

Modified Saez-Ballester scalar-tensor theory from $5D$ space-time

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(Dated: 14-Feb-2018)

In this paper, we bring together the five-dimensional Saez-Ballester (SB) scalar-tensor theory [1] and the induced-matter-theory (IMT) setting [2], to obtain a modified SB theory (MSBT) in four dimensions. Specifically, by using an intrinsic dimensional reduction procedure into the SB field equations in five-dimensions, a MSBT is obtained onto a hypersurface orthogonal to the extra dimension. This four-dimensional MSBT is shown to bear distinctive new features in contrast to the usual corresponding SB theory as well as to IMT and the Modified Brans-Dicke Theory (MBDT) [3]. In more detail, besides usual induced matter terms retrieved through the IMT, the MSBT scalar field is provided with additional physically distinct (namely, SB induced) terms as well as an intrinsic self-interacting potential (interpreted as consequence of the IMT process and the concrete geometry associated to the extra dimension). Moreover, our MSBT has four sets of field equations, with two sets having no analog in the standard SB scalar-tensor theory. It should be emphasized that the herein appealing solutions can emerge solely from the geometrical reductional process, from presence also of extra dimension(s) and not from any ad-hoc matter either in the bulk or on the hypersurface. Subsequently, we apply the herein MSBT to cosmology and consider an extended spatially flat FLRW geometry in a five-dimensional vacuum space-time. After obtaining the exact solutions in the bulk, we proceed to construct, by means of the MSBT setting, the corresponding dynamic, on the four-dimensional hypersurface. More precisely, we obtain the (SB) components of the induced matter, including the induced scalar potential terms. We retrieve two different classes of solutions. Concerning the first class, we show that the MSBT yields a barotropic equation of state for the induced perfect fluid. We then investigate vacuum, dust, radiation, stiff fluid and false vacuum cosmologies for this scenario and contrast the results with those obtained in the standard SB theory, IMT and BD theory. Regarding the second class solutions, we show that the scale factor behaves similar to a de Sitter (DeS) model. However, in our MSBT setting, this behavior is assisted by non-vanishing induced matter instead, without any a priori cosmological constant. Moreover, for all these solutions, we show that the extra dimension contracts with the cosmic time.

PACS numbers:

Keywords: Saez-Ballester scalar-tensor theory; induced-matter theory; extra dimensions; FLRW cosmology.

I. INTRODUCTION

Higher-dimensional models of the universe have been widely studied: Kaluza-Klein (KK) theories [4–6], ten-dimensional and eleven-dimensional supergravity [7–9] as well as string theories [10, 11]. In particular, Kaluza considered a five-dimensional space-time, under the following key assumptions [6], to bring together electromagnetism with general relativity (GR): (i) there is no ordinary matter in the higher dimensional space-time, (ii) geometrical quantities similar to those defined in GR were introduced and (iii) the cylinder condition for the extra coordinate was assumed, which implies that the derivatives with respect to the extra dimension must vanish. Following Kaluza's construction, subsequent compactified, projective [12–15] and noncompactified [2, 6, 16–18] versions emerged as well-known scenarios which have been thoroughly investigated, such that, at least, one of the above mentioned assumptions (i)-(iii) has been modified. Particularly, by introducing the noncompactified version of the KK theory to construct the so-called space-time-matter theory (or IMT), it has been attempted to obtain a firm basis concerning a suggestion by Einstein, who proposed that it is possible to transpose the “base wood” of the matter fields into the “pure marble” of the geometry in his general relativistic theory [18].

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Increasingly applications of gravitation and cosmology versions of KK theory, especially modern KK theories (namely, the IMT and the brane world scenarios [19, 20]) have motivated the investigation of embedding theorems and their corresponding generalized versions. In this respect, it is important to establish which embedding theorem of differential geometry should be considered to construct a higher dimensional model of gravity. Historically, within higher-dimensional physical settings, two embedding theorems have played a pivotal role [21]. The first one is the Janet-Cartan theorem [22], in which the embedding space is flat. The second one, which is particularly more powerful than the first one, is known as Campbell-Magaard (CM) theorem [23, 24], which has been employed to justify, geometrically, the IMT and its modifications [6, 25]. Subsequently, a few extended versions of the CM theorem have been produced. For instance, in [26, 27], the CM theorem is extended such that (instead of the Ricci-flatness condition) the embedding space is the Einstein space sourced with whether one (or more) scalar field(s) or a negative cosmological constant. Moreover, another extension of this theorem is obtained in [28], where it was shown that the restricted embedding *Ricci-flat* space-time can be replaced by a more general *Einstein space-time*. Furthermore, in [29], by starting from the Gauss-Codazzi equations, another modified proof has been prepared which can be applicable in the IMT as well as in membrane theory. We should mention that despite the widespread use in applying the CM theorem and its generalizations in physical scenarios, especially in noncompactified KK theories, a few problems have been mentioned [30] which has been interpreted as an inadequate ability of the CM theorem concerning the initial-value problem or non-occurrence of singularities [31]. As a defense concerning the mentioned problem, it has been emphasized that the CM theorem is a local theorem and there is no claim in it about solving such problems; on the other hand, it has been believed that answering to the problems such as the singularities as well as causality issues, will be possible in a complete theory of quantum gravity [31].

However, almost all of the mentioned efforts for generalizing the CM theorem have been done through in a local ¹ setting. In this respect, Katzourakis proved a global version of the CM theorem [32], see also [37–39]. Subsequently, this embedding theorem has been prepared allowing a new global insight to investigate the extra force and the particle mass [40].

In this paper, our focus will be on the latter and we will assume that the extra dimension is noncompact, embracing the motivations of [16, 18, 41, 42]. Furthermore, throughout this paper, as a distinctive novel feature, we will consider, instead of either GR or BD theory, the SB scalar-tensor theory [1] (as background setting) in a five dimensional space-time. Then, by using an appropriate reductional procedure, namely, the IMT algebraic framework, we will obtain the MSBT induced equations onto a four-dimensional hypersurface.

In the SB theory, in contrast to a BD theory [43] (as well as other G-varying theories, see [44] and references therein), the scalar field does *not* play the role of the (varying) G. Instead, it is taken as a dimensionless field, not having to satisfy restrictive constraints determined from observations [1]. Moreover, the strength of the coupling between the gravity and the scalar field is governed by a dimensionless parameter \mathcal{W} (see footnote 3). With such modifications, it has been shown that it is possible to address the cosmological missing matter problem [1]. In the context of cosmology, by assuming various line-elements, diverse implications from the SB theory have also been widely employed to obtain solutions in either four or five dimensions [45–52]. Furthermore, the Noether gauge symmetries of a generalized SB theory with a scalar potential, which has been added² by hand in [53], has been investigated.

Concerning IMT [2], it has been shown that matter in a four-dimensional space-time can be described from a purely geometric origin. More precisely, the properties of matter, at the macroscopic level, as observed in the four-dimensional space-time, would be attributed (within a IMT setting) to the dynamics of a large extra dimension [2, 6, 16, 18]. In cosmology, the IMT has been employed to construct concrete scenarios, which not only modify the corresponding solutions of GR, but also agree with recent cosmological observational data [18, 54, 55].

Applications of the IMT framework, in which the role of GR as a fundamental underlying theory is replaced by the Brans-Dicke (BD) scalar-tensor theory [43], have been formulated in [3, 56–59]. In the herein paper, we will show that the induced energy-momentum tensor (EMT) in MSBT is an extended version of that of IMT, implying significant differences from gravitational and cosmological perspectives. In particular, when the scalar field takes constant values we recover the effective equations of the IMT. Furthermore, we should note that there are two extra equations in our herein setting with no analog in the standard SB theory. As the conventional SB action is a modified version of GR with a canonical scalar field, our MSBT framework (with the intrinsic scalar potential) can also be employed to study inflationary scenarios as well as late time cosmological acceleration. These possible scenarios can be considered as the fundamental methods to obtain the results. The reason is that such possible solutions can be obtained from the geometrical implications of the reductional process due to extra dimension and not from any ad hoc matter elements.

