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Spline parameterization based nonlinear trajectory optimization along 4D waypoints

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Abstract. Flight trajectory optimization has become an important factor not only to reduce the operational costs (e.g.,, fuel and time related costs) of the airliners but also to reduce the environmental impact (e.g.,, emissions, contrails and noise etc.) caused by the airliners. So far, these factors have been dealt with in the context of 2D and 3D trajectory optimization, which are no longer efficient. Presently, the 4D trajectory optimization is required in order to cope with the current air traffic management (ATM). This study deals with a cubic spline approximation method for solving 4D trajectory optimization problem (TOP). The state vector, its time derivative and control vector are parameterized using cubic spline interpolation (CSI). Consequently, the objective function and constraints are expressed as functions of the value of state and control at the temporal nodes, this representation transforms the TOP into nonlinear programming problem (NLP). The proposed method is successfully applied to the generation of a minimum length optimal trajectories along 4D waypoints, where the method generated smooth 4D optimal trajectories with very accurate results.

Keywords: trajectory optimization; 4D waypoint navigation; spline parameterization; Nonlinear programming

1. Introduction

Trajectory optimization is a vital area in the aeronautic industry. This technique enables generating optimal trajectories for flight vehicles with consideration of fuel consumption, travel time, obstacle avoidance, and many other requirements. In current air traffic management (ATM) system, the operational requirements of flights so far have been dealt with in the context of 2D and 3D trajectory optimization. Which is inefficient and far from being optimal, as a result, it is reaching its operational and technological limits.

Recently, trajectory based operation (TBO) has been initiated by the single European sky ATM research (SESAR) program, where the flight plan deals with 4D trajectory optimization. Which improves flight efficiency as well as enhances the air traffic capacity at terminal space and ground handling capacity. The difference between a 3D trajectory and a 4D trajectory is that a 3D

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trajectory consists of three-dimensional points of the physical space (i.e., longitude, latitude and altitude) whereas a 4D trajectory is composed of three-dimensional space and the time at which the vehicle is scheduled to pass at the point.

The trajectory optimization problem (TOP) or optimal control problem (OCP) can be solved by various kind of numerical methods. However, these methods can be separated into two basic approaches: the indirect approach and the direct approach (Bryson and Ho 1975, Betts 1998, Rao 2009). The TOP is solved by the Pontryagin maximum principle (PMP) in indirect approach (Pontryagin 1962, Naidu 2003, Athans and Falb 2006, Betts 2010), where the original TOP is converted into boundary value problem (BVP) by formulating the first order necessary condition for optimality which derived from PMP. Generally, the indirect approach leads to more accurate results than the direct approach. However, in general, a rather good initial approximation of the co-state equation is required in order to convergence, which is quite difficult to guess as the physical meaning of co-state equation is not well established (Stryk and Bulirsch 1992). Besides, the boundary value problems (BVPs) that arise for many practical trajectory optimization problems in indirect approach, are quite difficult to solve, because of complex dynamics and constraints structure of the problem. It also demands computationally intensive iterative numerical procedures. Generally, the TOP results in a two-point boundary value problem (TPBVP) for a single phase and a multi-point boundary value problem (MPBVP) for a multiple phase problem.

On the other hand, the direct approach is based on the transformation of the original TOP into nonlinear programming problem (NLP) (Hull 1997, Bazaraa *et al.* 2006). Which is done by discretizing the state and control of an infinite dimensional problem into a finite dimensional problem. The resulting NLP is then solved numerically by the well-established optimization techniques. At present, the direct methods are widely used for solving TOPs, since not only these methods do not require an analytic expression for the necessary conditions of optimality, which can be a daunting task for complicated nonlinear dynamics, but also they tend to have better convergence properties over indirect methods. Another great advantage of direct methods is that they do not need an initial guess of co-states, as it is difficult to predict the time histories. The parameterization techniques have an important role in convergence and accuracy of the solution. Generally, the direct methods vary for the variables to be parameterized (i.e., state and control), in some methods only the control is parameterized (e.g., direct shooting and direct multiple shooting) in others both the state and control are parameterized (e.g., direct collocation).

The most known direct approaches are based on the Runge-Kutta scheme (Schwartz and Polak 1996, Dontchev *et al.* 2000) and direct collocation methods (Hargraves and Paris 1987, Enright and Conway 1992). The direct collocation methods approximate the state and control by piecewise polynomials at collocation points. Recently, some works have been presented for higher nonlinear dynamic system called a pseudospectral method which is a class of direct collocation method (Tohidi *et al.* 2013). The most common pseudo-spectral methods for solving TOP are Chebyshev pseudospectral method (CPM) (Bousson 2003, Fahroo and Ross 2002, Guoand Zhu 2013) and Gauss pseudospectral method (GPM) (Benson *et al.* 2006, Ma *et al.* 2016, Zhao *et al.* 2014). CPM is based on the approximation of both state and control by interpolating orthogonal polynomials at the Chebyshev nodes. On the other hand, the GPM approximate the state and control by interpolating orthogonal polynomials at the Legendre–Gauss (LG) nodes. However experimental results show that the approximation of controls by higher order polynomials give rise to excessive wavy curves for the states.

