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Equality of immanantal decomposable tensors, II

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Abstract

We state a necessary and sufficient condition for equality of two nonzero decomposable symmetrized tensors when the symmetrizer is associated with an irreducible character of the symmetric group of degree m.

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1. Introduction

Symmetry classes of tensors are extensions of the tensor product that involve a symmetry depending upon a group (finite) and an irreducible character. If V is vector space over $\mathbb{C}$, and λ is an irreducible character of S_m, we denote the symmetric class of tensors whose symmetry is defined by λ by V_λ^m. The most known symmetry classes associated with an irreducible character of S_m are the exterior power of V, $\wedge^m V$, and the symmetric power of V, $\vee^m V$.

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The study of decomposable elements

$$x_1 * \cdots * x_m, \quad (x_1, \ldots, x_m \in V)$$

of symmetry classes of tensors is an important topic in Multilinear Algebra since the knowledge of their properties is necessary for the understanding of symmetry classes.

Among others, two basic issues are particularly relevant and have been challenges in Multilinear Algebra for many years: The "vanishing question" that asks conditions on the vectors $x_1, \ldots, x_m$ that characterize the vanishing

$$x_1 * \cdots * x_m = 0,$$

and the "equality question" that asks for conditions on the vectors of V that characterize the equality

$$x_1 * \cdots * x_m = y_1 * \cdots * y_m,$$

where $x_1 * \cdots * x_m, y_1 * \cdots * y_m \neq 0$.

These conditions are well known for the tensor product. Indeed,

$$x_1 \otimes \cdots \otimes x_m = 0$$

if and only if one of the factors x_i is the zero vector, and the equality

$$x_1 \otimes \cdots \otimes x_m = y_1 \otimes \cdots \otimes y_m$$

holds (for nonzero decomposable tensors $x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m$) if the homologous factors are proportional (i.e. $y_i = c_i x_i, c_i \in \mathbb{C}$) and an extra normalization condition on the coefficients is fulfilled ($\prod_{i=1}^m c_i = 1$).

The answer to the corresponding question for the exterior power and symmetric power are also well known. We have $x_1 \wedge \cdots \wedge x_m = 0$ if and only if $(x_1, \ldots, x_m)$ are linearly dependent and $x_1 \vee \cdots \vee x_m = 0$ if and only if one of the vector x_i is zero.

Concerning the equality question (if $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ and are linearly independent), we have

$$x_1 \wedge \cdots \wedge x_m = y_1 \wedge \cdots \wedge y_m$$

if and only if $\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_m \rangle$ and the normalization condition $\det[a_{ij}] = 1$ holds, where $y_i = \sum_{j=1}^m a_{ij} x_j, i = 1, \ldots, m$.

For the symmetric power, the equality

$$x_1 \vee \cdots \vee x_m = y_1 \vee \cdots \vee y_m$$

(we are assuming $x_1, \ldots, x_m, y_1, \ldots, y_m$ are nonzero vectors) holds if and only if factors of the right side are proportional to factors of the left side (i.e., there exists a permutation $(i_1, \ldots, i_m)$ of the integers $(1, \ldots, m)$ such that $x_j = c_j y_{i_j}, j = 1, \ldots, m$) and the normalization property, $\prod_{i=1}^m c_i = 1$, satisfied.

Symmetry classes of tensors associated with irreducible characters of the full symmetric group are the most basic ones, and it is natural to start with them to solve the vanishing and the equality problems.

Indeed, the vanishing problem was completely solved by Gamas [2] for these symmetry classes. It was shown that if $x_1, \ldots, x_m$ are nonzero vectors the decomposable element $x_1 * \cdots * x_m$ of V_λ^m vanishes if and only if the rank partition of $x_1, \ldots, x_m$ does not dominate the conjugate partition of the partition associated with λ. This condition obviously extends the results for $\wedge^m V$ and $\vee^m V$, since the irreducible characters that define the symmetry $\wedge^m V$ and $\vee^m V$ have the associated partitions $(1, \ldots, 1)$ and (m) respectively.

The purpose of this paper is to present, in symmetry classes associated to irreducible characters of S_m, the completed description of the conditions on $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ that are necessary and sufficient for the equality

$$x_1 * \cdots * x_m = y_1 * \cdots * y_m$$

(assuming $x_1 * \cdots * x_m, y_1 * \cdots * y_m \neq 0$). These conditions extend the above refereed conditions for $\wedge^m V$ and $\vee^m V$ and can be viewed as a "mixture" of them. They also extend previous work done on this topic [7,4].

2. Preliminaries

Let m be a positive integer, and let Δ be a subset of $\{1, \ldots, m\}$. We use S_m to denote the symmetric group of degree m and S_Δ to denote the subgroup of S_m,

$$S_\Delta = \{\sigma \in S_m : \sigma(i) = i, i \notin \Delta\}.$$

Throughout this paper we denote by $|X|$ the cardinality of X, and we use Id_X to denote the identity map from X into X.

Let $\lambda = (\lambda_1, \cdots, \lambda_t)$, $\lambda_1 \geqslant \cdots \geqslant \lambda_t > 0$ be a partition of m. The existence of a bijective correspondence between the set of the partitions of m and the set of irreducible complex valued character of S_m is well known (see [11] or [12]). Then, we denote the irreducible character of S_m associated with the partition λ by λ. The greatest k such that $\lambda_k > 0$ is called the *length* of λ. The integers $\lambda_i, i = 1, \ldots, k$, where k is the length of λ are called *parts* of λ.

Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ and $\mu = (\mu_1, \ldots, \mu_r)$ be partitions of m. We say that λ *majorizes* μ (denoted by $\lambda \succeq \mu$) if $t \leqslant r$ and

$$\sum_{i=1}^{j} \lambda_i \geqslant \sum_{i=1}^{j} \mu_i, \quad j = 1, \ldots, t.$$

If $\lambda = (\lambda_1, \ldots, \lambda_t)$ is a partition of m, the sequence $\lambda' = (\lambda'_1, \ldots, \lambda'_{\lambda_1})$ where

$$\lambda'_i = |\{j : \lambda_j \geqslant i\}|, \quad i = 1, \ldots, \lambda_1,$$

is a partition of m called the *conjugate partition* of λ. Note that the length of λ' is λ_1.

Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a partition of m of length t, and let r be the number of distinct parts of λ'. Let $i_1, \ldots, i_r \in \{1, \ldots, \lambda_1\}$ satisfying

$$\lambda'_1 = \lambda'_{i_1} > \cdots > \lambda'_{i_r} = \lambda'_{\lambda_1}.$$

For each $s \in \{1, \ldots, r\}$ let

$$\Gamma_s = \{j : \lambda'_j = \lambda'_{i_s}\}.$$

We denote by $S_{\lambda'}$ the subgroup of S_{λ_1}

$$S_{\lambda'} = S_{\Gamma_1} \times \cdots \times S_{\Gamma_r},$$

where $\times$ denotes the direct product.

Let $\sigma \in S_{\lambda'}$ and let $k \in \{1, \ldots, r\}$. Since $\Gamma_i \cap \Gamma_j = \emptyset, i, j \in \{1, \ldots, r\}, i \neq j$, it is easy to see that

$$\sigma(\Gamma_1 \cup \cdots \cup \Gamma_k) = \sigma(\Gamma_1) \cup \cdots \cup \sigma(\Gamma_k) = \Gamma_1 \cup \cdots \cup \Gamma_k. \quad (1)$$

It is also easy to see that for all $i \in \{1, \ldots, \lambda'_1\}$ there exits $k \in \{1, \ldots, r\}$ such that

$$\{1, \ldots, \lambda_i\} = \Gamma_1 \cup \cdots \cup \Gamma_k. \quad (2)$$

We denote the Young diagram corresponding to λ by $[\lambda]$. In this paper we identify the boxes of $[\lambda]$ with integers of $\{1, \ldots, m\}$ and label the m boxes of $[\lambda]$ from left to right and from top to bottom.

A filling D of the diagram $[\lambda]$ with the integers $1, \ldots, m$ such that each integer occurs once will be called a *strict Young tableau*. The mapping σ that maps the integer $i (\in \{1, \ldots, m\})$ on the integer $\sigma(i)$ filling the box i of $[\lambda]$ is a permutation of m. We denote this strict Young tableau by $D_{\lambda,\sigma}$. If $D = D_{\lambda,\sigma}$ is a strict Young tableau, and $\nu \in S_m$, then we denote the strict Young tableau $D_{\lambda,\nu\sigma}$ by νD. We say that a strict Young tableau $D = D_{\lambda,\sigma}$ is *increasing by columns* (by rows) if the integers in each column (each row) of D are in increasing order. We say that a strict Young tableau is *standard* if it increases simultaneous by rows and by columns.