¹ The first global embedding into \mathbb{R}^N is due to John Nash [33] (moreover, as extension appeared in [34] and [35]) and a recent improvement has been prepared by Gunther [36].

² However, in the present paper, we show instead that, by admitting the SB theory into a five-dimensional space-time, it is not necessary to add such a potential by hand, but it can be elegantly retrieved solely from the geometry associated to the extra dimension.

As a realisation of how MBST can construct an innovative perspective in cosmology, we consider the extended version of the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) line-element in five dimensions. We show that the geometrically induced matter on the specific hypersurface can play the role of ordinary matter. Moreover, the extra dimension contracts in all of our cosmological solutions.

This paper is organized as follows. In Section II, we consider the SB field equations in five dimensions and then, by employing a dimensional reduction, extending the IMT procedure, we construct a modified version of the SB theory on a hypersurface. Moreover, the modified field equations include new dynamical ingredients, namely, an effective induced self-interacting SB scalar potential. In Section III, by assuming a five-dimensional vacuum space-time, which is described by the FLRW line-element, we obtain exact cosmological solutions for the SB field equations. By employing this MSBT-IMT framework, we will obtain in section IV the induced quantities onto the four-dimensional hypersurface. Then, we study the reduced four-dimensional cosmological solutions. Subsequently, we will show that there are two different classes of solutions. For the first class, we will explain that the resulted solutions can describe different states of matter fluids in the universe. Whereas the second class can describe a particular kind of matter which leads us to a DeS-like behavior for the scale factor. Finally, we present our conclusions in Section V.

II. DIMENSIONAL REDUCTION OF FIVE-DIMENSIONAL SAEZ-BALLESTER THEORY

In this section, by employing a specific dimensional reduction procedure for a five-dimensional SB theory, we obtain the effective field equations on a four-dimensional space-time.

The action for the five-dimensional SB theory, in analogy with the corresponding four-dimensional case [1], can be written as

$$\mathcal{S}^{(5)} = \int d^5x \sqrt{|\mathcal{G}^{(5)}|} \left[R^{(5)} - \mathcal{W}\phi^n \mathcal{G}^{ab} (\nabla_a \phi)(\nabla_b \phi) + \chi L_{\text{matt}}^{(5)} \right], \quad (2.1)$$

where ϕ is the dimensionless scalar field and \mathcal{W} and n are dimensionless parameters³; the Latin indices run from zero to four; $\mathcal{G}^{(5)}$ and $R^{(5)}$ are the determinant and curvature scalar associated with the five-dimensional metric \mathcal{G}_{ab} , respectively.

Here, we should emphasize two points: (i) In contrast to most of the scalar-tensor theories such as the BD theory [43], the scalar field does not have dimensions of inverse of the (five-dimensional) gravitational constant, but instead, it is dimensionless [1]. (ii) In the original setting of IMT [2, 6], it was assumed that there is no ordinary matter in the bulk. However, for the sake of formal generality, we consider a non-vanishing Lagrangian $L_{\text{matt}}^{(5)}$ (which is minimally coupled to the scalar field) describing ordinary matter⁴ in the five-dimensional space-time, for the sole reason of writing broader applicable expression.

Independently varying action (2.1) with respect to the metric and the scalar field gives the equations

$$G_{ab}^{(5)} = \mathcal{W}\phi^n \left[(\nabla_a \phi)(\nabla_b \phi) - \frac{1}{2} \mathcal{G}_{ab} (\nabla^c \phi)(\nabla_c \phi) \right] + \chi T_{ab}^{(5)} \quad (2.2)$$

and

$$2\phi^n \nabla^2 \phi + n\phi^{n-1} (\nabla_a \phi)(\nabla^a \phi) = 0, \quad (2.3)$$

where $\nabla^2 \equiv \nabla_a \nabla^a$; $T_{ab}^{(5)}$ and $G_{ab}^{(5)}$ stand for the EMT [of the ordinary matter fields] and the Einstein tensor in five-dimensional space-time, respectively. From equation (2.2), we obtain

$$R^{(5)} = \mathcal{W}\phi^n (\nabla_a \phi)(\nabla^a \phi) - \frac{2}{3} \chi T^{(5)}, \quad (2.4)$$

where $T^{(5)} = \mathcal{G}^{ab} T_{ab}^{(5)}$.

In what follows, by employing the specific reduction procedure, the five-dimensional field equations will be related to the corresponding ones on the four-dimensional hypersurface. We will show that effective matter and the induced

³ We should notice, in our herein work, the SB action is written in Planck units and that is the reason why the Einstein-Hilbert term does not have any coefficient and the coupling to matter is just $\chi = 8\pi$. When standard units are used, the scalar field could only be dimensionless if \mathcal{W} has dimensions of energy square.

⁴ In this paper, the “vacuum” space-time is used for a situation where there is not any type of ordinary matter.

scalar potential appearing in the four-dimensional field equations can emerge solely from the geometry of the fifth dimension. Let us be more precise. The effective field equations associated to the four-dimensional hypersurface will be derived by employing the five-dimensional SB field equations (2.2), (2.3), and the five-dimensional well-known metric [2, 6]

$$dS^2 = \mathcal{G}_{ab}(x^c)dx^a dx^b = g_{\mu\nu}(x^\alpha, l)dx^\mu dx^\nu + \epsilon\psi^2(x^\alpha, l)dl^2, \quad (2.5)$$

where l denotes a non-compact coordinate along the extra dimension, the Greek indices run from zero to 3, $\epsilon = \pm 1$ permits us to take the extra dimension as either time-like⁵ or space-like and ψ may depend on all the coordinates. It is usually supposed that the five-dimensional space-time is foliated by a family of four-dimensional hypersurfaces, Σ . For instance, we can set $l = l_0 = \text{constant}$ to get Σ_0 , whose intrinsic line-element is orthogonal to the five-dimensional unit vector

$$n^a = \frac{\delta_4^a}{\psi} \quad \text{where} \quad n_a n^a = \epsilon, \quad (2.6)$$

along the extra dimension [3, 57]. Consequently, we have the induced metric $g_{\mu\nu}$ on the hypersurface Σ_0 with the form

$$ds^2 = \mathcal{G}_{\mu\nu}(x^\alpha, l_0)dx^\mu dx^\nu \equiv g_{\mu\nu}dx^\mu dx^\nu. \quad (2.7)$$

Now, in order to get the four-dimensional part of the corresponding five-dimensional quantity, we can set $a \rightarrow \mu$ and $b \rightarrow \nu$ in equation (2.2). Consequently, we obtain

$$G_{\mu\nu}^{(5)} = \mathcal{W}\phi^n \left[(\mathcal{D}_\mu\phi)(\mathcal{D}_\nu\phi) - \frac{1}{2}g_{\mu\nu}(\mathcal{D}_\alpha\phi)(\mathcal{D}^\alpha\phi) \right] - \frac{\epsilon\mathcal{W}\phi^n}{2} \left(\frac{\phi}{\psi} \right)^2 g_{\mu\nu} + \chi T_{\mu\nu}^{(5)}, \quad (2.8)$$

where \mathcal{D}_α and the notation $\overset{*}{A}$ stand for the covariant derivative on the hypersurface (whose computation employs $g_{\mu\nu}$) and the derivative of any quantity A with respect to the extra coordinate l , respectively. Moreover, $\mathcal{D}^2 \equiv \mathcal{D}^\alpha \mathcal{D}_\alpha$ and the following relations have been employed

$$(\nabla^c\phi)(\nabla_c\phi) = (\mathcal{D}^\alpha\phi)(\mathcal{D}_\alpha\phi) + \epsilon \left(\frac{\phi}{\psi} \right)^2, \quad (2.9)$$

$$\nabla_\mu \nabla_\nu \phi = \mathcal{D}_\mu \mathcal{D}_\nu \phi + \frac{\epsilon \phi \overset{*}{g}_{\mu\nu}}{2\psi^2}, \quad (2.10)$$

$$\nabla^2 \phi = \mathcal{D}^2 \phi + \frac{(\mathcal{D}_\alpha\psi)(\mathcal{D}^\alpha\phi)}{\psi} + \frac{\epsilon}{\psi^2} \left[\overset{**}{\phi} + \overset{*}{\phi} \left(\frac{g^{\mu\nu} \overset{*}{g}_{\mu\nu}}{2} - \frac{\psi}{\psi} \right) \right], \quad (2.11)$$

$$\nabla_4 \nabla_4 \phi = \epsilon \psi (\mathcal{D}_\alpha \psi) (\mathcal{D}^\alpha \phi) + \overset{**}{\phi} - \frac{\psi}{\psi} \overset{*}{\phi}. \quad (2.12)$$