Aside from direct and indirect methods, dynamic programming (DP) is another well-established method to solve TOP in spite of having limitations of the menace of expanding grid and the curse of dimensionality (Bellman 1957). (Miyamoto *et al.* 2013) shows that the DP can be successfully used to solve fuel efficient optimal trajectory. In the early 2000s, a class of DP called iterative dynamic programming (IDP) was proposed, which solve the menace of expanding grid problem and showed better performance than the tradition DP (Luss 2000). Later on, the IDP was extended to single gridpoint dynamic programming (SGDP) which can be used to solve online TOP with accuracy (Bousson 2005). Recently, moving search space dynamic programming (MS-DP) was presented (Miyazawa *et al.* 2013) to reduce the computation time of DP.

Although a lot of the researches have been done on solving 2D and 3D trajectory optimization problem (TOP) only a few have been done to solve the 4D TOP (Bousson and Machado 2010, Mazzotta et al. 2017, Hagelauer and Mora-Camino 1998, Miyazawa et al. 2013), where mostly the pseudospectral method and dynamic programming were used. In this present paper, the 4D TOP was solved by direct optimal control approach where, the state and control are parameterized by cubic polynomials, that transform the original TOP into Nonlinear programming problem (NLP). Which can be solved numerically using well-established optimization techniques. This proposed method is successfully applied to the generation of a minimum length optimal trajectories from predefined 4D waypoints.

The paper is organized as follows: section 2 states the problem to be solved, then section 3 represents the proposed method of cubic spline parameterization of the state, its time derivative and control. Modelling of 4D navigation problem and nonlinear programming formulation is described in section 4, section 5 demonstrates the simulation and results of the proposed method on optimal air navigation 4D trajectory generation. Finally, conclusions and future work directions are presented in section 6.

2. Problem formulation

The main goal of this paper is to develop a method for solving the 4D trajectory optimization problem (TOP), using cubic spline parameterization of the state and control vector based on direct optimal control approach and to validate the proposed method by generating a minimum length optimal trajectory along predefined 4D waypoints. An example of a 4D trajectory is given in Fig. 1.

Most of the approaches consider the waypoints defined by tri-dimensional coordinate positions $P_j = (\lambda_j, \varphi_j, h_j)^T \quad j = 0,1,\ldots,n$ and do not consider the arrival time at the waypoint. By adding the

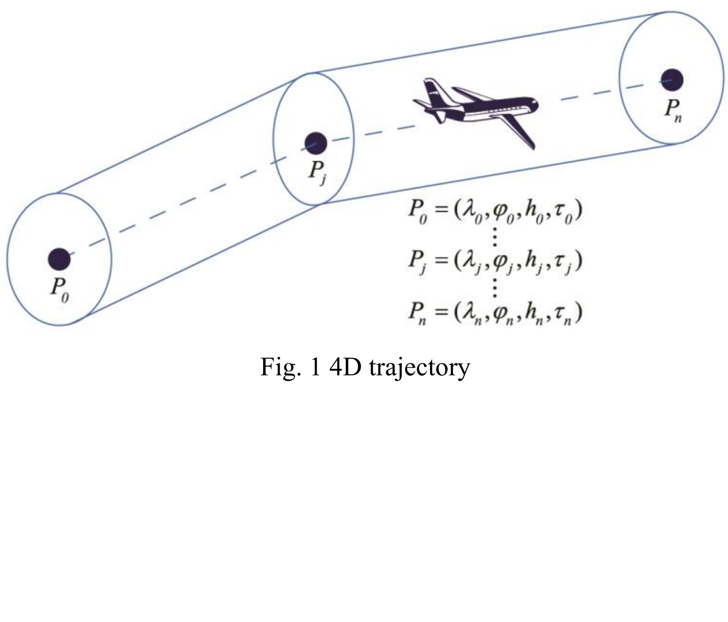

Fig. 1 4D trajectory

arrival time restriction to the tri-dimensional waypoint it is possible to define the 4D waypoints as $P_j = (\lambda_j, \varphi_j, h_j, \tau_j)^T$, where, $\lambda_j, \varphi_j, h_j, \tau_j$ are respectively longitude, latitude, altitude and arrival time at the waypoint P_j.

As trajectory generation requires a geocentric coordinates system, the 4D waypoints need to be transformed from the accustomed geodetic coordinate system to geocentric coordinates. Now to transform the geodetic coordinates $\lambda_j, \varphi_j, h_j$ the following equations need to be applied (Seemkooei 2002).

$$x_j = (N_j + h_j)\cos\varphi_j \cos\lambda_j \tag{1}$$

$$y_j = (N_j + h_j)\cos\varphi_j \sin\lambda_j \tag{2}$$

$$z_j = [N_j(1-e^2) + h_j]\sin\varphi_j \tag{3}$$

where, a is the Earth semi-major axis and e its eccentricity, the parameter N_j can be calculated as follows

$$N_j = \frac{\mathrm{a}}{\sqrt{1-e^2\sin^2\varphi_j}} \tag{4}$$

Now the 4D waypoints can be demonstrated in geocentric coordinates as follows

$$P_j = (x_j, y_j, z_j, \tau_j)^T \tag{5}$$

The problem to be solved is to navigate the aircraft along pre-defined 4D waypoints P_j (where, j=0,1,…,n) as in Eq. (5) from the initial waypoint P_0 to the final waypoint P_n while minimizing the total length of the trajectory. Next subsection describes the general trajectory optimization problem (TOP) formulation.

