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Chisquared and related inducing pivot variables: an application to orthogonal mixed models

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ABSTRACT
We use chi-squared and related pivot variables to induce probability measures for model parameters, obtaining some results that will be useful on the induced densities. As illustration we considered mixed models with balanced cross nesting and used the algebraic structure to derive confidence intervals for the variance components. A numerical application is presented.

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1. Introduction

Our main goal is to use pivot variables to induce probability measures for model parameters. The pivot variables used in this way will be called inducing pivot variables. Actually our approach has some similarity with that of Weerahandi (1993) and Weerahandi (1996).

We will be mainly interested in chi-squared and related pivot variables obtaining some results that will be useful on them, namely on the induced densities.

As a promising field of work we considered mixed models, since this type of model is widely used, see for example Hubert and Wijekoon (2006), Njuho and Milliken (2009), Ozkale (2009), Yang and Wu (2011) or Yang and Wu (2012).

More specifically, we considered mixed models with balanced cross nesting. The algebraic structure of these models has been studied, see Fonseca, Mexia, and Zmyślony (2006), using Commutative Jordan Algebras. As we shall see that algebraic structure leads to interesting possibilities for the use of the induced densities in deriving confidence intervals. Namely in an application example we simulated induced densities for variance components for which we had UMVUE estimators and there was a

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remarkable agreement between modes for these simulated induced densities and the UMVUE estimators.

The remainder of this article is arranged as follows. In the next section we introduce the concept of inducing pivot variable. Section 3 presents results on the inverse gamma and related distributions. Some results on empirical distributions are presented in Section 4. In Section 5 we have an application to mixed models. Namely we have a numerical example applied to real and simulated data. Some final remarks are presented in Section 6.

2. Inducing pivot variables

Pivot variables are functions of statistics and parameters with known distributions. For example, if $S \sim \gamma\chi^2_g$, then

$$Z = \frac{S}{\gamma} \tag{1}$$

is distributed as a central chi-square with g degrees of freedom, being therefore a pivot variable.

Now, let $\mathcal{B}^r$ be the σ-algebra of the borelian sets in $\mathbb{R}^r$, see Williams (1991), and the parameter space $\Theta \in \mathcal{B}^r$. According to Fonseca, Mexia, and Zmyślony (2007) the pivot variable

$$Z = g(Y, \theta) \tag{2}$$

is an inducing pivot variable if, for any realization y of Y the function

$$l(\theta|y) = g(y, \theta) \tag{3}$$

has an inverse measurable function $h(z|y)$ in $\mathcal{B}^r$.

Now, let P° be the probability measure associated to the distribution of the pivot variable, F°. The measurable functions $h(z|y)$, defined in $(\mathbb{R}^r, \mathcal{B}^r, P^\circ)$ and taking values in $\Theta \in \mathcal{B}^r$, define the probability measures

$$P_y(A) = P^\circ\left(l(A \cap \Theta|y)\right) \tag{4}$$

in $(\mathbb{R}, \mathcal{B}^r)$. Note that for any y

$$P_y(\Theta) = 1 \tag{5}$$

Consider now that the components $Z_1, ..., Z_r$ of the inducing pivot variable Z are independent and given by

$$Z_i = g_i(y, \theta_i), i = 1, ..., r \tag{6}$$

with $\theta_1, ..., \theta_r$ the components of θ. If

$$\Theta = \times_{i=1}^{r} \Theta_i \tag{7}$$

where $\times$ denotes the Cartesian product and $\Theta_i \in \mathcal{B}_i, i = 1, ..., r$, and the functions

$$l_i(\theta_i|y) = g_i(y, \theta_i), i = 1, ..., r \tag{8}$$

have measurable inverses $h_i(Z_i|y) \in \mathcal{B}_i, i = 1, ..., r$, we may induce in $(\mathbb{R}^r, \mathcal{B}^r)$ the probability measures

$$P_{y,i}(A) = P_i^{\circ}\left(l_i(A \cap \Theta_i)|y\right), i = 1, \ldots, r \tag{9}$$

where $P_i^{\circ}, i = 1, \ldots, r$ is the probability measure associated to the distribution of Z_i, $i = 1, \ldots, r$. Since these components are independent, taking $A_1, \ldots, A_r \in \mathcal{B}^r$, we get

$$P^{\circ}\left(\times_{i=1}^{r} l_i(A_i \cap \Theta_i)|y\right) = \prod_{i=1}^{r} P_i^{\circ}\left(l_i(A \cap \Theta_i)|y\right) \tag{10}$$

Thus, with $\bar{P}_y$ the product measure of the $P_{y,i}, i = 1, \ldots, r$, we have

$$\begin{aligned}\bar{P}_y(\times_{i=1}^{r} A_i) &= \prod_{i=1}^{r} P_{y,i}(A_i) = \prod_{i=1}^{r} P_i^{\circ}\left(l_i(A_i \cap \Theta_i)|y\right) = \\ &= P^{\circ}\left(\times_{i=1}^{r} l_i(A_i \cap \Theta_i)|y\right) = \\ &= P\left(l((\times_{i=1}^{r} A_i) \cap \Theta)|y\right)\end{aligned} \tag{11}$$

since

$$\begin{aligned}\times_{i=1}^{r}(A_i \cap \Theta_i) &= (\times_{i=1}^{r} A_i) \cap (\times_{i=1}^{r} \Theta_i) = \\ &= (\times_{i=1}^{r} A_i) \cap \Theta\end{aligned} \tag{12}$$

Therefore, the product measure $\bar{P}_y$ of the measures induced by the components is identical to P_y.

3. Inverse gamma and related distributions

We now consider the induced distributions obtained from pivot variables distributed as chi-squares.

3.1. Densities

Let $S_j \sim \gamma_j \chi^2_{g_j}$, $j = 1, \ldots, r$. So, we have the independent pivot variables

$$Z_j = \frac{S_j}{\gamma_j} \sim \chi^2_{g_j}, j = 1, \ldots, r \tag{13}$$

Since the inverse functions

$$h_j(Z_j, S_j) = \frac{S_j}{Z_j}, j = 1, \ldots, r \tag{14}$$

are measurable functions in $\mathcal{B}^r$, the Z_j, $j = 1, \ldots, r$ will be inducing pivot variables. We will now study the distributions and densities corresponding to the probability measures induced by them.

As it may be seen, for example in Witkovsk (2001), the random variables $\frac{1}{\chi^2_g}$ have inverse gamma distributions and densities. Since χ^2_g has density

$$f(x|g) = \frac{1}{2\Gamma\left(\frac{g}{2}\right)} \left(\frac{x}{2}\right)^{\frac{g}{2}-1} e^{-\frac{x}{2}}, x > 0 \tag{15}$$

making the transformation $x = \frac{S}{Z}$, we get the density

Table 1. Behavior of the function $f_0(y|g)$.

	0	Z_1	Z_0	Z_2
$f_0(Z\|g)$	↗	↗	↘	↘
$f_0'(Z\|g)$	+	+	–	–
$f_0''(Z\|g)$	+	–	–	+

$$f_0(Z|g) = k_g(s)Z^{-\frac{g+2}{2}}e^{-\frac{S}{2Z}}, Z > 0 \tag{16}$$

where $k_g(S) = \frac{S^{\frac{g}{2}}}{2^{\frac{g}{2}}\Gamma\left(\frac{g}{2}\right)}$. This density has derivatives

$$\begin{cases} f_0'(Z|g) = f_0(Z|g)\dfrac{S-(g+2)Z}{2Z^2}; Z > 0 \\ f_0''(Z|g) = f_0(Z|g)\dfrac{t(Z)}{4Z^4}; Z > 0 \end{cases} \tag{17}$$

where $t(Z) = S^2 - 2S(g+4)Z + (g+2)(g+4)Z^2$. Therefore

$$Z_0 = \frac{S}{g+2} > 0 \tag{18}$$

will be the only local extreme of $f_0(Z|g)$. Moreover, Z_0 will be a mode of $f_0(Z|g)$.