⁵ The idea of two-time physics has been highly motivated by the works by Itzhak Bars and his collaborators [60–63]. Indeed, the two-time idea in physical theories could provide some insight into one-time dynamics associated to the higher-dimensional frameworks. The second time has also been studied in GR when the quality of the universal parametric “historical time” has been investigated [64, 65]. It has been believed that [66] assigning time-like signature to the extra coordinate in the KK theories may receive the following constructive criticisms: (i) assuming the cylinder condition, by integrating over the extra coordinate in the $5D$ action, the resulted action takes the opposite sign with respect to that of Einstein, which is not correct. (ii) again, assuming a cylinder condition may yield tachyons. (iii) closed time curves might emerge. In our herein MSBT setting, similar to the MBDT and the IMT frameworks, as ϵ is chosen to take ± 1 , assuming the extra coordinate to be time-like is less restricted. In IMT and membrane theory, two-time physics has also been the main motivation of the investigation, see, e.g., [18, 67–69]. In what follows, let us mention some of their corresponding consequences. It has been shown that: as the second time-like coordinate is related to the (inertial) rest mass of a particle, there is no problem with the criticism (iii) [69]; the waves in the extra dimension, which oscillate around the hypersurface, can be described by assuming general two-time metric as given in [69]. Therefore, such models can be employed to describe multiply-imaged particles. As the MBDT [3] as well as our herein MSBT frameworks are the generalized versions of the IMT setting, we suggest that investigating the two-time physics, even by assuming the same metrics, can yield more generalized consequences.

In order to derive the SB effective field equations on the hypersurface, let us first construct the left hand side of the field equations, namely the Einstein tensor, on the hypersurface. Therefore, we have to relate both the Ricci tensor, $R_{\alpha\beta}^{(5)}$, and the Ricci curvature, $R^{(5)}$, to their corresponding quantities on the four-dimensional hypersurface. In this respect, we get

$$R_{\alpha\beta}^{(5)} = R_{\alpha\beta}^{(5)} - \frac{\mathcal{D}_\alpha \mathcal{D}_\beta \psi}{\psi} + \frac{\epsilon}{2\psi^2} \left[\frac{\psi g_{\alpha\beta}^*}{\psi} - g_{\alpha\beta}^{**} + g^{\lambda\mu} g_{\alpha\lambda}^* g_{\beta\mu}^* - \frac{1}{2} g^{\mu\nu} g_{\mu\nu}^* g_{\alpha\beta}^* \right], \quad (2.13)$$

$$R_{44}^{(5)} = -\epsilon\psi \mathcal{D}^2 \psi - \frac{1}{4} g^{\lambda\beta} g_{\lambda\beta}^* - \frac{1}{2} g^{\lambda\beta} g_{\lambda\beta}^{**} + \frac{1}{2\psi} g^{\lambda\beta} g_{\lambda\beta}^* \psi. \quad (2.14)$$

Moreover, by employing equations (2.2), (2.4) and (2.14), we can easily get

$$\frac{\mathcal{D}^2 \psi}{\psi} = -\frac{\epsilon}{2\psi^2} \left[g^{\lambda\beta} g_{\lambda\beta}^{**} + \frac{1}{2} g^{\lambda\beta} g_{\lambda\beta}^* - \frac{g^{\lambda\beta} g_{\lambda\beta}^* \psi}{\psi} \right] - \frac{\epsilon \mathcal{W} \phi^n (\phi)^2}{\psi^2} + \chi \left[\frac{T^{(5)}}{3} - \frac{\epsilon T_{44}^{(5)}}{\psi^2} \right], \quad (2.15)$$

where we have used $g^{\mu\beta} g^{\lambda\sigma} g_{\lambda\beta}^* g_{\mu\sigma}^* + g^{\mu\sigma} g_{\mu\sigma}^* = 0$. Using relations (2.14) and (2.15), the Ricci scalar in five-dimensional and four-dimensional space-times are related to each other as

$$R^{(5)} = R^{(4)} - \frac{\epsilon}{4\psi^2} \left[\left(g^{\alpha\beta} g_{\alpha\beta}^* \right)^2 + g^{\alpha\beta} g_{\alpha\beta}^* \right] + \frac{2\epsilon \mathcal{W} \phi^n (\phi)^2}{\psi^2} + 2\chi \left(\frac{\epsilon T_{44}^{(5)}}{\psi^2} - \frac{T^{(5)}}{3} \right). \quad (2.16)$$

Subsequently, by employing (2.13)-(2.16), the effective field equations (onto the 4D hypersurface) associated to our MSBT setting can be constructed. In the following three steps, by presenting appropriate interpretations, we would outline the MSBT field equations.

Firstly, the SB equations on the hypersurface can be constructed by using equations (2.8), (2.13) and (2.16), as

$$\begin{aligned} G_{\mu\nu}^{(4)} &= \chi \left(E_{\mu\nu} + T_{\mu\nu}^{[\text{SB}]} \right) + \mathcal{W} \phi^n \left[(\mathcal{D}_\mu \phi)(\mathcal{D}_\nu \phi) - \frac{1}{2} g_{\mu\nu} (\mathcal{D}_\alpha \phi)(\mathcal{D}^\alpha \phi) \right] - \frac{1}{2} g_{\mu\nu} V(\phi) \\ &\equiv \chi T_{\mu\nu}^{(4)[\text{eff}]} + \mathcal{W} \phi^n \left[(\mathcal{D}_\mu \phi)(\mathcal{D}_\nu \phi) - \frac{1}{2} g_{\mu\nu} (\mathcal{D}_\alpha \phi)(\mathcal{D}^\alpha \phi) \right] - \frac{1}{2} g_{\mu\nu} V(\phi). \end{aligned} \quad (2.17)$$

This consequently conveys the standard SB equations on the four-dimensional hypersurface which comprise an induced scalar potential, see equation (2.23). Moreover, in what follows, we should make clear a few points:

- The effects of the five-dimensional ordinary EMT on the hypersurface are denoted by $E_{\mu\nu}$, which is given by

$$E_{\mu\nu} \equiv T_{\mu\nu}^{(5)} - g_{\mu\nu} \left[\frac{T^{(5)}}{3} - \frac{\epsilon T_{44}^{(5)}}{\psi^2} \right]. \quad (2.18)$$

Obviously, if, at the beginning, we assume that there is no ordinary matter fields in the five-dimensional space-time [i.e., if we started by assuming $L_{\text{matt}}^{(5)} = 0$ in action (2.1)], then $E_{\mu\nu}$ vanishes.

- The quantity $T_{\mu\nu}^{[\text{SB}]}$ is an effective EMT for our MSBT setting, which comprises three components as

$$\chi T_{\mu\nu}^{[\text{SB}]} = T_{\mu\nu}^{[\text{IMT}]} + T_{\mu\nu}^{[\phi]} + \frac{1}{2} g_{\mu\nu} V(\phi), \quad (2.19)$$

where

$$\begin{aligned} T_{\mu\nu}^{[\text{IMT}]} &\equiv \frac{\mathcal{D}_\mu \mathcal{D}_\nu \psi}{\psi} - \frac{\epsilon}{2\psi^2} \left(\frac{\psi g_{\mu\nu}^*}{\psi} - g_{\mu\nu}^{**} + g^{\lambda\alpha} g_{\mu\lambda}^* g_{\nu\alpha}^* - \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta}^* g_{\mu\nu}^* \right) \\ &\quad - \frac{\epsilon g_{\mu\nu}}{8\psi^2} \left[g^{\alpha\beta} g_{\alpha\beta}^* + \left(g^{\alpha\beta} g_{\alpha\beta}^* \right)^2 \right], \end{aligned} \quad (2.20)$$

$$T_{\mu\nu}^{[\phi]} \equiv \left[\frac{\epsilon \mathcal{W}}{2} \phi^n \left(\frac{\phi}{\psi} \right)^2 \right] g_{\mu\nu}. \quad (2.21)$$

The first contribution in the effective EMT, i.e. $T_{\mu\nu}^{[\text{IMT}]}$, emerges from the fifth part of the metric (2.5) which is a consequence from the geometry of the extra dimension; while the second contribution $T_{\mu\nu}^{[\phi]}$ depends on the scalar field ϕ and its derivative with respect to the fifth coordinate.