2.1 Trajectory optimization problem (TOP)

A trajectory optimization problem (TOP) is to optimize a measure of performance (e.g., minimum time, minimum fuel consumption, obstacle avoidance etc.) over a trajectory of a vehicle while satisfying a set of constraints.

Considering a nonlinear system whose dynamics is modelled by a set of ordinary differential equations, which are referred to as the state or system equations, the TOP can be formulated as follows

$$\dot{X}(t) = \mathrm{f}[t, X(t), U(t)] \tag{6}$$

With $t \in \mathbb{R}$ is the time (time $t \in [t_0, t_f]$ is the independent variable), $X(t) \in \mathbb{R}^n$ is the state vector, $U(t) \in \Omega \subset \mathbb{R}^m$ is the control vector, Ω a compact domain of feasible controls, $f : \mathbb{R} \times \mathbb{R}^n \times \Omega$, a vector-valued function. Both the state and control are time t dependent.

The general TOP is to find the control U^* and the corresponding state trajectory X^* that optimizes the following performance index

$$J = \Phi[t_0, X(t_0), t_f, X(t_f)] + \int_{t_0}^{t_f} L[t, X(t), U(t)]dt \tag{7}$$

where, Φ and L are functional, t_0 and t_f are respectively the initial and final time. When only the first term of the performance index is present is referred to as the Mayer form, Φ is the Mayer term. If the performance index only involves an integral as the second term it is referred to as a problem of Lagrange, L is the Lagrange term. When both of the terms are present as it is in Eq. (7) it is called a problem of Bolza.

This TOP may be subject to the equality c_{eq}, and the inequality c_{inq} constraints on the state and the control along the trajectory

$$c_{eq}[X(t),U(t)] = 0 \tag{8}$$

$$c_{inq}[X(t),U(t)] \le 0 \tag{9}$$

It may also be subject to nonlinear boundary condition Ψ, which enforces restrictions on the initial and final states of the system

$$\Psi_{\min} \le \Psi[t_0, X(t_0), t_f, X(t_f)] \le \Psi_{\max} \tag{10}$$

3. Proposed method

This study solves the 4D trajectory optimization problem (TOP) using cubic spline parameterization. In this proposed method, the time-dependent dynamic variables (the state variables $X(t)$, its time derivative $\dot{X}(t)$ and the control variables $U(t)$) are approximate and parameterized using cubic polynomials, that transcribes the TOP into Nonlinear programming problem (NLP), then an NLP solver is used to solve the resulting NLP and to determine the 4D optimal trajectory. The cubic spline interpolation (CSI) and the parameterization of state, its derivative and control by CSI is described in the next subsections.

3.1 Cubic spline interpolation (CSI)

The cubic spline interpolation (CSI) is a widely used piecewise-polynomial approximation that uses third-degree polynomials between each successive pair of nodes. The CSI is piecewise continuous which ensures that the cubic interpolant is not only continuously differentiable on the nodes but also has a continuous second derivative.

As to the mathematical spline, the points are numerical data. The weights are the interpolant coefficients on the third-degree polynomials used to interpolate the data. These coefficients shape the curve so that it smoothly passes through each of the data points (Burden and Faires 2011, Wang and Ouyang 2013, Marsden 1974).

An example of CSI is given in Fig. 2, with the free boundary conditions (i.e.,, when the second derivative of the interpolant is zero at the endpoints) the spline is also known as natural spline. The curve of the spline of Fig. 2 can be visualized as if a flexible rod was forced to go through some data points.

The essential idea is to fit a piecewise function of the form

$$S_j(t) = \begin{cases} S_0(t) & if \;\; t_0 \le t \le t_1 \\ S_1(t) & if \;\; t_1 \le t \le t_2 \\ \vdots & \\ S_{n-1}(t) & if \;\; t_{n-1} \le t \le t_n \end{cases}$$

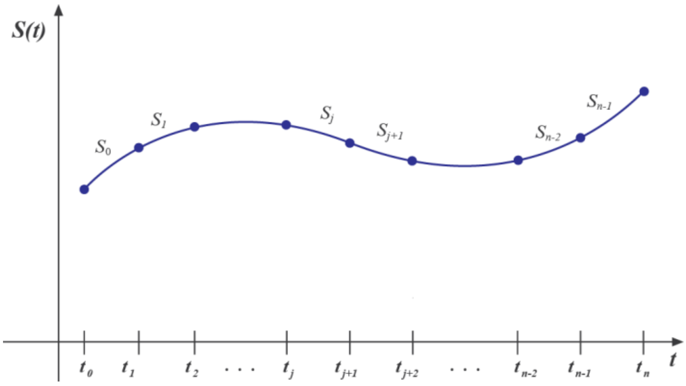

Fig. 2 Cubic spline interpolation (CSI)

where S_j is a third-degree polynomial of the form, for j=0,1,…n−1.