The zeros of $t(Z)$, which are inflection points of $f_0(Z|g)$, are

$$Z_i = \frac{S\left[(g+4) + (-1)^i\sqrt{2(g+4)}\right]}{(g+2)(g+4)}; i = 1, 2 \tag{19}$$

This points are equidistant of Z_0. If $Z_1 > 0$ we will have the monotony Table 1, where ↗ [↘] indicates increasing [decreasing] and the sign of the derivatives is indicated by + and –.

As it may be seen, inverse gamma densities are unimodal. We say that such densities belong to class $\mathcal{M}$.

Due to

$$\int_0^v u^m e^{-u} du = m!\left(1 - e^{-v}\sum_{j=0}^{m}\frac{m^{[j]}}{m!}v\right) \tag{20}$$

where $m^{[j]} = m\ldots(m-j+1)$, the induced distribution for $g = 2m$ will be

$$F_0(Z|2m) = e^{-\frac{S}{2Z}}\sum_{j=0}^{m-1}\frac{\left(\frac{S}{2Z}\right)^j}{j!}, Z > 0 \tag{21}$$

since, making the transformation $v = x^{-1}$ and $u = S\frac{v}{2}$, we will have

$$F_0(Z|g) = 1 - \frac{1}{\Gamma\left(\frac{g}{2}\right)}\int_0^{\frac{S}{2Z}} u^{\frac{g}{2}-1}e^{-u} du \tag{22}$$

Moreover, making the transformation $x = \frac{S}{2y}$ we have, for the l-th relative moment order to the origin,

Table 2. Behavior of function $m(Z|g)$.

	0	Z_0	$2Z_0$
$m(Z\|g)$	↗	↘	↘
$m'(y\|g)$	+	–	–
$m''(y\|g)$	–	–	+

$$\mu_l'(\cdot|g) = k_g(S)\int_0^{+\infty} x^{-\frac{g+2}{2}+l} e^{-\frac{S}{2x}} dx = \frac{S^l}{2^l}\frac{\Gamma\left(\frac{g}{2}-l\right)}{\Gamma\left(\frac{g}{2}\right)}; l < \frac{g}{2} \tag{23}$$

Namely we will get

$$\begin{cases} \mu_1(\cdot|g) = \dfrac{S}{2\left(\dfrac{g}{2}-1\right)} = \dfrac{S}{g-2}, g > 2 \\ \mu_2'(\cdot|g) = \dfrac{S^2}{4\left(\dfrac{g}{2}-1\right)\left(\dfrac{g}{2}-2\right)} = \dfrac{S^2}{(g-2)(g-4)}, g > 4 \end{cases} \tag{24}$$

Therefore, if $g > 4$,

$$\sigma^2(\cdot|g) = \frac{S^2}{(g-2)(g-4)} - \frac{S^2}{(g-2)^2} = \frac{2S^2}{(g-2)^2(g-4)} \tag{25}$$

We will now consider joint densities, establishing the

Proposition 3.1. *If $X_1, ..., X_m$ are independent variables, with unimodal densities $f_1(x_1), ..., f_m(x_m)$ and modes $\gamma_1, ..., \gamma_m$, their joint density $f(x^m)$ will be unimodal, with mode γ^m.*

Proof. We have

$$f(x^m) = \prod_{j=1}^{m} f_j(x_j) \le \prod_{j=1}^{m} f_j(\gamma_j) = f(\gamma^m)$$

for any x^m, which establishes the thesis. □

According to Proposition 3.1 the joint densities of the inverse gamma variables are unimodal.

Let

$$m(Z|g) = ln\left[f_0(Z|g)\right] \tag{26}$$

Thus the first and second derivative of $m(Z|g)$ will be given by

$$\begin{cases} m'(Z|g) = \dfrac{S-(g+2)Z}{2Z^2}; Z > 0 \\ m''(Z|g) = \dfrac{(g+2)Z-2S}{2Z^3}; Z > 0 \end{cases} \tag{27}$$

On Table 2 we may see the behavior of $m(Z|g)$, where, as before, ↗ [↘] indicates increasing [decreasing] and the sign of the derivatives is indicated by + and –.

If $r > 1$,

$$m(Z^r|g^r) = ln\left[\prod_{j=1}^{r} f(Z_j|g_j)\right] = \sum_{j=1}^{r} m(Z_j, g_j) \tag{28}$$

and thus

$$\begin{cases} \dfrac{\partial m(Z^r|g^r)}{\partial Z_j} = m'(Z_j|g_j) = \dfrac{S_j-(g_j+2)Z_j}{2Z_j^2}; j = 1, \ldots, r \\ \begin{cases} \dfrac{\partial^2 m(Z^r|g^r)}{\partial Z_j^2} = m''(Z_j|g_j) = \dfrac{(g_j+2)Z_j - 2S_j}{2Z_j^3}; j = 1, \ldots, r \\ \dfrac{\partial^2 m(Z^r|g^r)}{\partial Z_j \partial Z_l} = 0; j \neq l \end{cases} \end{cases} \tag{29}$$

Therefore we have the gradient

$$\mathbf{grad}\Big(m(Z|g)\Big) = \left[\frac{S_1-(g_1+2)Z_1}{2Z_1^2}, \ldots, \frac{S_r-(g_r+2)Z_r}{2Z_r^2}\right]' \tag{30}$$

and, representing by $D(a_1, \ldots, a_r)$ the diagonal matrix with principal elements $a_1, \ldots, a_r$, we have the hessian matrix

$$\mathbf{Hess}\Big(m(Z|g)\Big) = D\left(\frac{(g_1+2)Z_1-2S_1}{2Z_1^3}, \ldots, \frac{(g_r+2)Z_r-2S_r}{2Z_r^3}\right) \tag{31}$$

The only zero of the gradient, Z_0, has components $Z_{0,j} = \frac{S_j}{g_j+2}, j = 1, \ldots, r$. If $Z < 2Z_0 [> 2Z_0]$ the hessian matrix will be negative [positive] defined. Therefore, see Bazaraa, Sherali, and Shetty (1992), $m(Z|g)$ will be strictly concave [convex].

Now, consider

$$\nabla(a) = \{Z : m(Z|g) > a\} = \{Z : f(Z|g) > e^a\} \tag{32}$$

It may be seen that when $Z < 2Z_0 [> 2Z_0]$ and if $\nabla(a) \subset \times_{j=1}^{r} [0; 2Z_{0,j}], \nabla(a)$ will be a convex set. If $r = 2$, $f(Z|g)$ will have the level curves presented in Figure 1.