Secondly, we should derive the counterpart of equation (2.3) onto the four-dimensional space-time. By substituting relations (2.9) and (2.11) into equation (2.3), we obtain

$$2\phi^n \mathcal{D}^2 \phi + n\phi^{n-1} (\mathcal{D}_\alpha \phi) (\mathcal{D}^\alpha \phi) - \frac{1}{\mathcal{W}} \frac{dV(\phi)}{d\phi} = 0, \quad (2.22)$$

where

$$\frac{dV(\phi)}{d\phi} \equiv -\frac{2\mathcal{W}\phi^n}{\psi^2} \left\{ \psi (\mathcal{D}_\alpha \phi) (\mathcal{D}^\alpha \phi) + \frac{n\epsilon}{2} \left(\frac{\phi^{*2}}{\phi} \right) + \epsilon \left[\phi^{**} + \phi^* \left(\frac{g^{\mu\nu} g_{\mu\nu}^*}{2} - \frac{\psi^*}{\psi} \right) \right] \right\}. \quad (2.23)$$

It is important to note that, in our herein approach, the dimensional reduction procedure leads us to a differential equation to get the induced scalar potential. We should emphasize that, in most of the conventional scalar-tensor theories, such a potential has instead been added by hand to the action, see, e.g., [44, 53].

Finally, we would introduce the corresponding conservation equation as that obtained in the IMT. In this respect, by setting $a \rightarrow \alpha$ and $b \rightarrow D$ in equation (2.2), we obtain

$$G_{\alpha 4}^{(5)} = R_{\alpha 4}^{(5)} = \chi T_{\alpha 4}^{(5)} + \mathcal{W} \phi^n \phi^* (\mathcal{D}_\alpha \phi), \quad (2.24)$$

where, to get the first equality, we have used the metric (2.5). Again, using (2.5) gives [2]

$$G_{\alpha 4}^{(5)} = \psi P^{\beta}_{\alpha;\beta}, \quad (2.25)$$

where $P_{\alpha\beta}$ was defined as

$$P_{\alpha\beta} \equiv \frac{1}{2\psi} \left(g_{\alpha\beta}^* - g_{\alpha\beta} g^{\mu\nu} g_{\mu\nu}^* \right). \quad (2.26)$$

Consequently, using equations (2.24) and (2.25), we get a dynamical equation for $P_{\alpha\beta}$ as

$$\psi P^{\beta}_{\alpha;\beta} = \chi T_{\alpha 4}^{(5)} + \mathcal{W} \phi^n \phi^* (\mathcal{D}_\alpha \phi). \quad (2.27)$$

As we close this section, let us further elucidate a few concepts of the herein retrieved four-dimensional MSBT.

- The field equations associated to the five-dimensional SB theory, i.e., (2.2) and (2.3), with the metric (2.5), yield four sets of the effective field equations (2.15), (2.17), (2.22) and (2.27) on the hypersurface. Indeed, from the point of view of an observer in the four-dimensional space-time, who is not aware of the existence of the fifth dimension, equations (2.17) and (2.22) could be interpreted as the field equations associated to conventional SB theory in four dimensions. Such a description can be considered as a consequence of the generalized versions of the CM theorem [26–28]. More precisely, the field equations (2.17) and (2.22) could be obtained from the action

$$\mathcal{S}^{(4)} = \int d^4 x \sqrt{-g} \left[R^{(4)} - \mathcal{W} \phi^n g^{\alpha\beta} (\mathcal{D}_\alpha \phi) (\mathcal{D}_\beta \phi) - V(\phi) + \chi L_{\text{matt}}^{(4)} \right], \quad (2.28)$$

where

$$\sqrt{-g} \left(E_{\alpha\beta} + T_{\alpha\beta}^{[\text{SB}]} \right) \equiv 2\delta \left(\sqrt{-g} L_{\text{matt}}^{(4)} \right) / \delta g^{\alpha\beta}. \quad (2.29)$$

Moreover, we should note that equation (2.15) has no analog in the conventional SB theory, and the set of equations (2.27) is an extended version of the conservation law obtained in the IMT.

- It is straightforward to show that the effective EMT is covariantly conserved, i.e., $\mathcal{D}_\beta T_{\alpha}^{[\text{SB}]\beta} = 0$.
- In the particular case when $\phi = \text{constant}$ and $T_{\alpha 4}^{(5)} = 0$, equation (2.27) reduces to $P^{\beta}_{\alpha;\beta} = 0$. Moreover, if l is assumed to be a cyclic coordinate, then (2.27) reduces to an identity.

III. EXACT SOLUTIONS OF SB COSMOLOGY IN FIVE-DIMENSIONAL VACUUM SPACE-TIME

In this section, we shall work in a five-dimensional space-time and obtain exact cosmological solutions. Subsequently, in the next section, by using the MSBT setting, we will derive the effective EMT onto a four-dimensional hypersurface and will analyse the corresponding solutions.

Let us start by assuming a five-dimensional vacuum space-time described by an extended version of the spatially flat FLRW line-element as

$$dS^2 = -dt^2 + a^2(t) [dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2)] + \epsilon\psi^2(t)dl^2, \quad (3.1)$$

where t is the cosmic time and (r, θ, φ) are the spherical coordinates. Due to the space-time symmetries, we assume that a , ψ and ϕ depend only on the comoving time⁶

Therefore, employing the field equation (2.2) and (2.3) for the metric (3.1), in vacuum (please see the footnote 4), we obtain

$$\frac{\ddot{\phi}}{\phi} + \left[3 \left(\frac{\dot{a}}{a} \right) + \frac{n}{2} \left(\frac{\dot{\phi}}{\phi} \right) + \frac{\dot{\psi}}{\psi} \right] \frac{\dot{\phi}}{\phi} = 0, \quad (3.2)$$

$$\frac{\dot{a}}{a} \left(\frac{\dot{a}}{a} + \frac{\dot{\psi}}{\psi} \right) = \frac{\mathcal{W}}{6} \phi^n \dot{\phi}^2, \quad (3.3)$$

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}}{a} \left(\frac{\dot{a}}{a} + \frac{2\dot{\psi}}{\psi} \right) + \frac{\ddot{\psi}}{\psi} = -\frac{\mathcal{W}}{2} \phi^n \dot{\phi}^2, \quad (3.4)$$

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 = -\frac{\mathcal{W}}{6} \phi^n \dot{\phi}^2. \quad (3.5)$$

where “ $\dot{}$ ” denotes the derivative with respect to the cosmic time.