$$S_j(t) = a_j + b_j(t - t_j) + c_j(t - t_j)^2 + d_j(t - t_j)^3 \tag{11}$$

Given a function f defined on [a,b] and a set of nodes $a = t_0 < t_1 < \cdots < t_n = b$, a cubic spline interpolant S for f is a function that satisfies the following conditions:

(a) $S(t)$ is a cubic polynomial, denoted $S_j(t)$, on the subinterval [t_j,t_{j+1}] for each j=0,1,…n−1;

(b) $S_j(t_j) = f(t_j)$ and $S_j(t_{j+1}) = f(t_{j+1})$ for each j=0,1,…n−1;

(c) $S_{j+1}(t_{j+1}) = S_j(t_{j+1})$ for each j=0,1,…n−2;

(d) $S'_{j+1}(t_{j+1}) = S'_j(t_{j+1})$ for each j=0,1,…n−2;

(e) $S''_{j+1}(t_{j+1}) = S''_j(t_{j+1})$ for each j=0,1,…n−2;

(f) One of the following sets of boundary conditions is satisfied:

(i) $S''(t_0) = S''(t_n) = 0$ (natural (or free) boundary);

(ii) $S'(t_0) = f'(t_0)$ and $S'(t_n) = f'(t_n)$ (clamped boundary).

To construct a cubic spline between a pair of nodes, the 4 unknown interpolant coefficients a_j, b_j, c_j, and d_j need to be determined. A spline defined on an interval that is divided into n subintervals will require determining total $4n$ interpolant coefficients.

For simplicity, all the coefficients in Eq. (11) can be rewritten as follows

$$\begin{cases} a_j = S_j(t_j) = f(t_j) \\ b_j = \dfrac{(a_{j+1} - a_j)}{h_j} - \dfrac{h_j(2c_j + c_{j+1})}{3} \\ c_j = \dfrac{S''_j(t_j)}{2} \\ d_j = \dfrac{(c_{j+1} - c_j)}{3h_j} \end{cases} \tag{12}$$

where, $h_j = (t_{j+1} - t_j)$ is the interval with j=0,1,…n−1

3.2 State and control parameterization by cubic spline

According to the cubic spline interpolation (CSI) expression in Eq. (11), the state and the control vector can be parameterized respectively on the subinterval $t \in [t_j, t_{j+1}]$ for each j=0,1,…n−1

$$SX_j(t) = a_j + b_j(t-t_j) + c_j(t-t_j)^2 + d_j(t-t_j)^3 \tag{13}$$

$$SU_j(t) = a_j + b_j(t-t_j) + c_j(t-t_j)^2 + d_j(t-t_j)^3 \tag{14}$$

where, cubic spline interpolant $SX_j(t_j) = X(t_j)$ and $SU_j(t_j) = U(t_j)$, for the state and control vector $X(t_j) = X_j$ and $U(t_j) = U_j$. With the parameterization above, the derivative of the state vector, with respect to time can be approximated as

$$S\dot{X}_j(t) = b_j + 2c_j(t-t_j) + 3d_j(t-t_j)^2 \tag{15}$$

The interpolant coefficients a_j, b_j, c_j and d_j can be determined as in Eq. (12).

4. Modelling of 4D navigation problem

Normally, the system dynamics is modelled by a set of nonlinear equations of motion (EOMs). In this study, the three degrees of freedom (3DOF) EOMs are considered, where the state vector is represented by the position, velocity, flight path angle and heading of the flight vehicle.

The following differential equations are the dynamic model used to model the problem

$$\dot{x} = V\cos\gamma\cos\psi \tag{16}$$

$$\dot{y} = V\cos\gamma\sin\psi \tag{17}$$

$$\dot{z} = V\sin\gamma \tag{18}$$

$$\dot{V} = u_1 \tag{19}$$

$$\dot{\gamma} = u_2 \tag{20}$$

$$\dot{\psi} = u_3 \tag{21}$$

where, (x,y,z) are the geocentric coordinate system, the V, γ and ψ are respectively the velocity, flight path angle, and heading, the variables u_1,u_2 and u_3 are respectively the acceleration, the fight path angle rate and the heading rate. The state and control vector are respectively composed by $X = [x, y, z, V, \gamma, \psi]$ and $U = [u_1, u_2, u_3]$.

Due to aerodynamic, structural, and propulsive limitations, bound constraints are imposed on the state and control vectors as follows

$$V^{\min} \le V \le V^{\max} \tag{22}$$

$$\gamma^{\min} \le \gamma \le \gamma^{\max} \tag{23}$$

$$\psi^{\min} \le \psi \le \psi^{\max} \tag{24}$$

$$u_i^{\min} \le u_i \le u_i^{\max}, \quad i = 1,2,3 \tag{25}$$

4.1 Nonlinear programming formulation

As discussed in the previous section the proposed method transcribes the trajectory optimization problem (TOP) into a nonlinear programming problem (NLP). To do this the states, its derivatives, and control vectors are parameterized by the cubic spline interpolation (CSI) as shown in Eq. (13)-(15).