Consider now that there are some restrictions on the parameters: $\theta = (\theta_1, \ldots, \theta_r) \in \mathcal{D} \subset \mathbb{R}^r$. For example, variance components must to be non negative. In this case, taking

$$d = \int \underset{\mathcal{D}}{\cdots} \int f(Z|g) \prod_{j=1}^{r} dZ_j \tag{33}$$

the induced density will be given by

$$\begin{cases} f_0(Z|g; \mathcal{D}) = \dfrac{1}{d} f(Z|g); Z \in \mathcal{D} \\ f_0(Z|g; \mathcal{D}) = 0; Z \notin \mathcal{D} \end{cases} \tag{34}$$

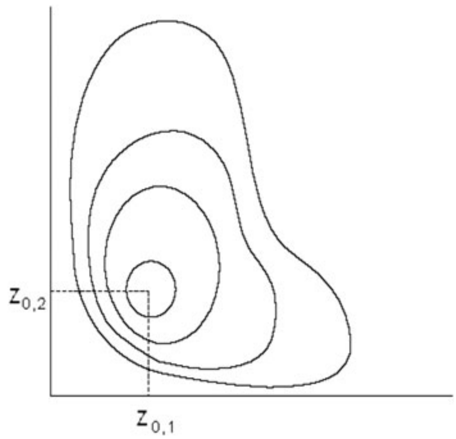


Figure 1. Level curves of $f(Z|g)$, when $r = 2$.

3.2. Linear combinations

In this section we will obtain analytic expressions for the densities of linear combinations of inverse gamma variables under certain conditions. Since the analytic treatment of these densities tend to be heavy we will consider an alternative approach.

Let us consider

$$V = \sum_{j=1}^{r} a_j Z_j = \sum_{j=1}^{r} a_j \frac{S_j}{\chi^2_{g_j}} \tag{35}$$

Proposition 3.2. *The linear combination of inverted gamma variables* $V = \sum_{j=1}^{r} a_j Z_j$ *have continuous densities* $f_V(v) > 0$ *whenever* $0 < F_V(v) < 1$.

Proof. We may rewrite V as $V = \sum_{i=1}^{d} b_i \dot{Z}_i$, keeping only the non null coefficients and taking $\dot{Z}_i = \frac{\dot{S}_i}{\chi^2_{\dot{g}_i}}, i = 1, ..., d$. These last variables have joint density $\prod_{i=1}^{d} f(Z_i|g_i)$. Since the transformation given by $\dot{Z}_1 = \frac{1}{b_1}(V - \sum_{i=2}^{d} b_i \dot{Z}_i)$ and by $\dot{Z}_i = U_i, i = 2, ..., d$, has jacobian $\frac{1}{b_1}$, the joint density of V and the U_i, $i = 2, ..., d$ will be

$$\frac{1}{|b_1|} f(v - \sum_{i=2}^{d} b_i u_i | \dot{g}_1) \prod_{i=2}^{d} f(u_i|\dot{g}_i)$$

So, V has density

$$f_V(v) = \int_0^{+\infty} ... \int_0^{+\infty} \frac{1}{|b|} f(v - \sum_{i=2}^{d} b_i u_i | \dot{g}_i) \prod_{i=2}^{d} f(u_i|\dot{g}_i) \prod_{i=2}^{d} du_i$$

which will be continuous and positive, since $f(u|\dot{g}_i) > 0$, for $u > 0$, $i = 1, ..., d$.

Thus, the distribution

$$F_V(v) = \int \overset{...}{_{\mathcal{D}(v)}} \int \prod_{i=1}^{d} f(Z_i|\dot{g}_i) \prod_{i=1}^{d} dZ_i$$

will be strictly increasing whenever $0 < F(v) < 1$. □

3.3. Densities for linear combinations

Making the transformation $x = \frac{a}{w}$ in

$$f_0(x|g) = \frac{1}{2\Gamma\left(\frac{g}{2}\right)}\left(\frac{x}{2}\right)^{\frac{g}{2}-1} e^{-\frac{x}{2}}; x > 0 \tag{36}$$

we obtain

$$f_0(w|g,a) = \frac{1}{2\Gamma\left(\frac{g}{2}\right)}\left(\frac{a}{2w}\right)^{\frac{g}{2}-1} e^{-\frac{a}{2w}}\frac{a}{w^2} = k(g,a)\frac{e^{-\frac{a}{2w}}}{w^{\frac{g}{2}+1}}; w > 0 \tag{37}$$

where $k(g,a) = \frac{\left(\frac{a}{2}\right)^{\frac{g}{2}}}{\Gamma\left(\frac{g}{2}\right)}$. Representing by $f_0^{(r)}$ the r-th order derivative of f_0, we get

$$f_0^{(1)}(w|g,a) = \frac{g(g+2)}{2a}\Big(f_0(w|g+4,a) - f_0(w|g+2,a)\Big) \tag{38}$$

as well as

$$f_0^{(r)}(w|g,a) = \sum_{j=r}^{2r} b_{r,j} f_0(w|g+2j,a); r = 0,1,\ldots \tag{39}$$

with $b_{1,1} = b_{1,2} = \frac{g(g+2)}{2a}$.

Since

$$f_0(w|g,a) \underset{w\to\pm\infty}{\longrightarrow} 0 \tag{40}$$

we get

$$f_0^{(r)}(w|g,a) \underset{w\to\pm\infty}{\longrightarrow} 0; r = 0,1,\ldots \tag{41}$$

Now, taking

$$f(z) = \int_0^z f_0(z-w|g_2,a_2) f_0(w|g_1,a_1)dw \tag{42}$$

according to (39) and (40) we get

$$f^{(r)}(z) = \int_0^z f_0^{(r)}(z-w|g_2,a_2) f_0(w|g_1,a_1)dw; r = 0,1,\ldots \tag{43}$$

see Taylor (1955), so that

$$f^{(r)}(0) = 0; r = 0,1,\ldots \tag{44}$$

Moreover, from (37) and (38), we also have

$$f_0^{(1)}(w|g,a) = \frac{1}{\Gamma\left(\frac{g}{2}\right)}\left(\frac{a}{2}\right)\frac{g}{2}\frac{e^{-\frac{a}{2w}}}{w^{\frac{g}{2}+1}}\left[\frac{a}{2w^2} - \frac{\frac{g}{2}+2}{2w}\right] \tag{45}$$

Thus

$$\begin{cases} f_0^{(1)}(w|g,a) \geq 0 \text{ if } & w \leq \dfrac{a}{g+2} \\ f_0^{(1)}(w|g,a) \leq 0 \text{ if } & w \geq \dfrac{a}{g+2} \end{cases} \tag{46}$$

If $z \leq \frac{a_2}{g_2+2}$ and $0 \leq w \leq z$ there will be $0 \leq z-w \leq z \leq \frac{a_2}{g_2+2}$ and so $f_0^{(1)}(z-w|g_2,a_2) > 0$. Thus, according to (43)

$$f^{(1)}(z) > 0; z \leq \frac{a_2}{g_2+2} \tag{47}$$

shifting the roles of $f_0(\cdot|g_1,a_1)$ and $f_0(\cdot|g_2,a_2)$ in expression (43) we will obtain

$$f^{(1)}(z) > 0; z \leq max\left\{\frac{a_1}{g_1+2}; \frac{a_2}{g_2+2}\right\} \tag{48}$$

Moreover, from the expressions (39) and (43), we get

$$f^{(r)}(z) = \sum_{j=r}^{2r} b_{r,j} \int_0^z f_0(z-w|g_2+2j,a_2) f_0(w|g_1,a_1)dw, r = 0,1,\ldots \tag{49}$$

Since

$$f_0(w|g,a) \leq f_0\left(\frac{a}{g+2}|g,a\right) = \frac{2\left(\frac{g}{2}+1\right)^{\frac{g}{2}+1}}{a_2\Gamma\left(\frac{g}{2}\right)} e^{-\left(\frac{g}{2}+1\right)} \tag{50}$$

we have, with a fixed d and $z > d$

$$\begin{aligned} |f^{(r)}(z)| &\leq \sum_{j=r}^{2r} |b_{r,j}| \int_0^z f_0(z-w|g_2+2j,a_2) f_0(w|g_1,a_1)dw \leq \\ &\leq \sum_{j=r}^{2r} |b_{r,j}| \Big(\int_0^d f_0(z-w|g_2+2j,a_2) f_0(w|g_1,a_1)dw + \\ &\quad + f_0\left(\frac{a_2}{g_2+2j+2}|g_2+2j,a_2\right)(1-F_0(d|g_1,a_1))\Big), r = 0,1,\ldots \end{aligned}$$

where $F_0(d|g_1,a_1)$ is the distribution of $f_0(w|g_1,a_1)$.

