Intersection of a Double Cone and a Line in the Split-Quaternions Context

R. Serôdio, P. D. Beites and José Vitória

Abstract

This is a work on an application of the real split-quaternions to Spatial Analytic Geometry. Concretely, the intersection of a double cone and a line, which can be the empty set, a point, two points or a line, is studied in the real split-quaternions setting.

Mathematics Subject Classification (2010). 15A66, 51P05, 83A05
Keywords. double cone, line, intersection, split-quaternion, event, direction

1 Introduction

Among the problems that can be approached in Spatial Analytic Geometry, some concern intersections of geometric 3D objects. For instance, the intersection of a cone and a line is one of them. Beyond its intrinsic mathematical interest, the latter problem is also relevant in other areas such as computer graphics, motion planning and collision detection.

There is a series of books, called *Graphics Gems*, that provides tools for the graphics community to face real programming issues. In [3], the intersection of a line and a cylinder was treated. An extension of this work appeared in [1], where the intersection of a line and a cone was addressed.

In the present text, we study the intersection of a double cone and a line in the real split-quaternions context. We collect some definitions, notations and results in section 2. Through the norm of the split-quaternion that gives the line direction, we characterize intersection conditions in section 3. Moreover, we present explicit formulas for computing the intersection if it is nonempty.

2 The Real Split-Quaternions

Let F be an arbitrary field and U be a finite-dimensional vector space over F.

The vector space U over F is a *semi-Euclidean* (or *pseudo-Euclidean*) *space* if $F = \mathbb{R}$ and U is equipped with a non-degenerate indefinite bilinear form. As in the positive definite case, the mentioned form is called the *inner product* and we denote it by $\langle \cdot, \cdot \rangle$.

If U is a semi-Euclidean space, then an element $x \in U$ is said to be *spacelike*, *lightlike* or *timelike* if $\langle x, x\rangle > 0$, $\langle x, x\rangle = 0$ or $\langle x, x\rangle < 0$, respectively. The *light cone* is the set of all lightlike elements. As in the definite case, two elements $x, y \in U$ are said to be *orthogonal* if $\langle x, y\rangle = 0$. So, the light cone consists of all elements that are orthogonal to themselves.

A vector space homomorphism $\varphi : U \to U$ is called an *involution* of U if, for all $u, v \in U$, $\varphi(\varphi(u)) = u$ and $\varphi(uv) = \varphi(v)\varphi(u)$.

The vector space U over F is an *algebra over* F if U is equipped with a bilinear map $s : U^2 \to U$, usually called *multiplication*. Given an algebra U, with multiplication denoted by juxtaposition, we now recall a few more concepts related to composition algebras assuming, from now on, that $\mathrm{ch}(F) \neq 2$.

U is a *composition algebra* over F if it is endowed with a nondegenerate quadratic form (the *norm*) $n : U \to F$ (that is, the associated symmetric bilinear form $(x, y) = \frac{1}{2}(n(x+y) - n(x) - n(y))$ is nondegenerate) which is multiplicative, *i.e.*, for any $x, y \in U$,

$$n(xy) \quad = \quad n(x)n(y).$$

Let U be a composition algebra over F. An element $z \in U$ is *isotropic* if $z \neq 0$ and $n(z) = 0$. A unital composition algebra U, that is, a composition algebra with identity e is a *Hurwitz algebra*. As proved in [4], the mapping defined by $x \mapsto \overline{x} = (x, e)e - x$ is an involution of U, called the *standard conjugation*, that satisfies $x\overline{x} (= \overline{x}x), x + \overline{x} \in Fe$ where Fe is the subspace of fixed elements under this involution. Furthermore, $n(x)$ and $tr(x) \in F$, respectively, the *norm* and the *trace* of x, are given by $x\overline{x} = n(x)e$ and $x + \overline{x} = tr(x)e$. An element $x \in U$ is *invertible* if and only if $n(x) \neq 0$. Moreover, $x^{-1} = \dfrac{\overline{x}}{n(x)}$ if x is invertible.

By the generalized Hurwitz theorem in [4], a 4-dimensional Hurwitz algebra over F is a (*generalized*) *quaternion algebra*, that is, an algebra over F with two generators i and j satisfying the relations $i^2 = a$, $j^2 = b$ and $ij = -ji$, with $a, b \in F \backslash \{0\}$. This algebra can be denoted as in [6], using the Hilbert symbol, by $\left(\frac{a,b}{F}\right)$. Setting $ij = k$, we have that $\left(\frac{a,b}{F}\right)$ is 4-dimensional over F, with basis $\{1, i, j, k\}$. Moreover, $k^2 = -ab$ and any two of the elements in $\{i, j, k\}$ anticommute.