Given a strict Young tableau D, the subgroup of S_m of the permutations ν such that D and νD have the same rows is called the *group of rows* of D and is denoted by $R(D)$. Similarly, we define the *group of columns* of D, $C(D)$.

Let $\mathbb{C}$ be the complex field, let V be a finite dimensional vector space over $\mathbb{C}$, and let $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ be linearly independent families of vectors of V that span the same subspace. If $y_i = \sum_{j=1}^{m} a_{ij} x_j$, $i = 1, \ldots, m$, the $m \times m$ matrix $[a_{ij}]$ is denoted by

$$M[y_1, \ldots, y_m | x_1, \ldots, x_m].$$

Let $\Gamma = \{i_1, \ldots, i_s\}$ and $\Delta = \{j_1, \ldots, j_s\}$ be subsets of $\{1, \ldots, m\}$ of cardinality s. Assume that $(x_i)_{i \in \Gamma}$ is linearly independent, and $\langle x_i : i \in \Gamma \rangle = \langle y_i : i \in \Delta \rangle$. We denote the $s \times s$ matrix $M[y_{j_1}, \ldots, y_{j_s} | x_{i_1}, \ldots, x_{i_s}]$ by

$$M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta|\Gamma].$$

If $\Gamma = \Delta$, we denote the $s \times s$ matrix $M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta|\Delta]$ by

$$M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta].$$

Definition 2.1. Let $(x_1, \ldots, x_m)$ be a family of nonzero vectors of V. We say that a collection $C = (C_1, \ldots, C_r)$, $(C_j = \{x_i : i \in \Delta_j\})_{j=1,\ldots,r}$ of subfamilies of $(x_1, \ldots, x_m)$, is a *coloring* of $(x_1, \ldots, x_m)$ if the following conditions hold:

(1) C_i is linearly independent, $i = 1, \ldots, r$;
(2) $\Delta_i \cap \Delta_j = \emptyset, i \neq j, i, j = 1, \ldots, r$;
(3) $\bigcup_{i=1}^{r} \Delta_i = \{1, \ldots, m\}$;
(4) $|\Delta_1| \geqslant \cdots \geqslant |\Delta_r|$.

The collection $(\Delta_1, \ldots, \Delta_r)$ is called the support of the coloring $C = (C_1, \ldots, C_r)$. The finite sequence $(|\Delta_1|, \ldots, |\Delta_r|)$ is a partition of m called the *shape* of C and denoted by

$$\text{shape}(C).$$

If shape$(C) = \mu$, we also say that C is a *μ-coloring* of $(x_1, \ldots, x_m)$.

Let $(x_1, \ldots, x_m)$ be a family of nonzero vectors of V. In [3] it was proved that, with respect to majorization, the set of the shapes of colorings of $(x_1, \ldots, x_m)$ has a maximum. This maximum partition is the *rank partition* of $(x_1, \ldots, x_m)$, and it will be denoted by

$$\rho(x_1, \ldots, x_m).$$

A ρ-coloring of $(x_1, \ldots, x_m)$, where ρ is the rank partition of $(x_1, \ldots, x_m)$, is called *factorization* of $(x_1, \ldots, x_m)$.

Denote by $\Gamma_{m,n}$ the set of the maps from $\{1, \ldots, m\}$ into $\{1, \ldots, n\}$. We denote the subset of $\Gamma_{m,n}$ of the strictly increasing maps by $Q_{m,n}$. If $A = [a_{ij}]$ is a $m \times n$ matrix, and if $\alpha \in \Gamma_{p,m}$ and $\beta \in \Gamma_{q,n}$, we use $A[\alpha|\beta]$ to denote $p \times q$ matrix whose (i, j) entry is $a_{\alpha(i)\beta(j)}, i = 1, \ldots, p, j = 1, \ldots, q$.

If V is an inner product vector space and if W is a subspace of V, we denote by $W^{\perp}$ the orthogonal complement of W. We denote by $\otimes^m V$ the mth tensor power of V, and we write $u_1 \otimes \cdots \otimes u_m$ for the tensor product (decomposable tensor) of $u_1, \ldots, u_m \in V$. An inner product (,) in V induces a unique inner product in $\otimes^m V$, also denoted by (,), satisfying

$$(u_1 \otimes \cdots \otimes u_m, v_1 \otimes \cdots \otimes v_m) = \prod_{i=1}^{m} (u_i, v_i) \tag{3}$$

for all $u_1, \ldots, u_m, v_1, \ldots, v_m \in V$.

For $\sigma \in S_m$, let $P(\sigma)$ be the unique linear operator of $\otimes^m V$ such that

$$P(\sigma)(u_1 \otimes \cdots \otimes u_m) = u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(m)}$$

for all $u_1, \ldots, u_m \in V$. An operator in the linear closure of $\{P(\sigma), \sigma \in S_m\}$ is called a *symmetrizer* and its image is called a *symmetry class of tensors*. The image of the decomposable tensor $u_1 \otimes \cdots \otimes u_m$ by a symmetrizer is called a *decomposable symmetrized tensor* and is denoted by

$$u_1 * \cdots * u_m.$$

Let D be a strict Young tableau. Denote by $P(D)$ the symmetrizer

$$P(D) = \sum_{\sigma \in R(D)} P(\sigma)$$

and by $N(D)$ the symmetrizer

$$N(D) = \sum_{\sigma \in C(D)} \epsilon(\sigma) P(\sigma).$$

The symmetrizer $E(D) = P(D)N(D)$ is the *Young symmetrizer* associated with D.

Theorem 2.1 [6]. *Let μ be a partition of m, and let D be a Young diagram associated with μ. Let $x_1, \ldots, x_m \in V$. Then*

$$E(D)(x_1 \otimes \cdots \otimes x_m) \neq 0$$

if and only if

$$N(D)(x_1 \otimes \cdots \otimes x_m) \neq 0.$$

If H is a subgroup of S_m and if λ is a complex irreducible character of H, the symmetrizer

$$T(H, \lambda) = \frac{\lambda(id)}{|H|} \sum_{\sigma \in H} \lambda(\sigma) P(\sigma)$$

is an orthogonal projection, and its range is the symmetry class denoted by $V_\lambda^m(H)$.

Observe that

$$P(D) = |R(D)| T(R(D), 1) \tag{4}$$

and

$$N(D) = |C(D)| T(C(D), \epsilon). \tag{5}$$

If $H = S_m$, we call $V_\lambda^m(H)$ an *immanantal symmetry class of tensors*, and we denote it by V_λ^m. A decomposable symmetrized tensor in V_λ^m is called an immanantal decomposable tensor or just a decomposable element of V_λ^m.

If $\lambda = \epsilon$, the alternating character, $V_\epsilon^m = \wedge^m V$, and $u_1 * \cdots * u_m$ is denoted by $u_1 \wedge \cdots \wedge u_m$. If $\lambda = 1$, the principal character, $V_1^m = \vee^m V$, and $u_1 * \cdots * u_m$ is denoted by $u_1 \vee \cdots \vee u_m$.

It is well known that $u_1 \wedge \cdots \wedge u_m \neq 0$ if and only if $u_1, \ldots, u_m$ are linearly independent vectors, and if a $\sigma \in S_m$, then

$$u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(m)} = \epsilon(\sigma) u_1 \wedge \cdots \wedge u_m.$$

It is also well known that given a basis $\{e_1, \ldots, e_n\}$ of V, the set

$$\{e_\alpha^\wedge : \alpha \in Q_{m,n}\},$$

where $e^{\wedge}_{\alpha}$ denotes $e_{\alpha(1)} \wedge \cdots \wedge e_{\alpha(m)}$ is a basis of $\wedge^m V$. This basis is orthogonal if $(e_1, \ldots, e_n)$ is an orthogonal basis of V. So, we can conclude that $\dim \wedge^k V = \binom{n}{m}$. By convention, $Q_{0,n}$=$\{\emptyset\}$ and $e^{\wedge}_{\emptyset} = 1$.

