Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 07–34

DISTRIBUTIONAL EQUATION FOR LAGUERRE-HAHN FUNCTIONALS ON THE UNIT CIRCLE

A. BRANQUINHO AND M.N. REBOCHO

ABSTRACT: Let u be a hermitian linear functional defined in the linear space of Laurent polynomials and F its corresponding Carathéodory function. We establish the equivalence between a Riccati differential equation with polynomial coefficients for F, $zAF' = BF^2 + CF + D$ and a distributional equation for u, $\mathcal{D}(Au) = B_1 u^2 + C_1 u + H_1 \mathcal{L}$, where $\mathcal{L}$ is the Lebesgue functional, and the polynomials B_1, C_1, D_1 are defined in terms of the polynomials A, B, C, D.

KEYWORDS: Hermitian functionals, measures on the unit circle, Carathéodory function, Laguerre-Hahn affine class on the unit circle, semi-classical functionals.
AMS SUBJECT CLASSIFICATION (2000): Primary 33C47, 42C05.

1. Introduction

In this paper we introduce the concept of Laguerre-Hahn class of hermitian functionals on the unit circle. Let u be a hermitian regular linear functional defined in the linear space of Laurent polynomials and F the corresponding Carathéodory function. The functional u (respectively, the corresponding Carathéodory function, F) is said to be *Laguerre-Hahn* if F satisfies a Riccati differential equation with polynomial coefficients,

$$zAF' = BF^2 + CF + D, \; A \not\equiv 0\,. \tag{1}$$

We shall call the set of all such functionals (respectively, Carathéodory function, F) the *Laguerre-Hahn class on the unit circle*. We remark that the Laguerre-Hahn class on the unit circle can be regarded as an extension of the Laguerre-Hahn class on the real line, studied in [5, 7, 8].

If $B \equiv 0$ in (1) we obtain the Laguerre-Hahn affine class on the unit circle (see [2, 3]). If $B \equiv 0$ and C, D specific polynomials, we obtain the semi-classical class on the unit circle (see [2, 6, 11]). Moreover, the Laguerre-Hahn class on the unit circle includes the class of second degree functionals on

Received November 15, 2007.
This work was supported by CMUC/FCT. The second author was supported by FCT, with grant ref. SFRH/BD/25426/2005.

the unit circle (see [4]) and includes, as we will see on section 3, the linear-fractional transformations of Carathéodory functions which are Laguerre-Hahn.

Analogously to what has been done on the real case (cf. [7, 8]), we establish the equivalence between (1) and a distributional equation for the corresponding u,

$$\mathcal{D}(Au) = B_1u^2 + C_1u + H_1\mathcal{L}\,, \tag{2}$$

where $\mathcal{L}$ is the Lebesgue functional, and B_1, C_1, D_1 polynomials defined in terms of A, B, C, D.

This paper is organized as follows. In section 2 we give the definitions and introduce some notations which will be used in the forthcoming sections. In section 3 we study the stability in the Laguerre-Hahn class, and give some examples of sequences of polynomial orthogonal with respect to a functional of Laguerre-Hahn type. In section 4 we state some auxiliary results which enable us to establish, in section 5, the equivalence between (1) and (2).

2. Preliminaries and notations

Let $\Lambda = \text{span}\,\{z^k : k \in \mathbb{Z}\}$ be the space of Laurent polynomials with complex coefficients, Λ' its algebraic dual space, $\mathbb{P} = \text{span}\,\{z^k : k \in \mathbb{N}\}$ the space of complex polynomials, and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (or, using the parametrization $z = e^{i\theta}$, $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi[\,\})$ the unit circle.

Let $u \in \Lambda'$ be a linear functional. We denote by $\langle u, f\rangle$ the action of u over $f \in \Lambda$.

Given the sequence of moments (c_n) of u, $c_n = \langle u, \xi^{-n}\rangle$, $n \in \mathbb{Z}$, $c_0 = 1$, the minors of the Toeplitz matrix are defined by

$$\Delta_{-1} = 1\,,\ \Delta_0 = c_0\,,\ \Delta_k = \begin{vmatrix} c_0 & \cdots & c_k \\ \vdots & & \vdots \\ c_{-k} & \cdots & c_0 \end{vmatrix}\,,\ k \in \mathbb{N}\,.$$

Definition 1 (cf. [10])**.** The linear functional u is:
- a) *hermitian* if $c_{-n} = \overline{c_n}, \forall n \geq 0$;
- b) *regular* or *quasi-definite* if $\Delta_n \neq 0, \forall n \geq 0$;
- c) *positive definite* if $\Delta_n > 0, \forall n \geq 0$.

As an example of hermitian functional we have the *Lebesgue functional*, $\mathcal{L}$, defined in terms of its moments by $\langle \mathcal{L}, \xi^{-n}\rangle = \delta_{0,n}$, $n \in \mathbb{Z}$.

