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Citation: AIP Conference Proceedings **1479**, 1694 (2012); doi: 10.1063/1.4756496
View online: http://dx.doi.org/10.1063/1.4756496
View Table of Contents: http://scitation.aip.org/content/aip/proceeding/aipcp/1479?ver=pdfcov
Published by the AIP Publishing

Linear and Quadratic Sufficiency and Commutativity

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Abstract. Given a mixed model let T be the orthogonal projection matrix on the range space spanned by the mean vector. If the model has variance- covariance matrix $\sigma^2 V$ we use commutative Jordan algebras to show that Ty is both linear sufficient and linear complete and that $Ty, y^{'}V^{+}y$ with V^{+} the Moore-Penrose inverse of V is quadratic sufficient whenever T and V commute.

Keywords: Linear sufficiency, quadratic sufficiency, orthogonal projection, linear completeness
AMS: 62H15,62J10

INTRODUCTION

Zmyślony (1978) showed that when the orthogonal projection matrix, OPM, T, on the range space Ω spanned by the mean vector $\mu = X\beta$ commutes with the variance-covariance matrix V, the least square estimators, LSE, of linear estimable functions $c^{'}\beta$ [or of estimable vectors] are best linear unbiased estimators, BLUE.

We now consider linear and quadratic sufficiency for models in which T and V commute. Following Mueller (1987), we say that:

- Ly is linearly sufficient if for every linear estimable function $c^{'}\beta$ the BLUE is given by $b^{'}LY$;
- $(LY; y^{'}Uy)$ is quadratically sufficient for the model $Y \sim Q(X\beta, \sigma^2 V)$; $\beta \in R^p, \sigma^2 > 0$, if Ly is linearly sufficiency and there exists a symmetric matrix A and a real α such that $y^{'}LALy + \alpha y^{'}Uy$ is a best quadratic unbiased estimator, BQUE, for $f\sigma^2$, with f known.

In the last expression, $Y \sim Q(X\beta, \sigma^2 V)$ indicates Y to be quasi-normal, this is itt's first four order moments are related in the same way as for normal vectors, with $\mu = X\beta$ and variance-covariance matrix $\sigma^2 V$. It is easy to see that the linear sufficiency of LY does not depend on the introduction of σ^2 in the definitions of the variance covariance matrix of Y. Besides these notions of sufficiency we will also consider, following again Mueller (1987), the linear completeness. Thus Ly is linearly complete if any linear function $c^{'}\beta$ with null mean value for every $\beta \in R^p$ is almost surely null.

In our study of quadratic completeness we will use commutative Jordan algebras, which we will discuss in the next section. Then we consider linear sufficiency and completeness and finally quadratic sufficiency.

COMMUTATIVE JORDAN ALGEBRAS

We restrict ourselves to commutative Jordan algebras, CJA, of symmetric matrices. There will be linear spaces constituted by symmetric matrices that commute and containing the squares of this matrices.

Seely (1971) showed that each CJA $\mathscr{A}$ has an unique basis constituted by pairwise orthogonal OPM, POOPM, the principal basis, $pb(\mathscr{A})$ of $\mathscr{A}$. According to Schott (1997), the matrices of a family $M = \{M_1, ..., M_w\}$ of symmetric matrices commute if and only if they are diagonalized by an orthogonal matrix P. Then we will have $M \subset \mathscr{V}(P)$ with $\mathscr{V}(P)$ the family of symmetric matrices diagonalized by P which is a CJA. Since intersection of CJA gives CJA (ref?), the intersection of all CJA that contain M will be a CJA, the CJA $\mathscr{A}(M)$ generated by $\mathscr{A}$.

With $Q = \{Q_1, ..., Q_m\} = pb(\mathscr{A})$, given $M \in \mathscr{A}$ we will have $M = \sum_{j=1}^{m} a_j Q_j$ it being easy to see that the MOORE-

Numerical Analysis and Applied Mathematics ICNAAM 2012
AIP Conf. Proc. 1479, 1694-1697 (2012); doi: 10.1063/1.4756496

PENROSE inverse of M is

$$M^+ = \sum_{j=1}^{m} a_j^+ Q_j,$$

putting $a_j^+ = a_j^{-1} \quad [0]$ when $a_j \neq 0 \quad [=0], j = 1, ..., m$. Thus any CJA contains the MOORE-PENROSE inverse of its matrices.

If $Q \in \mathscr{A}$ is an OPM it will be idempotent then $Q = \sum_{j=1}^{m} q_j Q_j$ with the $q_1, ..., q_m$ equal to 0 or 1, since the $Q_1, ..., Q_m$ are pairwise orthogonal and idempotent.