Theorem 2.2 [10]. *Let V be an n-dimensional vector space over $\mathbb{C}$, and let $(e_1, \ldots, e_n)$ be a basis of V. Let $(x_1, \ldots, x_k)$ be a family of linearly independent vectors of V, and let $T \in \mathbb{C}^{k\times n}$ satisfying*

$$x_i = \sum_{j=1}^{n} t_{ij} e_j, \quad i = 1, \ldots, k$$

Then

$$x_1 \wedge \cdots \wedge x_k = \sum_{\alpha \in Q_{k,n}} \det(T[\eta_k|\alpha]) e^{\wedge}_{\alpha},$$

where $\eta_k = (1, \ldots, k)$.

The next result gives a necessary and sufficient condition for an element $z \in \wedge^k V$ to be decomposable:

Corollary 2.1. *Let V be an n-dimensional vector space over $\mathbb{C}$, and let $\{e_1, \ldots, e_n\}$ be a basis of V. Let*

$$z = \sum_{\alpha \in Q_{k,n}} a_\alpha e^{\wedge}_{\alpha}$$

be an element of $\wedge^k V$. Then, z is decomposable if and only if there exists a $k \times n$ matrix A over $\mathbb{C}$ such that

$$a_\alpha = \det(A[\eta_k|\alpha]), \quad \alpha \in Q_{k,n}.$$

If $\Delta = \{i_1, \ldots, i_s\}$, $(i_1 < \cdots < i_s)$ is a subset of $\{1, \ldots, m\}$ we denote $x_{i_1} \wedge \cdots \wedge x_{i_s}$ by

$$\bigwedge_{i \in \Delta} x_i.$$

Let V be an n-dimensional vector space over $\mathbb{C}$. We denote the Grassmann algebra over V by $\wedge V$, that is, the 2^n-dimensional $\mathbb{C}$-algebra

$$\wedge V = \mathbb{C} \oplus V \oplus \wedge^2 V \oplus \cdots \oplus \wedge^n V.$$

Let $z, w \in \wedge V$, $z = z_0 + z_1 + \cdots + z_n$, and $w = w_0 + w_1 + \cdots + w_n$ where z_i, $w_i \in \wedge^i V$, $i = 1, \ldots, n$ and $z_0, w_0 \in \mathbb{C}$. We define an inner product in $\wedge V$ as follows:

$$(z, w) = \sum_{i=0}^{n} (z_i, w_i). \tag{6}$$

If $\sigma, \tau \in S_m$, we say that $\sigma > \tau$ if

$$(\sigma(1), \sigma(2), \dots, \sigma(m)) > (\tau(1), \tau(2), \dots, \tau(m))$$

by the lexicographic order. Based on the result of [9–p. 78], the following result was proved in [4]:

Proposition 2.1. *Let λ be a partition of m. Let $id = \sigma_1, \dots, \sigma_d$ be the permutations of S_m satisfying*

(1) $\sigma_i < \sigma_j$ *if* $i < j$, $i, j = 1, \dots, d$; *and*
(2) $D_{\lambda,\sigma_1}, \dots, D_{\lambda,\sigma_d}$ *are the standard strict Young tableaux associated with the Young diagram* $[\lambda]$.

Let $\nu \in S_m$. Define $P_i = P(\nu D_{\lambda,\sigma_i})$ and $N_i = N(\nu D_{\lambda,\sigma_i})$, $i = 1, \dots, m$. Let

$$e_{i,i} = \frac{\lambda(id)}{m!} M_i P_i N_i, \quad i = 1, \dots, d$$

where $M_1 = \mathrm{Id}_{\otimes^m V}$ and $M_i = \mathrm{Id}_{\otimes^m V} - e_{1,1} - \cdots - e_{i-1,i-1}$, $i = 2, \dots, d$. Then,

$$T(S_m, \lambda) = e_{1,1} + \cdots + e_{d,d}.$$

Moreover, $e_{1,1}, \dots, e_{d,d}$ are orthogonal idempotents.

Let $\mu = (\mu_1, \dots, \mu_r)$ be a partition of m, and let D be a strict Young tableau associated with the Young diagram $[\mu]$. Let k_{ij} be the integer filling the (i, j) box of D, and let θ be the permutation of S_m defined by the equalities

$$\theta(\mu_1 + \mu_2 + \cdots + \mu_{j-1} + t) = k_{j,t},$$

$j = 1, \dots, r$ and $t = 1, \dots, \mu_j$. Using the pair $(\nu, \otimes^m V)$ where $\nu = P(\theta^{-1}) \circ \otimes$, as a model of mth tensor power of V, we denote $\nu(x_1, \dots, x_m)$ by $x_1 \tilde{\otimes} \cdots \tilde{\otimes} x_m$ (or by $x_1 \otimes \cdots \otimes x_m$ if there are no ambiguities to avoid). Let $\Psi_i = \{k_{i,1}, \dots, k_{i,\mu_i}\}$, $j = 1, \dots, r$. It can be easily checked that if $H = S_{\Psi_1} \times \cdots \times S_{\Psi_r}$ and if $\lambda = \epsilon_{|H}$,

$$T(H, \lambda)(x_1 \tilde{\otimes} \cdots \tilde{\otimes} x_m) = \left(\bigwedge_{i \in \Psi_1} x_i \right) \otimes \cdots \otimes \left(\bigwedge_{i \in \Psi_r} x_i \right)$$

for all $x_1, \dots, x_m \in V$.

Since we will work with nonzero immanantal decomposable tensors, it is useful to say something about the vanishing of these tensors. In [2], Gamas obtained a necessary and sufficient condition for an immanantal decomposable tensor to be zero. This result was reformulated in [3].

Theorem 2.3. *In V_λ^m the immanantal decomposable tensor*

$$x_1 * \cdots * x_m$$

is nonzero if and only if there exists a λ'-coloring of the family $x_1, \ldots, x_m$; if and only if $\rho(x_1, \ldots, x_m) \succeq \lambda'$.

3. Irreducible subsets of the Grassmann Algebra

The objective of this section is to present two irreducible subsets of $\wedge V$ used in the proof of the main result. Throughout this paper, we identify the elements of $\wedge V$ with the 2^n-tuple of its coordinates on the basis $\{e_\alpha^\wedge : \alpha \in \bigcup_{i=0}^n Q_{i,n}\}$. We also view $\mathbb{C}^{2^n}$ as a topological space (with the Zariski topology).

Definition 3.1. A topological space X is called irreducible if, for any decomposition of $X = A_1 \cup A_2$ as union of closed subsets, we have $X = A_1$ or $X = A_2$. A subset X' of X is called irreducible if X', as a topological space with the induced topology, is irreducible.

Theorem 3.1 [8, p. 12]. *Let X' be a subset of a topological space X. Then, X' is irreducible if and only if the closure of X' is irreducible.*

The next result states a sufficient condition for the irreducibility of a subset of $\mathbb{C}^n$ with the Zarisky topology.

Proposition 3.1 [8, p. 15]. *Let $f_1, \ldots, f_n \in \mathbb{C}[T_1, \ldots, T_m]$ where $T_1, \ldots, T_m$ are independent indeterminates, and let U_0 be the subset of $\mathbb{C}^n$:*

$$U_0 = \{(f_1(t_1, \ldots, t_m), \ldots, f_n(t_1, \ldots, t_m)) : (t_1, \ldots, t_m) \in \mathbb{C}^m\} \tag{7}$$

The closure of U_0 in the Zariski topology is an irreducible affine algebraic variety.

We say that a subset U of $\mathbb{C}^n$ is parametrizable if there are polynomials $f_1, \ldots, f_n \in \mathbb{C}[T_1, \ldots, T_m]$ satisfying equality (7).