We may take d so that

$$\sum_{j=r}^{2r} |b_{r,j}| f_0\left(\frac{a_2}{g_2+2j+2}|g_2+2j,a_2\right)(1-F^0(d|g_1,a_1)) < \epsilon; r = 0,1,\ldots \tag{51}$$

for any $\epsilon > 0$. Thus, for $z > d$,

$$|f^{(r)}(z)| < \sum_{j=r}^{2r} |b_{r,j}| \int_0^d f_0(z-w|g_2+2j,a_2) f_0(w|g_1,a_1)dw + \epsilon; r = 0,1,\ldots \tag{52}$$

Since $f_0(z-w|g_2+2j,a_2) \xrightarrow[z\to+\infty]{} 0$, for any $w \in [0,d]$, we may apply, according to (50), the dominated convergence theorem to show that

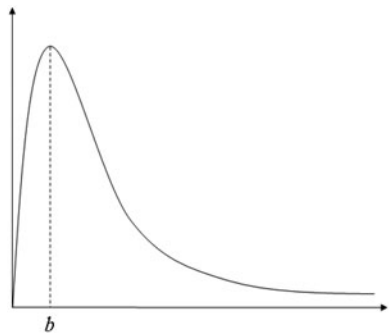


Figure 2. Unimodal density.

$$\sum_{j=r}^{2r} |b_{r,j}| \int_0^d f_0(z-w|g_2+2j, a_2) f_0(w|g_1, a_1)dw \underset{z \to +\infty}{\longrightarrow} 0; r = 0, 1, \ldots \tag{53}$$

and since ϵ is arbitrary we will have

$$f^{(r)}(z) \underset{z \to +\infty}{\longrightarrow} 0; r = 0, 1, \ldots \tag{54}$$

Summarizing, we have

$$\begin{cases} f^{(r)}(0) = 0; r = 0, 1, \ldots \\ f^{(1)}(z) > 0; 0 \leq z \leq max\left\{\dfrac{a_1}{g_1+2}; \dfrac{a_2}{g_2+2}\right\} \\ f^{(r)}(z) \underset{z \to +\infty}{\longrightarrow} 0; r = 0, 1, \ldots \end{cases}$$

A class of simple densities for which this conditions hold is that of the unimodal densities that are null at the origin and have support contained in $\mathbb{R}^+$. Let us admit that the mode is reduced to one point b and that the density increases [decreases] to the left [right] of that point. Then, the density is as shown in Figure 2. Let $\mathcal{M}$ be the class of these densities, so that $f^{(r)}(0) = 0$ and $f^{(r)}(z) \underset{z \to +\infty}{\longrightarrow} 0, r = 0, 1, \ldots$ and that exists z^+ so that, for $z < z^+$, $f^1(z) > 0$. Then, as we saw, the densities $f_0(\cdot|g, a)$ and the one obtained by convolution belong to class $\mathcal{M}$.

Let now

$$f = f_0(\cdot|g_{m+1}, a_{m+1}) \times f_{00} \tag{55}$$

where f_{00} is the convolution of the densities f_0 given by

$$f_{00} = *_{j=1}^m f_0(\cdot|g_j, a_j) \in \mathcal{M} \tag{56}$$

Note that f_{00} has order d moments defined if, and only if, every $f_0(\cdot|g_j, a_j), j = 1, \ldots, m$ has. That is,

$$\bigwedge_{j=1}^{m} g_j > 2d \tag{57}$$

where $\bigwedge_{j=1}^{m} g_j$ is the minimum of $g_1, ..., g_m$. Namely, the variance is defined for f_{00} if and only if $\bigwedge_{j=1}^{m} g_j > 4$.

4. Empiric distributions

Let $F(x)$ $[F_n(x)]$ be a continuous and strictly increasing function and x_p $[x_{n,p}]$ the corresponding quantile for probability p. Representing by $\overset{a.s.}{\rightarrow}$ almost surely convergence we have the

Proposition 4.1. *If $F(x)$ has a continuous density $f(x)$ and if $f(x) > 0$ whenever $0 < F(x) < 1$, then for any $\alpha \in]0;1[$ we have*

$$D_{n,\alpha} = Sup\{|x_{n,p} - x_p|; \frac{\alpha}{2} < p < 1 - \frac{\alpha}{2}\} \underset{n\to\infty}{\overset{a.s.}{\rightarrow}} 0$$

Proof. A ccording to Weierstrass theorem, f has a minimum $b > 0$ in the interval $[x_{\frac{\alpha'}{2}}; x_{\frac{1-\alpha'}{2}}]$. If

$$\frac{\alpha'}{2} < p - \frac{\epsilon}{b} < p + \frac{\epsilon}{b} < 1 - \frac{\alpha'}{2}$$

we have

$$F\left(x_p - \frac{\epsilon}{b}\right) < p - \epsilon < p + \epsilon < F\left(x_p + \frac{\epsilon}{b}\right)$$

So, when $D_{n,\alpha} < b$,

$$F_n\left(x_p - \frac{\epsilon}{b}\right) < p < F_n\left(x_p + \frac{\epsilon}{b}\right)$$

and we get

$$x_p - \frac{\epsilon}{b} < x_{n,p} < x_p + \frac{\epsilon}{b}$$

This establishes the thesis since from ϵ being arbitrary we may take $\alpha = \alpha' + 2\frac{\epsilon}{b}$ to get

$$|x_{n,p} - x_p| < \frac{\epsilon}{b}$$

whenever $\frac{\alpha}{2} < p < 1 - \frac{\alpha}{2}$. □

Taking independent and identically distributed (*iid*) variables $Z_j = (z_{1,j}, ..., z_{k,j}), j = 1, ..., r$, and a measurable function g, the $X_j = g(Z_j), j = 1, ..., r$ will constitute a sample for which we can apply the Glivenko-Cantelli theorem, see Loève (1960), as well as the Proposition 4.1, if the necessary conditions hold.

Let

$$X = C^T Z \tag{58}$$

So, the quantiles x_q will correspond to hyperplanes defined in $\mathbb{R}^k$ and, with $q_1 < q_2$, the section limited by the hyperplanes corresponding to x_{q_1} and x_{q_2} will have induced probability $q_2 - q_1$. It may be seen that if the distribution of Z has a continuous density, the distribution of X will have a continuous density too. If the density of Z is not null, when the respective distribution takes values between 0 and 1, the conditions of the Proposition 4.1 hold. So, we may estimate x_q and thus estimate the respective hyperplanes.

5. Application to mixed models

In this section we present an application of our approach considering mixed models.