Now

$$QMQ = \sum_{j=1}^{m} q_j a_j q_j Q_j = \sum_{j \in C} a_j Q_j,$$

with $C = \{j : q_j a_j q_j \neq 0\}$, and so

$$(QMQ)^+ = \sum_{j \in C} a_j^{-1} Q_j = QM^+Q$$

since we also have $C = \{j : q_j a_j^+ q_j \neq 0\}$.

LINEAR SUFFICIENCY AND COMPLETENESS

Let $N(W)$ be the nullity space of matrix W, then, according to Mueller (1987), Ly is linearly sufficient for the model with mean vector $X\beta$ and variance covariance matrix V if and only if $x_1 \cap x_2 \subset \triangle_1$, with

$$\begin{cases} x_1 = N(L) \\ x_2 = \Omega \oplus \triangle_1 \\ \triangle_1 = VN(X^{'}) \end{cases}$$

where $\oplus$ indicates direct sum of subspaces and

$$VN(X^{'}) = \{Vu; u \in N(X^{'})\}.$$

It is well known that, $N(X^{'}) = R(X)^{\perp}$ where $\perp$ indicates orthogonal complement, so we also have

$$\triangle_1 = VR(X)^{\perp} = \{Vz; z \perp R(X)\}.$$

Moreover, see again Mueller (1987), Ly is linearly complete if and only if

$$x_1 \cap x_2 = \triangle_1.$$

We now establish

Lemma 1 *We have $\triangle_1 = R(VT^c)$, with $T^c = I_n - T$.*

Proof: The thesis follows from $VN(X^{'}) = VR(X)^{\perp} = VR(T^c) = R(VT^c)$ since $R(X)^{\perp} = R(T)^{\perp} = R(T^c)$. □

Corollary 1 *When V and T commute, $x_2 = \Omega \boxplus \triangle_1$, where $\boxplus$ stands for orthogonal direct sum of subspaces.*

Proof: Since V and T [V and T^c] commute, $\triangle_1 = R(VT^c) = R(T^cV) \subset R(T^c) = \Omega^{\perp}$, thus $\Omega = R(T)$ and $\triangle_1$ will be orthogonal. □

Proposition 1 *When V and T commute, Ty will be linearly sufficient and linearly complete.*

Proof: We must have $x_1 = N(T) = \Omega^{\perp}$ thus, according to **Corollary 3.1** of **Lemma 1** $x_1 \cap x_2 = \triangle_1$, which establishes the thesis.$\square$

Besides this, see Scheffé (1959), $c^{'}\beta$ is estimable when and only when $c = X^{'}u$, so that $c^{'}\beta = u^{'}X\beta$. Now the LSE of $c^{'}\beta$ is $\widetilde{c^{'}\beta} = u^{'}X\widetilde{\beta}$, with $\widetilde{\beta} = (X^{'}X)^{+}X^{'}Y$. Since $T = X(X^{'}X)^{+}X^{'}$ we will have $\widetilde{c^{'}\beta} = u^{'}Ty$ which emphasizes the linear sufficiency of Ty. In this way, we relate the linear sufficiency of Ty with the Zmyślony (1978) result on LSE. Now if V depends on a vector σ^2 of variance components, so that

$$V = V(\sigma^2), \sigma^2 \in \ominus.$$

For V commuting with T we must have

$$TV(\sigma^2) = V(\sigma^2)T; \quad \sigma^2 \in \ominus.$$

For instance given the mixed model

$$y = \sum_{i=0}^{w} X_i \beta_i$$

with β_0 fixed and $\beta_1, ..., \beta_w$ independent, with null mean vectors and variance covariance matrices $\sigma_1^2 I_{c_1}, ..., \sigma_w^2 I_{c_w}$, y will have mean vectors and variance covariance matrices given by

$$\begin{cases} \mu = X_0\beta_0 \\ V = \sum_{i=1}^{w} \sigma_i^2 V_i; \quad \theta \in \ominus = \mathscr{X}_{i=1}^{w}[0; +\infty[\end{cases}$$

with $\mathscr{X}$ indicating cartesian product and $V_i = X_i X_i^{'}, i = 1, ..., w$, while the orthogonal projection matrix on the space spanned by μ will now be written as

$$T = X_0(X_0^{'}X_0)^{+}X_0^{'}.$$

For V and T to commute it is necessary and sufficient that

$$TV_i = V_i T, i = 1, ..., w.$$

This conditions holds namely for models with commutative orthogonal block structure, COBS, see Fonseca et al (2007). Then in these models Ty will be linearly sufficient and linearly complete.