Let $X = [x, y, z, V, \gamma, \psi]$ and $U = [u_1, u_2, u_3]$ be the coefficient vectors of the state and control respectively. Then expressing the trajectory optimization problem with performance index in Eq. (7) and such that the constraints in Eq. (8)-(10) satisfy at nodes t_0, t_1,…t_f.. The performance index of the 4D TOP for minimum length trajectory can be re-stated as

$$\min J = \int_0^{t_f} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{26}$$

where, the performance index J is chosen to minimize the length of the parametric curve of the 4D trajectory, and it is subjected to

$$\begin{cases} \dot{x} - V\cos\gamma\cos\psi = 0 \\ \dot{y} - V\cos\gamma\sin\psi = 0 \\ \dot{z} - V\sin\gamma = 0 \end{cases} \tag{27}$$

Velocity bounded constraint

$$V^{\min} \le V \le V^{\max} \quad \text{(m/s)} \tag{28}$$

Flight path bounded constraint

$$\gamma^{\min} \le \gamma \le \gamma^{\max} \quad \text{(rad)} \tag{29}$$

Heading bounded constraint

$$\psi^{\min} \le \psi \le \psi^{\max} \quad \text{(rad)} \tag{30}$$

In this paper the problems are considered where, the initial time t_0 and final time t_f are specified as well as the initial and final state of the problem, as shown below

The initial conditions are

$$[x(t_0), y(t_0), z(t_0)] = [x_0,\ y_0,\ z_0] \tag{31}$$

The terminal conditions are

$$[x(t_f), y(t_f), z(t_f)] = [x_f,\ y_f,\ z_f] \tag{32}$$

5. Simulation and result

This section presents the simulation and results of two case studies. In the first example, the takeoff phase of a typical commercial flight and in the second example, the cruise phase of a typical commercial flight was considered.

In both example, the trajectory optimization problem (TOP) was transformed into nonlinear programming (NLP) using the spline parameterization as described in the previous sections. Then the NLP solver *fmincon* function of the optimization toolbox of Matlab was used to solve the resulting NLP problem. All the analysis of the simulation has been done using Matlab 2016[a].

5.1 Example 1

This subsection presents the simulation and results of the first case scenario, where takeoff phase of a commercial flight was considered. Typically, the takeoff phase consists of total four segments, where the first, second and the final segments are climb segments and the third segment is the acceleration segment. When the aircraft reaches 35 ft above ground the first segment starts and the final segment ends when the aircraft is at a height of 1500 ft above the takeoff surface. This four segments of the flight can be represented by 5 waypoints. Table 1 shows the lists of predefined 4D waypoints in a takeoff phase of flight. Each waypoint is defined in geocentric coordinates system (x, y, z) and the desired arrival time t to reach it.

In this example the proposed method is successfully applied to optimize the minimum length 4D trajectory of the takeoff phase, the results are shown in Table 2. The second column of Table 2 shows the distance d between pair of waypoints for the nonoptimal original trajectory which was generated by interpolation between waypoints, the third column shows the distance *d_opt* between a pair of waypoints for the optimized minimum length trajectory, where *d_opt* is the optimized objective function J (parametric curve length) as shown in Eq. (26), and the fourth column of the table shows the difference in distances between the original nonoptimal and optimal trajectory.

Table 1 List of waypoints of the trajectory in takeoff phase

Waypoints	$x[m]$	y[m]	z[m]	$t[s]$
1	4914538.255	-789644.997	3974723.887	0
2	4913533.576	-788427.950	3976620.174	32
3	4912274.232	-787258.533	3979033.179	70
4	4909846.671	-780366.673	3983355.480	177
5	4908361.577	-777232.470	3987054.724	240

Table 2 Distance between waypoints

Waypoints	$d[m]$	*d_opt* [m]	[*d−d_opt*] [m]
1 - 2	2469.413	2467.081	2.332
2 - 3	2966.246	2962.444	3.802
3 - 4	8566.032	8489.586	76.446
4 - 5	5092.662	5070.812	21.850
Total	19094.353	18989.923	104.430

