MATRIX SYLVESTER EQUATIONS IN THE THEORY OF ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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Abstract. In this paper we characterize sequences of orthogonal polynomials on the unit circle whose Carathéodory function satisfies a Riccati differential equation with polynomial coefficients, in terms of matrix Sylvester differential equations. For the particular case of semi-classical orthogonal polynomials on the unit circle, it is derived a characterization in terms of first order linear differential systems.

1. Introduction

Let F be a Carathéodory function in the *Laguerre-Hahn class*, i.e., satisfying a Riccati differential equation with polynomial coefficients (see [4])

$$zAF' = BF^2 + CF + D\,,\ A \not\equiv 0\,. \tag{1}$$

A first approach to the analysis of Carathéodory functions satisfying this type of differential equations and to the analysis of its corresponding sequences of orthogonal polynomials was done by Alfaro and Marcellán in [2]. We remark that the Laguerre-Hahn class on the unit circle includes some well-known classes, such as the *Laguerre-Hahn affine class on the unit circle* (which corresponds to the case $B \equiv 0$ in (1)), the *semi-classical class on the unit circle* (which corresponds to the case $B \equiv 0$ and D a specific polynomial in (1)), and the *class of second degree functionals on the unit circle.* It also includes linear

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fractional transformations of Laguerre-Hahn Carathéodory functions (see [3, 4, 7, 8]).

The motivation for the study here presented comes from several applications related with orthogonal polynomials on the unit circle and also on the real line. In what concerns to the orthogonality on the real line we note the works of Magnus [18], Maroni [21, 22] and Hahn [15, 16]. In [18] Magnus used the theory of Laguerre-Hahn orthogonal polynomials (the "Riccati model") in the study of the convergence of Jacobi continued fractions. This was done, first, by considering a modified approximant which satisfies a Riccati differential equation and, then, by estimating the error behavior with the help of appropriate linear differential equations which are satisfied by a sequence of Laguerre-Hahn orthogonal polynomials (see [15, 16]). See also the example in [18, section 5], showing the use of the Riccati model in disordered systems analysis. In [21, 22], Maroni studies the Laguerre-Hahn class on the real line from an algebraic point of view, putting emphasis on the distributional equations for the corresponding forms defined in the linear space of real polynomials; some modifications that preserve the Laguerre-Hahn character are studied (in [4] the analogue of these results are established for Laguerre-Hahn functionals on the unit circle).

Let us now return to the orthogonality on the unit circle. Since the Laguerre-Hahn class on the unit circle contains linear fractional transformations of Carathéodory functions which satisfy Riccati type differential equations, then it is a suitable class to study some transformations related with the measure of orthogonality or with the orthogonal polynomials when one starts, for example, with Laguerre-Hahn affine orthogonal polynomials, or with orthogonal polynomials associated with second degree Carathéodory functions. Here are some examples:

a) shift perturbation of the reflection coefficients of the orthogonal polynomials (see [23]);

b) backward extension or modification of a finite number of places of the reflection coefficients of the orthogonal polynomials (see [23]);

c) rational perturbation of the measure of orthogonality (see [5, 6]).

In this paper we aim to obtain a characterization of the Laguerre-Hahn Carathéodory functions and a representation for the corresponding sequences of orthogonal polynomials on the unit circle. We will see that, also on the unit circle, the first order differential relations satisfied by Laguerre-Hahn orthogonal polynomials play an important role. In fact, a key result of our paper is the equivalence between (1) and the following matrix Sylvester differential equations for $Y_n = \begin{bmatrix} \phi_n & -\Omega_n \\ \phi_n^* & \Omega_n^* \end{bmatrix}$ and $\mathcal{Q}_n = \begin{bmatrix} -Q_n & Q_n^* \end{bmatrix}^T$,

$$
\begin{cases} zAY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C} \\ zA\mathcal{Q}_n' = \left(\mathcal{B}_n + (BF + C/2)\, I\right) \mathcal{Q}_n \,, n \in \mathbb{N} \,, \end{cases} \tag{2}
$$

where $\{\phi_n\}$, $\{\Omega_n\}$, and $\{Q_n\}$ are the corresponding sequences of orthogonal polynomials, of polynomials of the second kind, and functions of the second kind, respectively, $\mathcal{B}_n$ and $\mathcal{C}$ are matrices of order two with polynomial elements, and I is the identity matrix of order two (see Theorem 3).

As a consequence of the referred equivalence, we obtain a characterization of polynomials which are orthogonal with respect to a semi-classical weight, in terms of first order linear systems of differential equations (see Theorem 4). These systems are similar to the ones derived in [10, 17] (see also [19]). But here it is well to emphasize that, in those papers, the authors went further and studied the dynamics of the linear systems of differential equations subject to deformations of the semi-classical weight, thus showing the occurrence of Schlesinger systems as well as Painlevé equations.

The equivalence between (1) and (2) allow us to give a representation for $\{Y_n\}$ in terms of the solutions of two linear differential systems, $zA\mathcal{L}' = \mathcal{C}\mathcal{L}$ and $zA\mathcal{P}_n' = \mathcal{B}_n \mathcal{P}_n$, as $Y_n = \mathcal{P}_n \mathcal{L}^{-1}\,, \forall n \geq 1$ (see Theorem 5). Furthermore, the characterization for semi-classical polynomials previously obtained will help us to establish that the Carathéodory function F in (1) is a linear fractional transformation of a semi-classical Carathéodory function, say $\tilde{F}$ (see Theorem 6), and we give a representation for $\{Y_n\}$ in terms of the semi-classical orthogonal polynomials corresponding to $\tilde{F}$ (see Theorem 7).

This paper is organized as follows. In section 2 we give the definitions and state the basic results which will be used in the forthcoming sections. In section 3 we establish the equivalence between (1) and the matrix Sylvester differential equations (2). In section 4 we establish a characterization of semi-classical orthogonal polynomials on the unit circle in terms of first order linear differential systems. In section 5 we solve the matrix Sylvester differential equations from section 3, $zAY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}$, with the help of the the results previously obtained for semi-classical orthogonal polynomials. Thus, we determine a representation for the solution, Y_n, in terms of sequences of semi-classical orthogonal polynomials on the unit circle. Finally, in section 6, an example is presented.

2. Preliminary results

Let μ be a probability measure with infinite support on the unit circle $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi[\,\}$. The corresponding sequence of orthogonal polynomials, called *orthogonal polynomials on the unit circle* (with respect to μ), is defined by

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_n(e^{i\theta})\overline{\phi}_m(e^{-i\theta})\, d\mu(\theta) = h_n\delta_{n,m}\,,\ \ h_n \neq 0\,,\ \ n, m \in \mathbb{N}\,.$$

If μ is absolutely continuous with respect to $d\theta$, associated with a weight w, i.e., $d\mu(\theta) = w(\theta)\, d\theta$, then we say that $\{\phi_n\}$ is orthogonal with respect to w. If each ϕ_n is monic, then $\{\phi_n\}$ will be called a *monic orthogonal polynomial sequence*, and it will be denoted by MOPS.

Given a measure μ, the function F defined by

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta) \tag{3}$$

is a *Carathéodory function*, i.e., it is an analytic function in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $F(0) = 1$ and $\Re e(F) > 0$ for $|z| < 1$. The converse result also holds, since any Carathéodory function has a representation (3) for a unique probability measure μ on $\mathbb{T}$ (see, for example, [25]). In addition, it is well-known that $d\mu_r(\theta) = \Re e\, F(re^{i\theta})\, d\theta$ converge weakly to $d\mu$ when $r \uparrow 1$, $\lim_{r\uparrow 1} \Re e\, F(re^{i\theta}) = \Re e\, F(e^{i\theta})$ exists a.e. for $\theta \in [0, 2\pi]$, and if $d\mu(\theta) = w(\theta)d\theta + d\mu_s(\theta)$, with $d\mu_s$ the singular part, then

$$w(\theta) = \Re e\, F(e^{i\theta})\,.$$

Given a sequence of monic polynomials $\{\phi_n\}$ orthogonal with respect to μ, the *associated polynomials of the second kind* are given by

$$\Omega_0(z) = 1,\ \Omega_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\left(\phi_n(e^{i\theta}) - \phi_n(z)\right)\, d\mu(\theta)\,,\ \forall n \in \mathbb{N}\,,$$

and the *functions of the second kind* are given by

$$Q_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\phi_n(e^{i\theta})\, d\mu(\theta)\,,\ n = 0, 1, \ldots$$

We define the following matrices which will be used throughout the paper,

$$Y_n = \begin{bmatrix} \phi_n & -\Omega_n \\ \phi_n^* & \Omega_n^* \end{bmatrix},\ \mathcal{Q}_n = \begin{bmatrix} -Q_n \\ Q_n^* \end{bmatrix},\ \forall n \in \mathbb{N}\,, \tag{4}$$

where ϕ_n^* and Ω_n^* denote the reciprocal polynomial of ϕ_n and Ω_n, respectively, and $Q_n^*(z) = \Omega_n^*(z) - F(z)\phi_n^*(z)$. We recall that the reciprocal polynomial p^* of a polynomial p of exact degree n is defined by $p^*(z) = z^n\overline{p}(1/z)$.

The sequences $\{\phi_n\}$, $\{\Omega_n\}$ and $\{Q_n\}$ satisfy recurrence relations and coupled relations which we use in the matrix form, as given in the following theorem (see [14]).

