Confidence Regions for Variance Components Using Inducing Pivot Variables

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Abstract. The goal of this paper is to present what we think to be an interesting development of the concept of pivot variable. The inducing pivot variables induce probability measures which may be used to carry out inference.

As illustration of this approach we will show how to obtain confidence intervals for the variance components of mixed linear models

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INTRODUCTION

The goal of this paper is to present what we think to be an interesting development of the concept of pivot variable. These variables are functions of statistics and parameters with known distributions and induce probability measures which may be used to carry out inference.

In the next section we will show how pivot variables induce probability measures in parameter spaces. Then, in section 3, we will show how to use Monte Carlo methods to generate distributions and how to apply these to obtain confidence intervals and through duality, to test hypothesis, for the variance components of mixed linear models. This approach may be used whatever the degrees of freedom of the chi-square distributions. Thus there is no need to the degrees of freedom to be even, either in the numerator or in the denominator of generalized F statistics, see [2].

INDUCING PIVOT VARIABLES

In what follows we will use sufficient statistics to derive pivot variables. As already stated in the introduction, these variables are functions of statistics and parameters with known distributions.

For example, if S is distributed as the product by γ of a central chi-square with g degrees of freedom, $S \sim \gamma\chi^2_g$, then

$$Z = \frac{S}{\gamma} \tag{1}$$

is distributed as a central chi-square with g degrees of freedom, being therefore a pivot variable.

Now, let $\mathscr{B}_r$ be the σ-algebra of the borelian sets in $\mathbb{R}^r$, see [1], and the parameter space $\Theta \in \mathscr{B}_r$. According to [4] the pivot variable

$$\boldsymbol{Z} = g(\boldsymbol{Y}, \boldsymbol{\theta}) \tag{2}$$

is an inducing pivot variable if, for any realization $\boldsymbol{y}$ of $\boldsymbol{Y}$ the function

$$l(\boldsymbol{\theta}|\boldsymbol{y}) = g(\boldsymbol{y}, \boldsymbol{\theta}) \tag{3}$$

has an inverse measurable function $h(\boldsymbol{z}|\boldsymbol{y})$ in $\mathscr{B}_r$.

Now, let P° be the probability measure associated to the distribution of the pivot variable, F°. The measurable functions $\boldsymbol{h}(\boldsymbol{z}|\boldsymbol{y})$, defined in $(\mathbb{R}^r, \mathscr{B}_r, P^\circ)$ and taking values in $\Theta \in \mathscr{B}_r$, define the probability measures

$$P_{\boldsymbol{y}}(C) = P^\circ\left(\boldsymbol{l}(C \cap \Theta|\boldsymbol{y})\right) \tag{4}$$

in $(\mathbb{R}, \mathscr{B}_r)$. Note that for any $\boldsymbol{y}$

$$P_{\boldsymbol{y}}(\Theta) = 1 \tag{5}$$

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Consider now that the components $Z_1,...,Z_r$ of the inducing pivot variable $\boldsymbol{Z}$ are independent and given by

$$Z_i = g_i(\boldsymbol{y}, \theta_i), \;\; i = 1,...,r \tag{6}$$

with $\theta_1,...,\theta_r$ the components of $\boldsymbol{\theta}$. If besides this

$$\Theta = \times_{i=1}^{r} \Theta_i, \tag{7}$$

with $\Theta_i \in \mathscr{B}_i$, $i = 1,...,r$ and if the functions

$$l_i(\theta_i|\boldsymbol{y}) = g_i(\boldsymbol{y}, \theta_i), \;\; i = 1,...,r \tag{8}$$

have measurable inverses $h_i(z_i|\boldsymbol{y}) \in \mathscr{B}$, $i = 1,...,r$, we may induce in $(\mathbb{R},\mathscr{B})$ the probability measures

$$P_{\boldsymbol{y},i}(C) = P_i^{\circ}\left(l_i(C \cap \Theta_i)|\boldsymbol{y}\right), \;\; i = 1,...,r, \tag{9}$$

where P_i°, $i = 1,...,r$ is the probability measure associated to the distribution of Z_i, $i = 1,...,r$. Since these components are independent, taking $C_1,...,C_r \in \mathscr{B}$, we get

$$P^{\circ}\left(\times_{i=1}^{r} l_i(C_i \cap \Theta_i)|\boldsymbol{y}\right) = \prod_{i=1}^{r} P_i^{\circ}\left(l_i(C \cap \Theta_i)|\boldsymbol{y}\right). \tag{10}$$

Thus, with $\bar{P}_{\boldsymbol{y}}$ the product measure of the $P_{\boldsymbol{y},i}$, $i = 1,...,r$, we have

$$\begin{aligned} \bar{P}_{\boldsymbol{y}}(\times_{i=1}^{r} C_i) &= \prod_{i=1}^{r} P_{\boldsymbol{y},i}(C_i) = \prod_{i=1}^{r} P_i^{\circ}\left(l_i(C_i \cap \Theta_i)|\boldsymbol{y}\right) = \\ &= P^{\circ}\left(\times_{i=1}^{r} l_i(C_i \cap \Theta_i|\boldsymbol{y})\right) = \\ &= P\left(\boldsymbol{l}((\times_{i=1}^{r} C_i) \cap \Theta|\boldsymbol{y})\right), \end{aligned} \tag{11}$$

since

$$\begin{aligned} \times_{i=1}^{r}(C_i \cap \Theta_i) &= (\times_{i=1}^{r} C_i) \cap (\times_{i=1}^{r} \Theta_i) = \\ &= (\times_{i=1}^{r} C_i) \cap \Theta. \end{aligned} \tag{12}$$

Therefore, the product measure $\bar{P}_{\boldsymbol{y}}$ of the measures induced by the components is identical to $P_{\boldsymbol{y}}$.