Taking $F = \mathbb{R}$ and $a = -1$, $b = 1$, we obtain the real split-quaternion algebra $\left(\frac{-1,1}{\mathbb{R}}\right)$. In what follows, throughout the work, we denote this associative algebra and its multiplication by $\widehat{\mathbb{H}}$ and juxtaposition, respectively, and its identity 1 will be omitted most times. Notice that, for instance, $i + j$ is an isotropic element of $\widehat{\mathbb{H}}$.

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	1	$-i$
k	k	j	i	1

Table 1: Multiplication table of $\widehat{\mathbb{H}}$.

Let $p = p_0 + p_1 i + p_2 j + p_3 k \in \widehat{\mathbb{H}}$. If $p_0 = 0$ then p is called *pure split-quaternion*. The *scalar part* and the *vector part* of p are p_0 and $V_p = p_1 i + p_2 j + p_3 k$, respectively. The conjugate of p is $\overline{p} = p_0 - V_p$ and the norm of p is

$$n(p) \quad = \quad p_0^2 + p_1^2 - p_2^2 - p_3^2.$$

In this work, whenever convenient, the subspace $\widehat{\mathbb{H}}_0$, of the pure split-quaternions, of $\widehat{\mathbb{H}}$ is identified with the $(2+1)$-Minkowski space $\mathbb{R}^{2+1}$. This is the inner product space consisting of the real vector space $\mathbb{R}^3$ equipped with the Lorentz inner product

$$\langle x, y \rangle_L \quad = \quad -x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Notice that $\langle x, x\rangle_L = -n(x)$. The vector space may be regarded as a semi-normed vector space, provided that the (semi-) norm is given by

$$\|x\| \quad = \quad \sqrt{|\langle x, x\rangle_L|}.$$

The Lorentz cross product of $x, y \in \mathbb{R}^3$ is defined as follows, [2]:

$$x \wedge_L y \quad = \quad \begin{vmatrix} -i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

In the cited reference, it is proved that the identity

$$\langle x \wedge_L y, z \wedge_L w\rangle_L \quad = \quad -\begin{vmatrix} \langle x, z\rangle_L & \langle x, w\rangle_L \\ \langle y, z\rangle_L & \langle y, w\rangle_L \end{vmatrix}$$

holds in $\mathbb{R}^3$. In particular, taking $z = x$ and $w = y$ leads to the identity

$$n(x \wedge_L y) \quad = \quad n(x)n(y) - \langle x, y\rangle_L^2. \tag{1}$$

The multiplication of two split-quaternions $p = p_0 + V_p$ and $q = q_0 + V_q$ can be related to the Lorentz inner product and to the Lorentz cross product as follows: $pq = p_0q_0 + \langle V_p, V_q\rangle_L + p_0V_q + q_0V_p + V_p \wedge_L V_q$. In particular, if p and q are pure then

$$pq = \langle V_p, V_q\rangle_L + V_p \wedge_L V_q.$$

Lastly but importantly, for computational purposes, the algebra $\widehat{\mathbb{H}}$ can be identified with the real algebra $\mathbb{M}_{2\times 2}(\mathbb{R})$, this one equipped with the usual matrix multiplication. The identification is due to the known isomorphism $\psi : \widehat{\mathbb{H}} \to \mathbb{M}_{2\times 2}(\mathbb{R})$ defined by, [6]:

$$a + bi + cj + dk \mapsto \begin{bmatrix} a+d & b+c \\ c-b & a-d \end{bmatrix}.$$

3 Intersection of a Double Cone and a Line

In this section, we consider a line and a double cone, being the latter one a geometric figure made up of two right circular infinite cones placed apex to apex such that both share the same axis of symmetry. Of course, this double cone could be in any place of the 3-dimensional space, but, for the sake of convenience, we assume that the cone apexes are at the origin of the coordinates and that the axis of symmetry is the vertical axis. If this was not the case, we could always translate and/or rotate the cone such that these conditions are met. In order to study the intersection of the mentioned objects, we use the pure split-quaternions and find out that they form a convenient framework.