If u is a positive definite hermitian functional, then there exists a non-trivial probability measure μ supported on the unit circle such that

$$\langle u, \xi^{-n} \rangle = \int_0^{2\pi} \xi^{-n} d\mu(\theta)\,, \quad \xi = e^{i\theta}\,, \quad n \in \mathbb{Z}\,.$$

Definition 2. Let $\{\phi_n\}$ be a sequence of complex polynomials with $\deg(\phi_n) = n$ and u a hermitian linear functional. We say that $\{\phi_n\}$ is a *sequence of orthogonal polynomials* with respect to u (or $\{\phi_n\}$ is a sequence of orthogonal polynomials on the unit circle) if

$$\langle u, \phi_n(\xi)\overline{\phi_m}(1/\xi) \rangle = K_n \delta_{n,m}\,,\ \xi = e^{i\theta}\,,\ K_n \neq 0\,,\ n, m \in \mathbb{N}\,.$$

If for each $n \in \mathbb{N}$ the leading coefficient ϕ_n is 1, then $\{\phi_n\}$ is said to be a *sequence of monic orthogonal polynomials* and will be denoted by MOPS.

Remark . In the positive definite case, as the hermitian functional u has an integral representation in terms of a measure μ, we will also say that $\{\phi_n\}$ is orthogonal with respect to μ.

For a polynomial P with degree n, the reciprocal polynomial P^* is defined by $P^*(z) = z^n \overline{P}(1/z)$. It is well known (see [10]) that a given sequence of complex polynomials $\{\phi_n\}$ is orthogonal on the unit circle if, and only if, $\{\phi_n\}$ satisfy a recurrence relation of the following type,

$$\phi_n(z) = z\phi_{n-1}(z) + a_n \phi_{n-1}^*(z), \quad n \geq 1\,, \tag{3}$$

with $|a_n| \neq 1$, and initial conditions $\phi_0(z) = 1,\ \phi_{-1}(z) = 0$.

Given a sequence of monic orthogonal polynomials $\{\phi_n\}$ with respect to u, the *sequence of associated polynomials of the second kind* $\{\Omega_n\}$ is defined by

$$\Omega_0(z) = 1\,, \quad \Omega_n(z) = \langle u_\theta, \frac{e^{i\theta} + z}{e^{i\theta} - z} \left(\phi_n(e^{i\theta}) - \phi_n(z)\right) \rangle,\ n = 1, 2, \ldots\,,$$

and verify $\phi_n^*(z)\Omega_n(z) + \phi_n(z)\Omega_n^*(z) = 2K_n z^n$, for all $n = 1, 2, \ldots$ (see for instance [10]).

We consider the formal series associated with the hermitian linear functional u,

$$F_u(z) = c_0 + 2\sum_{k=1}^{+\infty} c_k z^k\,,\ |z| < 1\,, \tag{4}$$

$$F_u(z) = -c_0 - 2\sum_{k=1}^{+\infty} c_{-k} z^{-k}\,,\ |z| > 1\,. \tag{5}$$

Since, for each $\theta \in [0, 2\pi[$, the following expansions take place,

$$\frac{e^{i\theta}+z}{e^{i\theta}-z} = 1 + 2\sum_{k=1}^{+\infty}(e^{i\theta})^{-k}z^k, \ |z|<1\,,$$
$$\frac{e^{i\theta}+z}{e^{i\theta}-z} = -1 - 2\sum_{k=1}^{+\infty}(e^{i\theta})^{k}z^{-k}, \ |z|>1$$

then, taking into account the definition of the moments of u, formally we can write

$$\langle u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = F_u(z)\,, \quad z \in \mathbb{C}\setminus\mathbb{T} \tag{6}$$

where $\langle u_\theta, \ . \ \rangle$ denotes the action of u over the variable θ, $\theta \in [0, 2\pi[$. Thus, we will also say that the series in (4), (5) formally correspond to the function F_u defined by (6).

In the positive definite case, F_u is the *Carathéodory function* corresponding to u, and is represented by

$$F_u(z) = \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(\theta)\,, \quad z \in \mathbb{C}\setminus\mathbb{T}\,,$$

where μ is the probability measure associated with u.

Next we see some operations in the linear space Λ'. Given $u \in \Lambda'$, $f, g \in \Lambda$, the functionals fu and $\mathcal{D}u$, both elements of Λ', are defined by

$$\langle fu, p\rangle = \langle u, fp\rangle\,, \quad \langle \mathcal{D}u, p\rangle = -i\langle u, \xi p'\rangle\,, \quad p \in \Lambda$$

thus, $\langle \mathcal{D}(gu), p\rangle = -i\langle u, \xi g p'\rangle$, $p \in \Lambda$. We remark that if u is hermitian, then $\mathcal{D}u$ is also hermitian.