5.1. Variance components

Given the mixed model

$$Y = \sum_{i=0}^{w} X_i \beta_i \tag{59}$$

where β_0 is fixed and the $\beta_1, ..., \beta_w$ are independent with null mean vectors and variance-covariance matrices $\sigma_1^2 I_{c_1}, ..., \sigma_w^2 I_{c_w}$.

The mean vector and variance-covariance matrix of Y will be

$$\begin{cases} \mu = X_0 \beta_0 \\ V = \sum_{i=1}^{w} \sigma_i^2 M_i \end{cases} \tag{60}$$

with $M_i = X_i X_i^T, i = 1, ..., w$.

Let T be the orthogonal projection matrix on the range space of X_0. When matrices $M_1, ..., M_w$ and T commute we have, see Fonseca, Mexia, and Zmyślony (2008),

$$\begin{cases} T = \sum_{j=1}^{z} Q_j \\ M_i = \sum_{j=1}^{m} b_{i,j} Q_j \end{cases} \tag{61}$$

where the $Q_1, ..., Q_m$ are pairwise orthogonal orthogonal projection matrices. Then

$$V = \sum_{j=1}^{m} \gamma_j Q_j \tag{62}$$

with

$$\gamma_j = \sum_{i=1}^{w} b_{i,j} \sigma_i^2 \tag{63}$$

Moreover, taking

$$\sigma^2 = \begin{bmatrix} \sigma_1^2 \\ \vdots \\ \sigma_w^2 \end{bmatrix}; \quad \gamma(1) = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_z \end{bmatrix}; \quad \gamma(2) = \begin{bmatrix} \gamma_{z+1} \\ \vdots \\ \gamma_m \end{bmatrix} \tag{64}$$

and considering for matrix $B = [b_{i,j}]$ the partition

$$B = [B(1) \quad B(2)] \tag{65}$$

where $B(1)$ has z columns, we have

$$\gamma(l) = B^T(l)\sigma^2, l = 1, 2 \tag{66}$$

When the row vectors of $B(2)$ are linearly independent we have

$$\sigma^2 = C\gamma(2) \tag{67}$$

with C the MOORE-PENROSE inverse of $B^T(2)$.

Taking $C = [c_{i,j}]$, let C_i^+ and C_i^- be the sets of column indexes of the positive and negative elements of the i-th row of matrix C. Thus, with $\dot{m} = m - z$ and $\dot{\gamma}_j = \gamma_{j+z}, j = 1, \ldots, \dot{m}$,

$$\sigma_i^2 = \sum_{j=1}^{\dot{m}} c_{i,j}\dot{\gamma}_j = (\sigma_i^2)^+ - (\sigma_i^2)^-, i = 1, \ldots, w \tag{68}$$

where

$$\begin{cases} (\sigma_i^2)^+ = \sum_{j\in C_i^+} c_{i,j}\dot{\gamma}_j, i = 1, \ldots, w \\ (\sigma_i^2)^- = \sum_{j\in C_i^-} |c_{i,j}|\dot{\gamma}_j, i = 1, \ldots, w \end{cases} \tag{69}$$

with $|\cdot|$ describing the determinant.

These results are interesting since, with

$$S_j = Y^T Q_{j+z} Y, j = 1, \ldots, \dot{m} \tag{70}$$

we have the unbiased estimators

$$\dot{\gamma}_j = \frac{S_j}{g_j}, j = 1, \ldots, \dot{m} \tag{71}$$

where

$$g_j = \text{rank}(Q_{j+z}), j = 1, \ldots, \dot{m} \tag{72}$$

Thus we also will have unbiased estimators for the $\sigma_i^2, i = 1, \ldots, w$ and their positive and negative parts.

The distribution of S_j, $j = 1, \ldots, \dot{m}$, in (70) depends on the distribution of Y. For example, if Y is distributed as a t-Student distribution with g degrees of freedom, t_g, then S_j has a non-central F distribution, $F(|1, g, \delta)$, where δ is the non-centrality parameter and 1 and g are the degrees of freedom. However, when we deal with linear models it is usual to assume that Y is normal distributed, which means that in this case we have

$$Y \sim N(\mu, V) \tag{73}$$

and, consequently,

$$S_j \sim \dot{\gamma}_j \chi^2_{g_j}, \quad j = 1, \ldots, \dot{m} \tag{74}$$

This is, S_j is distributed as the product by $\dot{\gamma}_j$ of a central chi-square with g_j degrees of freedom, $j = 1, \ldots, \dot{m}$. So due to (13) and (14) the $Z_j = \frac{S_j}{\dot{\gamma}_j}, j = 1, \ldots, \dot{m}$ will be inducing pivot variables.

Now we may induce probability measures for the γ_j, $i = 1, \ldots, \dot{m}$, using large samples $\{G_1, \ldots, G_n\}$, with $G_{u,j} \sim \chi^2_{g_j, p_j}, j = 1, \ldots, \dot{m}, u = 1, \ldots, n$, and derive from these secondary samples $\{Z_1, \ldots, Z_n\}$ in which Z_u has components

$$Z_{u,j} = \frac{S_j}{G_{u,j}} \tag{75}$$

$j = 1, \ldots, \dot{m}, u = 1, \ldots, n$. Note that $Z_{1,j}, \ldots, Z_{n,j}$ are *iid*, with distribution induced by Z_j, $j = 1, \ldots, \dot{m}$.

5.2. Numerical application

5.2.1. Application to real data

We considered an experiment in which two groups of two clones, grown side by side, of the caste Touriga Nacional. The origins of the two groups were distinct. In this case there is a fixed effects factor (location) that crosses with another fixed effects factor (origin) which nests a random effects factor (clone). The yields, in Kg, are presented in Table 3. These data were presented in Fonseca, Mexia, and Zmyślony (2003) in an application to a grapevine experiment.

The model for the observations vector

$$Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_{90} \end{bmatrix} \tag{76}$$

is given by

$$Y = X\beta_0 + X_1\beta_1 + X_2\beta_2 + e \tag{77}$$

where β_0 is a fixed vector, $rank(X) = 12$ and $\beta_1 \sim \mathcal{N}(0, \sigma_1^2 I_{18})$, $\beta_2 \sim \mathcal{N}(0, \sigma_2^2 I_6)$ and $e \sim \mathcal{N}(0, \sigma^2 I_{90})$ are independent.

Following Fonseca, Mexia, and Zmyślony (2003) we obtained the statistics S_j, $j = 1, \ldots, 6$, given by

$$\begin{cases} S_1 = 0.6748 \\ S_2 = 9.6105 \\ S_3 = 6.2590 \\ S_4 = 3.4464 \\ S_5 = 3.4158 \\ S_6 = 26.4333 \end{cases} \tag{78}$$

where the indexes $j = 1, 2, 3$ correspond to the 1st, 2nd and 3rd factors, $j = 4, 5$ correspond to the interactions between the 1st and 2nd factor and the 1st and 3rd factor and $j = 6$ corresponds to the error.