Consider the $(2+1)$-Minkowski space with the two horizontal axis chosen to be spatial dimensions while the vertical axis is time. The split-quaternions i, j, k are the unit vectors in the t, x, y axis, respectively (Figure 1). For the sake of simplicity, in what follows, we write $\langle\cdot,\cdot\rangle$ and $\wedge$ instead of $\langle\cdot,\cdot\rangle_L$ and $\wedge_L$, respectively.

Each point in the Minkowski space is usually called an *event*. In what follows, through a vectorization of the affine Minkowski space, the end-point of a position vector (with respect to the origin of the coordinates) is identified with that vector.

Definition 3.1. *Let $a, b \in \widehat{\mathbb{H}}_0$ with $b \neq 0$. A* **line** *is the set of events (pure split-quaternions) $\{a + \lambda b \ : \lambda \in \mathbb{R}\}$, where b gives the line direction and a is an event in the Minkowski space.*

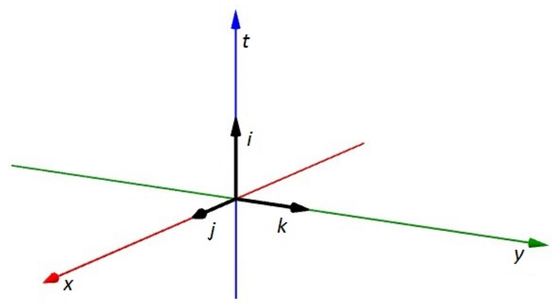

Figure 1: The $(2+1)$-Minkowski space with the unit vectors i, j, k represented.

Definition 3.2. *The* **double cone** *is the set of events (pure split-quaternions)* $\{ti + xj + yk \ : \ x^2 + y^2 = t^2, \ x, y, t \in \mathbb{R}\}$.

Observe that any double cone can fit this definition after an appropriate scaling. For this reason, we only consider this double cone and an arbitrary line.

The upper cone is known as the *future light cone* and the lower one as the *past light cone*. If an event is inside the future light cone, we will call it a *future event* and if it is inside the past light cone, we will call it a *past event* (Figure 2). Thus, if a is a future event, then $tr(a) > 0$ and $n(a) > 0$. Likewise, if it is a past event, then $tr(a) < 0$ but still $n(a) > 0$. In any case, if a line passes through a future or past event, then it must intersect the double cone.

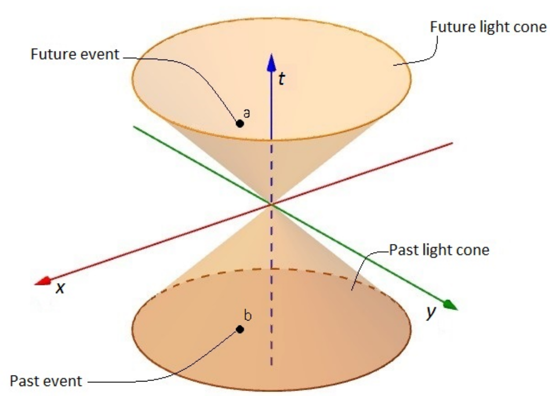

Figure 2: A light cone diagram.

Given a vector b, the set of all events orthogonal to b will form a plane known as a *separation* plane. Given an event a, if b is timelike, lightlike or spacelike, then the Lorentz inner product of a with b is negative, zero or positive when a is above, over or below the separation plane of b, respectively. Moreover, if b is spacelike, then the Lorentz inner product of a with b is negative if a and b are on the same side with respect to the separation plane of b, zero if a is over the separation plane of b, and positive otherwise.

If a line is timelike, i.e., with a timelike vector direction, its separation plane contains no lightlike event, [5]. In other words, a timelike vector will never be orthogonal to a lightlike one.

Lemma 3.3. *Let $u \in \widehat{\mathbb{H}}_0$ be timelike. If $u_\perp \neq 0$ is orthogonal to u, then $u_\perp$ is spacelike.*

Proof. Let $u \in \widehat{\mathbb{H}}_0$ be timelike and $u_\perp$ belong to the separation plane of u. Then $n(u) > 0$ and $\langle u, u_\perp \rangle = 0$.

For each $u_\perp$ considered, there exists an α such that $v = u_\perp + \alpha u$ is lightlike, i.e, $n(v) = 0$.