We consider the *generating function of the moments for a hermitian functional,* u, defined in $\mathbb{C}\setminus\mathbb{T}$ by

$$\mathcal{F}_u(z) = \sum_{n=0}^{+\infty} c_n z^n\,, \ |z|<1\,, \quad \mathcal{F}_u(z) = -\sum_{n=1}^{+\infty} c_{-n} z^{-n}\,, \ |z|>1\,.$$

Then the following relation between the Carathéodory function and the generating function of the moments holds,

$$\mathcal{F}_u(z) = \frac{F_u(z)+1}{2}\,, \ \ z \in \mathbb{C}\setminus\mathbb{T}\,. \tag{7}$$

The next definition is the analogue of the definition given in [8] for the real case.

Definition 3. Let $u, v \in \Lambda'$ be hermitian linear functionals and $\mathcal{F}_u, \mathcal{F}_v$ the corresponding generating functions of the moments. The *product of* u *and* v, uv, is the functional defined in terms of its moments by

$$(uv)_n = \sum_{\substack{\nu+k=n \\ \mathrm{sgn}(\nu)=\mathrm{sgn}(k)=\mathrm{sgn}(n)}} u_\nu v_k \,, \qquad n \in \mathbb{Z}\,, \tag{8}$$

that satisfies

$$\mathcal{F}_u(z)\,\mathcal{F}_v(z) = \mathcal{F}_{uv}(z) \tag{9}$$

Remark . The functional uv defined by (8) is hermitian.

As a consequence, we get the following result.

Lemma 1. *Let* u *be a hermitian linear functional and* F_u *the corresponding formal Carathéodory function. Then,*

$$(F_u)^2 = 2F_{u^2} - 2F_u + 1\,. \tag{10}$$

Proof: Put $u = v$ in (9) and get, after using (7) that $F_{u^2} + 1 = ((F_u)^2 + 2F_u + 1)/2$, and the required follows. ■

3. Stability in the Laguerre-Hahn class on the unit circle. Examples.

We begin this section studying the stability of the Laguerre-Hahn Carathéodory functions under a special kind of linear-fractional transformations.

Theorem 1. *Let* F_1 *be a Carathéodory function satisfying a Riccati differential equations with polynomials coefficients,*

$$A_1 F_1' = B_1 F_1^2 + C_1 F_1 + D_1,\; A_1 \not\equiv 0, \tag{11}$$

and F_2 *a linear-fractional transformation of* F_1 *of the following type*

$$F_2 = \frac{\alpha_1 - \beta_1 F_1}{-\alpha_2 + \beta_2 F_1},\;\; \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{P},\;\; \alpha_1\beta_2 - \alpha_2\beta_1 \not\equiv 0\,.$$

Then, F_2 *satisfies*

$$A_2 F_2' = B_2 F_2^2 + C_2 F_2 + D_2\,, \tag{12}$$

with

$$\begin{aligned}
A_2 &= -(\alpha_1\beta_2 - \alpha_2\beta_1)A_1 \not\equiv 0\,, && (13)\\
B_2 &= (\alpha_2\beta_2' - \alpha_2'\beta_2)A_1 + \alpha_2^2 B_1 + \alpha_2\beta_2 C_1 + \beta_2^2 D_1\,, && (14)\\
C_2 &= (\alpha_2\beta_1' + \alpha_1\beta_2' - \alpha_2'\beta_1 - \alpha_1'\beta_2)A_1 \\
&\qquad + 2\alpha_1\alpha_2 B_1 + (\alpha_1\beta_2 + \alpha_2\beta_1)C_1 + 2\beta_1\beta_2 D_1\,, && (15)\\
D_2 &= (\alpha_1\beta_1' - \alpha_1'\beta_1)A_1 + \alpha_1^2 B + \alpha_1\beta_1 C_1 + \beta_1^2 D_1\,. && (16)
\end{aligned}$$

Thus, if F_2 is a Carathéodory function, then F_2 is Laguerre-Hahn.

Proof: We have $F_2 = \dfrac{\alpha_1 - \beta_1 F_1}{-\alpha_2 + \beta_2 F_1}$ or equivalently $F_1 = \dfrac{\alpha_1 + \alpha_2 F_2}{\beta_1 + \beta_2 F_2}$. By substituting $F_1 = \dfrac{\alpha_1 + \alpha_2 F_2}{\beta_1 + \beta_2 F_2}$ in (11) we obtain (12) with coefficients given by (13)-(16). Thus, the assertion follows. ■

Next we see some examples of Laguerre-Hahn sequences.