Corollary 3.1. *The parametrizable subsets of $\mathbb{C}^n$ are irreducible.*

We denote by $\mathfrak{F}(V)$ the subset of $\wedge V$

$$\mathfrak{F}(V) = \left\{\sum_{i=1}^n u_1 \wedge \cdots \wedge u_i : u_1, \ldots, u_n \in V\right\}.$$

Theorem 3.2. *The set $\mathfrak{F}(V)$ is an irreducible subset of $\mathbb{C}^{2^n}$.*

Proof. In order to prove that $\mathfrak{F}(V)$ is irreducible, we only have to show that $\mathfrak{F}(V)$ is parametrizable. Denote by T and $\mathbb{T}$, respectively, the following scalar matrix and the matrix of indeterminates:

$$T = \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nn} \end{pmatrix}, \quad t_{ij} \in \mathbb{C}, \quad i, j = 1, \ldots, n,$$

$$\mathbb{T} = \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{pmatrix}$$

Define $\mathfrak{h}_{0,\emptyset} = 0$. For each $i \in \{1, \ldots, n\}$ and for each $\alpha \in Q_{i,n}$, let $\mathfrak{h}_{i,\alpha}$ be the polynomial of $\mathbb{C}[T_{11}, \ldots, T_{1n}, \ldots, T_{n1}, \ldots, T_{nn}]$ defined by

$$\mathfrak{h}_{i,\alpha} =: \det(\mathbb{T}[\eta_i|\alpha]).$$

Observe that, if $u_k = \sum_{j=1}^{n} t_{kj} u_j$, $k = 1, \ldots, n$, then

$$u_1 \wedge \cdots \wedge u_i = \sum_{a \in Q_{i,n}} \mathfrak{h}_{i,\alpha}(T) e_\alpha^\wedge, \quad i = 1, \ldots, n.$$

Consider the map

$$\Phi : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathfrak{F}(V)$$

$$(t_{11}, \ldots, t_{1n}, \ldots, t_{n1}, \ldots, t_{nn}) \to \sum_{i=1}^{n} \left(\sum_{\alpha \in Q_{i,n}} \mathfrak{h}_{i,\alpha}(T) e_\alpha^\wedge \right).$$

Using Corollary 2.1, we can easily see that Φ is well defined, i.e., $\mathrm{Im}\Phi \subseteq \mathfrak{F}(V)$. Since we are identifying the elements of $\wedge V$ with their 2^n-tuple of the coordinates in the base $\{e_\alpha^\wedge : \alpha \in \bigcup_{i=0}^{n} Q_{i,n}\}$, we conclude that the range of Φ is

$$\mathrm{Im}\,\Phi = \Big\{ (\mathfrak{h}_{i,\alpha}(t_{11}, \ldots, t_{1n}, \ldots, t_{n1}, \ldots, t_{nn}))_{\substack{i=0,\ldots,n \\ \alpha \in Q_{i,n}}} : (t_{11}, \ldots, t_{1n}, \ldots, t_{n1}, \ldots, t_{nn}) \in \mathbb{C}^{n^2} \Big\}.$$

If we prove that Φ is onto, we conclude, by Corollary 3.1, that $\mathfrak{F}(V)$ is an irreducible set of $\wedge V$. Let $z \in \mathfrak{F}(V)$. Then there exist $v_1, \ldots, v_n \in$V such that

$$z = \sum_{i=1}^{n} v_1 \wedge \cdots \wedge v_i.$$

For all $i \in \{1, \ldots, n\}$ let

$$v_i = \sum_{j=1}^{n} a_{ij} e_j,$$

and let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

By Theorem 2.2, we have

$$v_1 \wedge \cdots \wedge v_i = \sum_{\alpha \in Q_{i,n}} \det(A[\eta_i|\alpha]) e_\alpha^\wedge,$$

where $\eta_i = (1, \ldots, i)$.

Therefore,

$$\begin{aligned} \Phi(a_{11}, \ldots, a_{n1}, \ldots, a_{1n}, \ldots, a_{nn}) &= \sum_{i=1}^{n} \sum_{\alpha \in Q_{i,n}} \mathfrak{h}_{i,\alpha}(A) e_\alpha^\wedge \\ &= \sum_{i=1}^{n} v_1 \wedge \cdots \wedge v_i \\ &= z, \end{aligned}$$

and the proof is complete. □

Let W be a subspace of V such that $\dim W = k < n$. Let $j \in \{1, \ldots, n\}$. We denote by $\mathfrak{F}_{j,W}(V)$ the set of the $z \in \mathfrak{F}(V)$ for which there exist $u_1, \ldots, u_n \in V$ satisfying

$$z = \sum_{i=1}^{n} u_1 \wedge \cdots \wedge u_i,$$

and $u_j \in W^\perp$.

Theorem 3.3. *The set $\mathfrak{F}_{j,W}(V)$ is an irreducible subset $\mathbb{C}^{2^n}$.*

Proof. We prove that $\mathfrak{F}_{j,W}(V)$ is parametrizable. The result follows by Corollary 3.1.

Denote by T' and $\mathbb{T}'$, respectively, the following scalar matrix and the matrix of indeterminates:

$$T' = \begin{pmatrix} t_{11} & \cdots & t_{1k} & t_{1k+1} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{jk+1} & \cdots & t_{jn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ t_{n1} & \cdots & t_{nk} & t_{nk+1} & \cdots & t_{nn} \end{pmatrix},$$

and

$$\mathbb{T}' = \begin{pmatrix} T_{11} & \cdots & T_{1k} & T_{1k+1} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & T_{jk+1} & \cdots & T_{jn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nk} & T_{nk+1} & \cdots & T_{nn} \end{pmatrix}.$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal base of V such that

$$W = \langle e_1, \ldots, e_k \rangle. \tag{8}$$

Therefore, if $v \in W^\perp$, we have

$$v = \sum_{t=k+1}^{n} a_t e_t.$$

Consider the map

$$\Phi_1 : \mathbb{C}^n \times \cdots \times \mathbb{C}^{n-k} \times \cdots \times \mathbb{C}^n \to \mathfrak{F}_{j,W}(V)$$

such that

$$\Phi_1(t_{11}, \ldots, t_{1n}, \ldots, t_{jk+1}, \ldots, t_{jn}, \ldots, t_{n1}, \ldots, t_{nn}) = \sum_{i=1}^{n} \left(\sum_{\alpha \in Q_{i,n}} \det(T'[\eta_i|\alpha]) e_\alpha^\wedge \right).$$

Define $\mathfrak{g}_{0,\emptyset} = 0$. For each $i \in \{1, \ldots, n\}$ and each $\alpha \in Q_{i,n}$, we use $\mathfrak{g}_{i,\alpha}$ to denote the polynomial of $\mathbb{C}[T_{11}, \ldots, T_{1n}, \ldots, T_{jk+1}, \ldots, T_{jn}, \ldots, T_{n1}, \ldots, T_{nn}]$

$$\mathfrak{g}_{i,\alpha} := \det(\mathbb{T}'[\eta_i|\alpha]).$$

By identifying the elements of $\wedge V$ with their 2^n-tuple of coordinates in the basis $\{e_\alpha^\wedge : \alpha \in \bigcup_{i=0}^n Q_{i,n}\}$, we conclude that the image of Φ_1 is

$$\mathrm{Im}\Phi_1 = \Big\{ (\mathfrak{g}_{i,\alpha}(t_{11}, \ldots, t_{1n}, \ldots, t_{jk+1}, \ldots, t_{jn}, \ldots t_{n1}, \ldots, t_{nn}))_{\substack{i=0,\ldots,n \\ \alpha \in Q_{i,n}}} \\ (t_{11}, \ldots, t_{1n}, \ldots, t_{jk+1}, \ldots, t_{jn}, \ldots, t_{n1}, \ldots, t_{nn}) \in \mathbb{C}^{n^2-k} \Big\}.$$

After proving that Φ_1 is onto, we conclude that $\mathfrak{F}_{j,W}(V)$ is parametrizable, and so, by Corollary 3.1, $\mathfrak{F}_{j,W}(V)$ is an irreducible subset of $\wedge V$.

Let $w \in \mathfrak{F}_{j,W}(V)$. Then w belongs to $\mathfrak{F}(V)$, and so, there exist $w_1, \ldots, w_n \in V$ such that

$$w = \sum_{i=1}^{n} w_1 \wedge \cdots \wedge w_i,$$

and $w_j \in W^\perp$.