Theorem 1 (cf. [12, 13, 24])**.** *Let F be a Carathéodory function and $\{\phi_n\}$, $\{\Omega_n\}$, and $\{Q_n\}$ the corresponding MOPS on the unit circle, the sequence of associated polynomials of the second kind, and the sequence of the functions of the second kind, respectively. Let $\{Y_n\}$ and $\{\mathcal{Q}_n\}$ be the sequences defined in (4). Then, the following relations hold, $\forall n \in \mathbb{N}$,*

$$Y_n = \mathcal{A}_n Y_{n-1}, \ \ \mathcal{A}_n = \begin{bmatrix} z & a_n \\ \overline{a}_n z & 1 \end{bmatrix}, \tag{5}$$

$$\mathcal{Q}_n = Y_n \begin{bmatrix} -F \\ 1 \end{bmatrix}, \tag{6}$$

with $a_n = \phi_n(0)$, $Y_0 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.
Moreover, $\forall n \in \mathbb{N}$,

$$\phi_n^*(z)\Omega_n(z) + \phi_n(z)\Omega_n^*(z) = 2h_n z^n, \tag{7}$$

$$\phi_n^*(z)Q_n(z) + \phi_n(z)Q_n^*(z) = 2h_n z^n, \tag{8}$$

with $h_n = \prod_{k=1}^{n}(1-|a_k|^2)$.

Let $H_0(z) = \sum_{j=0}^{+\infty} b_j z^j, \ |z|<1, \ \ H_\infty = \sum_{j=0}^{+\infty} b_j z^{-j}, \ |z|>1$. We will write $H_0(z) = \mathcal{O}(z^k)$ or $H_\infty(z) = \mathcal{O}(z^{-k})$ if $b_0 = \cdots = b_{k-1} = 0$, $k \in \mathbb{N}$.

Corollary 1. *Let $\{\phi_n\}$ be a MOPS on the unit circle and $\{Q_n\}$ be the corresponding sequence of functions of the second kind. Then, $\forall n \in \mathbb{N}$,*

$$\begin{aligned} Q_n(z) &= 2h_n z^n + \mathcal{O}(z^{n+1}), \ |z|<1, \\ Q_n(z) &= 2a_{n+1}h_n z^{-1} + \mathcal{O}(z^{-2}), \ |z|>1, \\ Q_n^*(z) &= 2\overline{a}_{n+1}h_n z^{n+1} + \mathcal{O}(z^{n+2}), \ |z|<1 \\ Q_n^*(z) &= 2h_n + \mathcal{O}(z^{-1}), \ |z|>1 \end{aligned}$$

with $a_{n+1} = \phi_{n+1}(0)$, $h_n = \prod_{k=1}^{n}(1-|a_k|^2)$.

Corollary 2. *Let $\{\phi_n\}$ be a MOPS on the unit circle and $\{\Omega_n\}$ be the corresponding sequence of associated polynomials of the second kind. Then, the following holds:*
a) If there exists $k \in \mathbb{N}$ such that $\phi_k(\alpha) = \Omega_k(\alpha) = 0$, then $\alpha = 0$;
b) If there exists $k \in \mathbb{N}$ such that $\phi_k(\alpha) = Q_k(\alpha) = 0$, then $\alpha = 0$.

Theorem 2 (Geronimus, [11])**.** *Given a sequence of complex numbers (a_n) satisfying $|a_n| < 1$, $\forall n \in \mathbb{N}$, let $\{\phi_n\}$ and $\{\Omega_n\}$ be the sequences of polynomials defined by the recurrence relation (5), and let F be the corresponding Carathéodory function. Then, the sequence defined for $n \geq 1$, by*

$$\frac{\Omega_n^*(z)}{\phi_n^*(z)} = 1 + \frac{-2\overline{a}_1 z}{1+\overline{a}_1 z} - \frac{\dfrac{\overline{a}_2}{\overline{a}_1} z(1-|a_1|^2)}{1+\dfrac{\overline{a}_2}{\overline{a}_1} z} - \cdots - \frac{\dfrac{\overline{a}_{n+1}}{\overline{a}_n} z(1-|a_n|^2)}{1+\dfrac{\overline{a}_{n+1}}{\overline{a}_n} z},$$

converges uniformly to $F(z)$, on compact subsets of $\mathbb{D}$.

3. Characterization in terms of matrix Sylvester differential equations

Hereafter, I denotes the identity matrix of order two.

Theorem 3. *Let F be a Carathéodory function and $\{Y_n\}$ and $\{\mathcal{Q}_n\}$ the corresponding sequences defined by (4). The following statements are equivalent:*
a) F satisfies the differential equation with polynomial coefficients

$$zAF' = BF^2 + CF + D\,; \tag{9}$$

b) $\{Y_n\}$ and $\{\mathcal{Q}_n\}$ satisfy the Sylvester differential equations

$$zAY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C} \tag{10}$$

$$zA\mathcal{Q}_n' = \left(\mathcal{B}_n + (BF + C/2)\, I\right) \mathcal{Q}_n\,,\ n \in \mathbb{N}\,, \tag{11}$$

where $\mathcal{B}_n$ are matrices of bounded degree polynomials,

$$\mathcal{B}_n = \begin{bmatrix} l_{n,1} & -\Theta_{n,1} \\ -\Theta_{n,2} & l_{n,2} \end{bmatrix}, \tag{12}$$

and

$$\mathcal{C} = \begin{bmatrix} C/2 & -D \\ B & -C/2 \end{bmatrix}. \tag{13}$$

Proof. $a) \Rightarrow b)$.
Let F satisfy (9). Firstly we obtain (10). This will be done by dividing

the proof in two parts: in the first part we deduce the equations

$$\begin{cases} zA\Omega_n' = (l_{n,1}+C/2)\Omega_n - D\phi_n + \Theta_{n,1}\Omega_n^* \\ zA\phi_n' = (l_{n,1}-C/2)\phi_n + B\Omega_n - \Theta_{n,1}\phi_n^* \end{cases} \tag{14}$$

and in the second part we deduce the equations

$$\begin{cases} zA(\Omega_n^*)' = (l_{n,2}+C/2)\Omega_n^* + D\phi_n^* + \Theta_{n,2}\Omega_n \\ zA(\phi_n^*)' = (l_{n,2}-C/2)\phi_n^* - B\Omega_n^* - \Theta_{n,2}\phi_n \end{cases} \tag{15}$$

where $l_{n,1}, l_{n,2}, \Theta_{n,1}, \Theta_{n,2}$ are polynomials whose degrees are bounded by a number independent of n. These two systems of equations can be written in the matrix form (10), with $\mathcal{B}_n$ and $\mathcal{C}$ given by (12) and (13), respectively.

First part. If we substitute $F = \dfrac{Q_n}{\phi_n} - \dfrac{\Omega_n}{\phi_n}$ (cf. (6)) in $zAF' = BF^2 + CF + D$ we obtain

$$zA\left(\frac{Q_n}{\phi_n} - \frac{\Omega_n}{\phi_n}\right)' = B\left(\frac{Q_n}{\phi_n} - \frac{\Omega_n}{\phi_n}\right)^2 + C\left(\frac{Q_n}{\phi_n} - \frac{\Omega_n}{\phi_n}\right) + D\,,$$

i.e.,

$$\begin{aligned} zA\left(\frac{Q_n}{\phi_n}\right)' - B\frac{Q_n}{\phi_n}\left(\frac{Q_n}{\phi_n} - 2\frac{\Omega_n}{\phi_n}\right) - C\frac{Q_n}{\phi_n} \\ = zA\left(\frac{\Omega_n}{\phi_n}\right)' + B\left(\frac{\Omega_n}{\phi_n}\right)^2 - C\left(\frac{\Omega_n}{\phi_n}\right) + D\,. \end{aligned}$$

Therefore we have

$$\left\{ zA\left(\frac{\Omega_n}{\phi_n}\right)' + B\left(\frac{\Omega_n}{\phi_n}\right)^2 - C\left(\frac{\Omega_n}{\phi_n}\right) + D \right\}\phi_n^2 = \tilde{\Theta}_n \tag{16}$$

with

$$\tilde{\Theta}_n = \left\{ zA\left(\frac{Q_n}{\phi_n}\right)' - B\frac{Q_n}{\phi_n}\left(\frac{Q_n}{\phi_n} - 2\frac{\Omega_n}{\phi_n}\right) - C\frac{Q_n}{\phi_n} \right\}\phi_n^2\,.$$

From (16) it follows that $\tilde{\Theta}_n$ is a polynomial. From the asymptotic expansion of Q_n in $|z|<1$ (see Corollary 1), and since the left side of (16) is a polynomial, we get

$$\tilde{\Theta}_n(z) = z^n\tilde{\Theta}_n^1(z)\,,$$

with $\tilde{\Theta}_n^1$ a polynomial. From the asymptotic expansion of Q_n in $|z|>1$ (see Corollary 1) it follows that $\tilde{\Theta}_n^1$ has bounded degree,

$$\deg(\tilde{\Theta}_n^1) = \max\{\deg(zA)-2, \deg(B)-1, \deg(C)-1\},\ \forall n\in\mathbb{N}\,.$$

Thus, (16) becomes

$$\left\{ zA\left(\frac{\Omega_n}{\phi_n}\right)' + B\left(\frac{\Omega_n}{\phi_n}\right)^2 - C\left(\frac{\Omega_n}{\phi_n}\right) + D \right\}\phi_n^2 = z^n\tilde{\Theta}_n^1\,.$$

Using (7) in the previous equation we obtain

$$\left\{ zA\left(\frac{\Omega_n}{\phi_n}\right)' + B\left(\frac{\Omega_n}{\phi_n}\right)^2 - C\left(\frac{\Omega_n}{\phi_n}\right) + D \right\}\phi_n^2 = \Theta_{n,1}(\phi_n\Omega_n^* + \Omega_n\phi_n^*)\,,$$

where $\Theta_{n,1} = \tilde{\Theta}_n^1/(2h_n)\,.$

Consequently, $\forall n \in \mathbb{N}$,

$$\left\{ zA\Omega_n' - \frac{C}{2}\Omega_n + D\phi_n - \Theta_{n,1}\Omega_n^* \right\}\phi_n$$
$$= \left\{ zA\phi_n' + \frac{C}{2}\phi_n - B\Omega_n + \Theta_{n,1}\phi_n^* \right\}\Omega_n\,.$$