Corollary 1. *Let $\{\phi_n\}$ be a MOPS on $\mathbb{T}$, $\{\Omega_n\}$ the sequence of associated polynomials of the second kind and F, F_1 the corresponding Carathéodory functions. If F is Laguerre-Hahn and satisfies $zAF' = BF^2 + CF + D$, with $A \not\equiv 0$, then F_1 is Laguerre-Hahn and satisfies $zAF_1' = -DF_1^2 - CF_1 - B$.*

Proof: It is well known that $\{\Omega_n\}$ is orthogonal with respect to $F_1 = 1/F$, with F the Carathéodory function associated to $\{\phi_n\}$ (cf. for example [9]). Thus, the assertion follows from Theorem 1. ■

Definition 4 (Peherstorfer [9])**.** Let $\{\phi_n\}$ be a MOPS on the unit circle, (a_n) the corresponding sequence of reflection coefficients and $N \in \mathbb{N}$. The sequence $\{\phi_n^N\}$ defined by (3) with $\phi_n^N(0) = a_{n+N}$, $n = 0, 1, \ldots$ is said to be the *sequence of associated polynomials of $\{\phi_n\}$ of order N.*

Let $\{\phi_n\}$ be a MOPS with respect to a Carathéodory function F and $\{\Omega_n\}$ be the sequence of orthogonal polynomials of the first kind. In [9] it is established that $\{\phi_n^N\}$ is orthogonal with respect to the Carathéodory function F^N given by

$$F^N = \frac{(\Omega_N - \Omega_N^*) + (\phi_N + \phi_N^*)F}{(\Omega_N + \Omega_N^*) + (\phi_N - \phi_N^*)F}\,, \tag{17}$$

and $(\Omega_N - \Omega_N^*)(\phi_N - \phi_N^*) - (\phi_N + \phi_N^*)(\Omega_N + \Omega_N^*) = -2K_N z^N \not\equiv 0\,.$

Corollary 2. *Let F be a Carathéodory function, $\{\phi_n\}$ the corresponding MOPS, $\{\Omega_n\}$ the sequence of associated polynomials of the second kind and F^N the Carathéodory function corresponding to the sequence of associated polynomials of $\{\phi_n\}$ of order N, $\{\phi_n^N\}$. If F is Laguerre-Hahn and satisfies*

$$zAF' = BF^2 + CF + D, A \not\equiv 0\,,$$

then F^N is Laguerre-Hahn and satisfies

$$A_N \left(F^N\right)' = B_N \left(F^N\right)^2 + C_N F^N + D_N\,,$$

with $A_N = 4K_N z^{N+1} A \not\equiv 0$ and

$$\begin{aligned} B_N &= zA\left\{(\Omega_N + \Omega_N^*)'(\phi_N - \phi_N^*) - (\phi_N - \phi_N^*)'(\Omega_N + \Omega_N)^*\right\} \\ &\quad + (\Omega_N + \Omega_N^*)^2 B - (\Omega_N + \Omega_N^*)(\phi_N - \phi_N^*)C + (\phi_N - \phi_N^*)^2 D\,, \\ C_N &= zA\left\{(\Omega_N - \Omega_N^*)(\phi_N - \phi_N^*)' - (\Omega_N + \Omega_N^*)'(\phi_N + \phi_N^*)\right\} \\ &\quad - 2(\Omega_N^2 - (\Omega_N^*)^2)B + 2(\phi_N\Omega_N + \phi_N^*\Omega_N^*)C - 2(\phi_N^2 - (\phi_N^*)^2)D\,, \\ D_N &= (\Omega_N - \Omega_N^*)^2 B - (\Omega_N - \Omega_N^*)(\phi_N + \phi_N^*)C + (\phi_N + \phi_N^*)^2 D\,. \end{aligned}$$

Proof: From (17) it follows that $F = \dfrac{(\Omega_N + \Omega_N^*)F^N - (\Omega_N - \Omega_N^*)}{(\phi_N + \phi_N^*) - (\phi_N - \phi_N^*)F^N}$, and using Theorem 1 the asssertion follows. ■

4. Auxiliary results

Next we establish results that will be used in next section. We consider hermitian linear functionals $u \in \Lambda$ and F_u the corresponding formal Carathéodory function defined by (6).

Lemma 2 (cf. [2]). *Let u be a hermitian linear functional and A, B polynomials. Then, for all $z \in \mathbb{C} \setminus \mathbb{T}$,*

$$\langle B(\xi)u, \frac{\xi + z}{\xi - z}\rangle = P_{u,B}(z) + B(z)F_u(z)\,, \tag{18}$$

$$A(z)F_u'(z) = -A'(z)F_u(z) + Q_{u,A}(z) + \frac{1}{iz}\langle \mathcal{D}(Au), \frac{\xi + z}{\xi - z}\rangle \tag{19}$$

where $P_{u,B}$, $Q_{u,A}$ are polynomials with $\deg(P_{u,B}) = \deg(B)$, $\deg(Q_{u,A}) = \deg(A) - 1$, *defined by*

$$P_{u,B}(z) = \langle u, \frac{\xi + z}{\xi - z}\left(B(\xi) - B(z)\right)\rangle\,, \tag{20}$$

$$Q_{u,A}(z) = -A'(z) - \langle u, 2\xi \sum_{k=2}^{\deg(A)} \frac{A^{(k)}(z)}{k!}(\xi - z)^{k-2}\rangle \tag{21}$$

Remark . From (20) and (21) follows $P_{u,B+\tilde{B}} = P_{u,B} + P_{u,\tilde{B}}\,, \quad P_{u,\alpha B} = \alpha P_{u,B}\,,$ with $\alpha \in \mathbb{C}$ and $B, \tilde{B} \in \mathbb{P}$.