Assuming normality, from (74), we have $S_j \sim \dot{\gamma}_j \chi^2_{g_j}, j = 1, \dots, 6$, where $g_1 = 2, g_2 = 1, g_3 = 4, g_4 = 2, g_5 = 8$ and $g_6 = 72$. Following Fonseca, Mexia, and Zmyślony (2003) again, we obtained the UMVUE for the variance components given by

$$\begin{cases} \tilde{\sigma}_1^2 = 0.07585 \\ \tilde{\sigma}_2^2 = 0.01197 \end{cases} \tag{79}$$

Now, from (68) and (69) we may write

$$\begin{cases} \tilde{\sigma}_1^2 = (\tilde{\sigma}_1^2)^+ - (\tilde{\sigma}_1^2)^- \\ \tilde{\sigma}_2^2 = (\tilde{\sigma}_2^2)^+ - (\tilde{\sigma}_2^2)^- \end{cases} \tag{80}$$

where

$$\begin{cases} (\tilde{\sigma}_1^2)^+ = \frac{1}{15}\tilde{\dot{\gamma}}_3; (\tilde{\sigma}_1^2)^- = \frac{1}{15}\tilde{\dot{\gamma}}_5 \\ (\tilde{\sigma}_2^2)^+ = \frac{1}{5}\tilde{\dot{\gamma}}_5; (\tilde{\sigma}_2^2)^- = \frac{1}{5}\tilde{\dot{\gamma}}_6 \end{cases} \tag{81}$$

and thus have

$$\begin{cases} (\tilde{\sigma}_1^2)^+ = 0.10432; (\tilde{\sigma}_1^2)^- = 0.02847 \\ (\tilde{\sigma}_2^2)^+ = 0.08540; (\tilde{\sigma}_2^2)^- = 0.07343 \end{cases} \tag{82}$$

Then, we used the mathematical software Maple 12 to generate sets G_u, $u = 1, 2, 3$, with components $G_{i,u} \sim \chi^2_{g_u}$, $u = 1, 2, 3$, $i = 1, \dots, 10000$, with $g_1 = 4, g_2 = 8, g_3 = 72$, and derived from these samples W_k, $k = 1, 2$, with components

$$\begin{cases} W_{i,1}{}^+ = \frac{1}{15}Z_{i,1}; W_{i,1}{}^- = \frac{1}{15}Z_{i,2}; i = 1, \dots, 10000 \\ W_{i,2}{}^+ = \frac{1}{5}Z_{i,2} - Z_{i,3}; W_{i,2}{}^- = \frac{1}{5}Z_{i,3}; i = 1, \dots, 10000 \end{cases} \tag{83}$$

where $Z_{i,u} = \frac{S_u}{G_{i,u}}$, $u = 1, 2, 3$, $i = 1, \dots, 10000$.

The densities corresponding to the probability measures induced by W_1 and W_2 are presented in Figures 3 and 4, respectively, as well as the corresponding UMVUE, which is represented by a black dot with coordinates $(\tilde{\sigma}_k^2, 0)$, $k = 1, 2$.

These graphs were built from histograms. The breadths of the classes are 0.025544, for Figure 3 and 0.018950 for Figure 4. The central point of the modal class for $\tilde{\sigma}_1^2$ and $\tilde{\sigma}_2^2$ are 0.0336 and 0.0065, respectively.

As it may be seen in (80), the estimators of the variance components have a positive and a negative part. The empirical joint densities induced for both parts are represented in Figures 5 and 6, as well as the black point, which coordinates in the *xoy* plane are

Table 3. Yields in Kg.

	Origin 1			Origin 2		
Location	Clone 1	Clone 2	Clone 2	Clone 1	Clone 2	Clone 3
1	3.00	1.00	1.10	1.75	1.10	1.05
	1.85	1.10	1.50	3.50	1.05	1.25
	0.75	1.00	1.80	2.50	0.50	2.00
	1.35	1.60	1.45	2.00	1.05	1.50
	1.45	1.50	1.25	0.65	1.25	2.10
2	1.80	1.60	0.85	2.00	1.20	1.00
	0.70	1.75	0.65	3.00	1.35	2.70
	2.50	0.50	0.55	2.55	1.20	2.15
	1.70	1.35	0.90	3.00	0.30	2.10
	0.40	1.10	0.90	2.65	2.50	2.70
3	1.05	0.75	0.90	1.60	1.05	1.60
	1.50	0.65	0.90	3.05	1.95	1.10
	1.15	0.90	0.55	0.25	2.00	2.05
	0.85	0.85	0.70	1.66	2.20	1.50
	1.15	1.05	0.35	2.65	2.35	3.00

$((\tilde{\sigma}_k^2)^+, (\tilde{\sigma}_k^2)^-, h)$, $k = 1, 2$, where the value of h varies from chart to chart, so that it is visible on top of the bars.

It can be seen that, using the real data presented in Table 3, the densities presented in Figures 3 and 4 and the joint densities in Figures 5 and 6 are all unimodal with the UMVUE near the mode.

5.2.2. Application to simulated data

After using real data we repeated the study with simulated data. We simulated five times the components, Y_r, $r = 1, \dots, 90$, of the observations vector in (76) obtaining new observation vectors

$$\dot{Y}_s = \begin{bmatrix} \dot{Y}_{1,s} \\ \vdots \\ \dot{Y}_{90,s} \end{bmatrix} \tag{84}$$

$s = 1, \dots, 5$

In a first approach we used the Normal distribution, with

$$\dot{Y}_{r,1} \sim N\left(Y_r, \frac{Y_r}{3}\right) \tag{85}$$

$r = 1, \dots 90$. In a second and third approach, we simulated the components of Y using the non-central Student's t distribution, $t_{p,\delta}$, where p is the number of degrees of freedom and δ the non-centrality parameter. In the second approach we considered $p = 5$ and $\delta_r = 10 \times Y_r, r = 1, \dots, 90$, whereas in the third approach we considered $p = 50$ and again $\delta_r = 10 \times Y_r, r = 1, \dots, 90$. So

$$\begin{cases} \dot{Y}_{r,2} \sim t_{5,10\times Y_r} \\ \dot{Y}_{r,3} \sim t_{50,10\times Y_r} \end{cases} \tag{86}$$

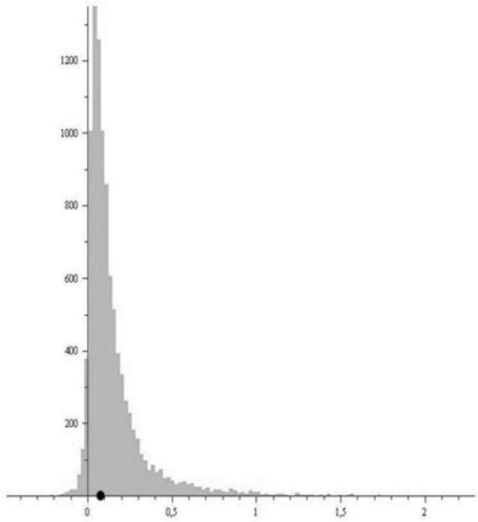


Figure 3. Location of the *UMVUE* on the empirical density of the 3^{rd} factor.

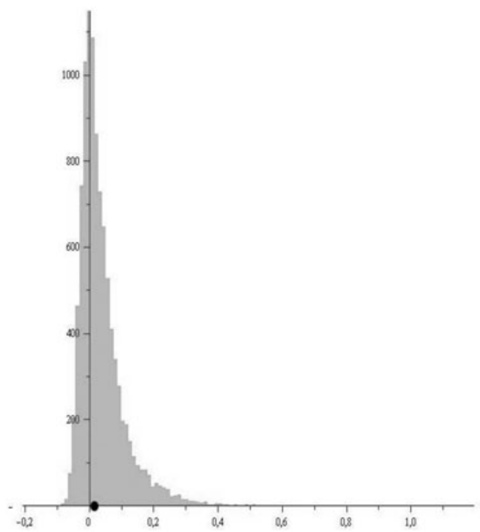


Figure 4. Location of the *UMVUE* on the empirical density of the interaction $1^{st} \times 3^{rd}$ factor.