Then, for all $i \in \{1, \dots, n\} - \{j\}$,

$$w_i = \sum_{t=1}^{n} a_{it} e_t, \quad \text{and} \quad w_j = \sum_{t=k+1}^{n} a_{jt} e_t,$$

and bearing in mind Theorem 2.2, we have

$$\begin{aligned} w &= \sum_{i=1}^{n} \left(\sum_{\alpha \in Q_{in}} \det(A[\eta_i|\alpha]) e_\alpha^\wedge \right) \\ &= \sum_{i=1}^{n} \left(\sum_{\alpha \in Q_{in}} \mathfrak{g}_{i,\alpha}(A) e_\alpha^\wedge \right) \end{aligned}$$

where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1k} & a_{1k+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{jk+1} & \cdots & a_{jn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} & a_{nk+1} & \cdots & a_{nn} \end{pmatrix}.$$

By definition of Φ_1, we obtain

$$\Phi_1(a_{11}, \dots, a_{1n}, \dots, a_{jk+1}, \dots, a_{jn}, \dots, a_{n1}, \dots, a_{nn}) = w,$$

and the proof is complete. □

We end this section with the following result:

Theorem 3.4 [1]. *Let $x_1, \dots, x_k$ and $y_1, \dots, y_k$ be families of linearly independent vectors of the inner product vector space V. Then, $x_1 \wedge \cdots \wedge x_k$ is orthogonal to $y_1 \wedge \cdots \wedge y_k$ if and only if*

$$\langle x_1, \dots, x_k \rangle^\perp \cap \langle y_1, \dots, y_k \rangle \neq \{0\}.$$

4. Equality of decomposable symmetrized tensors

Finding conditions for equality of decomposable symmetrized tensors was an open problem in the study of the symmetry class of tensors. This paper addresses conditions for equality of immanantal decomposable tensors. If λ is a linear character of S_m (that is, $\lambda = \epsilon$ or $\lambda = 1$), then the conditions have been known for some time [10,12]:

Theorem 4.1 [10,12]. *Assume that* $x_1 \wedge \ldots \wedge x_m \neq 0$. *Then*,

$$x_1 \wedge \cdots \wedge x_m = y_1 \wedge \cdots \wedge y_m$$

if and only if

(1) $\langle x_1, \ldots, x_m \rangle = \langle y_1, \ldots, y_m \rangle$; *and*
(2) $\det M[y_1, \ldots, y_m | x_1, \ldots, x_m] = 1$.

Theorem 4.2 [10,12]. *Assume that* $x_1 \vee \ldots \vee x_m \neq 0$. *Then*,

$$x_1 \vee \cdots \vee x_m = y_1 \vee \cdots \vee y_m$$

if and only there exists $c_1, \ldots, c_m \in \mathbb{C}$ *and* $\sigma \in S_m$ *such that*

(1) $y_i = c_i x_{\sigma(i)}, i = 1, \ldots, m$; *and*
(2) $\prod_{i=1}^{m} c_i = 1$.

If λ is not a linear character of S_m, the problem of finding conditions for equality of immanantal decomposable tensors has not been solved. However, some partial results have appeared in recent years. The main theorem proved in [4] is a necessary and sufficient condition for equality of immanantal decomposable tensors $x_1 * \cdots * x_m = y_1 * \cdots * y_m$ with the assumption that the rank partition of $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ is λ'.

Theorem 4.3 [4]. *Let* $(x_1, \ldots, x_m)$ *and* $(y_1, \ldots, y_m)$ *be families of nonzero vectors of* V *with rank partition* λ'. *Then*,

$$x_1 * \cdots * x_m = y_1 * \cdots * y_m$$

if and only if the following conditions hold:

(1) *The set of supports of the factorizations of* $(x_1, \ldots, x_m)$ *is equal to the sets of supports of the factorizations of* $(y_1, \ldots, y_m)$; *and*
(2) *If* $(\Delta_1, \ldots, \Delta_{\lambda_1})$ *is the support of a factorization of* $(x_1, \ldots, x_m)$ *then*

$$\langle x_i : i \in \Delta_j \rangle = \langle y_i : i \in \Delta_j \rangle, \quad j = 1, \ldots, \lambda_1,$$

and

$$\prod_{i=1}^{\lambda_1} \det M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta_i] = 1.$$

A necessary condition for equality of immanantal decomposable tensors is given in [5] without any constraint on the families of vectors $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$.

Theorem 4.4 [5]. *Let* λ *be an irreducible character of* S_m, *and let* $(x_1, \ldots, x_m)$ *and* $(y_1, \ldots, y_m)$ *be families of nonzero vectors of* V. *If*

$$x_1 * \cdots * x_m = y_1 * \cdots * y_m \neq 0,$$

then every support of a λ'*-coloring of* $(x_1, \ldots, x_m)$ *is also a support of a* λ'*-coloring of* $(y_1, \ldots, y_m)$.

In a previous paper we state a necessary and sufficient condition for equality of two nonzero decomposable symmetrized tensors when the symmetrizer is associated with an irreducible character of the symmetric group of degree m with the form $(p, \ldots, p)$.

Theorem 4.5 [7]. *Let* $(x_1, \ldots, x_m)$ *and* $(y_1, \ldots, y_m)$ *be families of nonzero vectors of* V, *and let* $\lambda = (p, \ldots, p)$ *be an irreducible character of* S_m. *Assume that* $x_1 * \cdots * x_m \neq 0$. *Then,*

$$x_1 * \cdots * x_m = y_1 * \cdots * y_m$$

if and only if the following conditions hold:

(1) *The set of supports of the* λ'*-colorings of* $(x_1, \ldots, x_m)$ *is equal to the set of supports of the* λ'*-colorings of* $(y_1, \ldots, y_m)$; *and*
(2) *If* $(\Delta_1, \ldots, \Delta_p)$ *is the support of a* λ'*-coloring of* $(x_1, \ldots, x_m)$, *then there exists* $\sigma \in S_p$ *such that*

$$\langle x_i : i \in \Delta_j \rangle = \langle y_i : i \in \Delta_{\sigma(j)} \rangle, \quad j = 1, \ldots, \lambda_1,$$

and

$$\prod_{i=1}^{p} \det M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta_{\sigma(i)} | \Delta_i] = 1.$$

The next result is the main theorem of this paper and generalizes the previous theorem to all irreducible characters of S_m.

Theorem 4.6. *Let* $(x_1, \ldots, x_m)$ *and* $(y_1, \ldots, y_m)$ *be families of nonzero vectors of* V, *and let* $\lambda = (\lambda, \ldots, \lambda_t)$ *be an irreducible character of* S_m. *Assume that* $x_1 * \cdots * x_m \neq 0$. *Then,*

$$x_1 * \cdots * x_m = y_1 * \cdots * y_m$$

if and only if the following conditions hold:

(1) *The set of supports of the* λ'*-colorings of* $(x_1, \ldots, x_m)$ *is equal to the set of supports of the* λ'*-colorings of* $(y_1, \ldots, y_m)$; *and*
(2) *If* $(\Delta_1, \ldots, \Delta_{\lambda_1})$ *is a support of a* λ'*-coloring of* $(x_1, \ldots, x_m)$, *then there exists* $\sigma \in S_{\lambda'}$ *such that*

$$\langle x_i : i \in \Delta_j \rangle = \langle y_i : i \in \Delta_{\sigma(j)} \rangle, \quad j = 1, \ldots, \lambda_1,$$

and

$$\prod_{i=1}^{\lambda_1} \det M[y_1, \ldots, y_m|x_1, \ldots, x_m][\Delta_{\sigma(i)}|\Delta_i] = 1.$$

Proof. Assume that

$$x_1 * \cdots * x_m = y_1 * \cdots * y_m.$$

The first condition is satisfied by Theorem 4.4, so we only have to prove the second condition. We do this by induction. Let $(\Delta_1, \ldots, \Delta_{\lambda_1})$ be the support of a λ'-coloring of $(x_1, \ldots, x_m)$. We start the proof by showing that there exists $t_1 \in \{1, \ldots, \lambda_1\}$ such that

$$\langle x_i : i \in \Delta_1\rangle = \langle y_i : i \in \Delta_{t_1}\rangle.$$

If $\dim V = \lambda_1'$, then

$$\langle x_i : i \in \Delta_1\rangle = \langle y_i : i \in \Delta_1\rangle.$$

Assume that $\dim V > \lambda_1'$, and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V such that

$$\langle x_i : i \in \Delta_1\rangle = \langle e_1, \ldots, e_{\lambda_1'}\rangle.$$

We denote by $\bar{e}_\alpha^\wedge$ the unit tensor

$$\bar{e}_\alpha^\wedge := \frac{1}{\|e_\alpha^\wedge\|} e_\alpha^\wedge.$$

Let D be a strict Young tableau associated with $[\lambda]$, whose ith column is Δ_i, $i = 1, \ldots, \lambda_1$. Then, by Proposition 2.1 and Theorem 2.1,

$$\begin{aligned} E(D)(x_1 \otimes \cdots \otimes x_m) &= \frac{m!}{\lambda(id)} e_{11}(x_1 * \cdots * x_m) \\ &= \frac{m!}{\lambda(id)} e_{11}(y_1 * \cdots * y_m) \\ &= E(D)(y_1 \otimes \cdots \otimes y_m) \neq 0, \end{aligned} \tag{9}$$

where $e_{1,1} = \frac{\lambda(id)}{m!} E(D)$.