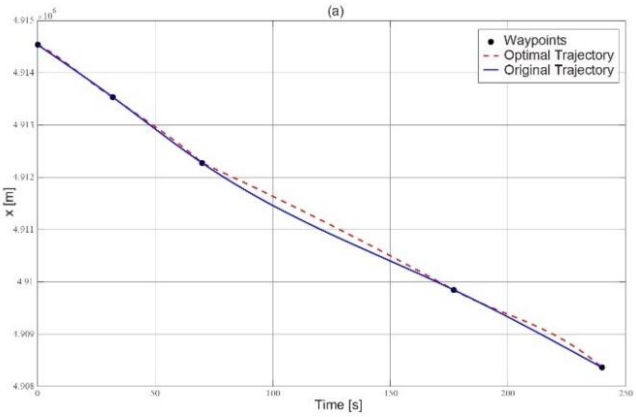

Fig. 3 x-axis vs time

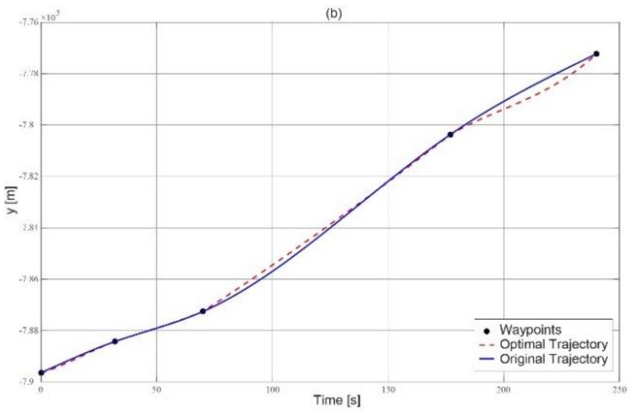

Fig. 4 y-axis vs time

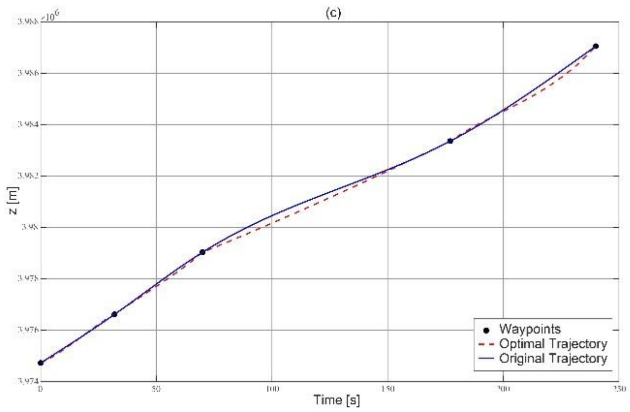

Fig. 5 z-axis vs time

Table 2 shows the total distance from initial to terminal waypoint for the nonoptimal original trajectory which is 19094.353 meters. It can be observed that the aircraft needs to fly 104.43 meters less if it follows the optimal minimum length trajectory which is 18989.923 meters in

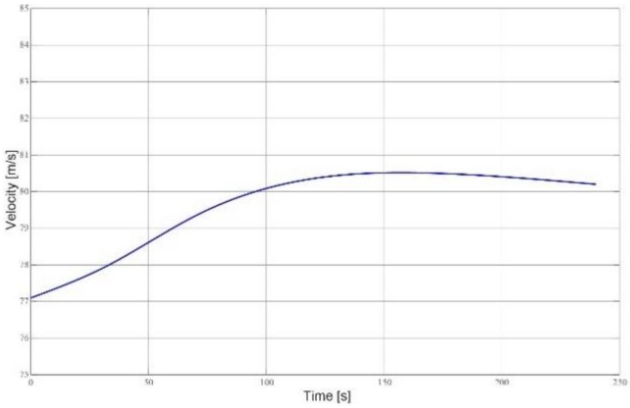

Fig. 6 Velocity vs time of optimal trajectory

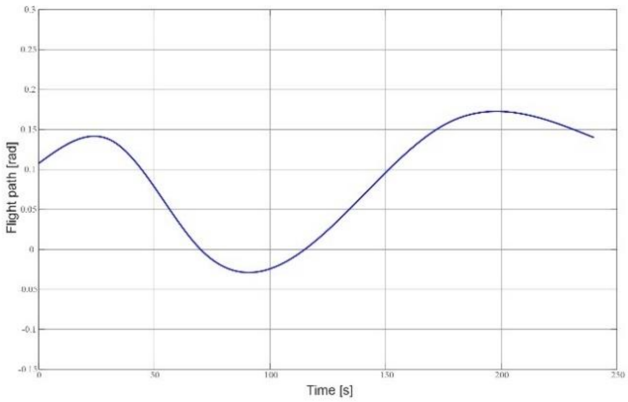

Fig. 7 Flight path angle vs time of optimal trajectory

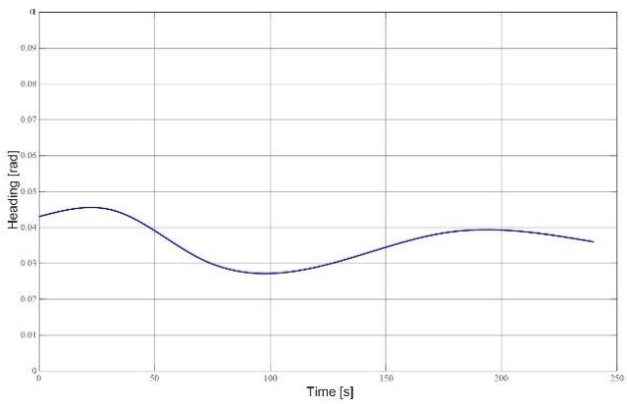

Fig. 8 Heading angle vs time of optimal trajectory

length. The optimal trajectory reduces the total distance by 0.55% over the nonoptimal original trajectory.