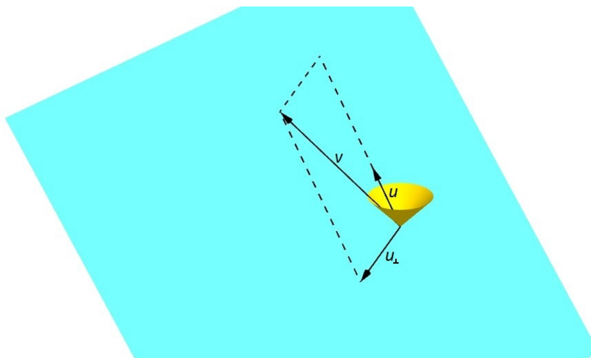

Figure 3: Representation of u, $u_\perp$ and the lightlike vector v.

Hence,

$$\begin{aligned} n(v) &= -\langle v, v \rangle \\ &= -\langle u_\perp, u_\perp \rangle - 2\alpha \langle u_\perp, u \rangle - \alpha^2 \langle u, u \rangle \\ &= n(u_\perp) + \alpha^2 n(u). \end{aligned}$$

But $n(v) = 0$. Thus, $n(u_\perp) + \alpha^2 n(u) = 0$ which implies that $n(u_\perp) = -\alpha^2 n(u)$. Since $n(u) > 0$, we conclude that $n(u_\perp) < 0$ and $u_\perp$ is spacelike. □

Lemma 3.4. *Let $u \in \widehat{\mathbb{H}}_0 \backslash \{0\}$ be lightlike. If $u_\perp \neq 0$ is orthogonal to u, then $u_\perp$ will never be timelike, i.e., $n(u_\perp) \leq 0$.*

Proof. Let us demonstrate by contradiction. Suppose that given $u \in \widehat{\mathbb{H}}_0$ lightlike there exists a timelike $u_\perp$ orthogonal to u. If this is the case, then u is orthogonal to $u_\perp$ and, by Lemma 3.3, u must be spacelike, which is a contradiction since u is, by hypothesis, lightlike. □

Observe that if u is spacelike, then an orthogonal vector $u_\perp$ may be spacelike, lightlike or timelight. For example, given the spacelike vector $u = (1, 1, 1)$, any orthogonal vector $u_\perp = (x, y, t)$ must satisfy the equation $t = x + y$. The vectors $v_1 = (1, 1, 2)$, $v_2 = (1, -1, 0)$ and $v_3 = (1, 0, 1)$ satisfy the referred equation but $n(v_1) > 0$, $n(v_2) < 0$ and $n(v_3) = 0$, which implies that v_1, v_2 and v_3 is timelike, spacelike and lightlike, respectively.

Proposition 3.5. *Let u, v be two events such that at least one of them is timelike. Then $n(u \wedge v) < 0$.*

Proof. Let us suppose, without loss of generality, that u is timelike. We can decompose v such that $v = \alpha u + \beta u_\perp$, where $u_\perp$ is a convenient vector belonging to the separation plane of u. Hence,

$$\begin{aligned} n(v) &= -\langle v, v \rangle \\ &= -\alpha^2 \langle u, u \rangle - \beta^2 \langle u_\perp, u_\perp \rangle \\ &= \alpha^2 n(u) + \beta^2 n(u_\perp), \end{aligned}$$

and

$$\begin{aligned}\langle u, v\rangle &= \alpha\langle u, u\rangle \\ &= -\alpha n(u).\end{aligned}$$

From (1), we obtain

$$\begin{aligned}n(u \wedge v) &= n(u)n(v) - \langle u, v\rangle^2 \\ &= \alpha^2 n^2(u) + \beta^2 n(u)n(u_\perp) - \alpha^2 n^2(u) \\ &= \beta^2 n(u)n(u_\perp).\end{aligned}$$

As, by hypothesis, $n(u) > 0$ and, from Lemma 3.3, $n(u_\perp) < 0$, then $n(u \wedge v) < 0$. □

Theorem 3.6. *Let $\mathfrak{L}$ be a line passing at an event a and with direction $b \neq 0$.*

1) If $n(b) \neq 0$, then the line $\mathfrak{L}$ intersects the double cone at 0, 1 or 2 points whenever $n(a \wedge b)$ is positive, zero or negative, respectively. The intersection points are given by

$$s = a + \frac{< a, b > \pm\sqrt{-n(a \wedge b)}}{n(b)} b.$$

2) If $n(b) = 0$, then the line $\mathfrak{L}$ intersects the double cone at 0, 1 or an infinite number of points.

(i) If a belongs to the separation plane of b

(α) and $n(a) = 0$, then the line $\mathfrak{L}$ intersects the double cone at an infinite number of points, i.e., all the points of the line belong to the double cone;

(β) and $n(a) \neq 0$, then the line $\mathfrak{L}$ does not intersect the double cone.