Among the coupled non-linear field equations (3.2)-(3.5), only three of them are independent and we have three unknown quantities $a(t)$, $\phi(t)$ and $\psi(t)$. From equation (3.2), we get a constant of motion as

$$a^3 \dot{\phi} \phi^{\frac{n}{2}} \psi = c_1, \quad (3.6)$$

where c_1 is a constant of integration. Moreover, using (3.3) and (3.4), we get

$$\frac{\mathcal{W}}{6} \phi^n \dot{\phi}^2 - \frac{\dot{a}}{a} \left[\frac{\dot{a}}{a} - 2 \left(\frac{\dot{\psi}}{\psi} \right) \right] + \frac{\ddot{\psi}}{\psi} = 0. \quad (3.7)$$

Now, by employing (3.5) and (3.7) to remove $\frac{\mathcal{W}}{6} \phi^n \dot{\phi}^2$, we obtain

$$3 \left(\frac{\dot{a}}{a} \right) \left(\frac{\dot{\psi}}{\psi} \right) + \frac{\ddot{\psi}}{\psi} = 0. \quad (3.8)$$

From the above equation, we conclude that $a^3 \dot{\psi}$ is another constant of motion. Namely, we have

$$a^3 \dot{\psi} = c_2, \quad (3.9)$$

where c_2 is another constant of integration. Combining (3.6) and (3.9), we obtain

$$\psi = \begin{cases} \psi_0 \exp \left(\frac{2\beta}{n+2} \phi^{\frac{n+2}{2}} \right) & \text{for } n \neq -2, \\ \psi_0 \phi^\beta & \text{for } n = -2, \end{cases} \quad (3.10)$$

⁶ In this work, let us leave aside the modified results which can be retrieved by considering the following cases: (i) taking the scale factors and the scalar field which depend also on the spatial coordinates, specially l , (ii) considering ordinary matter fields in five-dimensional space-time. (By performing the calculations without (ii), the proposal “replacing the base wood of matter by the pure marble of geometry” is satisfied [10].), (iii) Obtaining the solutions associated to the other values of the spatial curvature constant.

where $\beta \equiv \frac{c_2}{c_1}$ and ψ_0 is a constant of integration. By employing (3.10) in (3.3), we get

$$a = \begin{cases} a_0 \exp\left(\frac{2\gamma}{n+2} \phi^{\frac{n+2}{2}}\right) & \text{for } n \neq -2, \\ a_0 \phi^\gamma & \text{for } n = -2, \end{cases} \quad (3.11)$$

where a_0 is an integration constant and γ was defined as

$$\gamma \equiv \frac{1}{2} \left(-\beta \pm \sqrt{\beta^2 + \frac{2\mathcal{W}}{3}} \right). \quad (3.12)$$

By substituting ψ and a from (3.10) and (3.11) into (3.6), we get

$$\begin{cases} \dot{\phi} \phi^{\frac{n}{2}} \exp\left[\frac{2(3\gamma+\beta)}{n+2} \phi^{\frac{n+2}{2}}\right] = \frac{c_1}{a_0^3 \psi_0} & \text{for } n \neq -2, \\ \dot{\phi} \phi^{(3\gamma+\beta)-1} = \frac{c_1}{a_0^3 \psi_0} & \text{for } n = -2. \end{cases} \quad (3.13)$$

Each of the above differential equations can give two different solutions whether the quantity $3\gamma + \beta$ vanishes or not. Integrating, disregarding the values of n , we get two different solutions as presented in the following subsections.

A. Case I: $3\gamma + \beta \neq 0$

For this case, from (3.13), we obtain

$$\begin{cases} \phi^{\frac{n+2}{2}} = \ln \left[\frac{c_1(3\gamma+\beta)}{a_0^3 \psi_0} (t - t_i) \right]^{\frac{n+2}{2(3\gamma+\beta)}} & \text{for } n \neq -2, \\ \phi(t) = \left[\frac{c_1(3\gamma+\beta)}{a_0^3 \psi_0} (t - t_i) \right]^{\frac{1}{3\gamma+\beta}} & \text{for } n = -2, \end{cases} \quad (3.14)$$

where t_i is an integration constant. Consequently, by substituting (3.14) into relations (3.10) and (3.11), we can rewrite a and ψ in terms of the cosmic time as

$$a(t) = a_0 \left[\frac{c_1(3\gamma + \beta)}{a_0^3 \psi_0} (t - t_i) \right]^{\frac{\gamma}{3\gamma+\beta}}, \quad (3.15)$$

$$\psi(t) = \psi_0 \left[\frac{c_1(3\gamma + \beta)}{a_0^3 \psi_0} (t - t_i) \right]^{\frac{\beta}{3\gamma+\beta}}. \quad (3.16)$$

We notice that the above relations do not depend on n .

B. Case II: $3\gamma + \beta = 0$

In this case, using (3.12), we can express γ and \mathcal{W} versus β as

$$\gamma = -\frac{\beta}{3}, \quad \mathcal{W} = -\frac{4}{3}\beta^2. \quad (3.17)$$

Using (3.13), we get

$$\begin{cases} \phi^{\frac{n+2}{2}} = \frac{c_1(n+2)}{2a_0^3 \psi_0} (t - t_i) & \text{for } n \neq -2, \\ \phi(t) = \exp \left[\frac{c_1}{a_0^3 \psi_0} (t - t_i) \right] & \text{for } n = -2. \end{cases} \quad (3.18)$$

Therefore, from relations (3.10), (3.11) and (3.18), we obtain a and ψ , in terms of cosmic time, as

$$a(t) = a_0 \exp \left[-\frac{c_1 \beta}{3a_0^3 \psi_0} (t - t_i) \right], \quad (3.19)$$

$$\psi(t) = \psi_0 \exp \left[\frac{c_1 \beta}{a_0^3 \psi_0} (t - t_i) \right]. \quad (3.20)$$

Again, we should note that, when the scale factors a and ψ are expressed in terms of the cosmic time, likewise Case I, they are independent of n .

IV. REDUCED SAEZ-BALLESTER COSMOLOGY IN FOUR DIMENSIONS

In this section, by employing the MSBT setting outlined in section II, we investigate exact cosmological solutions generated on the four-dimensional hypersurface, which will be compared with corresponding ones obtained from GR, IMT, standard SB theory and the MBDT.

It is straightforward to show that the non-vanishing components of the effective EMT (2.19) associated to the line-element (3.1), on the four-dimensional hypersurface Σ_0 , are given by

$$\chi T_0^{0[\text{SB}]} = -\frac{\ddot{\psi}}{\psi} + \frac{1}{2}V(\phi), \quad (4.1)$$

$$\chi T_i^{i[\text{SB}]} = -\frac{\dot{a}\dot{\psi}}{a\psi} + \frac{1}{2}V(\phi), \quad (4.2)$$

where $i = 1, 2, 3$ (with no sum) and the induced scalar potential is obtained by employing (2.23). As ϕ and ψ only depend on the cosmic time, we get

$$\left. \frac{dV}{d\phi} \right|_{\Sigma_0} = 2\mathcal{W}\phi^n \dot{\phi} \left(\frac{\dot{\psi}}{\psi} \right). \quad (4.3)$$

On the other hand, using (3.10), we get

$$\left. \frac{dV}{d\phi} \right|_{\Sigma_0} = \begin{cases} 2\beta\mathcal{W} \left(\phi^{\frac{n}{2}} \dot{\phi} \right)^2 \phi^{\frac{n}{2}} & \text{for } n \neq -2, \\ 2\beta\mathcal{W} \left(\frac{\dot{\phi}}{\phi} \right)^2 \phi^{-1} & \text{for } n = -2. \end{cases} \quad (4.4)$$

In order to obtain the quantity in the parentheses of relations (4.4), we should use (3.13). However, as we get different solutions for the cases $3\gamma + \beta \neq 0$ and $3\gamma + \beta = 0$, in similarity to the previous section, we prefer to proceed the analysis in two separated subsections as follows.

A. Case I: $3\gamma + \beta \neq 0$

In this case, by employing relations (3.13) into corresponding relations in (4.4), after integrating the resulted equations, we obtain for the induced potential (which we emphasize is not added by hand and solely emerges through the redaction process)

$$V(\phi) = \begin{cases} V_0 \exp \left[\frac{-4(3\gamma+\beta)}{n+2} \phi^{\frac{n+2}{2}} \right] & \text{for } n \neq -2, \\ V_0 \phi^{-2(3\gamma+\beta)} & \text{for } n = -2, \end{cases} \quad (4.5)$$

where V_0 is given by

$$V_0 \equiv -\frac{\mathcal{W}c_1^2\beta}{a_0^6\psi_0^2(3\gamma+\beta)}. \quad (4.6)$$

Finally, by replacing the scalar field from the corresponding relations in (3.14), the induced scalar potential can be given versus the cosmic time as

$$V(t) = -\frac{\mathcal{W}\beta}{(3\gamma + \beta)^3}(t - t_i)^{-2}, \quad (4.7)$$

which is independent of n .