Lemma 3 (cf. [1])**.** *Let $u \in \Lambda'$ be a hermitian linear functional and A a polynomial. Then, $(A(z) + \overline{A}(1/z))u$ is hermitian.*

Next lemma is a generalization of Theorem 3.4 of [1].

Lemma 4. *Let u be a hermitian linear functional. If there exist polynomials A, B, C, H such that $\mathcal{D}(Au) = Bu^2 + Cu + H\mathcal{L}$, with $\mathcal{L}$ the Lebesgue functional, then*

$$\mathcal{D}(A + \overline{A})u = (B + \overline{B})u^2 + (C + \overline{C})u + (H + \overline{H})\mathcal{L} \tag{22}$$

Conversely, if (22) holds, then u satisfies the distributional equation with polynomial coefficients

$$\mathcal{D}(A_1 u) = B_1 u^2 + (C_1 + isA_1)u + H_1\mathcal{L} \tag{23}$$

where $s = \max\{\deg(A), \deg(B), \deg(C), \deg(H)\}$, *and the polynomials A_1, B_1, C_1, H_1 are given by* $A_1(z) = z^s(A(z) + \overline{A}(1/z))\,,$ $B_1(z) = z^s(B(z) + \overline{B}(1/z))\,,$ $C_1(z) = z^s(C(z) + \overline{C}(1/z))\,,$ $H_1(z) = z^s(H(z) + \overline{H}(1/z))\,.$

Proof: If $\mathcal{D}(Au) = Bu^2 + Cu + H\mathcal{L}$, then

$$\langle \mathcal{D}(Au), \xi^k\rangle = \langle Bu^2 + Cu + H\mathcal{L}, \xi^k\rangle,\ \forall k \in \mathbb{Z}\,.$$

Applying conjugates follows

$$\langle \mathcal{D}(\overline{A}u), \xi^{-k}\rangle = \langle \overline{B}u^2 + \overline{C}u + \overline{H}\mathcal{L}, \xi^{-k}\rangle,\ \forall k \in \mathbb{Z}\,.$$

Therefore, we obtain

$$\langle \mathcal{D}((A + \overline{A})u), \xi^n\rangle = \langle (B + \overline{B})u^2 + (C + \overline{C})u + (H + \overline{H})\mathcal{L}, \xi^n\rangle,\ \forall n \in \mathbb{Z}\,,$$

and (22) follows.

Conversely, if u satisfies (22), then we successively get for all $k \in \mathbb{Z}$

$$\langle \mathcal{D}((A+\overline{A})u), \xi^k \rangle \;=\; \langle (B+\overline{B})u^2, \xi^k \rangle + \langle (C+\overline{C})u, \xi^k \rangle + \langle (H+\overline{H})\mathcal{L}, \xi^k \rangle \,,$$

$$-ik\langle u, (A(\xi)+\overline{A}(1/\xi))\xi^k \rangle = \langle u^2, (B(\xi)+\overline{B}(1/\xi))\xi^k \rangle + \langle (C(\xi)+\overline{C}(1/\xi))\xi^k \rangle + \langle \mathcal{L}, (H(\xi)+\overline{H}(1/\xi))\xi^k \rangle \,.$$

Let $s = \max\{\deg(A), \deg(B), \deg(C), \deg(H)\}$, then the last equation is given by

$$-ik\langle u, \xi^s(A(\xi)+\overline{A}(1/\xi))\xi^{k-s} \rangle = \langle u^2, \xi^s(B(\xi)+\overline{B}(1/\xi))\xi^{k-s} \rangle + \langle u, \xi^s(C(\xi)+\overline{C}(1/\xi))\xi^{k-s} \rangle + \langle \mathcal{L}, \xi^s(H(\xi)+\overline{H}(1/\xi))\xi^{k-s} \rangle, \ \forall k \in \mathbb{Z} \tag{24}$$

If we write $m = k - s$ and define A_1, B_1, C_1, H_1 as in the statement of the lemma, the equation (24) is given by

$$-i(s+m)\langle u, A_1(\xi)\xi^m \rangle = \langle u^2, B_1(\xi)\xi^m \rangle + \langle u, C_1(\xi)\xi^m \rangle + \langle \mathcal{L}, H_1(\xi)\xi^m \rangle, \ \forall m \in \mathbb{Z} \,,$$

and so,

$$-im\langle u, A_1(\xi)\xi^m \rangle = \langle u^2, B_1(\xi)\xi^m \rangle + \langle u, (C_1(\xi)+isA_1(\xi))\xi^m \rangle + \langle \mathcal{L}, H_1(\xi)\xi^m \rangle, \ \forall m \in \mathbb{Z} \,.$$