We distinguish the following cases (see Corollary 2):
a) ϕ_n and Ω_n have no common roots, $\forall n \in \mathbb{N}$, i.e., $\phi_n(0) \neq 0, \forall n \in \mathbb{N}\,;$
b) there exists a finite number of indexes $k \in \mathbb{N}$ such that ϕ_k and Ω_k have common roots, i.e., $\phi_k(0) = \Omega_k(0) = 0$ for a finite number of k's;
c) there exists $n_0 > 1$ such that $\phi_n(0) = 0$, $\forall n \geq n_0\,.$

Case a) If ϕ_n and Ω_n have no common roots, $\forall n \in \mathbb{N}$, then we conclude that there exists a polynomial $l_{n,1}$ such that

$$\left\{\begin{array}{l} zA\Omega_n' - \frac{C}{2}\Omega_n + D\phi_n - \Theta_{n,1}\Omega_n^* = l_{n,1}\Omega_n\,, \\ zA\phi_n' + \frac{C}{2}\phi_n - B\Omega_n + \Theta_{n,1}\phi_n^* = l_{n,1}\phi_n\,,\ \forall n \in \mathbb{N}\,, \end{array}\right. \tag{17}$$

and we obtain (14). Moreover, $l_{n,1}$ has bounded degree,

$$\deg(l_{n,1}) = \max\{\deg(zA) - 1, \deg(B), \deg(C), \deg(D)\},\ \forall n \in \mathbb{N}\,.$$

Case b) We first assume that $\phi_1(0) \neq 0, \ldots, \phi_{k-1}(0) \neq 0$, and k is the first index such that $\phi_k(0) = 0$. Thus, ϕ_n and Ω_n have no common roots for $n = 1, \ldots, k-1$. From case a), equations (17) hold for $n = 1, \ldots, k-1$. Let us write (17) to $k-1$ and multiply by z, to obtain

$$\left\{\begin{array}{l} z^2A\Omega_{k-1}' - \frac{C}{2}z\Omega_{k-1} + Dz\phi_{k-1} - z\Theta_{k-1,1}\Omega_{k-1}^* = l_{k-1,1}z\Omega_{k-1}\,, \\ z^2A\phi_{k-1}' + \frac{C}{2}z\phi_{k-1} - Bz\Omega_{k-1} + z\Theta_{k-1,1}\phi_{k-1}^* = l_{k-1,1}z\phi_{k-1}\,. \end{array}\right.$$

By substituting

$$\Omega_k(z) = z\Omega_{k-1}(z),\ \Omega_k^*(z) = \Omega_{k-1}^*(z),\ z\Omega_{k-1}'(z) = \Omega_k'(z) - \Omega_{k-1}(z)$$

and

$$\phi_k(z) = k\phi_{k-1}(z),\ \phi_k^*(z) = \phi_{k-1}^*(z),\ z\phi_{k-1}'(z) = \phi_k'(z) - \phi_{k-1}(z)$$

in previous equations, it follows that

$$\begin{cases} zA\Omega_k' - \frac{C}{2}\Omega_k + D\phi_k - z\Theta_{k-1,1}\Omega_k^* = (l_{k-1,1} + A)\Omega_k\,, \\ zA\phi_k' + \frac{C}{2}\phi_k - B\Omega_k + z\Theta_{k-1,1}\phi_k^* = (l_{k-1,1} + A)\,\phi_k\,, \end{cases}$$

and we obtain (14) to $n = k$ with $l_{k,1} = l_{k-1,1} + A$ and $\Theta_{k,1} = z\Theta_{k-1,1}$.

Furthermore, if $\phi_{k+1}(0) = \cdots = \phi_{k+k_0}(0) = 0,\ \phi_{k+k_0+1}(0) \neq 0$ to some $k_0 \in \mathbb{N}$, then, using the same method as before, the differential relations (14) are obtained for $n = k+1, \ldots, k+k_0$, with

$$l_{n,1} = l_{k-1,1} + (n-k+1)A\,,\ \ \Theta_{n,1} = z^{n-k+1}\Theta_{k-1,1}\,,\ \ n = k+1, \ldots, k+k_0\,.$$

Case c) If $\phi_n(0) = 0,\ \forall n \geq n_0$, then ϕ_n and Ω_n are polynomials of the Bernstein-Szegő type,

$$\phi_n(z) = z^{n-n_0+1}\phi_{n_0-1}(z)\,,\ \ \Omega_n(z) = z^{n-n_0+1}\Omega_{n_0-1}(z)\,.$$

Applying the same method as before, we conclude that equations (14) hold, $\forall n \in \mathbb{N}$, and, for $n \geq n_0$, $l_{n,1}$ and $\Theta_{n,1}$ are given by

$$l_{n,1} = l_{n_0-1} + (n - n_0 + 1)A\,,\ \Theta_{n,1} = z^{n-n_0+1}\Theta_{n_0-1,1}\,.$$

Second part. If we substitute $F = \dfrac{\Omega_n^*}{\phi_n^*} - \dfrac{Q_n^*}{\phi_n^*}$ (cf. (6)) in $zAF' = BF^2 + CF + D$ and proceed as in the first part, we obtain (15) with polynomials $\Theta_{n,2}$ and $l_{n,2}$ such that

$$\deg(\Theta_{n,2}) = \max\{\deg(zA) - 1, \deg(B), \deg(C)\},\ \forall n \in \mathbb{N}\,,$$

$$\deg(l_{n,2}) = \max\{\deg(zA) - 1, \deg(B), \deg(C), \deg(D)\},\ \forall n \in \mathbb{N}\,.$$

Let us now obtain (11). Taking derivatives on $Q_n = \Omega_n + \phi_n F$ and $Q_n^* = \Omega_n^* - \phi_n^* F$ (cf. (6)) we obtain

$$\begin{aligned} zAQ_n' &= zA\Omega_n' + zA\phi_n' F + zAF'\phi_n\,, \\ zA(Q_n^*)' &= zA(\Omega_n^*)' - zA(\phi_n^*)' F - zAF'\phi_n^*\,. \end{aligned}$$

If we use (9), (14) and (15) in the previous equations, then (11) follows.

$b) \Rightarrow a)$.

Taking into account (6), $\mathcal{Q}_n = Y_n \begin{bmatrix} -F \\ 1 \end{bmatrix}, \forall n \in \mathbb{N}$, the equation (11) is equivalent to

$$zAY_n' \begin{bmatrix} -F \\ 1 \end{bmatrix} + zAY_n \begin{bmatrix} -F' \\ 0 \end{bmatrix} = \mathcal{B}_n Y_n \begin{bmatrix} -F \\ 1 \end{bmatrix} + (BF + C/2)Y_n \begin{bmatrix} -F \\ 1 \end{bmatrix}\,.$$

From (10) it follows that

$$(\mathcal{B}_n Y_n - Y_n \mathcal{C}) \begin{bmatrix} -F \\ 1 \end{bmatrix} + zAY_n \begin{bmatrix} -F' \\ 0 \end{bmatrix} = \mathcal{B}_n Y_n \begin{bmatrix} -F \\ 1 \end{bmatrix} + (BF + C/2) Y_n \begin{bmatrix} -F \\ 1 \end{bmatrix},$$

i.e.,

$$Y_n \left(zA \begin{bmatrix} -F' \\ 0 \end{bmatrix} - \mathcal{C} \begin{bmatrix} -F \\ 1 \end{bmatrix} \right) = (BF + C/2) Y_n \begin{bmatrix} -F \\ 1 \end{bmatrix}.$$

Taking into account that Y_n is nonsingular, we obtain

$$zA \begin{bmatrix} -F' \\ 0 \end{bmatrix} - \mathcal{C} \begin{bmatrix} -F \\ 1 \end{bmatrix} = (BF + C/2) \begin{bmatrix} -F \\ 1 \end{bmatrix}.$$

Since $\mathcal{C}$ is given by (13), $zAF' = BF^2 + CF + D$ follows. □

The following formula for $\operatorname{tr}(\mathcal{B}_n)$ was given in [20] for a particular case of a semi-classical sequence of orthogonal polynomials on the unit circle.

Corollary 3. *Under the conditions of the previous theorem, the matrices $\mathcal{B}_n$ given by (12) satisfy*

$$zA\mathcal{A}_n' = \mathcal{B}_n \mathcal{A}_n - \mathcal{A}_n \mathcal{B}_{n-1}, \; n \geq 2, \tag{18}$$

$$\operatorname{tr}(\mathcal{B}_n) = nA, \; n \in \mathbb{N}, \tag{19}$$

$$\det(\mathcal{B}_n) = \det(\mathcal{B}_1) - A \sum_{k=1}^{n-1} l_{k,2}, \; n \geq 2, \tag{20}$$

where $\operatorname{tr}(\mathcal{B}_n)$ *and* $\det(\mathcal{B}_n)$ *denote the trace and the determinant of* $\mathcal{B}_n$, *respectively, and*

$$\det(\mathcal{B}_1) = A\left(2zA\overline{a}_1 - h_1(D+B) + C(|a_1|^2+1)\right)/(2\,h_1) + BD - C^2/4, \tag{21}$$

$a_1 = \phi_1(0), \; h_1 = 1 - |a_1|^2$.