VARIANCE COMPONENTS

Given the mixed model

$$\boldsymbol{y} = \sum_{i=0}^{w} \boldsymbol{X}_i \boldsymbol{\beta}_i, \tag{13}$$

where $\boldsymbol{\beta}_0$ is fixed and the $\boldsymbol{\beta}_1,...,\boldsymbol{\beta}_w$ are independent with null mean vectors and variance-covariance matrices $\sigma_1^2 \boldsymbol{I}_{c_1},...,\sigma_w^2 \boldsymbol{I}_{c_w}$, the mean vector and variance-covariance matrix of $\boldsymbol{y}$ will be

$$\begin{cases} \boldsymbol{\mu} = \boldsymbol{X}_0 \boldsymbol{\beta}_0 \\ \boldsymbol{V} = \sum_{i=1}^{w} \sigma_i^2 \boldsymbol{M}_i \end{cases}, \tag{14}$$

with $\boldsymbol{M}_i = \boldsymbol{X}_i \boldsymbol{X}_i'$, $i = 1,...,w$.

When matrices $\boldsymbol{M}_1,...,\boldsymbol{M}_w$ and, $\boldsymbol{T}$, the orthogonal projection matrix on the range space of $\boldsymbol{X}_0$, commute we have, see [5],

$$\begin{cases} \boldsymbol{T} = \sum_{j=1}^{z} \boldsymbol{Q}_j \\ \boldsymbol{M}_i = \sum_{j=1}^{m} b_{i,j} \boldsymbol{Q}_j \end{cases}, \tag{15}$$

where the $\boldsymbol{Q}_1,...,\boldsymbol{Q}_m$ are pairwise orthogonal orthogonal projection matrices. Then

$$\boldsymbol{V} = \sum_{j=1}^{m} \gamma_j \boldsymbol{Q}_j, \tag{16}$$

whith

$$\gamma_j = \sum_{i=1}^{w} b_{i,j}\sigma_i^2. \tag{17}$$

Moreover, taking

$$\boldsymbol{\sigma}^2 = \begin{bmatrix} \sigma_1^2 \\ \vdots \\ \sigma_w^2 \end{bmatrix}; \quad \boldsymbol{\gamma}(1) = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_z \end{bmatrix}; \quad \boldsymbol{\gamma}(2) = \begin{bmatrix} \gamma_{z+1} \\ \vdots \\ \gamma_m \end{bmatrix}, \tag{18}$$

and considering for matrix $\boldsymbol{B} = [b_{i,j}]$ the partition

$$\boldsymbol{B} = [\boldsymbol{B}(1) \quad \boldsymbol{B}(2)], \tag{19}$$

where $\boldsymbol{B}(1)$ has z columns, we have

$$\boldsymbol{\gamma}(l) = \boldsymbol{B}'(l)\boldsymbol{\sigma}^2, \;\; l = 1,2. \tag{20}$$

When the row vectors of $\boldsymbol{B}(2)$ are linearly independent we have

$$\boldsymbol{\sigma}^2 = \boldsymbol{C}\boldsymbol{\gamma}(2), \tag{21}$$

with $\boldsymbol{C}$ the MOORE-PENROSE inverse of $\boldsymbol{B}'(2)$.

Taking $\boldsymbol{C} = [c_{i,j}]$, let C_i^+ and C_i^- be the sets of column indexes of the positive and negative elements of the i-th row of matrix $\boldsymbol{C}$. Thus, with $\dot{m} = m - z$ and $\dot{\gamma}_j = \gamma_{j-z}$

$$\sigma_i^2 = \sum_{j=1}^{\dot{m}} c_{i,j}\dot{\gamma}_j = (\sigma_i^2)^+ - (\sigma_i^2)^-, \;\; i = 1,...,w, \tag{22}$$

where

$$\begin{cases} (\sigma_i^2)^+ = \sum_{j\in C_i^+} c_{i,j}\gamma_j, \;\; i = 1,...,w \\ (\sigma_i^2)^- = \sum_{j\in C_i^-} |c_{i,j}|\gamma_j, \;\; i = 1,...,w \end{cases}. \tag{23}$$

These results are interesting since, with

$$S_j = \boldsymbol{y}'\boldsymbol{Q}_{j+z}\boldsymbol{y}, \;\; j = 1,...,\dot{m} \tag{24}$$

we have the unbiased estimators

$$\dot{\gamma}_j = \frac{S_j}{g_j}, \;\; j = 1,...,\dot{m}, \tag{25}$$

where

$$g_j = \operatorname{rank}(\boldsymbol{Q}_{j+z}), \;\; j = 1,...,\dot{m}. \tag{26}$$

Thus we also will have unbiased estimators for the σ_i^2, $i = 1,...,w$ and their positive and negative parts. When normality is assumed,

$$S_j \sim \dot{\gamma}_j\chi^2_{g_j}, \;\; j = 1,...,\dot{m}, \tag{27}$$

this is, S_j is distributed as the product by $\dot{\gamma}_j$ of a central chi-square with g_j degrees of freedom, $j = 1,...,\dot{m}$.