From the definition of $\mathcal{D}$, the previous equation is equivalent to

$$\langle \mathcal{D}(A_1 u), \xi^m \rangle = \langle B_1 u^2, \xi^m \rangle + \langle (C_1+isA_1)u, \xi^m \rangle + \langle H_1\mathcal{L}, \xi^m \rangle, \ \forall m \in \mathbb{Z} \,,$$

and we get (23). ■

Lemma 5. *Let u be a hermitian linear functional and F_u the corresponding formal Carathéodory function. If F_u satisfies $zAF_u' = BF_u^2 + CF_u + D$ in $\mathbb{C} \setminus \mathbb{T}$, then u satisfies the distributional equation*

$$\langle \mathcal{D}(Au) + L(\xi)u - 2iB(\xi)u^2, \frac{\xi+z}{\xi-z} \rangle = iH(z) \tag{25}$$

with

$$L = i\left(-zA' + 2B - C\right) \,, \; H = B - 2P_{u^2,B} + P_{u,-zA'+2B-C} - zQ_{u,A} + D. \tag{26}$$

Proof: If we use (19) and (10) in $zAF_u' = BF_u^2 + CF_u + D$ we get

$$(-zA' + 2B - C)F_u - i\langle \mathcal{D}(Au), \frac{\xi+z}{\xi-z}\rangle = 2BF_{u^2} + B + D - zQ_{u,A} \qquad (27)$$

On the other hand, from (18) and (19) we know that,

$$\begin{aligned}(-zA' + 2B - C)F_u &\\ &= \langle(-\xi A'(\xi) + 2B(\xi) - C(\xi))u, \frac{\xi+z}{\xi-z}\rangle - P_{u,-zA'+2B-C}(z)\\ BF_{u^2} &= \langle B(\xi)u^2, \frac{\xi+z}{\xi-z}\rangle - P_{u^2,B}\,,\end{aligned}$$

and substituting these two equations in (27) we get

$$\begin{aligned}&\langle \mathcal{D}(Au) + i\,(-\xi A' + 2B - C)\,u - 2iBu^2, \frac{\xi+z}{\xi-z}\rangle\\ &\qquad\qquad = i\,(B - 2P_{u^2,B} + P_{u,-zA'+2B-C} - zQ_{u,A} + D)\ .\end{aligned}$$

Thus, we get (25) with L, H given in (26). ■

Theorem 2. *Let u be a hermitian linear functional and F_u the corresponding formal Carathéodory function. If F_u satisfies*

$$zAF_u' = BF_u^2 + CF_u + D,\ |z| < 1$$

then u satisfies the distributional equation with polynomial coefficients

$$\mathcal{D}(A_1u) = B_1u^2 + (isA_1 - L_1)u + H_1\mathcal{L}\,,$$

where $s = \max\{\deg(A), \deg(B), \deg(L), \deg(H)\}$, and the polynomials A_1, B_1, L_1, H_1 are given by $A_1(z) = z^s(A(z) + \overline{A}(1/z))$, $B_1(z) = z^s(2iB(z) + \overline{2iB}(1/z))$, $L_1(z) = z^s(L(z) + \overline{L}(1/z))$, $H_1(z) = z^s(iH(z) + \overline{iH}(1/z))$, and L, H are defined in (26)*.*

Proof: If F_u satisfies $zAF_u' = BF_u^2 + CF_u + D,\ |z| < 1$, then, from Lemma 5, we obtain (25), $\langle \mathcal{D}(Au) + L(\xi)u - 2iB(\xi)u^2, \frac{\xi+z}{\xi-z}\rangle = iH(z)$. Applying conjugates and the transformation $Z = 1/z$ to previous equation we get $\langle \mathcal{D}(\overline{A}u) + \overline{L}u - \overline{2iB}u^2, \frac{\overline{\xi}+1/z}{\overline{\xi}-1/z}\rangle = -i\overline{H}(1/z)$. Since $-\frac{1/\xi+1/z}{1/\xi-1/z} = \frac{\xi+z}{\xi-z}$, we obtain

$$\langle \mathcal{D}(\overline{A}u) + \overline{L}u - \overline{2iB}u^2, \frac{\xi+z}{\xi-z}\rangle = i\overline{H}(1/z)\,. \qquad (28)$$

Summing (25) with (28) we get

$$\langle \mathcal{D}\left((A+\overline{A})u\right)+(L+\overline{L})u-(2iB+\overline{2iB})u^2, \frac{\xi+z}{\xi-z}\rangle = i(H(z)+\overline{H}(1/z))\,.$$

Therefore, if we compute the moments of the hermitian functional

$$\mathcal{D}\left((A+\overline{A})u\right)+(L+\overline{L})u-(2iB+\overline{2iB})u^2$$

(using the asymptotic expansion of $\frac{\xi+z}{\xi-z}$ in $|z|<1$ and in $|z|>1$) we get

$$\mathcal{D}\left((A+\overline{A})u\right)+(L+\overline{L})u-(2iB+\overline{2iB})u^2 = (iH+\overline{iH})\mathcal{L}\,.$$

From Lemma 4 we obtain the required functional equation. ■

If we take $B=0$ in previous Theorem we obtain the result for the Laguerre-Hahn affine class.