Proof. To obtain (18) we take derivatives on $Y_n = \mathcal{A}_n Y_{n-1}$ and substitute $Y_n' = \mathcal{A}_n' Y_{n-1} + \mathcal{A}_n Y_{n-1}'$ in (10), $zAY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}$. Therefore, we get

$$zA\mathcal{A}_n' Y_{n-1} + zA\mathcal{A}_n Y_{n-1}' = \mathcal{B}_n Y_n - Y_n \mathcal{C}.$$

Using (10) for $n-1$ in the previous equation we get

$$zA\mathcal{A}_n' Y_{n-1} + \mathcal{A}_n (\mathcal{B}_{n-1} Y_{n-1} - Y_{n-1}\mathcal{C}) = \mathcal{B}_n Y_n - Y_n \mathcal{C}.$$

Using the recurrence relation (5) we obtain

$$zA\mathcal{A}_n' Y_{n-1} + \mathcal{A}_n (\mathcal{B}_{n-1} Y_{n-1} - Y_{n-1}\mathcal{C}) = \mathcal{B}_n \mathcal{A}_n Y_{n-1} - \mathcal{A}_n Y_{n-1} \mathcal{C},$$

i.e.,

$$zA\mathcal{A}_n' Y_{n-1} = \left(\mathcal{B}_n\mathcal{A}_n - \mathcal{A}_n\mathcal{B}_{n-1}\right) Y_{n-1}\,.$$

Since Y_n is nonsingular, for all $n \in \mathbb{N}$ and $z \neq 0$, we obtain (18).

To deduce (19) we use equations (14) and (15),

$$\begin{cases} zA\phi_n' + C/2\phi_n - B\Omega_n + \Theta_{n,1}\phi_n^* = l_{n,1}\phi_n \\ zA\Omega_n' - C/2\Omega_n + D\phi_n - \Theta_{n,1}\Omega_n^* = l_{n,1}\Omega_n \\ zA(\Omega_n^*)' - C/2\Omega_n^* - D\phi_n^* - \Theta_{n,2}\Omega_n = l_{n,2}\Omega_n^* \\ zA(\phi_n^*)' + C/2\phi_n^* + B\Omega_n^* + \Theta_{n,2}\phi_n = l_{n,2}\phi_n^*\,. \end{cases}$$

If we multiply the previous equations by Ω_n^*, ϕ_n^*, ϕ_n and Ω_n, respectively, we obtain, after summing,

$$zA\left(\phi_n'\Omega_n^* + \phi_n(\Omega_n^*)' + (\phi_n^*)'\Omega_n + \phi_n^*\Omega_n'\right) = \left(l_{n,1} + l_{n,2}\right)\left(\phi_n\Omega_n^* + \phi_n^*\Omega_n\right)\,,$$

i.e.,

$$zA\left(\phi_n\Omega_n^* + \phi_n^*\Omega_n\right)' = \left(l_{n,1} + l_{n,2}\right)\left(\phi_n\Omega_n^* + \phi_n^*\Omega_n\right)\,.$$

Thus,

$$zA\left(\phi_n\Omega_n^* + \phi_n^*\Omega_n\right)' = \operatorname{tr}(\mathcal{B}_n)\left(\phi_n\Omega_n^* + \phi_n^*\Omega_n\right)\,.$$

If we use (7) in the previous equation then we get (19).

We now establish (20). From (18) we obtain, for $n \geq 2$,

$$\det(\mathcal{B}_n\mathcal{A}_n) = \det(zA\mathcal{A}_n' + \mathcal{A}_n\mathcal{B}_{n-1})\,.$$

Taking into account that $\mathcal{B}_n$ is given by (12) and $\mathcal{A}_n = \begin{bmatrix} z & a_n \\ \overline{a}_n z & 1 \end{bmatrix}$, we obtain

$$\det(\mathcal{B}_n)\det(\mathcal{A}_n) = z(1 - |a_n|^2)\left(\det(\mathcal{B}_{n-1}) + A\, l_{n-1,2}\right),\ \forall n \geq 2\,.$$

Since $\det(\mathcal{A}_n) = z(1 - |a_n|^2)$, then the last equation is equivalent, if $z \neq 0$, to

$$\det(\mathcal{B}_n) = \det(\mathcal{B}_{n-1}) + A\, l_{n-1,2}\,,\ \forall n \geq 2\,.$$

Consequently, we obtain (20). Moreover, if we compute $\det(\mathcal{B}_1)$ by taking $n = 1$ in (10), we obtain (21). □

Remark . (18) is equivalent to the following equations, for all $n \in \mathbb{N}$,

$$\begin{cases} a_n l_{n,1} - \Theta_{n,1} = -z\Theta_{n-1,1} + a_n l_{n-1,2} \\ z l_{n,1} - \overline{a}_n z\Theta_{n,1} = z l_{n-1,1} - a_n\Theta_{n-1,2} + zA \\ -z\Theta_{n,2} + \overline{a}_n z l_{n,2} = \overline{a}_n z l_{n-1,1} - \Theta_{n-1,2} + \overline{a}_n zA \\ -a_n\Theta_{n,2} + l_{n,2} = -\overline{a}_n z\Theta_{n-1,1} + l_{n-1,2}\,. \end{cases} \tag{22}$$

4. A characterization of semi-classical orthogonal polynomials on the unit circle

In this section we derive a characterization for sequences of semi-classical orthogonal polynomials on the unit circle.

Definition 1 (cf. [26])**.** Let μ be a measure supported on the unit circle given by $d\mu(\theta) = w(\theta)\, d\theta + \sum_{k=1}^{N} \lambda_k \delta_{z_k}$, where $\lambda_k \geq 0\,, |z_k| = 1\,, k = 1, ..., N,\ N \in \mathbb{N}$. The weight w (or the measure μ) is *semi-classical* if there exist polynomials A, C such that

$$\frac{dw(\theta)/d\theta}{w(\theta)} = \frac{C(z)}{A(z)}\,,\ z = e^{i\theta}. \tag{23}$$

The corresponding sequence of orthogonal polynomials is called *semi-classical.*

For our purposes, we will consider the analytic continuation of the weight w to an annulus $\{z : \epsilon_1 < |z| < \epsilon_2\}$ and, in order to simplify the notation, we still denote this analytic continuation by $w = w(z)$. Thus, the equation (23) is now equivalent to

$$\frac{w'(z)}{w(z)} = \frac{-i\, C(z)}{zA(z)} \quad ('{:=}\, d/dz). \tag{24}$$

It is well-known that the corresponding Carathéodory function satisfies a first order linear differential equation

$$zA(z)F'(z) = -i\, C(z)F(z) + D(z)\,,$$

where D is a polynomial (see [26]). Moreover, the converse holds for a specific polynomial D depending on A, C (see [3, 7]).

We will need the lemma that follows (see [9]).

Lemma 1. *Let X and M be matrix functions of order two such that $X' = M\,X$. Then,*

$$(\det(X))' = \mathrm{tr}(M)\ \det(X)\,. \tag{25}$$

The theorem that follows is a generalization of a result for semi-classical orthogonal polynomials on the real line established in [19]. Moreover, it shows that the necessary condition given in [3, Theorem 5] for a MOPS on the unit circle to be semi-classical is also sufficient.

Theorem 4. *Let $\{\phi_n\}$ be a MOPS with respect to a weight w, $\{Q_n\}$ be the sequence of functions of the second kind, and* $\widehat{Y}_n = \begin{bmatrix} \phi_n & -Q_n/w \\ \phi_n^* & Q_n^*/w \end{bmatrix}$,

$\forall n \geq 1$. *Then,*

$$w(z) = K e^{\int_{z_1}^{z} \frac{C(t)}{tA(t)} dt}, \ K \in \mathbb{C}, \tag{26}$$

if, and only if, $\widehat{Y}_n$ *satisfies*

$$zA\widehat{Y}_n' = (\mathcal{B}_n - C/2\, I)\widehat{Y}_n, \ \forall n \in \mathbb{N}, \tag{27}$$

where $\mathcal{B}_n$ *is the matrix associated with the equation* $zAF' = CF + D$ *satisfied by the corresponding Carathéodory function.*

Proof. Let w satisfy $w'/w = C/(zA)$ and let the corresponding F satisfy $zAF' = CF + D$.

From Theorem 3 the following two equations hold,

$$zA \begin{bmatrix} -Q_n'/w \\ (Q_n^*)'/w \end{bmatrix} = (\mathcal{B}_n + C/2\, I) \begin{bmatrix} -Q_n/w \\ Q_n^*/w \end{bmatrix}, \tag{28}$$

$$zA \begin{bmatrix} \phi_n \\ \phi_n^* \end{bmatrix}' = (\mathcal{B}_n - C/2\, I) \begin{bmatrix} \phi_n \\ \phi_n^* \end{bmatrix}, \tag{29}$$

where $\mathcal{B}_n$ are the matrices associated with $zAF' = CF + D$. Moreover, as $w'/w = C/(zA)$, then

$$zA \begin{bmatrix} -Q_n/w \\ Q_n^*/w \end{bmatrix}' = zA \begin{bmatrix} -Q_n'/w \\ (Q_n^*)'/w \end{bmatrix} - C\, I \begin{bmatrix} -Q_n/w \\ Q_n^*/w \end{bmatrix}. \tag{30}$$

If we substitute (28) in (30) we get

$$zA \begin{bmatrix} -Q_n/w \\ Q_n^*/w \end{bmatrix}' = (\mathcal{B}_n - C/2\, I) \begin{bmatrix} -Q_n/w \\ Q_n^*/w \end{bmatrix}. \tag{31}$$

Finally, from (29) and (31), the differential system (27) follows.

We now prove the converse.

If $\widehat{Y}_n = \begin{bmatrix} \phi_n & -Q_n/w \\ \phi_n^* & Q_n^*/w \end{bmatrix}$ satisfies (27) then, from Lemma 1, we obtain

$$(\det(\widehat{Y}_n))' = \frac{\operatorname{tr}(\mathcal{B}_n - C/2\, I)}{zA} \det(\widehat{Y}_n).$$

From (8) we get $\det(\widehat{Y}_n) = 2h_n z^n/w$, thus the last equation is equivalent to

$$\frac{w'}{w} = \frac{nA - \operatorname{tr}(\mathcal{B}_n - C/2\, I)}{zA}.$$

If we use $\operatorname{tr}(\mathcal{B}_n) = nA$ (cf. (19)) in the previous equation then we get $\frac{w'}{w} = \frac{C}{zA}$, and we conclude that w is given by (26). $\square$

5. Solutions of the Sylvester differential equations

In this section we solve the Sylvester differential equations (10), $zAY_n' = \mathcal{B}_nY_n - Y_n\mathcal{C}$, $\forall n \in \mathbb{N}$. The result that comes next is a particular case of a result on matrix Riccati equations known as Radon's Lemma (see [1]).