Let $u_1, \ldots, u_n$ be a family of vectors of V, and let $u_1', \ldots, u_m'$ be a family of vectors taken from $u_1, \ldots, u_n$ choosing $u_j' = u_i$ if j belongs to ith row of D.

Obviously, using (9), we get

$$\begin{aligned} &(u_1' \otimes \cdots \otimes u_m', E(D)(x_1 \otimes \cdots \otimes x_m)) \\ &\quad = (u_1' \otimes \cdots \otimes u_m', E(D)(y_1 \otimes \cdots \otimes y_m)), \end{aligned} \tag{10}$$

and, since $T(R(D), 1)$ is an orthogonal projection, we get from (10) (using equalities (4) and (5)) the equality

$$\begin{aligned} &(T(R(D), 1)(u_1' \otimes \cdots \otimes u_m'), T(C(D), \epsilon)(x_1 \otimes \cdots \otimes x_m)) \\ &\quad = (T(R(D), 1)(u_1' \otimes \cdots \otimes u_m'), T(C(D), \epsilon)(y_1 \otimes \cdots \otimes y_m)). \end{aligned}$$

Bearing in mind the way we have chosen the vectors $u'_1, \ldots, u'_m$, we can see that, for all $\sigma \in R(D)$, we have

$$P(\sigma)(u'_1 \otimes \cdots \otimes u'_m) = u'_1 \otimes \cdots \otimes u'_m.$$

So

$$T(R(D), 1)(u'_1 \otimes \cdots \otimes u'_m) = u'_1 \otimes \cdots \otimes u'_m,$$

and, since $T(C(D), \epsilon)$ is an orthogonal projection, we obtain

$$\begin{aligned}&(T(C(D), \epsilon)(u'_1 \otimes \cdots \otimes u'_m), T(C(D), \epsilon)(x_1 \otimes \cdots \otimes x_m))\\ &\quad = (T(C(D), \epsilon)(u'_1 \otimes \cdots \otimes u'_m), T(C(D), \epsilon)(y_1 \otimes \cdots \otimes y_m)).\end{aligned}$$

Then,

$$\begin{aligned}&\left(\left(\bigwedge_{i=1}^{\lambda'_1} u_i\right) \otimes \cdots \otimes \left(\bigwedge_{i=1}^{\lambda'_{\lambda_1}} u_i\right), \left(\bigwedge_{i\in\Delta_1} x_i\right) \otimes \cdots \otimes \left(\bigwedge_{i\in\Delta_{\lambda_1}} x_i\right)\right)\\ &\quad = \left(\left(\bigwedge_{i=1}^{\lambda'_1} u_i\right) \otimes \cdots \otimes \left(\bigwedge_{i=1}^{\lambda'_{\lambda_1}} u_i\right), \left(\bigwedge_{i\in\Delta_1} y_i\right) \otimes \cdots \otimes \left(\bigwedge_{i\in\Delta_{\lambda_1}} y_i\right)\right),\end{aligned}$$

that is,

$$\prod_{j=1}^{\lambda_1} \left(\bigwedge_{i=1}^{\lambda'_j} u_i, \bigwedge_{i\in\Delta_j} x_i\right) = \prod_{j=1}^{\lambda_1} \left(\bigwedge_{i=1}^{\lambda'_j} u_i, \bigwedge_{i\in\Delta_j} y_i\right). \tag{11}$$

From now on we assume, for all $j = 1, \ldots, \lambda_1$, that

$$\bigwedge_{i\in\Delta_j} x_i, \bigwedge_{i\in\Delta_j} y_i \in \wedge V,$$

and denote by $\tilde{u}$ the element of $\wedge V$,

$$\tilde{u} = \sum_{i=1}^{n} u_1 \wedge \cdots \wedge u_i.$$

By (6) we conclude

$$\left(\tilde{u}, \bigwedge_{i\in\Delta_j} x_i\right) = \left(\bigwedge_{i=1}^{\lambda'_j} u_i, \bigwedge_{i\in\Delta_j} x_i\right), \quad j = 1, \ldots, \lambda_1,$$

and, using (11), we can write

$$\prod_{j=1}^{\lambda_1} \left(\tilde{u}, \bigwedge_{i\in\Delta_j} x_i\right) = \prod_{j=1}^{\lambda_1} \left(\tilde{u}, \bigwedge_{i\in\Delta_j} y_i\right). \tag{12}$$

Assume that

$$\tilde{u} \in \left\langle \bigwedge_{i\in\Delta_1} x_i \right\rangle^{\perp}.$$

Then, by (12), we have

$$\prod_{j=1}^{\lambda_1} \left(\tilde{u}, \bigwedge_{i\in\Delta_j} y_i \right) = 0. \tag{13}$$

For all $j = 1, \ldots, \lambda_1$ let

$$\bigwedge_{i\in\Delta_j} y_i = v_j + w_j, \tag{14}$$

where $v_j \in \langle \bigwedge_{i\in\Delta_1} x_i \rangle$, and $w_j \in \left\langle \bigwedge_{i\in\Delta_1} x_i \right\rangle^{\perp}$.

Let $k \in \{1, \ldots, \lambda_1\}$ be

$$k = |\{t : \lambda'_1 = \lambda'_t\}|.$$

Then $\Gamma_1 = \{1, \ldots, k\}$. From (13) and (14) we conclude that

$$(\tilde{u}, w_1) \ldots (\tilde{u}, w_k) \prod_{j=k+1}^{\lambda_1} \left(\tilde{u}, \bigwedge_{i\in\Delta_j} y_i \right) = 0, \tag{15}$$

for all $\tilde{u} \in \mathfrak{F}(V) \cap \langle \bigwedge_{i\in\Delta_1} x_i \rangle^{\perp}$.

Let $W = \langle x_i : i \in \Delta_1 \rangle$. As we have defined before, $\mathfrak{F}_{\lambda'_1, W}(V)$ is the subset of $\mathfrak{F}(V)$ of the elements

$$\tilde{u} = \sum_{i=1}^{n} u_1 \wedge \cdots \wedge u_i,$$

such that

$$u_{\lambda'_1} \in \langle x_i : i \in \Delta_1 \rangle^{\perp}.$$

By Theorem 3.4 and by (6) we have

$$\mathfrak{F}_{\lambda'_1, W}(V) \subseteq \mathfrak{F}(V) \cap \left\langle \bigwedge_{i\in\Delta_1} x_i \right\rangle^{\perp}. \tag{16}$$

For all $j = 1, \ldots, k$, let $\mathscr{P}_j$ be the hyperplane of $\wedge V$,

$$\langle w_j \rangle^{\perp},$$

and for all $j = k + 1, \ldots, \lambda_1$, let $\mathscr{S}_j$ be the hyperplane of $\wedge V$,

$$\left\langle \bigwedge_{i\in\Delta_j} y_i \right\rangle^{\perp}.$$

Using (15) and (16) we obtain

$$\mathfrak{F}_{\lambda_1',W}(V) \subseteq \mathscr{P}_1 \cup \cdots \cup \mathscr{P}_k \cup \mathscr{S}_{k+1} \cup \cdots \cup \mathscr{S}_{\lambda_1}.$$

Therefore,

$$\begin{aligned}\mathfrak{F}_{\lambda_1',W}(V) &= \mathfrak{F}_{\lambda_1',W}(V) \cap (\mathscr{P}_1 \cup \cdots \cup \mathscr{P}_k \cup \mathscr{S}_{k+1} \cup \cdots \cup \mathscr{S}_{\lambda_1})\\ &= (\mathfrak{F}_{\lambda_1',W}(V) \cap \mathscr{P}_1) \cup \cdots \cup (\mathfrak{F}_{\lambda_1',W}(V) \cap \mathscr{P}_k) \cup \\ &\quad\ (\mathfrak{F}_{\lambda_1',W}(V) \cap \mathscr{S}_{k+1}) \cup \cdots \cup (\mathfrak{F}_{\lambda_1',W}(V) \cap \mathscr{S}_{\lambda_1}).\end{aligned}$$