Now, we can calculate the non-vanishing components of the induced matter for this case. By employing relations (3.15), (3.16) and (4.7) into (4.1) and (4.2), it is easy to show that

$$\rho_{\text{SB}} = \frac{\beta [\mathcal{W} - 6\gamma(3\gamma + \beta)]}{2\chi(3\gamma + \beta)^3} (t - t_i)^{-2}, \quad (4.8)$$

$$p_{\text{SB}} = -\frac{\beta [\mathcal{W} + 2\gamma(3\gamma + \beta)]}{2\chi(3\gamma + \beta)^3} (t - t_i)^{-2}, \quad (4.9)$$

where $\rho_{\text{SB}} \equiv T_0^{0[\text{SB}]}$ and $p_{\text{SB}} = p_i \equiv T_i^{i[\text{SB}]}$ are the induced energy density and isotropic pressures, respectively. We see that these expressions, referring to geometrical induced effective matter, are independent of n and yield a effective (induced) barotropic equation of state, associated with such induced perfect fluid for this case. Namely,

$$p_{\text{SB}} = W_{\text{SB}}\rho_{\text{SB}}, \quad \text{where} \quad W_{\text{SB}} = -\frac{\mathcal{W} + 2\gamma(3\gamma + \beta)}{\mathcal{W} - 6\gamma(3\gamma + \beta)}. \quad (4.10)$$

By employing relations (3.12), (4.8) and (4.9), it is easy to show that the induced matter (on the four-dimensional hypersurface) is conserved; namely, $\dot{\rho}_{\text{SB}} + 3H(\rho_{\text{SB}} + p_{\text{SB}}) = 0$.

In the rest of this subsection, we will concentrate on particular cases for the equation of state parameter for $3\gamma + \beta \neq 0$. After obtaining the properties of the reduced quantities as well as geometrical induced matter and scalar potential, we will compare the results with those resulted from the standard theories and observational data.

1. (Effective) dust cosmologies

For a relativistic pressureless fluid, by setting $W_{\text{SB}} = 0$ in (4.10) and using (3.12), we get⁷ $\gamma = -\frac{2\beta}{3}$ and $\mathcal{W} = -\frac{4\beta^2}{3}$. Consequently, employing (3.14), (3.15), (3.16), (4.8), (4.9) and (4.5), we obtain

$$\begin{aligned} ds^2 &= dS^2|_{\Sigma_o} = -dt^2 + a_0^2 \left[\frac{-c_1\beta}{a_0^3\psi_0}(t - t_i) \right]^{\frac{4}{3}} [dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2)], \\ \psi(t) &= \psi_0 \left[\frac{-c_1\beta}{a_0^3\psi_0}(t - t_i) \right]^{-1}, \\ \phi^{\frac{n+2}{2}} &= \ln \left[\frac{-c_1\beta}{a_0^3\psi_0}(t - t_i) \right]^{-\frac{n+2}{2\beta}}, \quad V(\phi) = V_0 \exp \left[\frac{4\beta}{n+2} \phi^{\frac{n+2}{2}} \right] \quad \text{for } n \neq -2, \\ \phi(t) &= \left[\frac{-c_1\beta}{a_0^3\psi_0}(t - t_i) \right]^{-\frac{1}{\beta}}, \quad V(\phi) = V_0 \phi^{2\beta} \quad \text{for } n = -2, \\ V(t) &= -\frac{4}{3}(t - t_i)^{-2}, \\ \rho_{\text{SB}}(t) &= \frac{8}{3\chi}(t - t_i)^{-2}. \end{aligned} \quad (4.11)$$

where V_0 is given by

$$V_0 \equiv -\frac{1}{3} \left(\frac{2\beta c_1}{a_0^3\psi_0} \right)^2. \quad (4.12)$$

⁷ In what follows, for each case, the static cosmological solution with $\gamma = 0$ is the trivial result. We should note that we will not consider this special case in this paper.

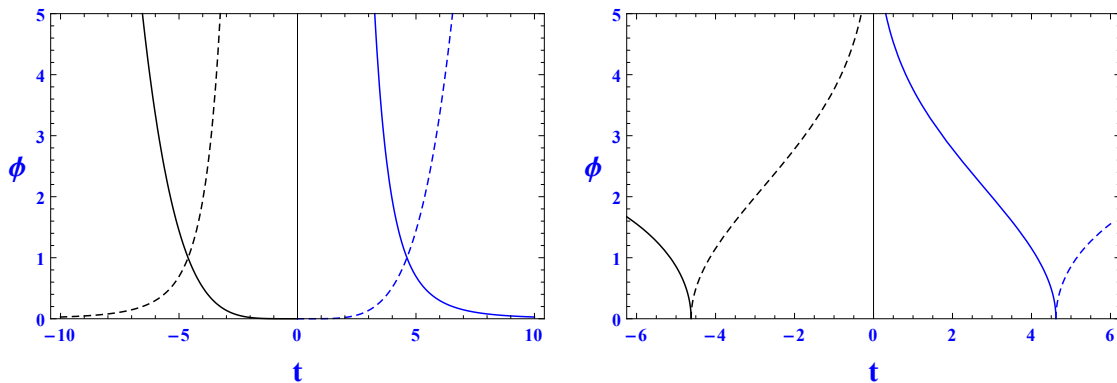


Figure 1: The time behavior of the scalar field for $n = -2$ (left panel) and $n = 2$ (right panel) associated to the dust cosmologies. We have taken $t_i = 0$, $a_0 = \psi_0 = 1$ and $c_1\beta = \pm\sqrt{3}/8$. Such that the upper and lower signs are associated to the curves plotted in the intervals $t < 0$ and $t > 0$, respectively. Moreover, the dashed and solid curves correspond to $c_1 > 0$ and $c_1 < 0$, respectively. .

We observe that for a matter-dominated case, our model yields two disconnected branches⁸

However, for both of the branches, by assuming $a_0^3\psi_0 > 0$, the time behavior of the scalar field depends not only on the sign of the $c_1\beta$ but also on the signs of c_1 and β . We have shown that for different values of n , dependent on the signs of c_1 and β , the scalar field can whether contracts or expands with the cosmic time. For instance, in figure 1, we have plotted the behavior of the scalar field versus cosmic time for $n = \pm 2$.

Let us focus on the branch II in which $t > t_i$. In this case, we have shown that: (i) the scale factor a is always in a decelerating expansion regime, which is inapplicable to the present epoch. (ii) However, the extra dimension and the induced energy density contract with the cosmic time, which are desirable behaviors for induced matter theories. (iii) The time behavior of the scalar field depends not only on the integration constants but also on n . Such that for $n \neq -2$, the cosmic time must be restricted as $t \geq t_i > 0$; namely, the big bang singularity is removed. Whilst, for $n = -2$, the universe has a big bang singularity.

2. (Effective) vacuum cosmologies

One of the crucial questions in studying cosmological scenarios in the context of scalar-tensor theories is whether non-trivial vacuum solutions exist or not. For instance, in the context of the BD theory, such solutions are interesting because when the time asymptotically goes to zero, for wide ranges of the varying BD coupling parameter, the (non-vacuum) FLRW solutions approach the vacuum ones [73]. Moreover, by studying the cosmological non-static vacuum solutions (say, for spatially flat FLRW metric), we can address the validity of the Birkhoff theorem and Mach's principle in the corresponding context.

In what follows, we would study a particular case in which the geometrical induced matter vanishes onto the four-dimensional hypersurface.

In order to investigate the vacuum solution, from relations (4.8) and (4.9), we find that the only allowed solution is obtained by setting $\beta = 0$, such that by employing (3.12), we get $\gamma = \pm\sqrt{W/6}$. Consequently, the solutions (3.14),

⁸ The pre-big bag cosmology [70] has been inspired by superstring theory. Such a scenario has also been constructed by a class of solutions associated to the low-energy string theory, see, e.g., the Nariai solution associated to the dust fluid in the context of the BD theory [44, 71, 72], such that for one of them (branch I), the scale factor contracts when $t < t_i$ and for the other (branch II) it expands for $t > t_i$. Whilst, for the branch I, the fifth dimension expands and for the branch II, it contracts. We should note that the mentioned behaviors directly depend on the integration constants c_1 , β , a_0 and ψ_0 . We have shown that, by assuming $a_0^3\psi_0 > 0$, the sign of the $c_1\beta$ determines which branch is obtained. Namely, $c_1\beta > 0$ yields the branch I, while $c_1\beta < 0$ yields the branch II.