Theorem 5. *Let F be a Carathéodory function satisfying $zAF' = BF^2 + CF + D$ and $\mathcal{B}_n, \mathcal{C}$ be the corresponding matrices given by (12) and (13), respectively. Let $G \subset \mathbb{C}$ be a domain not containing the zeros of zA, and $z_0 \in G$. If $\mathcal{L}$ ($\mathcal{L}$ nonsingular) and $\mathcal{P}_n$ satisfy, $\forall n \in \mathbb{N}$,*

$$\begin{cases} zA(z)\mathcal{L}'(z) = \mathcal{C}(z)\mathcal{L}(z) \\ \mathcal{L}(z_0) = I \end{cases} \tag{32}$$

and

$$\begin{cases} zA(z)\mathcal{P}_n'(z) = \mathcal{B}_n(z)\mathcal{P}_n(z) \\ \mathcal{P}_n(z_0) = Y_n(z_0) \end{cases} \tag{33}$$

then the corresponding sequence $\{Y_n\}$ associated with F, defined in (4), has the following representation in G,

$$Y_n(z) = \mathcal{P}_n(z)\,\mathcal{L}^{-1}(z)\,, \forall n \in \mathbb{N}. \tag{34}$$

Proof. To $zAF' = BF^2+CF+D$ we associate (10), $zAY_n' = \mathcal{B}_nY_n - Y_n\mathcal{C}$, with $\mathcal{B}_n$ and $\mathcal{C}$ given by (12) and (13), respectively (see Theorem 3). Let $\mathcal{L}$ and $\mathcal{P}_n$ satisfy (32) and (33), respectively. Let us see that that $Y_n = \mathcal{P}_n\mathcal{L}^{-1}$ is the solution of $zAY_n' = \mathcal{B}_nY_n - Y_n\mathcal{C}$. Taking into account that

$$zA(\mathcal{P}_n\mathcal{L}^{-1})' = zA\mathcal{P}_n'\mathcal{L}^{-1} + zA\mathcal{P}_n(\mathcal{L}^{-1})'$$

and $(\mathcal{L}^{-1})' = -\mathcal{L}^{-1}\mathcal{L}'\mathcal{L}^{-1}$, from (33) we get

$$zA(\mathcal{P}_n\mathcal{L}^{-1})' = \mathcal{B}_n\mathcal{P}_n\mathcal{L}^{-1} - zA\mathcal{P}_n\mathcal{L}^{-1}\mathcal{L}'\mathcal{L}^{-1}\,.$$

Using (32) in the previous equation we get

$$zA(\mathcal{P}_n\mathcal{L}^{-1})' = \mathcal{B}_n\mathcal{P}_n\mathcal{L}^{-1} - \mathcal{P}_n\mathcal{L}^{-1}\mathcal{C}\mathcal{L}\mathcal{L}^{-1}\,,$$

i.e., $Y_n = \mathcal{P}_n\mathcal{L}^{-1}$ satisfies $zAY_n' = \mathcal{B}_nY_n - Y_n\mathcal{C}$, and the assertion follows. □

Remark . The solution of (32) is given by $\mathcal{L}(z) = L(z)L^0$, with L a fundamental matrix of the differential system (32) satisfying $zAL' = \mathcal{C}L$, and $L^0 = L(z_0)^{-1}$. The solution of (33) is given by $\mathcal{P}_n(z) = P_n(z)P_n^0$, with P_n a fundamental matrix of (33) satisfying $zAP_n' = \mathcal{B}_nP_n$, and P_n^0 satisfying $P_n(z_0)P_n^0 = Y_n(z_0)$, i.e., $P_n^0 = (P_n(z_0))^{-1}Y_n(z_0)$. Thus, if we substitute $\mathcal{L}$ and $\mathcal{P}_n$, given as above, in (34), the solution of the Sylvester differential equations (10) becomes

$$Y_n(z) = P_n(z) E_n L^{-1}(z) \tag{35}$$

with

$$E_n = (P_n(z_0))^{-1} Y_n(z_0) L(z_0)\,. \tag{36}$$

5.1. **Solution of (32).** We search for a matrix L of order 2 satisfying $zA(z)L'(z) = \mathcal{C}(z)L(z)$, with $\mathcal{C}$ given in (13).

Lemma 2. *Let L be a fundamental matrix of solutions of (32). Then,* $\det(L(z)) = \det(L(z_0))$.

Proof. From Lemma 1 (cf. (25)) we have

$$(\det(L))' = \frac{\operatorname{tr}(\mathcal{C})}{zA} \det(L)\,.$$

Since $\operatorname{tr}(\mathcal{C}) = 0$, it follows that $(\det(L))' = 0$, i.e.,

$$\det(L) = c,\ c \in \mathbb{C}\,.$$

Thus, $\det(L(z)) = \det(L(z_0))$, for some $z_0 \in \mathbb{C}$. □

Lemma 3. *Let $\mathcal{C}$ be the matrix defined by (13). Then,*

(a) $\mathcal{C}^2 = \beta\, I,\ \ \beta = (C/2)^2 - BD$ *;*
(b) *The eigenvalues of $\mathcal{C}$ are* $\pm\sqrt{\beta}$;
(c) $V_{\sqrt{\beta}} = \operatorname{span}\{\begin{bmatrix} D & C/2 - \sqrt{\beta}\end{bmatrix}^T\}$ *is the eigenspace corresponding to $\sqrt{\beta}$ and* $V_{-\sqrt{\beta}} = \operatorname{span}\{\begin{bmatrix} D & C/2 + \sqrt{\beta}\end{bmatrix}^T\}$ *is the eigenspace corresponding to $-\sqrt{\beta}$.*

In what follows, L_1, L_2 are column vectors of size 2 .

Lemma 4. *Let $L = [L_1\ L_2]$ be a fundamental matrix of (32). Then,*

$$zAL_1' = \sqrt{\beta} L_1 + zAc_1 V_{-\sqrt{\beta}}\,, \tag{37}$$

$$zAL_2' = -\sqrt{\beta} L_2 + zAc_2 V_{\sqrt{\beta}}\,, \tag{38}$$

with c_1, c_2 functions.

Proof. From (32) it follows that

$$(\mathcal{C} + \sqrt{\beta}\ I) \left(L_1' - \frac{\sqrt{\beta}}{zA} L_1 \right) = 0_{2\times 1}\,, \tag{39}$$

$$(\mathcal{C} - \sqrt{\beta}\ I) \left(L_2' + \frac{\sqrt{\beta}}{zA} L_2 \right) = 0_{2\times 1}\,. \tag{40}$$

Since the eigenvalues of $\mathcal{C}$ are $\pm\sqrt{\beta}$, and the corresponding eigenvectors are $V_{\sqrt{\beta}}$ and $V_{\sqrt{-\beta}}$, from (39) and (40) we obtain, respectively,

$$L_1' - \frac{\sqrt{\beta}}{zA} L_1 = c_1(z)\, V_{-\sqrt{\beta}}$$
$$L_2' + \frac{\sqrt{\beta}}{zA} L_2 = c_2(z)\, V_{\sqrt{\beta}}$$

where c_1, c_2 are functions. Thus, (37) and (38) follow. □

5.2. **Solution of (33).** We search for matrices P_n of order two satisfying, for each $n \in \mathbb{N}$,

$$zAP_n' = \mathcal{B}_n P_n \,. \tag{41}$$

Hereafter we will consider $z_1 \in \mathbb{C}$ and $\tilde{C}$ a polynomial such that $\int_{z_1}^{z} \frac{\tilde{C}/2}{tA} dt$ is defined (in suitable domains).

Lemma 5. *Let $\mathcal{B}_n$ be the matrices given in (12), let $A, \tilde{C}$ be polynomials. $\tilde{P}_n$ is a solution of*

$$zA\tilde{P}_n' = (\mathcal{B}_n - \tilde{C}/2\, I)\tilde{P}_n \tag{42}$$

if, and only if, $P_n = e^{\int_{z_1}^{z} \frac{\tilde{C}/2}{tA} dt}\, \tilde{P}_n$ is a solution of (41).

Proof. Let $\tilde{P}_n$ be a solution of (42). Since

$$zA(e^{\int_{z_1}^{z} \frac{\tilde{C}/2}{tA} dt}\, \tilde{P}_n)' = \frac{\tilde{C}}{2} e^{\int_{t_1}^{z} \frac{\tilde{C}/2}{tA} dt}\, \tilde{P}_n + zA\tilde{P}_n'\, e^{\int_{t_1}^{z} \frac{\tilde{C}/2}{tA} dt} \,,$$

and $\tilde{P}_n$ satisfies (42), then we obtain

$$zA(e^{\int_{t_1}^{z} \frac{\tilde{C}/2}{tA} dt}\, \tilde{P}_n)' = \mathcal{B}_n \tilde{P}_n e^{\int_{z_1}^{z} \frac{\tilde{C}/2}{tA} dt},$$

thus $P_n = e^{\int_{t_1}^{z} \frac{\tilde{C}/2}{tA} dt}\, \tilde{P}_n$ satisfies (41). Analogously one proves the converse. □

Taking into account the previous lemma, we will solve (41) searching for a solution $\{P_n\}$ given by $P_n = e^{\int_{z_1}^{z} \frac{\tilde{C}/2}{tA} dt}\, \tilde{P}_n \,,\ n \in \mathbb{N}$, where $\tilde{P}_n$ satisfies (42). Furthermore, we will search for $\tilde{P}_n$ given by $\tilde{P}_n = \begin{bmatrix} \tilde{\phi}_n & -\tilde{Q}_n/\tilde{w} \\ (\tilde{\phi}_n)^* & \tilde{Q}_n^*/\tilde{w} \end{bmatrix}$, $\forall n \in \mathbb{N}$, where $\{\tilde{\phi}_n\}$ is a MOPS on the unit circle with respect to a weight function $\tilde{w}$, and $\{\tilde{Q}_n\}$ is the corresponding sequence of functions of the second kind.