Corollary 3. *Let u be a hermitian linear functional. If F_u satisfies*

$$zAF_u' = CF_u + D \quad \textit{for} \quad |z|<1\,,$$

then u satisfies

$$\mathcal{D}(A_1 u) = (isA_1 - L_1)u + H_1\mathcal{L}\,,$$

where $s=\max\{\deg(A),\deg(L),\deg(H)\}$, and A_1, B_1, L_1, H_1, L, H are given by $A_1(z)=z^s(A(z)+\overline{A}(1/z))$, $L_1(z)=z^s(L(z)+\overline{L}(1/z))$, $H_1(z)=z^s(iH+\overline{iH}(1/z))$, $L(z)=i\,(-zA'(z)-C(z))$, $H(z)=P_{u,-zA'-C}(z)-zQ_{u,A}(z)+D(z)$.

5. The characterization theorem

Theorem 3. *Let u be a hermitian linear functional and F_u the corresponding formal Carathéodory function. If u satisfies $\mathcal{D}(Au)=Bu^2+Cu+\xi H\mathcal{L}$, where $\mathcal{L}$ is the Lebesgue functional, then F_u satisfies the differential equations, in $\mathbb{C}\setminus\mathbb{T}$,*

$$zAF_u' = -\frac{iB}{2}F_u^2+(-zA'-iB-iC)F_u+\frac{iB}{2}-iP_{u^2,B}+zQ_{u,A}-iP_{u,C}+E\,, \tag{29}$$

with $E(z)=-2izH(z)\,\mathbb{I}(z)$, where $\mathbb{I}(z)=\begin{cases}1 & , \quad |z|<1\\ 0 & , \quad |z|>1\end{cases}$.

Conversely, if F_u satisfies the differential equations (29), then u satisfies the distributional equation

$$\mathcal{D}(Au)=Bu^2+Cu+\xi H\mathcal{L}\,, \tag{30}$$

where $\mathcal{L}$ is the Lebesgue functional.

Proof: If we substitute $\mathcal{D}(Au) = Bu^2 + Cu + \xi H\mathcal{L}$ in (19) we get

$$\begin{aligned} & zA(z)F_u'(z) \\ &= -zA'(z)F_u + zQ_{u,A}(z) - i\langle Bu^2, \frac{\xi+z}{\xi-z}\rangle - i\langle Cu, \frac{\xi+z}{\xi-z}\rangle - i\langle \xi H\mathcal{L}, \frac{\xi+z}{\xi-z}\rangle \,. \end{aligned}$$

From (18) follows

$$\begin{aligned} zA(z)F_u' = -zA'(z)F_u + zQ_{u,A}(z) - i\left(P_{u^2,B} + B(z)F_{u^2}\right) \\ - i\left(P_{u,C} + C(z)F_u\right) - i\langle \xi H\mathcal{L}, \frac{\xi+z}{\xi-z}\rangle \,. \quad (31) \end{aligned}$$

Since $\langle \xi H\mathcal{L}, \frac{\xi+z}{\xi-z}\rangle = 2zH(z)\,\mathbb{I}(z)$ then, in $|z|<1$, (31) is equivalent to

$$\begin{aligned} zA(z)F_u' = -iB(z)F_{u^2} + (-zA'(z) - iC(z))F_u \\ + zQ_{u,A}(z) - iP_{u^2,B}(z) - iP_{u,C}(z) - 2izH(z) \end{aligned}$$

and, in $|z|>1$, (31) is equivalent to

$$\begin{aligned} zA(z)F_u' = -iB(z)F_{u^2} + (-zA'(z) - iC(z))F_u \\ + zQ_{u,A}(z) - iP_{u^2,B}(z) - iP_{u,C}(z) \,. \end{aligned}$$

Using (10) we get (29).

We now prove the converse. If F satisfies

$$\begin{aligned} zAF_u' = -\frac{iB}{2}F_u^2 + (-zA' - iB - iC)F_u + \frac{iB}{2} - iP_{u^2,B} \\ + zQ_{u,A}(z) - iP_{u,C}(z) + E(z), \ |z|<1 \,, \end{aligned}$$

then, from (25), we get

$$\langle V, \frac{\xi+z}{\xi-z}\rangle = 2zH\,\mathbb{I}(z) \quad \text{with} \quad V = \mathcal{D}(Au) - Bu^2 - Cu \,.$$

Using the asymptotic expansion of $\dfrac{\xi+z}{\xi-z}$, in $|z|<1$ and in $|z|>1$, the two previous equations are, respectively, equivalent to

$$\langle V, 1+2\sum_{k=1}^{+\infty}\xi^{-k}z^{k}\rangle \;=\; 2zH(z)\,,\quad |z|<1\,, \tag{32}$$

$$\langle V, -1-2\sum_{k=1}^{+\infty}\xi^{k}z^{-k}\rangle \;=\; 0\,,\quad |z|>1\,. \tag{33}$$