$r = 1, \ldots 90$. In a fourth approach we used the non-central Chi-square distribution, $\chi^2_{p,\delta}$, where p is the number of degrees of freedom and δ the non-centrality parameter, with

$$\dot{Y}_{r,4} \sim \chi^2_{5,\,Y_r} \tag{87}$$

$r = 1, \ldots 90$. Finally, in a fifth approach we used the Cauchy distribution *Cauchy*$(t,\ s)$, with location parameter $t = 3 \times Y_r$ and scale parameter $s = 0.3$. So

$$\dot{Y}_{r,5} \sim Cauchy(3 \times Y_r,\ \ 0.3) \tag{88}$$

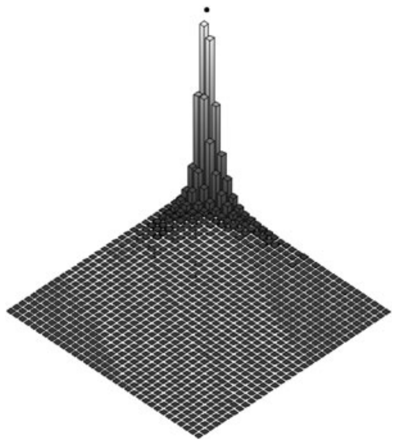


Figure 5. Location of the *UMVUE* on the bidimensional empirical density of the 3^{rd} factor.

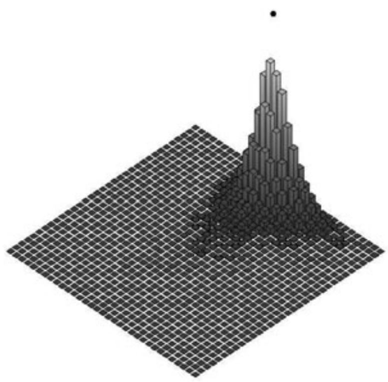


Figure 6. Location of the *UMVUE* on the bidimensional empirical density of the interaction $1^{st} \times 3^{rd}$ factor.

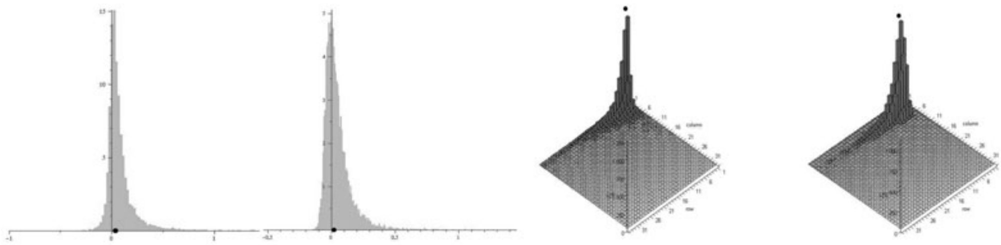
$r = 1, \ldots 90$. The data is presented in Table A1, in the Appendix. In Table 4 are presented the corresponding sum of squares and the UMVUE for the variance components, for the five simulations.

As above, we used the mathematical software Maple 12 to generate sets G_u, $u = 1, 2, 3$, with components $G_{i,u} \sim \chi^2_{g_u}$, $u = 1, 2, 3$, $i = 1, \ldots, 10000$, with $g_1 = 4, g_2 = 8, g_3 = 72$, and derived from these samples W_k, $k = 1, 2$, with components as in (83), for each simulation. The densities corresponding to the probability measures induced by W_k, $k = 1, 2$, and the empirical joint densities induced for the positive and negative parts, ${W_{i,1}}^+$ and ${W_{i,1}}^-$, are presented in the next figures. The figures also contain the corresponding UMVUE, which is represented by a black dot. As before, the coordinates of the black dot are $(\tilde{\sigma}^2_k, 0)$, $k = 1, 2$, for the two-dimensional charts and $((\tilde{\sigma}^2_k)^+, (\tilde{\sigma}^2_k)^-, h)$, $k = 1, 2$, for the three-dimensional charts. The value of h varies from chart to chart, so that it is visible on top of the bars.

Table 4. Sums of squares and UMVUE for the variance components.

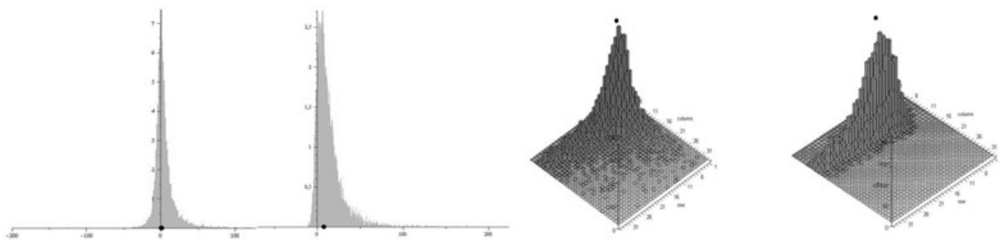
Simulations:	1^{st}	2^{nd}	3^{rt}	4^{th}	5^{th}
S_1	4.491	442.383	629.271	174.755	72.965
S_2	5.019	760.097	602.456	351.899	29.410
S_3	41.671	4188.057	3347.747	1860.036	273.154
$\tilde{\sigma}_1^2$	0.03303	1.03891	5.46738	−0.01991	0,97100
$\tilde{\sigma}_2^2$	0.00972	7.36894	5.76210	3.63071	−0,02351

Simulation 1:



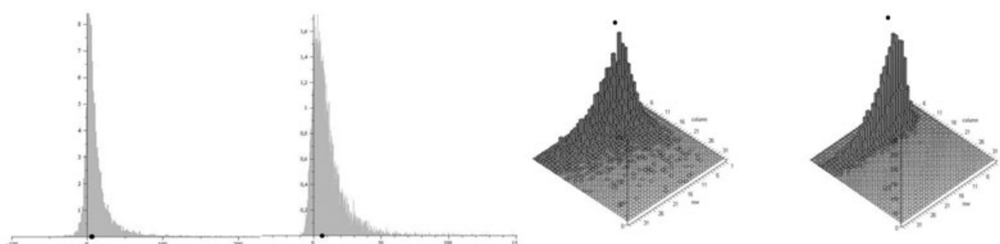
3^{rd} factor $1^{st} \times 3^{rd}$ factor 3^{rd} factor $1^{st} \times 3^{rd}$ factor

Simulation 2:



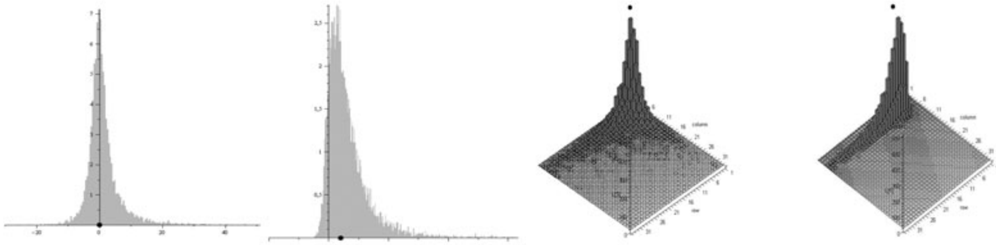
3^{rd} factor $1^{st} \times 3^{rd}$ factor 3^{rd} factor $1^{st} \times 3^{rd}$ factor