Let us remark that, using the same arguments as the ones used in the proof of Theorem 4, from $zA\tilde{P}_n' = (\mathcal{B}_n - \tilde{C}/2\,I)\tilde{P}_n$ we get

$$\frac{\tilde{w}'}{\tilde{w}} = \frac{nA - \operatorname{tr}(\mathcal{B}_n - \tilde{C}/2\,I)}{zA}\,,$$

and since $\operatorname{tr}(\mathcal{B}_n) = nA$, there follows $\dfrac{\tilde{w}'}{\tilde{w}} = \dfrac{\tilde{C}}{zA}$, thus

$$\tilde{w} = K\, e^{\int_{z_1}^{z} \frac{\tilde{C}}{tA}dt}\,,\ K \in \mathbb{C}\,. \tag{43}$$

Henceforth,

$$P_n = e^{\int_{z_1}^{z} \frac{\tilde{C}/2}{tA}dt} \begin{bmatrix} \tilde{\phi}_n & -\tilde{Q}_n/\tilde{w} \\ (\tilde{\phi}_n)^* & \tilde{Q}_n^*/\tilde{w} \end{bmatrix}, n \in \mathbb{N}\,, \tag{44}$$

with $\tilde{w}$ given by (43).

Remark . According with Theorem 4, $\tilde{P}_n = \begin{bmatrix} \tilde{\phi}_n & -\tilde{Q}_n/\tilde{w} \\ (\tilde{\phi}_n)^* & \tilde{Q}_n^*/\tilde{w} \end{bmatrix}$ satisfies $zA\tilde{P}_n' = (\tilde{\mathcal{B}}_n - \tilde{C}/2\,I)\tilde{P}_n$, where $\tilde{\mathcal{B}}_n$ is associated with the equation for the corresponding Carathéodory function, say $zA\tilde{F}' = \tilde{C}\tilde{F} + \tilde{D}$, thus depending on $A, \tilde{C}, \tilde{D}$. On the other hand, $\mathcal{B}_n$ of (42) depend on A, B, C, D. As it will be seen in Lemma 7, this is possible because the polynomials B, C, D depend on $\tilde{C}, \tilde{D}$.

Lemma 6. *Let F be a Carathéodory function satisfying $zAF' = BF^2 + CF + D$ and $\{\phi_n\}$ the corresponding MOPS. For all $n \in \mathbb{N}$, let P_n be a fundamental matrix of the corresponding differential system (33). If P_n is given by (44), where $\{\tilde{\phi}_n\}$ is the MOPS with respect to the weight $\tilde{w}$, then the following equations hold:*

$$P_n = \tilde{\mathcal{A}}_n P_{n-1},\ \ \tilde{\mathcal{A}}_n = \begin{bmatrix} z & \tilde{a}_n \\ \bar{\tilde{a}}_n z & 1 \end{bmatrix},\ n \in \mathbb{N}\,, \tag{45}$$

$$zA\tilde{\mathcal{A}}_n' = \mathcal{B}_n\tilde{\mathcal{A}}_n - \tilde{\mathcal{A}}_n\mathcal{B}_{n-1}\,,\ n \geq 2\,. \tag{46}$$

Proof. (45) is a consequence of the recurrence relations for $\{\tilde{P}_n\}$ (see Theorem 1),

$$\tilde{P}_n = \tilde{\mathcal{A}}_n\tilde{P}_{n-1},\ \ \tilde{\mathcal{A}}_n = \begin{bmatrix} z & \tilde{\phi}_n(0) \\ \overline{\tilde{\phi}_n(0)}z & 1 \end{bmatrix},\ n \in \mathbb{N}\,,$$

We now establish (46). Since P_n satisfies $zAP_n' = \mathcal{B}_n P_n$, then by substituting $P_n = \tilde{\mathcal{A}}_n P_{n-1}$ in the previous equation, there follows

$$zA\tilde{\mathcal{A}}_n' P_{n-1} + \tilde{\mathcal{A}}_n zAP_{n-1}' = \mathcal{B}_n\tilde{\mathcal{A}}_n P_{n-1}\,,\ n \geq 2\,.$$

Using $zAP'_{n-1} = \mathcal{B}_{n-1}P_{n-1}$ in the last equation we get

$$zA\tilde{\mathcal{A}}'_n P_{n-1} + \tilde{\mathcal{A}}_n\mathcal{B}_{n-1}P_{n-1} = \mathcal{B}_n\tilde{\mathcal{A}}_nP_{n-1}\,.$$

Thus,

$$(zA\tilde{\mathcal{A}}'_n + \tilde{\mathcal{A}}_n\mathcal{B}_{n-1})P_{n-1} = \mathcal{B}_n\tilde{\mathcal{A}}_nP_{n-1}\,.$$

Since P_n is nonsingular ($\det(P_n) \neq 0$, $\forall n \in \mathbb{N}$, $\forall z \neq 0$) then

$$zA\tilde{\mathcal{A}}'_n + \tilde{\mathcal{A}}_n\mathcal{B}_{n-1} = \mathcal{B}_n\tilde{\mathcal{A}}_n$$

follows, and we obtain (46). □

Remark . From (18) and (46) we get the equations

$$zA(\mathcal{A}_n - \tilde{\mathcal{A}}_n)' = \mathcal{B}_n(\mathcal{A}_n - \tilde{\mathcal{A}}_n) - (\mathcal{A}_n - \tilde{\mathcal{A}}_n)\mathcal{B}_{n-1}\,,\ n \geq 2\,.$$

Hence,

$$\begin{cases} \overline{\lambda}_n\Theta_{n,1} = \lambda_n\Theta_{n-1,2} \\ \lambda_n l_{n,1} = \lambda_n l_{n-1,2} \\ \overline{\lambda}_n\Theta_{n-1,1} = \lambda_n\Theta_{n,2} \\ \overline{\lambda}_n l_{n,2} - \overline{\lambda}_n l_{n-1,1} = \overline{\lambda}_n zA \end{cases} \tag{47}$$

where $\lambda_n = a_n - \tilde{a}_n,\ a_n = \phi_n(0),\ \tilde{a}_n = \tilde{\phi}_n(0)\,,\ \forall n \in \mathbb{N}\,.$

Hereafter we will denote linear fractional transformations $T(F) = \dfrac{a+bF}{c+dF}$ by $T_{(a,b;c,d)}(F)\,.$

Theorem 6. *Let F be a Carathéodory function satisfying $zAF' = BF^2 + CF + D$, and $\{\phi_n\}$ be the corresponding MOPS. Let P_n, $n \in \mathbb{N}$, be a fundamental matrix of the differential system (33) given by (44), and $\tilde{F}$ be the corresponding Carathéodory function. Then, there exists a unique linear fractional transformation, $T_{(a,b;c,d)}$, with $a, b, c, d \in \mathbb{P}$ and $ad - bc \not\equiv 0$, such that $F = T_{(a,b;c,d)}(\tilde{F})$.*

Proof. To prove that F is a linear fractional transformation of $\tilde{F}$, we begin by establishing that the reflection coefficients of $\{\phi_n\}$ and $\{\tilde{\phi}_n\}$, i.e., $a_n = \phi_n(0)$ and $\tilde{a}_n = \tilde{\phi}_n(0)$, differ only in a finite number of indexes.

Let us write $\lambda_n = a_n - \tilde{a}_n$, $\forall n \in \mathbb{N}$. First we establish that $\mathcal{Z} = \{n \in \mathbb{N} : \lambda_n \neq 0\}$ is a finite set. In fact, if $\mathcal{Z}$ was not finite, for example, $\mathcal{Z} \equiv \mathbb{N}$, then $\lambda_n \neq 0$, $\forall n \in \mathbb{N}$. But from (47) we would obtain

$$l_{n,1} = l_{n-1,2},\ \forall n \in \mathbb{N}\,.$$

Substituting in (22), we would obtain

$$\Theta_{n,1} = z\Theta_{n-1,1},\ \forall n \in \mathbb{N}\,,$$

hence

$$\Theta_{n,1} = z^n \Theta_{1,1}, \forall n \in \mathbb{N}\,.$$

But this is a contradiction to the fact that $\deg(\Theta_n)$ is bounded. Therefore, $\mathcal{Z} \not\equiv \mathbb{N}$. On the other hand, if we consider, without loss of generality, the case

$$\begin{cases} a_n = \tilde{a}_n\,, & n = 1, 2, \ldots, n_0\,, \\ a_n \neq \tilde{a}_n\,, & n \geq n_0\,, \end{cases}$$

then we will obtain the same conclusion.

To conclude that F is a rational transformation of $\tilde{F}$ of the referred type, we take into account its representation in continued fraction given in Theorem 2. To establish the uniqueness of $T_{(a,b;c,d)}$ we remind that the inverse of $T_{(a,b;c,d)}\,,\ \ ad - bc \neq 0$, is given by $T_{(a,-c;-b,d)}$. Therefore, if T_1 and T_2 are two linear fractional transformations such that $T_1(\tilde{F}) = T_2(\tilde{F})$, then the composition $T_2^{-1} \circ T_1$ satisfies $(T_2^{-1} \circ T_1)(\tilde{F}) = \tilde{F}$, and thus we obtain $T_2^{-1} \circ T_1 = id$, i.e., $T_1 = T_2$. Hence, the uniqueness of T is established. □

5.3. **Determination of the polynomial $\tilde{C}$.** In what follows we determine the polynomial $\tilde{C}$ which defines $\{P_n\}$ given in (44).