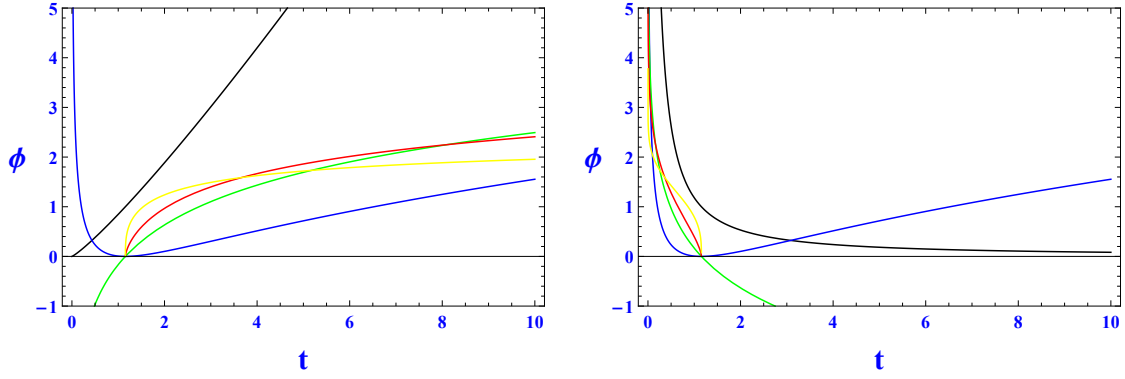


Figure 2: The time behavior of the scalar field for $n = -2$ (black curve), $n = -1$ (blue curve), $n = 0$ (green curve), $n = 1$ (red curve) and $n = 4$ (yellow curve) associated to the vacuum cosmologies. The left and right panels correspond to the positive and negative values of both c_1 and γ , respectively. Moreover, we have taken $t_i = 0$ for all cases.

(3.15), (3.16) and (4.5) reduce to

$$\begin{aligned}
 ds^2 &= dS^2|_{\Sigma_o} = -dt^2 + \left[\frac{3c_1\gamma}{\psi_0}(t - t_i) \right]^{\frac{2}{3}} \left[dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \right], \\
 \psi(t) &= \psi_0 = \text{constant}, \\
 \phi^{\frac{n+2}{2}} &= \ln \left[\frac{3c_1\gamma}{a_0^3\psi_0}(t - t_i) \right]^{\frac{n+2}{6\gamma}} \quad \text{for } n \neq -2, \\
 \phi(t) &= \left[\frac{3c_1\gamma}{a_0^3\psi_0}(t - t_i) \right]^{\frac{1}{3\gamma}} \quad \text{for } n = -2, \\
 V(t) &= 0,
 \end{aligned} \tag{4.13}$$

where, for the canonical (scalar field) case, which is obtained by setting $n = 0$, we get

$$\phi = \ln \left[\frac{3c_1\gamma}{a_0^3\psi_0}(t - t_i) \right]^{\frac{1}{3\gamma}}, \tag{4.14}$$

while the other quantities do not change.

Comparing the relations associated to the scalar fields in (4.11) and (4.13), we find that by setting⁹ $\gamma \mapsto -\tilde{\beta}/3$ in the latter, we get the same functions of the scalar field in the former. Consequently, by choosing suitable initial values, figures 1 can also show the behaviors of the scalar fields associated to the vacuum solution.

Again, we would focus on the time range where $t > t_i$. Similar to the previous case, for $n \neq -2$, the relation associated to the scalar field indicates that the allowed values for the cosmic time is $t \geq t_i > 0$. Therefore, for this case, the relation for the scale factor shows that the universe cannot be started with vanishing size. Namely, for this case, the big bang singularity is removed. Whereas, for the case $n = -2$, the quantity $t - t_i$ can vanish and therefore the universe has a big bang singularity. In figure 2, we have plotted the behaviors of the scalar field versus cosmic time for differed vales of n and of the integration constants. For all values of n , the scale factor of the universe decelerates with cosmic time. We should note that as $a(t) \propto t^{1/3}$, disregarding the scalar field, this solution does not correspond to the general relativistic spatially flat FLRW universe in vacuum, which is Minkowski space; whilst, it is similar to the corresponding one in the context of BD theory, i.e., Ohanlon-Tupper solution, when the BD coupling parameter tends to infinity.

⁹ We should note that the β in this case is not exactly equal to that in the dust case.

3. (Effective) radiation cosmologies

By setting $W_{\text{SB}} = \frac{1}{3}$ in (4.10) and using (3.12), we get $\gamma = -\beta$ and consequently $\mathcal{W} = 0$. Therefore, (2.1) reduces to the Einstein-Hilbert action for GR in five dimensions. Consequently, as we have assumed that there is an ordinary matter in a five-dimensional space-time, all the equations in section II reduce to those associated to the generalized version of IMT. We should note that, to get the correct set of solutions, we can start from equations (3.2)-(3.5) by setting $\phi = \text{constant}$.

For our model, it is straightforward to show that the associated solutions of this effective radiation case are

$$\begin{aligned}
 ds^2 &= dS^2|_{\Sigma_o} = -dt^2 + a_0^2 \left(\frac{-2\beta c_1}{a_0^3 \psi_0} \right) (t - t_i) [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)], \\
 \psi(t) &= \psi_0 \left[\frac{-2\beta c_1}{a_0^3 \psi_0} (t - t_i) \right]^{-\frac{1}{2}}, \\
 \phi(t) &= \phi_0 = \text{constant}, \quad V(\phi) = 0, \\
 \rho_{\text{SB}} &= 3p_{\text{SB}}(t) = \frac{3}{4\chi} (t - t_i)^{-2} \propto \frac{1}{a^4}, \\
 R^{(4)} &= 6(\dot{H} + 2H^2) = 0.
 \end{aligned} \tag{4.15}$$

Namely, the solution corresponds to the vanishing Ricci curvature. Moreover, we should note that the solution is independent of \mathcal{W} and the extra dimension contracts as the cosmic time increases. By comparing this particular solution obtained in our herein modified SB setting with those obtained in the context of the standard BD theory, we find that it is exactly the same as the Nariai solution [44, 71, 72] associated to the radiative fluid for a spatially flat FLRW universe.

4. (Effective) stiff fluid and false vacuum

For the case of stiff fluid, by solving $W_{\text{SB}} = 1$ in (4.10) and using (3.12), we find that the only acceptable solution is $\beta = 0$ which is the same value of the vacuum case already investigated. Namely, in our model, it is impossible to obtain a stiff fluid with $\rho_{\text{SB}} = p_{\text{SB}} \neq 0$.

For the case of false vacuum, by setting $W_{\text{SB}} = -1$, we get either $\gamma = 0$ or $3\gamma + \beta = 0$. The former gives the static universe solution and it is not of interest and the latter will be investigated in the next subsection.