Writing $zH(z)=h_1z+\cdots+h_lz^l$ then, from (32) we get

$$\langle V,1\rangle=0\,,\quad \langle V,\xi^{-k}\rangle=h_k\,,\ k=1,\ldots,l\,,\quad \langle V,\xi^{-k}\rangle=0\,,\ k\geq l+1 \tag{34}$$

and, from (33) we get

$$\langle V,1\rangle=0\,,\quad \langle V,\xi^{k}\rangle=0\,,\ k\geq 1\,. \tag{35}$$

Finally, from (34) and (35), we conclude that $V=zH\mathcal{L}$ and (30) holds. ■

If we do $B=0$ in previous Theorem we get the following result to the Laguerre-Hahn affine class (see [2, 3]).

Corollary 4. *Let u be a hermitian linear functional and F_u the corresponding formal Carathéodory function. If u satisfies $\mathcal{D}(Au)=Cu+\xi H(\xi)\mathcal{L}$, then F_u is such that*

$$zA(z)F_u'(z)=\left(-zA'(z)-iC(z)\right)F_u(z)+zQ_{u,A}(z)-iP_{u,C}(z)+E(z)\,, \tag{36}$$

for $z\in\mathbb{C}\setminus\mathbb{T}$, with $E(z)=-2izH(z)\,\mathbb{I}(z)$.
Conversely, if F_u satisfies (36), then u satisfies $\mathcal{D}(Au)=Cu+\xi H(\xi)\mathcal{L}$.

As a consequence of previous Theorem, we get a characterization for semi-classical functionals (see [2]).

Corollary 5. *Let u be a hermitian linear functional and F the corresponding formal Carathéodory function, satisfying a differential equation with polynomial coefficients*

$$zA(z)F'(z)=(-zA'(z)-iC(z))F(z)+P(z)\,,\quad z\in\mathbb{C}\setminus\mathbb{T}\,. \tag{37}$$

A necessary and sufficient condition for u to be semi-classical and satisfy $\mathcal{D}(Au)=Cu$ is that the polynomial P of (37) satisfy $P=zQ_{u,A}-iP_{u,C}$, where $Q_{u,A}$ and $P_{u,C}$ are given by (20) and (21).

References

[1] M. Alfaro and L. Moral, *Quasi-orthogonality on the unit circle and semi-classical forms*, Portugaliæ Mathematica, **51** (1) (1994), 47-62.

[2] A. Branquinho and M.N. Rebocho, *Characterizations of Laguerre-Hahn affine orthogonal polynomials on the unit circle*, to appear in J. Comput. Anal. Appl.

[3] A. Cachafeiro and C. Pérez, *A study of the Laguerre-Hahn affine functionals on the unit circle*, J. Comput. Anal. Appl., **6** (2) (2004), 107-123.

[4] A. Cachafeiro and C. Pérez, *Second degree functionals on the unit circle*, Integral Transforms and Special Functions, **15** (4) (2004), 281-294.

[5] A. Magnus, *Riccati acceleration of the Jacobi continued fractions and Laguerre-Hahn orthogonal polynomials*, in "Padé Approximation and its Applications", Proc. Bad Honnef 1983, Lect. Notes in Math. **1071** (H. Werner and H.T. Bunger, eds.), Springer-Verlag, Berlin, 1984, 213-230.

[6] F. Marcellán, *Orthogonal polynomials and Toeplitz matrices: Some applications*, in "Seminario Matemático García Galdeano" (M. Alfaro, ed.), Univ. Zaragoza, 1989, 31-57.

[7] P. Maroni, *Le calcul des formes linéaires et les polynômes orthogonaux semi-classiques*, in Lect. Notes in Math., vol. 1329, Springer Verlag, Berlin, 1988, 279-290.

[8] P. Maroni, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques.* In "Orthogonal polynomials and their application" (C. Brezinski, L. Gori and A. Ronveux, eds.) J.C.Baltzer AG. Basel IMACS Annals on Computing and Applied Mathematics, **9** (1991), 95-130.

[9] F. Peherstorfer, *A special class of polynomials orthogonal on the unit circle including the associated polynomials*, Constr. Approx., **12** (2) (1996), 161-185.

[10] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Providence Rhode Island, 1975 (fourth edition).

[11] C. Tasis, *Propriedades diferenciales de los polinomios ortogonales relativos a la circunferencia unidad*, PhD Thesis, Universidad de Cantabria, 1989 (in spanish).

A. Branquinho
CMUC, Department of Mathematics, University of Coimbra, Largo D. Dinis, 3001-454 Coimbra, Portugal.

E-mail address: ajplb@mat.uc.pt
URL: http://www.mat.uc.pt/∼ajplb

M.N. Rebocho
Department of Mathematics, University of Beira Interior, 6201-001 Covilhã, Portugal, CMUC, University of Coimbra, 3001-454 Coimbra, Portugal.

E-mail address: mneves@mat.ubi.pt