QUADRATIC SUFFICIENCY

Given the model

$$Y \sim Q(X\beta, \sigma^2 V),$$

$(Ly, y^{'}Uy)$ enjoys quadratic sufficiency if and only if, see Mueller (1987),

$$x_1 \cap x_2 = \triangle_1 \cap \triangle_2 \cap \triangle_3,$$

with x_1, x_2 and $\triangle_1$ defined as above and

$$\begin{cases} \triangle_2 = N(X^{'}U) \\ \triangle_3 = N(I_n - \alpha VU) \end{cases}$$

The equality between subspaces is assumed to hold for the α that is considered in the definition of quadratic sufficiency. When T and V commute, taking $T^c = I_n - T$, considering T^c the complement subspace of matrix T, the matrices of a family $M = \{T, T^c, V\}$ commute and generate a CJA $\mathscr{A} = \mathscr{A}(M)$ that contains $V^{+}, Q = VV^{+}, Q_1 = TQ$ and $Q_2 = T^cV$. Matrices Q, Q_1 and Q_2 are OPM and Q_1 and Q_2 are pairwise orthogonal.

We now establish

Lemma 2 *When T and V commute and we take $L = T$ and $U = V^+$, we have*

$$\left\{\begin{array}{l} x_1 = N(T) = \Omega^{\perp} \\ x_2 = \Omega \boxplus \bigtriangledown_2 \\ \triangle_1 = \bigtriangledown_2 \\ \triangle_2 = \bigtriangledown_1^{\perp} \end{array}\right.$$

and

$$\triangle_3 = \left\{\begin{array}{ll} 0, & \text{if} \quad \alpha \neq 1 \\ \bigtriangledown_1^{'} \boxplus \bigtriangledown_2, & \text{if} \quad \alpha = 1 \end{array}\right.$$

with $\bigtriangledown_1 = R(Q^{'})$ and $\bigtriangledown_2 = R(Q_2)$.

Proof: We already saw that taking $L = T$ we have $x_1 = \Omega^{\perp}$ and that, according to **Lemma 1**, $\triangle_1 = R(VT^c)$. Since $R(V) = R(Q)$ we also will have

$$\triangle_1 = R(VT^c) = R(T^cV) = T^cR(V) = T^cR(Q) = R(T^cQ) = R(Q_2) = \bigtriangledown_2,$$

thus Ω and $\triangle_1 = \bigtriangledown_2$ are orthogonal so $x_2 = \Omega \boxplus \bigtriangledown_2$. Next we have

$$\triangle_2 = N(X^{'}V^+) = R(V^+X)^{\perp} = (V^+R(X))^{\perp} = (V^+R(T))^{\perp} = R(V^+T)^{\perp} =$$

$$= R(TV^+)^{\perp} = (TR(V^+))^{\perp} = (TR(Q))^{\perp} = (R(TQ))^{\perp} = R(Q_1)^{\perp} = \bigtriangledown_1^{\perp}.$$

Lastly we point out that, with $Q^c = I_n - Q$ we have

$$I_n - \alpha VV^+ = I_n - \alpha Q = Q^c + (1-\alpha)Q$$

which is invertible when $\alpha \neq 1$. When $\alpha = 1$, we have $I_n - Q = Q^c$ and $N(I_n - Q) = R(Q) = Q^c = \bigtriangledown_1 \boxplus \bigtriangledown_2$, which completes the proof.□

We now may establish the

Proposition 2 *When T and V commute, taking $\alpha = 1$, $(Ty, y^{'}V^+y)$ is quadratic complete for the model $Y \sim Q(X\beta, \sigma^2 V)$.*

Proof: The thesis follows from, according to **Lemma 2**

$$\left\{\begin{array}{l} x_1 \cap x_2 = \bigtriangledown_2 \\ \triangle_1 \cap \triangle_2 \cap \triangle_3 = \bigtriangledown_2 \end{array}\right.$$

since $\bigtriangledown_2$ being a subspace of $\bigtriangledown_1^{\perp}$ and of $(\bigtriangledown_1 \boxplus \bigtriangledown_2)$. □

ACKNOWLEDGEMENTS

The authors are grateful to the anonymous referees for their useful comments and remarks. This work was partially supported by the center of Mathematics, University of Beira Interior, under the project PEst-OE/MAT/UI0212/2011.

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