So, we have the independent pivot variables

$$Z_j = \frac{S_j}{\gamma_j} \sim \chi^2_{g_j}, \;\; j = 1,...,\dot{m}. \tag{28}$$

Since the inverse functions

$$h_j(Z_j,S_j) = \frac{S_j}{Z_j}, \;\; j = 1,...,\dot{m}. \tag{29}$$

are mensurable functions in $\mathscr{B}$, the Z_j, $j = 1,...,\dot{m}$ will be inducing pivot variables.

Now, we may induce probability measures for the γ_j, $i=1,...,\dot{m}$, using large samples $\{\boldsymbol{G}_1,...,\boldsymbol{G}_n\}$, with $G_{u,j} \sim \chi^2_{g_j}$, $j=1,...,\dot{m}$, $u=1,...,n$, and derive from these secondary samples $\{\boldsymbol{Z}_1,...,\boldsymbol{Z}_n\}$ in which $\boldsymbol{Z}_u$ has components

$$\boldsymbol{Z}_{u,j} = \frac{S_j}{G_{u,j}}, \tag{30}$$

$j=1,...,\dot{m}$, $u=1,...,n$. Note that $Z_{1,j},...,Z_{n,j}$ are independent and identically distributed, with distribution $\dot{\mathscr{F}}_j$ associated to the probability measure induced by Z_j, $j=1,...,\dot{m}$.

Let $F_{n,j}$, $j=1,...,\dot{m}$ be the empirical distribution of the sample $\{Z_{1,j},...,Z_{n,j}\}$ and $x_{n,p}$ the F_n quantile for probability p.

Representing by $\xrightarrow{a.s.}$ almost surely convergence and by x_p the quantile for probability p, we have the

Proposition 1

If $F(x)$ has a continuous density $f(x)$ and if $f(x) > 0$ whenever $0 < F(x) < 1$, then for any $\alpha \in]0;1[$ we have

$$D_{n,\alpha} = Sup\left\{|x_{n,p} - x_p|; \frac{\alpha}{2} < p < 1 - \frac{\alpha}{2}\right\} \xrightarrow[n\to\infty]{a.s.} 0$$

Proof. According to Weierstrass theorem, f has a minimum $b > 0$ in the interval $[x_{\frac{\alpha'}{2}}; x_{\frac{1-\alpha'}{2}}]$. If $\frac{\alpha'}{2} < p - \frac{\varepsilon}{b} < p + \frac{\varepsilon}{b} < 1 - \frac{\alpha'}{2}$ we have $F(x_p - \frac{\varepsilon}{b}) < p - \varepsilon < p + \varepsilon < F(x_p + \frac{\varepsilon}{b})$. So, when $D_{n,\alpha} < b$, $F_n(x_p - \frac{\varepsilon}{b}) < p < F_n(x_p + \frac{\varepsilon}{b})$ we get $(x_p - \frac{\varepsilon}{b}) < x_{n,p} < x_p + \frac{\varepsilon}{b})$. This establishes the thesis since from ε being arbitrary we may take $\alpha = \alpha' + 2\frac{\varepsilon}{b}$ to get $|x_{n,p} - x_p| < \frac{\varepsilon}{b}$ whenever $\frac{\alpha}{2} < p < 1 - \frac{\alpha}{2}$. $\square$

Now, using Proposition 1, we may estimate the quantiles $x_{j,p}$ of F_j from the quantiles $x_{n,j,q}$ of $F_{n,j}$, $j=1,...,\dot{m}$. Therefore, we may construct confidence intervals $[\dot{x}_{n,j,\frac{\alpha}{2}}; \dot{x}_{n,j,1-\frac{\alpha}{2}}]$; $[0; \dot{x}_{n,j,1-\alpha}]$ and $[\dot{x}_{n,j,\alpha}; +\infty[$ for the γ_j, $j=1,...,\dot{m}$, with (estimated) confidence level $1-\alpha$. These confidence intervals allow us to test hypothesis

$$H_{0,j} : \gamma_j = \gamma_{j,0} \tag{31}$$

against

$$H_{1,j} : \gamma_j \neq \gamma_{j,0}; \;\; \gamma_j > \gamma_{j,0} \;\text{ and }\; \gamma_j < \gamma_{j,0}. \tag{32}$$

We reject the test hypothesis if $\gamma_{j,0}$ is not contained in the corresponding confidence interval. Thus, by duality, we obtain tests with (approximate) confidence level α.

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