Lemma 7. *Under the conditions of the previous theorem, let F be a Carathéodory function satisfying $zAF' = BF^2 + CF + D$, let $\tilde{C}$ be a polynomial which defines a weight $\tilde{w}$ given by (43), and $\tilde{F}$ the Carathéodory function associated with $\tilde{w}$. Let $T_{(\alpha_1,-\beta_1;-\alpha_2,\beta_2)}\,,\ \ \alpha_i, \beta_i \in \mathbb{P},\ i = 1, 2,\ \alpha_1\beta_2 - \alpha_2\beta_1 \not\equiv 0$, such that $F = T(\tilde{F})$. Let us consider the first order linear differential equation for $\tilde{F}$,*

$$zA\tilde{F}' = \tilde{C}\tilde{F} + \tilde{D}\,,\ \tilde{D} \in \mathbb{P}\,. \tag{48}$$

Then, the following relations hold:

$$B = (\alpha_2\beta_2' - \alpha_2'\beta_2)zA + \alpha_2\beta_2\tilde{C} + \beta_2^2\tilde{D}\,, \tag{49}$$

$$C = (\alpha_2\beta_1' + \alpha_1\beta_2' - \alpha_2'\beta_1 - \alpha_1'\beta_2)zA + (\alpha_1\beta_2 + \alpha_2\beta_1)\tilde{C} + 2\beta_1\beta_2\tilde{D}\,, \tag{50}$$

$$D = (\alpha_1\beta_1' - \alpha_1'\beta_1)zA + \alpha_1\beta_1\tilde{C} + \beta_1^2\tilde{D}\,, \tag{51}$$

where we have considered, without lost of generality, $\alpha_2\beta_1 - \alpha_1\beta_2 = 1$.

Proof. Since $\tilde{w}'/\tilde{w} = \tilde{C}/(zA)$ (cf. (43)), then $\tilde{w}$ is semi-classical. Therefore, the corresponding $\tilde{F}$ satisfies (48), with $\tilde{D}$ a polynomial (see [3, 7]).

Let us write $F = \dfrac{\alpha_1 - \beta_1 \tilde{F}}{-\alpha_2 + \beta_2 \tilde{F}}$, i.e., $\tilde{F} = \dfrac{\alpha_1 + \alpha_2 F}{\beta_1 + \beta_2 F}$. Using $\tilde{F} = \dfrac{\alpha_1 + \alpha_2 F}{\beta_1 + \beta_2 F}$ in (48), it follows that

$$zA(\alpha_2\beta_1 - \alpha_1\beta_2)F' = B_2F^2 + C_2F + D_2\,, \tag{52}$$

with

$$\begin{aligned}
B_2 &= (\alpha_2\beta_2' - \alpha_2'\beta_2)zA + \alpha_2\beta_2\tilde{C} + \beta_2^2\tilde{D}\,,\\
C_2 &= (\alpha_2\beta_1' + \alpha_1\beta_2' - \alpha_2'\beta_1 - \alpha_1'\beta_2)zA + (\alpha_1\beta_2 + \alpha_2\beta_1)\tilde{C} + 2\beta_1\beta_2\tilde{D}\,,\\
D_2 &= (\alpha_1\beta_1' - \alpha_1'\beta_1)zA + \alpha_1\beta_1\tilde{C} + \beta_1^2\tilde{D}\,.
\end{aligned}$$

Hence, F satisfies $zAF' = BF^2 + CF + D$ and (52), thus it follows that

$$\frac{zA(\alpha_2\beta_1 - \alpha_1\beta_2)}{zA} = \frac{B_2}{B} = \frac{C_2}{C} = \frac{D_2}{D}\,.$$

Therefore, if $\alpha_2\beta_1 - \alpha_1\beta_2 = 1$, then

$$B = B_2,\ C = C_2,\ D = D_2,$$

and (49)-(51) follow. $\square$

According with Theorem 6, for each polynomial $\tilde{C}$ defining a weight $\tilde{w}$ by (43) and $\{P_n\}$ as in (44), there exists a unique linear fractional transformation T such that $F = T(\tilde{F})$, with $\tilde{F}$ the Carathéodory function associated with $\tilde{w}$. In this issue, we pose the question: being $\tilde{C}_1$ and $\tilde{C}_2$ polynomials (defining weights of the same type as in (43)) and $\tilde{F}_1$, $\tilde{F}_2$ the corresponding Carathéodory functions such that F is a linear fractional transformation of $\tilde{F}_i$, $i = 1, 2$, to obtain relations between $\tilde{C}_1$ and $\tilde{C}_2$. The next lemma gives us an answer.

Lemma 8. *Under the same conditions of the previous lemma, let F be a Carathéodory function satisfying $zAF' = BF^2 + CF + D$. Let $\tilde{C}_1, \tilde{C}_2$ be polynomials defining semi-classical weights of the type (43), and let F_1 and F_2 be the corresponding Carathéodory functions, non rational, satisfying*

$$zAF_1' = \tilde{C}_1F_1 + \tilde{D}_1\,, \tag{53}$$

$$zAF_2' = \tilde{C}_2F_2 + \tilde{D}_2\,. \tag{54}$$

Let $T_1 = T_{(\alpha_1,-\beta_1;-\alpha_2,\beta_2)}$, $T_2 = T_{(\gamma_1,-\eta_1;-\gamma_2,\eta_2)}$ be the transformations such that $T_1(F_1) = F$, $T_2(F_2) = F$. If we assume, without loss of *generality, that* $\alpha_2\beta_1-\alpha_1\beta_2=1,\ \gamma_2\eta_1-\gamma_1\eta_2=1$, *then the following relations take place:*

$$(\alpha_2\beta_2'-\alpha_2'\beta_2)zA+\alpha_2\beta_2\tilde{C}_1+\beta_2^2\tilde{D}_1 = (\gamma_2\eta_2'-\gamma_2'\eta_2)zA+\gamma_2\eta_2\tilde{C}_2+\eta_2^2\tilde{D}_2\,, \tag{55}$$

$$(\alpha_2\beta_1'+\alpha_1\beta_2'-\alpha_2'\beta_1-\alpha_1'\beta_2)zA+(\alpha_1\beta_2+\alpha_2\beta_1)\tilde{C}_1+2\beta_1\beta_2\tilde{D}_1 = (\gamma_2\eta_1'+\gamma_1\eta_2'-\gamma_2'\eta_1-\gamma_1'\eta_2)zA+(\gamma_1\eta_2+\gamma_2\eta_1)\tilde{C}_2+2\eta_1\eta_2\tilde{D}_2\,, \tag{56}$$

$$(\alpha_1\beta_1'-\alpha_1'\beta_1)zA+\alpha_1\beta_1\tilde{C}_1+\beta_1^2\tilde{D}_1 = (\gamma_1\eta_1'-\gamma_1'\eta_1)zA+\gamma_1\eta_1\tilde{C}_2+\eta_1^2\tilde{D}_2\,. \tag{57}$$

Proof. Since $F=T_1(F_1)$ with F_1 satisfying (53), from previous lemma we obtain

$$B=(\alpha_2\beta_2'-\alpha_2'\beta_2)zA+\alpha_2\beta_2\tilde{C}_1+\beta_2^2\tilde{D}_1\,,$$
$$C=(\alpha_2\beta_1'+\alpha_1\beta_2'-\alpha_2'\beta_1-\alpha_1'\beta_2)zA+(\alpha_1\beta_2+\alpha_2\beta_1)\tilde{C}_1+2\beta_1\beta_2\tilde{D}_1\,,$$
$$D=(\alpha_1\beta_1'-\alpha_1'\beta_1)zA+\alpha_1\beta_1\tilde{C}_1+\beta_1^2\tilde{D}_1\,.$$

Also, since $F=T_2(F_2)$ with F_2 satisfying (54), from previous lema we obtain

$$B=(\gamma_2\eta_2'-\gamma_2'\eta_2)zA+\gamma_2\eta_2\tilde{C}_2+\eta_2^2\tilde{D}_2\,,$$
$$C=(\gamma_2\eta_1'+\gamma_1\eta_2'-\gamma_2'\eta_1-\gamma_1'\eta_2)zA+(\gamma_1\eta_2+\gamma_2\eta_1)\tilde{C}_2+2\eta_1\eta_2\tilde{D}_2\,,$$
$$D=(\gamma_1\eta_1'-\gamma_1'\eta_1)zA+\gamma_1\eta_1\tilde{C}_2+\eta_1^2\tilde{D}_2\,.$$

Therefore, (55)-(57) follow. □

We now state the main result of this section, a representation formulae for $\{Y_n\}$, defined in (4), associated with a Carathéodory function F that satisfies $zAF'=BF^2+CF+D$.

Theorem 7. *Let F be a Carathéodory function satisfying $zAF'=BF^2+CF+D\,,\ A,B,C,D\in\mathbb{P}$, and let $\{Y_n\}$ be the corresponding sequence given by (4). Then, there exists a polynomial $\tilde{C}$ (defined by Lemmas 8 and 9), and a weight $\tilde{w}=K\,e^{\int_{z_1}^{z}\frac{\tilde{C}}{tA}dt}\,,\ K\in\mathbb{C}$, such that*

$$Y_n=\begin{bmatrix}\sqrt{\tilde{w}}\tilde{\phi}_n & -\tilde{Q}_n/\sqrt{\tilde{w}}\\ \sqrt{\tilde{w}}\tilde{\phi}_n^* & \tilde{Q}_n^*/\sqrt{\tilde{w}}\end{bmatrix}E_n\,L^{-1}\,,\quad n\in\mathbb{N}\,,$$

where $\{\tilde{\phi}_n\}$ is the MOPS with respect to $\tilde{w}$, $\{\tilde{Q}_n\}$ is the sequence of functions of the second kind associated with $\{\tilde{\phi}_n\}$, E_n are the matrices defined in (36), and L is a fundamental matrix of (32).