(ii) If a doesn't belong to the separation plane of b, then the line $\mathfrak{L}$ intersects the double cone at 1 point given by

$$s = a + \frac{n(a)}{2 < a, b >} b.$$

Proof. Consider the line $\mathfrak{L}$ that passes through an event a and has direction $b \neq 0$. The generic point of $\mathfrak{L}$ can be represented by $X = a + \lambda b$, with $\lambda \in \mathbb{R}$. The intersection of $\mathfrak{L}$ with the double cone consists of the points of this line that belong to the double cone. But all the points of the double cone are lightlike. Hence, if s is such a point, then $\langle s, s\rangle = 0$. But

$$\langle s, s\rangle = \langle a, a\rangle + 2\lambda \langle a, b\rangle + \lambda^2 \langle b, b\rangle .$$

Since $\langle a, a\rangle = -n(a)$ and $\langle b, b\rangle = -n(b)$, we obtain

$$\lambda^2 n(b) - 2\lambda \langle a, b\rangle + n(a) = 0. \tag{2}$$

If $n(b) \neq 0$, then, by (1),

$$\begin{aligned}\lambda &= \frac{\langle a, b\rangle \pm \sqrt{\langle a, b\rangle^2 - n(a)n(b)}}{n(b)} \\ &= \frac{\langle a, b\rangle \pm \sqrt{-n(a \wedge b)}}{n(b)}.\end{aligned}$$

From here we conclude that $\mathfrak{L}$ intersects the double cone at 0, 1 or 2 points whenever $n(a \wedge b)$ is positive, zero or negative, respectively. In the two latter cases, the intersection points of $\mathfrak{L}$ with the double cone are given by

$$s = a + \frac{\langle a, b\rangle \pm \sqrt{-n(a \wedge b)}}{n(b)} b.$$

If $n(b) = 0$, then, from (2), we obtain

$$-2\lambda \langle a, b\rangle + n(a) = 0. \tag{3}$$

If the event a belongs to the separation plane of b, then $\langle a, b\rangle = 0$. Hence, the points of $\mathfrak{L}$ belong to the double cone if, and only if, $n(a) = 0$. In this case, all points of the line belong to the double cone. If this is not the case, then the intersection is the empty set.

If the event a does not belong to the separation plane of b, then $\langle a, b\rangle \neq 0$ and the solution of equation (3) is

$$\lambda = \frac{n(a)}{2\langle a, b\rangle}.$$

In this case, the intersection point of $\mathfrak{L}$ with the double cone is given by

$$s = a + \frac{n(a)}{2\langle a, b\rangle} b.$$

□

Observe that if the event a or the direction b are timelike, then, by Proposition 3.5, $n(a \wedge b) < 0$ and the line intersects the double cone at least at one point.

Acknowledgements

The research of R. Serôdio and P. D. Beites was supported by Fundacão para a Ciência e a Tecnologia (Portugal), project PEst-OE/MAT/UI0212/2014. P. D. Beites was also supported by Ministerio de Economía y Competitividad (Spain), project MTM2013-45588-C3-1-P.

References

[1] C.-K. Shene, Computing the Intersection of a Line and a Cone. In Alan Paeth (editor), *Graphics Gems V*, AP Professional, 1995, 227–231.

[2] J. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Springer, 2006.

[3] J. M. Cychosz and W. N. Waggenspack Jr., Intersecting a Ray with a Cylinder. In Paul Heckbert (editor), *Graphics Gems IV*, AP Professional, 1994, 356–365.

[4] N. Jacobson, Composition Algebras and their Automorphisms, *Rendiconti del Circolo Matematico di Palermo* **7** (1958), 55–80.

[5] R. Goldblatt, *Orthogonality and Spacetime Geometry*, Springer-Verlag, 1987.

[6] T. Y. Lam, *The Algebraic Theory of Quadratic Forms*, W. A. Benjamin, 1973.

R. Serôdio
Centro de Matemática e Aplicações (CMA-UBI) and Departamento de Matemática, Universidade da Beira Interior, Covilhã, Portugal
E-mail address: rserodio@ubi.pt

P. D. Beites
Centro de Matemática e Aplicações (CMA-UBI) and Departamento de Matemática, Universidade da Beira Interior, Covilhã, Portugal
E-mail address: pbeites@ubi.pt

José Vitória
Departamento de Matemática, Universidade de Coimbra, Coimbra, Portugal
E-mail address: jvitoria@mat.uc.pt