Since $\mathfrak{F}_{\lambda_1',W}(V)$ is an irreducible subset of $\wedge V$ (Theorem 3.3) and since $\mathfrak{F}_{\lambda_1',W}(V) \cap \mathscr{P}_j,\ j = 1, \ldots, k$ and $\mathfrak{F}_{\lambda_1',W}(V) \cap \mathscr{S}_t,\ t = k+1, \ldots, \lambda_1$ are closed sets of $\mathfrak{F}_{\lambda_1',W}(V)$, in the topology induced by the Zariski topology of $\mathbb{C}^{2^n}$, there exists $t \in \{1, \ldots, k\}$ such that

$$\mathfrak{F}_{\lambda_1',W}(V) = \mathfrak{F}_{\lambda_1',W}(V) \cap \mathscr{P}_t,$$

or there exists $r \in \{k+1, \ldots, \lambda_1\}$ such that

$$\mathfrak{F}_{\lambda_1',W}(V) = \mathfrak{F}_{\lambda_1',W}(V) \cap \mathscr{S}_r.$$

Assume that an $r \in \{k+1, \ldots, \lambda_1\}$ exists such that $\mathfrak{F}_{\lambda_1',W}(V) = \mathfrak{F}_{\lambda_1',W}(V) \cap \mathscr{S}_r$. Then

$$\left(\tilde{u}, \bigwedge_{i \in \Delta_r} y_i\right) = 0 \tag{17}$$

for all $\tilde{u} \in \mathfrak{F}_{\lambda_1',W}(V)$.

Since $\lambda_1' > \lambda_r'$, we can choose n vectors of V, $u_1, \ldots, u_n$, such that

$$\bigwedge_{i=1}^{\lambda_r'} u_i = \bigwedge_{i \in \Delta_r} y_i,$$

and $u_{\lambda_1'} \in \langle x_i : i \in \Delta_1\rangle^{\perp}$.

Let

$$\tilde{u} = \sum_{i=1}^{n} u_1 \wedge \cdots \wedge u_i.$$

Then $\tilde{u} \in \mathfrak{F}_{\lambda_1',W}(V)$, and so, by (17),

$$\begin{aligned}\left(\tilde{u}, \bigwedge_{i \in \Delta_r} y_i\right) &= \left(\bigwedge_{i=1}^{\lambda_r'} u_i, \bigwedge_{i \in \Delta_r} y_i\right)\\ &= \left(\bigwedge_{i \in \Delta_r} y_i, \bigwedge_{i \in \Delta_r} y_i\right)\\ &= 0.\end{aligned}$$

This is a contradiction, since $\bigwedge_{i\in\Delta_r} y_i \neq 0$. So

$$\mathfrak{F}_{\lambda'_1,W}(V) \neq \mathfrak{F}_{\lambda'_1,W}(V) \cap \mathscr{S}_r,$$

for all $r \in \{k+1, \ldots, \lambda_1\}$. Then there exists $t \in \Gamma_1$ such that

$$\mathfrak{F}_{\lambda'_1,W}(V) = \mathfrak{F}_{\lambda'_1,W}(V) \cap \mathscr{P}_t.$$

Therefore,

$$(\tilde{u}, w_t) = 0$$

for all $\tilde{u} \in \mathfrak{F}_{\lambda'_1,W}(V)$.

Since $w_t \in \wedge^{\lambda'_1} V$, we can conclude, using Theorem 4.1, that

$$(z, w_t) = 0$$

for all decomposable elements $z \in \langle \bigwedge_{i\in\Delta_1} x_i\rangle^\perp$, now viewed as a subspace of $\wedge^{\lambda'_1} V$. Since $w_t \in \langle \bigwedge_{i\in\Delta_1} x_i\rangle^\perp$ and $\{e^\wedge_\alpha : \alpha \in Q_{\lambda'_1,n} - \{\eta_{\lambda'_1}\}\}$ is an orthogonal basis for $\langle \bigwedge_{i\in\Delta_1} x_i\rangle^\perp$ formed by decomposable tensors, we conclude that

$$w_t = 0.$$

Therefore,

$$\bigwedge_{i\in\Delta_t} y_i = v_t$$

where $v_t \in \langle \bigwedge_{i\in\Delta_1} x_i\rangle$, and then

$$\bigwedge_{i\in\Delta_t} y_i = d_1 \bigwedge_{i\in\Delta_1} x_i$$

where $d_1 = \det M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta_t|\Delta_1]$. By applying Theorem 4.1, we conclude that

$$\langle x_i : i \in \Delta_1\rangle = \langle y_i : i \in \Delta_t\rangle.$$

Let $l \in \{2, \ldots, \lambda_1 - 1\}$ and assume that for all $j \in \{1, \ldots, l\} \cap \Gamma_s$ there exists $t_j \in (\{1, \ldots, \lambda_1\} - \{t_1, \ldots, t_{j-1}\}) \cap \Gamma_s$ such that

$$\langle x_i : i \in \Delta_j\rangle = \langle y_i : i \in \Delta_{t_j}\rangle.$$

Then $d_2, \ldots, d_l$ exists such that

$$\bigwedge_{i\in\Delta_{t_j}} y_i = d_j \bigwedge_{i\in\Delta_j} x_i, \quad j = 2, \ldots, l, \tag{18}$$

where $d_j = \det M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta_{t_j}|\Delta_j]$, $j = 2, \ldots, l$.

Since for every $\tilde{u} \in \mathfrak{F}(V)$

$$\prod_{j=1}^{\lambda_1} \left(\tilde{u}, \bigwedge_{i\in\Delta_j} x_i\right) = \prod_{j=1}^{\lambda_1} \left(\tilde{u}, \bigwedge_{i\in\Delta_j} y_i\right),$$

we obtain from (18)

$$\left(\tilde{u}, \bigwedge_{i\in\Delta_1} x_i\right)\ldots\left(\tilde{u}, \bigwedge_{i\in\Delta_l} x_i\right)\left(\prod_{j=l+1}^{\lambda_J}\left(\tilde{u}, \bigwedge_{i\in\Delta_j} x_i\right)\right.$$
$$\left. -d_1\ldots d_l \prod_{\substack{j=1\\ j\notin\{t_1,\ldots,t_l\}}}^{\lambda_1}\left(\tilde{u}, \bigwedge_{i\in\Delta_j} y_i\right)\right) = 0. \tag{19}$$

For $j \in \{l+1,\ldots,\lambda_1\}$ let

$$\bigwedge_{i\in\Delta_j} x_i = \sum_{\alpha\in Q_{\lambda'_j,n}} a_{j\alpha} e_\alpha^{-\wedge},$$

and for $j \in \{1,\ldots,\lambda_1\} - \{t_1,\ldots,t_l\}$ let

$$\bigwedge_{i\in\Delta_j} y_i = \sum_{\alpha\in Q_{\lambda'_j,n}} b_{j\alpha} e_\alpha^{-\wedge}.$$

Let z be an arbitrary element of $\wedge V$. Then

$$z = x_0 + \left(\sum_{\alpha\in Q_{1,n}} x_{1\alpha}\bar{e}_\alpha^{\wedge}\right) + \cdots + \left(\sum_{\alpha\in Q_{n,n}} x_{n\alpha}\bar{e}_\alpha^{\wedge}\right),$$

and so, for $j \in \{l+1,\ldots,\lambda_1\}$,

$$\left(z, \bigwedge_{i\in\Delta_j} x_i\right) = \sum_{\alpha\in Q_{\lambda'_j,n}} \overline{a_{j\alpha}} x_{j\alpha},$$

and for $j \in \{1,\ldots,\lambda_1\} - \{t_1,\ldots,t_l\}$,

$$\left(z, \bigwedge_{i\in\Delta_j} y_i\right) = \sum_{\alpha\in Q_{\lambda'_j,n}} \overline{b_{j\alpha}} x_{j\alpha}.$$

Let $\mathscr{H}$ be the algebraic variety of $\wedge V$ defined by the polynomial of $\mathbb{C}[X_{j\alpha} : j = 1,\ldots,n, \alpha \in Q_{j,n}]$.

