.3304e-09	5.5859e-10

Table 1: Comparison of the present method with the moving finite element method in [20]
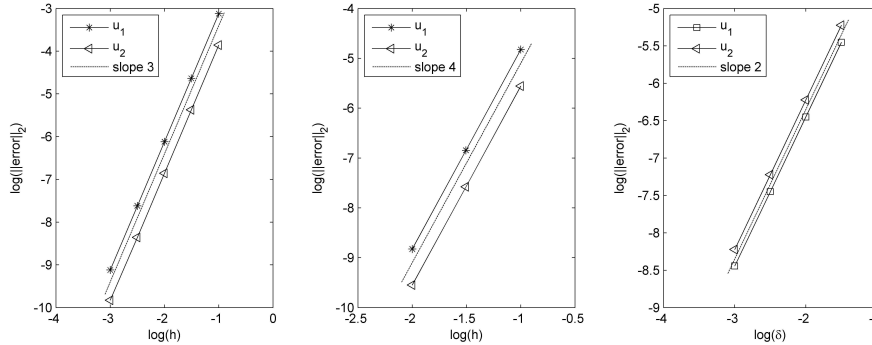
Figure 3: Study of the convergence for h with approximations of degree 2 (left) and 3 (center), and for δ (right).

5.2   Example 2

As a second example, we choose to simulate the second example presented in [20]. This will permit us to compare the present method with an adaptive one. Consider problem (3) with $ne = 2$ and Q_t defined by

$$\alpha(t) = \sqrt{2/3} - \sqrt[3]{t + (2/3)^{3/2}}, \quad \beta(t) = 1 - \alpha(t), \quad 0 \le t \le 1.$$

The diffusion coefficients are

$$a_1(r, s) = 2 - \frac{1}{1 + s^2}, \quad a_2(r, s) = e^{-r^2},$$

and the reaction forces are

$$f_1(x, t) = \frac{0.1x}{(1 + t)^4}, \quad f_2(x, t) = \frac{e^{-x^2}}{(1 + t)^6}.$$

The initial conditions u_{10} and u_{20} are the natural spline functions of degree three that interpolate the points $\{(0, 0), (0.2, 1), (0.5, 0.5), (1, 0)\}$ and $\{(0, 0), (0.6, 0.65), (0.8, 1), (1, 0)\}$, respectively. The approximate solutions were obtained with four finite elements ($h = 0.25$), $\delta = 10^{-3}$ and $k = 4$. The obtained solutions in the fixed domain are plotted in Figure 4.

The pictures in Figure 5 represent the obtained solutions in the moving boundary domain, after applying the inverse transformation $\tau^{-1}(y, t)$. In this example, initially, each population occupies mainly one region opposite from the other population. As the time increases the two populations expands to all the domain and decreases very quickly.

The pictures are similar to those in [20] and the numerical comparisons between the two methods show that the methods are similar. However, due to the fact that in [20] an adaptive mesh was used, initially the difference between the methods is greater in the areas where the solution has a higher slope, but this difference become less significant as time grows.
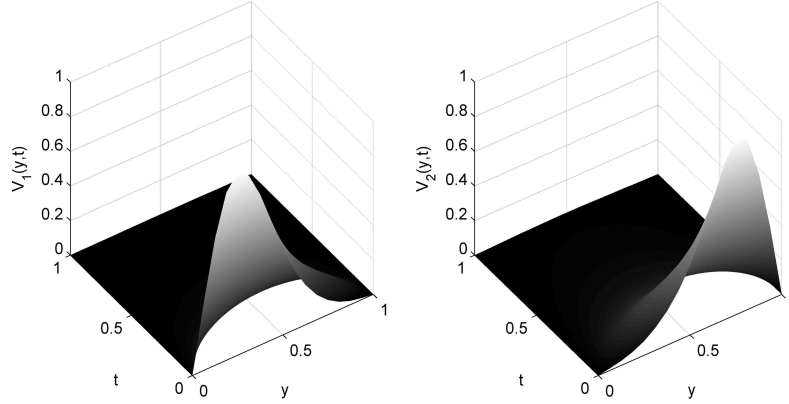
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Figure 4: Evolution in time of the approximated solution in the fixed boundary problem for v_1 (left) and v_2 (right).
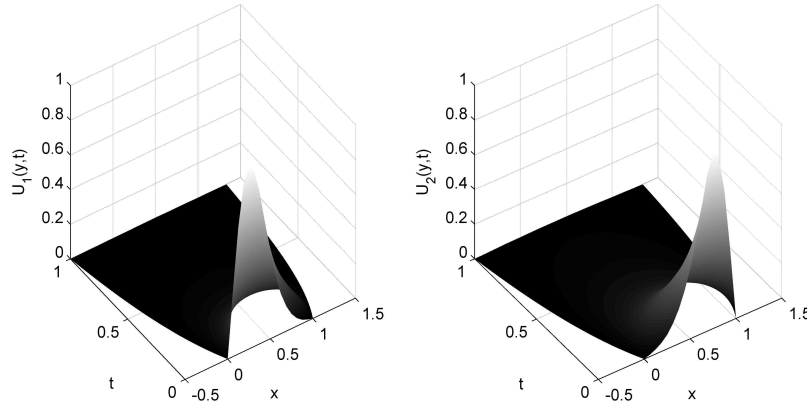

Figure 5: Evolution in time of the approximated solution in the moving boundary problem for u_1 (left) and u_2 (right).

6   Conclusions

We proved optimal rates of convergence for a linearized Crank-Nicolson-Galerkin finite element method with piecewise polynomial of arbitrary degree basis functions in space when applied to a system of nonlocal parabolic equations. Some numerical experiments were presented, considering different functions a, f, α and β. The numerical results are in accordance with the theoretical results and are similar in accuracy to results obtained by other methods.

Acknowledgements

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