Simulation 3:



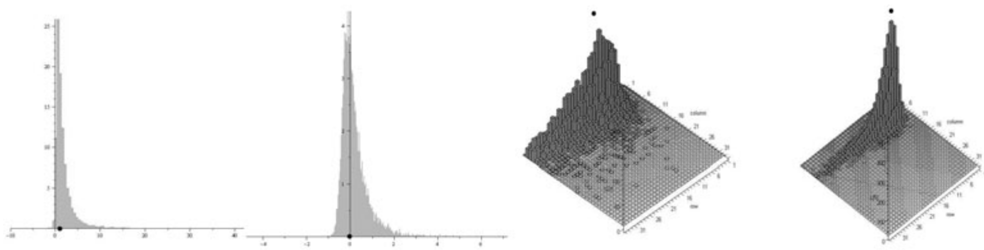
3^{rd} factor $1^{st} \times 3^{rd}$ factor 3^{rd} factor $1^{st} \times 3^{rd}$ factor

Simulation 4:



3^{rd} factor $1^{st} \times 3^{rd}$ factor 3^{rd} factor $1^{st} \times 3^{rd}$ factor

Simulation 5:



3^{rd} factor $1^{st} \times 3^{rd}$ factor 3^{rd} factor $1^{st} \times 3^{rd}$ factor

It can be seen that the densities and the joint densities are all unimodal with the UMVUE near the mode, as already noted for the real data presented in Table 3. These results clearly point to the robustness of the method we presented, since we obtained similar results to those of the normal case with observations generated from the t-distribution, Chi-square distribution and Cauchy distribution.

6. Final remarks

In this work we have combined two lines of inference based on the use of complete sufficient statistics for orthogonal mixed linear models. The first of these lines, initiated by Seely (1970), Seely (1971), Seely and Zyskind (1971), Seely (1972) and Seely (1977), leads to obtaining UMVUE. The second line, where the inducing pivot variables may be included, see Weerahandi (1993) and Weerahandi (1996), uses sufficient statistics to induce densities in the parametric space.

The combination of the two lines was based on the use of UMVUE to validate the simulations based on the induced densities. Note that this validation was based on a qualitative study of these densities, which has highlighted its unimodality. Moreover, the bidimensional graphs of the empiric densities are in accordance with the qualitative study of these densities. Furthermore, the observed proximity between the point whose coordinates are the UMVUE and the mode validates the numerical results. This validation is important because it allows the safe use of induced densities, through methods of Monte-Carlo, to obtain confidence regions, and by duality to carry out tests.

Note that these techniques are applicable whatever the degrees of freedom of the chi-squares. Thus, it is not necessary to have even degrees of freedom in the denominator or the numerator of the generalized F statistics, see Fonseca, Mexia, and Zmyślony (2002). Moreover, the problem of disturbing parameters is automatically overcome through the use of induced densities.

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Appendix

Table A1. Simulated yields in Kg.

1^{st} simulation		2^{nd} simulation		3^{rd} simulation		4^{th} simulation		5^{th} simulation	
3.56	2.13	21.98	26.32	32.98	16.95	17.09	8.76	8.90	3.64
1.15	3.30	25.75	32.77	17.51	42.82	32.51	3.63	5.59	10.56
0.72	1.42	6.65	27.28	6.56	28.06	3.90	5.96	2.52	7.63
1.87	2.24	16.27	9.53	10.83	18.31	5.11	12.21	4.59	5.59
1.39	0.41	19.56	5.26	12.21	6.37	5.19	8.77	3.65	2.03
1.15	1.34	17.88	28.05	18.15	24.07	11.36	19.87	5.89	6.02
1.36	3.51	9.72	23.90	6.49	35.48	3.06	6.20	2.12	9.37
2.02	2.51	32.76	27.80	23.69	23.18	3.54	2.28	10.17	7.37
1.89	3.20	9.24	18.92	20.07	36.52	2.07	4.27	4.61	8.80
0.55	1.72	5.61	21.58	3.69	23.60	2.27	3.58	1.09	7.51
1.38	1.90	7.30	16.92	8.72	17.57	1.29	8.91	3.34	5.98
1.27	4.27	10.69	27.53	15.86	23.75	17.83	2.04	3.94	9.14
1.39	0.21	18.51	3.26	12.24	2.27	2.88	2.22	3.67	0.83
0.81	0.84	9.76	9.16	8.06	14.71	4.12	4.92	2.17	4.80
1.32	1.81	11.99	23.54	13.55	27.04	7.26	2.95	3.44	7.94
1.06	0.54	5.13	10.06	8.25	10.89	3.02	1.94	3.37	0.83
1.41	0.28	9.55	12.49	11.96	11.33	10.19	3.46	6.15	3.10
1.07	0.26	11.97	3.87	11.74	5.83	1.73	5.24	3.05	1.23
1.60	0.92	16.92	26.33	18.05	8.07	4.26	2.02	4.92	2.85
1.55	1.35	14.89	11.46	15.21	14.23	1.29	4.33	4.09	3.91
2.48	1.71	15.98	9.23	14.76	12.44	2.74	2.99	4.31	3.79
1.37	1.19	19.57	14.32	20.77	15.59	6.73	9.26	1.78	3.44
0.41	0.97	4.36	8.25	3.98	10.86	2.49	13.64	1.75	3.96
0.96	0.44	16.66	2.34	15.58	1.76	2.80	5.44	4.50	0.35
1.24	2.26	8.87	21.84	9.70	23.06	1.90	12.52	3.96	6.65
0.97	1.17	11.58	13.90	8.21	11.60	4.55	9.11	4.31	3.06
0.90	2.71	5.97	20.18	6.77	21.81	4.02	14.67	3.23	5.71
0.95	2.27	9.45	23.25	10.44	21.96	4.73	12.96	2.70	6.11
1.37	2.99	4.35	25.62	7.44	25.27	3.24	2.76	2.33	6.57
1.28	2.30	10.28	16.85	8.66	21.82	1.59	12.73	3.25	6.92
0.76	0.82	10.18	17.08	11.50	9.35	6.45	3.30	2.48	2.29
0.79	0.86	27.17	15.60	18.85	13.91	5.89	5.84	4.23	3.01
1.84	1.97	42.32	27.82	16.10	17.29	9.72	15.37	5.11	6.01
2.05	1.18	16.77	8.67	15.22	13.53	1.79	3.67	4.21	4.44
1.42	1.79	13.74	22.20	15.84	21.24	6.30	3.60	2.99	6.22
1.45	1.18	11.01	21.86	12.65	9.42	10.72	7.85	2.51	3.86
0.78	3.53	7.74	27.84	7.27	31.17	5.96	8.81	2.33	8.13
0.66	1.07	3.34	26.15	6.75	18.84	18.52	2.75	1.65	6.33
0.78	2.07	18.98	28.32	9.73	22.63	4.72	16.85	2.85	6.52
1.71	2.58	5.39	23.84	8.10	23.17	9.52	13.88	3.04	8.43
0.65	2.39	9.41	21.64	8.62	17.29	7.51	13.01	2.72	5.19
1.33	1.40	9.89	8.24	8.92	10.65	3.70	5.20	3.02	3.56
0.45	2.77	4.07	13.76	3.62	23.14	8.51	2.36	0.53	6.20
0.49	1.76	12.01	12.11	6.10	14.58	9.46	7.54	1.82	4.49
0.52	3.10	6.86	34.56	3.80	30.84	1.68	4.99	0.72	8.57