$$\prod_{j=l+1}^{\lambda_1}\left(\sum_{\alpha\in Q_{\lambda'_j,n}} \overline{a_{j\alpha}} X_{j\alpha}\right) - d_1\ldots d_l \prod_{\substack{j=1\\ j\notin\{t_1,\ldots,t_l\}}}^{\lambda_1}\left(\sum_{\alpha\in Q_{\lambda'_j,n}} \overline{b_{j\alpha}} X_{j\alpha}\right),$$

and let $\mathscr{R}_j$ be the hyperplane of $\wedge V$,

$$\left\langle \bigwedge_{i\in\Delta_j} x_i \right\rangle^{\perp}.$$

By using (19) we conclude

$$\mathfrak{F}(V) \subseteq (\mathscr{R}_1 \cup \cdots \cup \mathscr{R}_l \cup \mathscr{H}),$$

so

$$\begin{aligned}\mathfrak{F}(V) &= \mathfrak{F}(V) \cap (\mathscr{R}_1 \cup \cdots \cup \mathscr{R}_l \cup \mathscr{H})\\ &= (\mathfrak{F}(V) \cap \mathscr{R}_1) \cup \cdots \cup (\mathfrak{F}(V) \cap \mathscr{R}_l) \cup (\mathfrak{F}(V) \cap \mathscr{H}).\end{aligned}$$

Since

$$\bigwedge_{i \in \Delta_j} x_i \neq 0,$$

we have

$$\left(\bigwedge_{i \in \Delta_j} x_i, \bigwedge_{i \in \Delta_j} x_i\right) \neq 0,$$

therefore, there exists $\tilde{u} \in \mathfrak{F}(V)$ such that

$$\left(\tilde{u}, \bigwedge_{i \in \Delta_j} x_i\right) \neq 0.$$

Hence,

$$\mathfrak{F}(V) \cap \mathscr{R}_j \neq \mathfrak{F}(V), \quad j = 1, \ldots, l,$$

and since $\mathfrak{F}(V)$ is an irreducible subset of $\wedge V$ (Theorem 3.2), we conclude that

$$\mathfrak{F}(V) = \mathfrak{F}(V) \cap \mathscr{H},$$

that is,

$$\prod_{j=l+1}^{\lambda_1} \left(\tilde{u}, \bigwedge_{i \in \Delta_j} x_i\right) = d_1 \cdots d_l \prod_{\substack{j=1\\ j \notin \{t_1,\ldots,t_l\}}}^{\lambda_1} \left(\tilde{u}, \bigwedge_{i \in \Delta_j} y_i\right)$$

for all $\tilde{u} \in \mathfrak{F}(V)$. Assume $\tilde{u}$ is such that

$$\tilde{u} \in \left\langle \bigwedge_{i \in \Delta_{l+1}} x_i \right\rangle^{\perp}$$

and $l + 1 \in \Gamma_q$. As before, we can conclude that a $t_{l+1} \in \{1, \ldots, \lambda_l\} - \{t_1, \ldots, t_l\}$ exists such that

$$\bigwedge_{i \in \Delta_{t_{l+1}}} y_i = d_{l+1} \bigwedge_{i \in \Delta_{l+1}} x_i,$$

where $d_{l+1} = \det M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta_{t_{l+1}} | \Delta_{l+1}]$. Using Theorem 4.1 we obtain

$$\langle x_i : i \in \Delta_{l+1}\rangle = \langle y_i : i \in \Delta_{t_{l+1}}\rangle.$$

Therefore, $t_{l+1} \in \Gamma_q$. So, by induction, we conclude that there exists $\sigma \in S_{\lambda'}$ such that

$$\langle x_i : i \in \Delta_j\rangle = \langle y_i : i \in \Delta_{\sigma(j)}\rangle, \quad j = 1, \ldots, \lambda_1.$$

Denote the integer in the box (i, j) of D by k_{ij}. Let σ' be the permutation of S_m defined by

$$\sigma'(k_{i,j}) = k_{i\sigma(j)}.$$

Bearing in mind (1) and (2) it is easy to see that $\sigma' \in R(D)$.

Then, since $P(D)P(\sigma') = P(D)$,

$$\begin{aligned}
&\frac{1}{|C(D)|}E(D)(y_1 \otimes \cdots \otimes y_m)\\
&\quad = P(D)\left(\left(\bigwedge_{i\in\Delta_1} y_i\right) \otimes \cdots \otimes \left(\bigwedge_{i\in\Delta_{\lambda_1}} y_i\right)\right)\\
&\quad = P(D)P(\sigma')\left(\left(\bigwedge_{i\in\Delta_{\sigma(1)}} y_i\right) \otimes \cdots \otimes \left(\bigwedge_{i\in\Delta_{\sigma(\lambda_1)}} y_i\right)\right)\\
&\quad = \prod_{i=1}^{\lambda_1} \det M[y_1, \ldots, y_m|x_1, \ldots, x_m][\Delta_{\sigma(i)}|\Delta_i]\\
&\qquad \times P(D)\left(\left(\bigwedge_{i\in\Delta_1} x_i\right) \otimes \cdots \otimes \left(\bigwedge_{i\in\Delta_{\lambda_1}} x_i\right)\right)\\
&\quad = \prod_{i=1}^{\lambda_1} \det M[y_1, \ldots, y_m|x_1, \ldots, x_m][\Delta_{\sigma(i)}|\Delta_i]\\
&\qquad \times \frac{1}{|C(D)|}E(D)(x_1 \otimes \cdots \otimes x_m),
\end{aligned}$$

and, since

$$E(D)(x_1 \otimes \cdots \otimes x_m) = E(D)(y_1 \otimes \cdots \otimes y_m) \neq 0,$$

we obtain

$$\prod_{i=1}^{\lambda_1} \det M[y_1, \ldots, y_m|x_1, \ldots, x_m][\Delta_{\sigma(i)}|\Delta_i] = 1.$$

Assume now that the conditions (1) and (2) are satisfied. We prove that

$$x_1 * \cdots * x_m = y_1 * \cdots * y_m,$$

showing that

$$E(D)(x_1 \otimes \cdots \otimes x_m) = E(D)(y_1 \otimes \cdots \otimes y_m)$$

for all standard strict Young tableaux associated to $[\lambda]$.

Let D be a standard strict Young tableaux associated to $[\lambda]$. If the columns of D are not a support of a λ'-coloring of $x_1, \ldots, x_m$, then by (1) they are not a support of a λ'-coloring of $y_1, \ldots, y_m$, and so we have

$$E(D)(x_1 \otimes \cdots \otimes x_m) = E(D)(y_1 \otimes \cdots \otimes y_m) = 0.$$

If the columns of D are a support of a λ'-coloring of $x_1, \ldots, x_m$, using the adequate model for the tensor product we have

$$\begin{aligned}
&\frac{1}{|C(D)|} E(D)(y_1 \otimes \cdots \otimes y_m) \\
&\quad = P(D)\left(\left(\bigwedge_{i \in \Delta_1} y_i\right) \otimes \cdots \otimes \left(\bigwedge_{i \in \Delta_{\lambda_1}} y_i\right)\right) \\
&\quad = P(D)P(\sigma')\left(\left(\bigwedge_{i \in \Delta_{\sigma(1)}} y_i\right) \otimes \cdots \otimes \left(\bigwedge_{i \in \Delta_{\sigma(\lambda_1)}} y_i\right)\right) \\
&\quad = \prod_{i=1}^{\lambda_1} \det M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta_{\sigma(i)} | \Delta_i] \\
&\qquad \times P(D)\left(\left(\bigwedge_{i \in \Delta_1} x_i\right) \otimes \cdots \otimes \left(\bigwedge_{i \in \Delta_{\lambda_1}} x_i\right)\right) \\
&\quad = \prod_{i=1}^{\lambda_1} \det M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta_{\sigma(i)} | \Delta_i] \\
&\qquad \times \frac{1}{|C(D)|} E(D)(x_1 \otimes \cdots \otimes x_m),
\end{aligned}$$

and since

$$\prod_{i=1}^{\lambda_1} \det M[y_1, \ldots, y_m | x_1, \ldots, x_m][\Delta_{\sigma(i)} | \Delta_i] = 1,$$

we conclude that

$$E(D)(x_1 \otimes \cdots \otimes x_m) = E(D)(y_1 \otimes \cdots \otimes y_m).$$

Finally, bearing in mind the Proposition 2.1, we get

$$e_{ii}(x_1 \otimes \cdots \otimes x_m) = e_{ii}(y_1 \otimes \cdots \otimes y_m), \quad i = 1, \ldots, d,$$

where d is the number of the standard strict Young tableaux associated with $[\lambda]$, and so,

$$x_1 * \cdots * x_m = y_1 * \cdots * y_m,$$

and the proof is complete. □

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