Proof. These equations are a direct application of Theorem 5, namely (35). □

6. Example

Let us consider the sequence of Jacobi orthogonal polynomials on the unit circle, $\{\phi_n\}$, with parameters $\alpha = \beta$, $\tilde{F}$ the corresponding Carathéodory function. Let $\{\Omega_n\}$ be the sequence of associated polynomials of the second kind and F be the corresponding Carathéodory function. F satisfies (see [4])

$$z(z^2-1)F' = -2\alpha c_0(z^2-1)F^2 - 2\alpha(z^2+1)F\,,$$

where c_0 is the moment of order zero of the Jacobi measure on the unit circle.

Taking into account Theorem 6, firstly we will solve the following differential systems:

$$z(z^2-1)L'(z) = \begin{bmatrix} -\alpha(z^2+1) & 0 \\ -2\alpha c_0(z^2-1) & \alpha(z^2+1) \end{bmatrix} L(z)\,, \tag{58}$$

$$z(z^2-1)P_n' = \mathcal{B}_n P_n\,. \tag{59}$$

In what follows we consider a complex domain G such that $\{0,1,-1\} \subsetneq G$, and a z_0 in G.

Lemma 9. *The fundamental matrix of solutions of (58) is given by*

$$L(z) = z^{-\alpha}(z^2-1)^{\alpha}$$
$$\times \begin{bmatrix} z^{2\alpha}(z^2-1)^{-2\alpha} & z^{2\alpha}(z^2-1)^{-2\alpha} \\ 1 - 2\alpha c_0 \int_{z_1}^{z} t^{2\alpha-1}(t^2-1)^{-2\alpha}dt & 1 - 2\alpha c_0 \int_{z_2}^{z} t^{2\alpha-1}(t^2-1)^{-2\alpha}dt \end{bmatrix}$$

with $z_1 \neq z_2$.

Now we obtain a solution of (59). Takin into account Theorem 4, henceforth we will consider $\tilde{C}$ as polynomial and we will solve (59) searching for a solution $\tilde{P}_n$ given by (44), $P_n = e^{\int_{z_1}^{z} \frac{\tilde{C}/2}{tA}dt} \begin{bmatrix} \tilde{\phi}_n & -\tilde{Q}_n/\tilde{w} \\ \tilde{\phi}_n^* & \tilde{Q}_n^*/\tilde{w} \end{bmatrix}$, $\forall n \in \mathbb{N}$, with $A = z^2-1$, $\{\tilde{\phi}_n\}$ the MOPS with respect to $\tilde{w}$, $\{\tilde{Q}_n\}$ the corresponding sequence of functions of the second kind, and $\tilde{w} = Ke^{\int_{z_1}^{z} \frac{\tilde{C}}{tA}dt}$.

On the other hand, F is a linear fractional transformation of $\tilde{F}$ given by $F = 1/\tilde{F}$ (see, for example, [23, 25]), with $\tilde{F}$ satisfying (see [26])

$$z(z^2-1)\tilde{F}' = 2\alpha(z^2+1)\tilde{F} + 2\alpha c_0(z^2-1)\,.$$

Therefore, by Lemma 7, $\tilde{C} = 2\alpha(z^2+1)$ follows, and consequently we obtain $\tilde{w} = \left((z^2-1)/z\right)^{2\alpha}$.

From Theorem 7, the following representation for $Y_n = \begin{bmatrix} \phi_n & -\Omega_n \\ \phi_n^* & \Omega_n^* \end{bmatrix}$ holds:

$$Y_n\ K = \begin{bmatrix} \tilde{\phi}_n & -\left((z^2-1)/z\right)^{-2\alpha} \tilde{Q}_n \\ (\tilde{\phi}_n)^* & \left((z^2-1)/z\right)^{-2\alpha} (\tilde{Q}_n)^* \end{bmatrix} E_n$$
$$\times \begin{bmatrix} 1 - 2\alpha c_0 \int_{z_2}^{z} t^{2\alpha-1}(t^2-1)^{-2\alpha} dt & -z^{2\alpha}(z^2-1)^{-2\alpha} \\ -1 + 2\alpha c_0 \int_{z_1}^{z} t^{2\alpha-1}(t^2-1)^{-2\alpha} dt & z^{2\alpha}(z^2-1)^{-2\alpha} \end{bmatrix}, \quad n \in \mathbb{N}$$

where $K = 2\alpha c_0 \displaystyle\int_{z_1}^{z_2} t^{2\alpha-1}(t^2-1)^{-2\alpha}\, dt\,,\ \ E_n = (P_n(z_0))^{-1} Y_n(z_0) L(z_0)$.

References

[1] H. Abou-Kandil, G. Freiling, V. Ionescu, and G. Jank, *Matrix Riccati Equations in Control and Systems Theory*, Birhäuser, 2003.

[2] M. Alfaro and F. Marcellán, *Recent trends in orthogonal polynomials on the unit circle*, in "Orthogonal Polynomials and their applications", (C. Brezinski, L. Gori and A. Ronveaux Eds.) J.C. Baltzer A.G. Basel IMACS Ann. Comput. Appl. Math., **9** (1-4), (1991), 3-14.

[3] A. Branquinho and M.N. Rebocho, *Characterizations of Laguerre-Hahn affine orthogonal polynomials on the unit circle*, J. Comput. Anal. Appl. **10** (2) (2008), 229-242.

[4] A. Branquinho and M.N. Rebocho, *Distributional equation for Laguerre-Hahn functionals on the unit circle*, J. Comput. Appl. Math. **233** (2009) 634-642.

[5] A. Cachafeiro, F. Marcellán, and C. Pérez, *Lebesgue perturbation of a quasi-definite Hermitian functional. The positive-definite case.*, Linear Algebra and its Applications 369 (2003), 235-250.

[6] A. Cachafeiro, F. Marcellán, and C. Pérez, *Orthogonal polynomials with respect to the sum of an arbitrary measure and a Bernstein-Szegő measure*, Advances in Computational Mathematics, **26** (1-3) (2007), 81-104.

[7] A. Cachafeiro and C. Pérez, *A study of the Laguerre-Hahn affine functionals on the unit circle*, J. Comput. Anal. Appl. **6** (2) (2004), 107-123.

[8] A. Cachafeiro and C. Pérez, *Second degree functionals on the unit circle*, Integral Transforms and Special Functions, **15** (4) (2004), 281-294.

[9] J. Favard, *Cours D'Analyse de l'Ecole Polytechnique*, Tome II, Théorie des équations, Gauthier-Villars, Paris, 1962.

[10] P.J. Forrester and N.S. Witte, *Bi-orthogonal Polynomials on the Unit Circle, Regular Semi-Classical Weights and Integrable Systems*, Constr. Approx. **24** (2) (2006), 201-237.

[11] J. Geronimus, *On the trignometric moment problem*, Annals of Mathematics **47** (4) (1946), 742-761.

[12] Ya.L. Geronimus, *Polynomials orthogonal on a circle and interval*, vol 18, International Series on Applied Mathematics, Consultants Bureau, New York, 1961.

[13] Ya.L. Geronimus, *Polynomials orthogonal on a circle and their applications*, American Mathematical Society Translations, Series 1, Vol. 3, Providence R. I., 1962.

[14] L. Golinskii and P. Nevai, *Szegő difference equations, transfer matrices and orthogonal polynomials on the unit circle*, Commun. Math. Phys. **223** (2001), 223-259.

[15] W. Hahn, *Über lineare Differentialgleichungen, daren Lösungen einer Rekursionsformel genügen*, Math. Nachrichten 4 (1951) 1-11: II: ibid.7 (1952) 85-104.

[16] W. Hahn, *On Differential Equations for Orthogonal Polynomials*, Funkcialaj Ekvacioj **21** (1978), 1-9.

[17] M. E. H. Ismail and N. S. Witte, *Discriminants and functional equations for polynomials orthogonal on the unit circle*, J. Approx. Theory, **110** (2) (2001), 200-228.

[18] A.P. Magnus, *Riccati acceleration of the Jacobi continued fractions and Laguerre-Hahn orthogonal polynomials*, 213-230, in "Padé Approximation and its Applications, Proc., Bad Honnef 1983", Lect. Notes in Math. 1071 (H. Werner and H.T. Bunger, eds.), Springer Verlag, Berlin, 1984.

[19] A.P. Magnus, *Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials*, J. Comput. Appl. Math. **57** (1995), 215-237.

[20] A.P. Magnus, *Freud equations for Legendre polynomials on a circular arc and solution of the Grünbaum-Delsarte-Janssen-Vries problem*, J. Approx. Theory **139** (1-2) (2006), 75-90.

[21] P. Maroni, *Le calcul des formes linéaires et les polynômes orthogonaux semi-classiques*, Lect. Notes in Math. 1329, Springer Verlag, Berlin, Heidelberg, New York, 1988, 279-290.

[22] P. Maroni, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques*, em "Orthogonal Polynomials and their applications", (C. Brezinski, L. Gori and A. Ronveaux Eds.) J.C. Baltzer A.G. Basel IMACS Ann. Comput. Appl. Math., **9** (1-4) (1991), 95-130.

[23] F. Peherstorfer, *A special class of polynomials orthogonal on the unit circle including the associated polynomials*, Constr. Approx. **12** (2) (1996), 161-185.

[24] F. Peherstorfer and R. Steinbauer, *Characterization of orthogonal polynomials with respect to a functional*, J. Comput. Appl. Math. **65** (1995), 339-355.

[25] B. Simon, *Orthogonal Polynomials on the Unit Circle. Part 1: Classical Theory*, Amer. Math. Soc. Colloq. Publ. Volume 54, Part 1. Amer. Math. Soc. Providence, Rhode Island, 2005.

[26] C. Tasis, *Propriedades diferenciales de los polinomios ortogonales relativos a la circunferencia unidad*, PhD Thesis, Universidad de Cantabria, 1989 (in spanish).

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