The Figs. 3-5 represent the nonoptimal original trajectory as the solid blue line and the optimal trajectory as the dashed red line for the geocentric x-axis, y-axis, and z-axis vs time, respectively. The solid black circles are the waypoints. It can be seen from the figures that the generated trajectories are smooth and they pass through all the predefined waypoints. They trajectories are also within the bounded constraints of aircraft limit as the trajectories were approximated by the cubic polynomials which not only ensures the continuity for the trajectory but also for its derivatives.

The Figs. 6-8 represent the velocity, flight path angle, and heading angle vs time of the minimum length optimal trajectory respectively. In these figures, as it was predicted the velocity, flight path angle, and heading angle satisfies the boundaries constraints which were imposed on them Eqs. (22)-(24). They also show constant behavior along the trajectory, which means the trajectory is smooth.

5.2 Example 2

This subsection shows the simulation and results of the optimized minimum length 4D trajectory for an aircraft flying from a given initial position to a terminal position in cruise phase, passing through some specified 4D waypoint coordinates at given scheduled times. Each waypoint is defined in geocentric coordinates system (x, y, z) and the desired arrival time t at that waypoint is also specified. Table 3 shows the lists of predefined 4D waypoints in a cruise phase of flight. Where, in this study total five 4D waypoints were chosen.

In this specific mission to optimize the minimum length 4D trajectory, the proposed method achieves a solution for the problem as shown in Table 4. Where, d is the distance between pair of waypoints for the nonoptimal original trajectory, d_opt is the distance between a pair of waypoints for the optimized minimum length trajectory.

As it is seen from Table 4, the total distance from initial to terminal waypoint for the original trajectory is 170466.26 meters and for the optimal minimum length trajectory is 168841.80 meters.

Table 3 List of waypoints of the trajectory in cruise phase

Waypoints	x [m]	y [m]	z [m]	t [s]
1	4922679.656	-827307.389	3973823.239	0
2	4910474.780	-866732.905	3980948.665	180
3	4891961.555	-895195.506	3998888.720	344
4	4882512.256	-937572.651	4000554.483	528
5	4858467.696	-969054.126	4022122.186	720

Table 4 Distance between waypoints

Waypoints	d [m]	d_opt [m]	[$d-d\_opt$] [m]
1 - 2	42317.00	41882.00	435.00
2 - 3	38585.17	38404.92	180.25
3 - 4	43667.86	43450.47	217.39
4 - 5	45896.23	45104.42	791.82
Total	170466.26	168841.80	1624.50

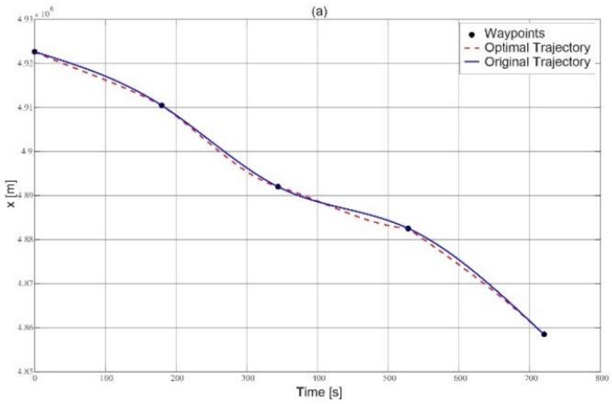

Fig. 9 x-axis vs time

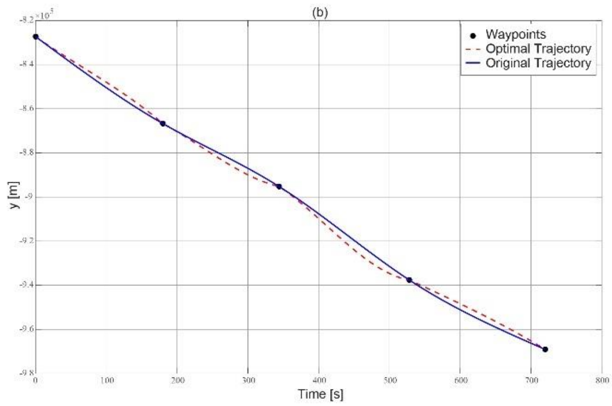

Fig. 10 y-axis vs time

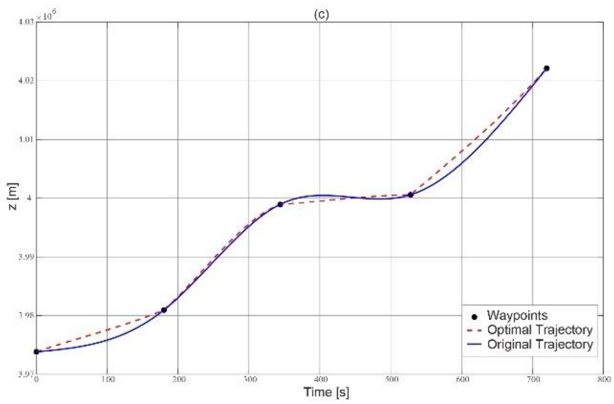

Fig. 11 z-axis vs time

So, the optimal trajectory reduces the total distance by 1624.50 meters from the original trajectory, which reduces the total distance by 0.95% over the original trajectory.

The Figs. 9-11 show the geocentric coordinates x-axis, y-axis, and z-axis vs time, respectively

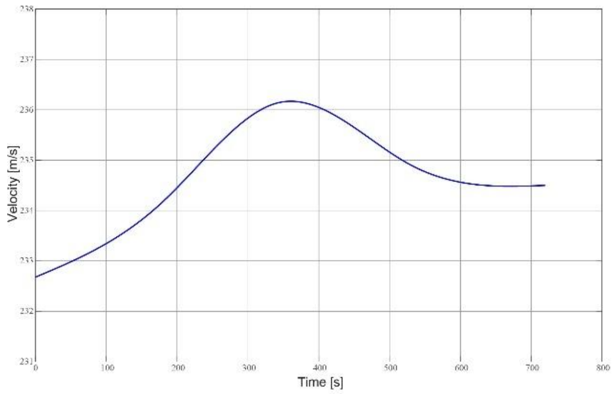

Fig. 12 Velocity vs time of optimal trajectory

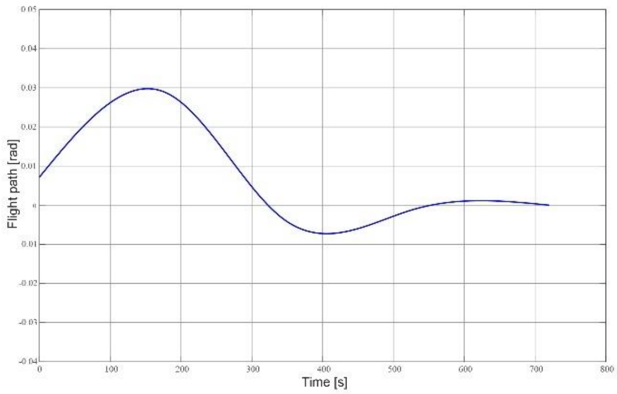

Fig. 13 Flight path angle vs time of optimal trajectory

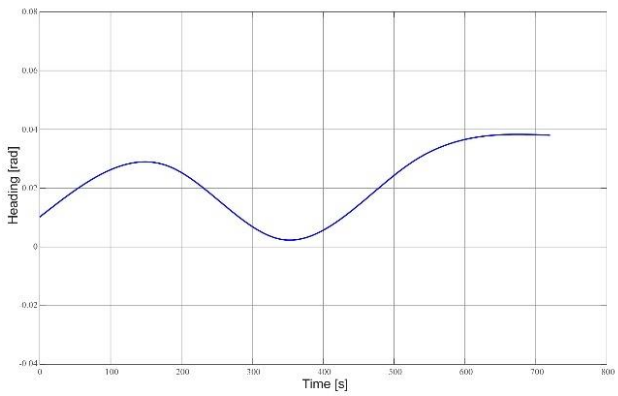

Fig. 14 Heading angle vs time of optimal trajectory

of the cruise phase of flight. The optimal trajectory is represented by the dashed red line, nonoptimal trajectory is represented by the blue line and the waypoints are represented by the black circles. The Figs. 12-14 represent respectively the velocity, flight path angle, and heading angle vs time of the optimal trajectory. As it was predicted the figures show constant behaviour along the trajectory, which implies the trajectory is smooth. It can also be observed from the figures that the velocity, flight path angle, and heading angle are within the bounded constraints of aircraft limit.

6. Conclusions

This paper proposes a method for optimizing 4D trajectory from pre-defined 4D waypoints. In this method, the trajectory optimization problem (TOP) was solved by direct optimal control approach based on cubic spline parameterization of the state, its derivatives and control vector, which transformed the original infinite-dimensional TOP to finite-dimensional nonlinear programming problem (NLP), then the resulting NLP was solved numerically by well-established NLP solver.

In this paper, the proposed method was applied to generate an optimal minimum length trajectory along pre-defined 4D waypoints for takeoff and cruise phase of flight. The analysis results show promising potential of the method for generating the 4D optimal trajectory with the following properties:

• The proposed method generated smooth 4D optimal trajectory with accuracy, which not only ensures the continuity for the trajectory but also for its derivatives. In the examples of generating minimum length optimal trajectory from predefined 4D waypoints in the takeoff and cruise phase of flight, the method was able to reduce the total length by 0.55% and 0.95% respectively than the corresponding original nonoptimal trajectories.

• The proposed method has the advantage of requiring less computation space and time, which means it can be used successfully to solve online 4D trajectory optimization problem with high precision.

Although the proposed method generated a smooth 4D optimal trajectory for minimum length as the objective, the research work will be extended where fuel saving, minimum time and other operational requirements are the main concerns of the mission.

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