B. Case II: $3\gamma + \beta = 0$

In this case, from (3.13), for all values of n , we obtain

$$\phi^{\frac{n}{2}} \dot{\phi} = \frac{c_1}{a_0^3 \psi_0}. \tag{4.16}$$

In order to get the induced scalar potential, we substitute (4.16) into (4.4), which yields

$$\left. \frac{dV}{d\phi} \right|_{\Sigma_o} = \begin{cases} 2\beta \mathcal{W} \left(\frac{c_1}{a_0^3 \psi_0} \right)^2 \phi^{\frac{n}{2}} & \text{for } n \neq -2, \\ 2\beta \mathcal{W} \left(\frac{c_1}{a_0^3 \psi_0} \right)^2 \phi^{-1} & \text{for } n = -2. \end{cases} \tag{4.17}$$

Integrating equations (4.17) gives

$$V(\phi) = \begin{cases} -\frac{16\beta}{3(n+2)} \left(\frac{c_1 \beta}{a_0^3 \psi_0} \right)^2 \phi^{\frac{n+2}{2}} & \text{for } n \neq -2, \\ -\frac{8\beta}{3} \left(\frac{c_1 \beta}{a_0^3 \psi_0} \right)^2 \ln \phi & \text{for } n = -2, \end{cases} \tag{4.18}$$

where we have used (3.17) and, without loss of generality, we have assumed that the constants of integration vanish. Consequently, by substituting each relation of (3.18) into its corresponding case in (4.18), we get the induced scalar

potential, versus cosmic time, as

$$V(t) = -\frac{8}{3} \left(\frac{c_1 \beta}{a_0^3 \psi_0} \right)^3 (t - t_i), \quad (4.19)$$

which does not depend on n . Now, by employing relations (3.19), (3.20), (4.1), (4.2) and (4.18), the non-vanishing components of the induced matter are given by

$$\rho_{\text{SB}} = \frac{1}{\chi} \left(\frac{c_1 \beta}{a_0^3 \psi_0} \right)^2 \left[\frac{4}{3} \left(\frac{c_1 \beta}{a_0^3 \psi_0} \right) (t - t_i) + 1 \right], \quad (4.20)$$

$$p_{\text{SB}} = \frac{1}{\chi} \left(\frac{c_1 \beta}{a_0^3 \psi_0} \right)^2 \left[-\frac{4}{3} \left(\frac{c_1 \beta}{a_0^3 \psi_0} \right) (t - t_i) + \frac{1}{3} \right], \quad (4.21)$$

where, again, we have used (3.17). By employing (3.12), (4.20) and (4.21), it is easy to show that $\dot{\rho}_{\text{SB}} + 3H(\rho_{\text{SB}} + p_{\text{SB}}) = 0$. Namely, the induced EMT for this case is also conserved.

Moreover, it is straightforward to show that

$$\begin{aligned} ds^2 &= dS^2|_{\Sigma_0} = -dt^2 + a_0^2 \exp \left[\frac{2c_1 \tilde{\beta}}{3a_0^3 \psi_0} (t - t_i) \right] [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)], \\ \psi(t) &= \psi_0 \exp \left[-\frac{c_1 \tilde{\beta}}{a_0^3 \psi_0} (t - t_i) \right], \\ \phi^{\frac{n+2}{2}} &= \frac{c_1 (n+2)}{2a_0^3 \psi_0} (t - t_i) \quad \text{for} \quad n \neq -2, \\ \phi(t) &= \exp \left[\frac{c_1}{a_0^3 \psi_0} (t - t_i) \right] \quad \text{for} \quad n = -2, \\ V(t) &= \frac{8}{3} \left(\frac{c_1 \tilde{\beta}}{a_0^3 \psi_0} \right)^3 (t - t_i), \\ \rho_{\text{SB}} &= \frac{1}{\chi} \left(\frac{c_1 \tilde{\beta}}{a_0^3 \psi_0} \right)^2 \left[-\frac{4}{3} \left(\frac{c_1 \tilde{\beta}}{a_0^3 \psi_0} \right) (t - t_i) + 1 \right], \\ p_{\text{SB}} &= \frac{1}{\chi} \left(\frac{c_1 \tilde{\beta}}{a_0^3 \psi_0} \right)^2 \left[\frac{4}{3} \left(\frac{c_1 \tilde{\beta}}{a_0^3 \psi_0} \right) (t - t_i) + \frac{1}{3} \right], \end{aligned} \quad (4.22)$$

where $\tilde{\beta} \equiv -\beta$. The above relations indicate that at a fixed time as $t = t_i$ the universe commences to expand from a nonsingular value a_0 . Moreover, at that fixed time, the energy density and pressure take constant values as $\rho_{\text{SB}} = 3p_{\text{SB}} = \frac{1}{\chi} \left(\frac{c_1 \tilde{\beta}}{a_0^3 \psi_0} \right)^2$.

In order to describe the evolution of the universe at subsequent times, let us consider a few quantities by looking at the integration constants. As we are interested in the solutions in which the fifth dimension contracts by cosmic time [6], therefore, (3.9) yields $c_2 = -c_1 \tilde{\beta} < 0$. On the other hand, let us assume that both a_0 and ψ_0 take positive values, thus, $\frac{c_1 \tilde{\beta}}{\psi_0 a_0^3}$ should take positive values.

By considering relation (4.20), the above arguments can be in accordance with $\rho_{\text{SB}} > 0$ provided that

$$0 < \frac{4}{3} \left(\frac{c_1 \tilde{\beta}}{a_0^3 \psi_0} \right) \tau_{\text{f}} < 1, \quad (4.23)$$

where $\tau_{\text{f}} \equiv t_{\text{f}} - t_i$ is an arbitrary fixed time and the inequality in the left hand side arises from our assumptions discussed above. The condition (4.23) does not give any constraint for $\frac{c_1 \tilde{\beta}}{a_0^3 \psi_0}$ at early times. However, for large τ_{f} , it indicates that $\frac{c_1 \tilde{\beta}}{a_0^3 \psi_0} \ll 1$. However, as this quantity is a constant and cannot evolve with time, we conclude that it must take very small values at all times. We should note that these considerations have been obtained in the geometrized units where $\chi = 8\pi$.

By employing the above assumptions associated to the integration constants, we conclude that when the cosmic time increases, the induced energy density decreases. Moreover, the scale factor of the universe accelerates exponentially, which is similar to the one obtained in GR for the spatially flat universe, i.e., the de Sitter model. However, there is a crucial difference between our model and the de Sitter model. Namely, in our model we have $\rho_{\text{SB}} + p_{\text{SB}} = \frac{4}{3\chi} \left(\frac{c_1 \beta}{a_3^3 \psi_0} \right)^2 \neq 0$. Moreover, for this case, the scalar field (for $n \neq -2$) and the induced scalar potential increase when time grows. While $\phi(t)$ (only for $n = -2$) and $\psi(t)$ contract with the cosmic time, which is a sought feature in the context of induced-matter theories.

V. SUMMARY AND DISCUSSIONS

In this paper, by employing a dimensional reduction procedure for a five-dimensional SB theory, we have constructed a MSBT in four dimensions. The resulting effective EMT, namely $T_{\mu\nu}^{[\text{SB}]}$, contains three parts: $T_{\mu\nu}^{[\text{IMT}]}$ (which is produced from the fifth dimensional part of the metric) is geometrically induced on a four dimensional hypersurface; $T_{\mu\nu}^{[\phi]}$, which has no analog in the conventional SB theory and depends also on the scalar field and its derivative with respect to the fifth coordinate; finally, we have an induced scalar potential, which contributes to the SB field equations. In the MSBT, as in the GR and the standard SB theory, the effective EMT satisfies a prevalent conservation law. As it is seen in our herein model as well as in [74], the modified construction of some scalar-tensor theories provide simpler and more convenient methods for obtaining comprehensive set of exact cosmological solutions, without taking any *ansatz*, onto the hypersurface, with respect to the corresponding standard models.

It has been believed that the IMT is supported by the CM theorem. Moreover, the CM theorem has been extended to the class of embeddings in which the embedding space-time is an Einstein space-time rather than a Ricci-flat one, see, e.g., [26–28]. In [27], instead of symmetric cases, they have used a general treatment with full Einstein-scalar field system. Also, they have provided general discussions concerning Cauchy as well as initial value problems. These generalized versions of the CM theorem might protect our herein approach in constructing the MSBT setting. Concerning the CM theorem, two points also should be added: (i) it has been claimed that as this theorem is only for the analytic functions, thus it is unsuitable to employ in noncompactified theories [30]. However, it has been argued that as the analytic functions may only be unsuitable to study the topological defects rather than investigating the cosmological models where the manifold is global [40]. (ii) The assumption of locality in the CM theorem was removed and a global extension of it was proved [32].

Let us emphasize a few similarities and differences of the MSBT with respect to the standard four dimensional SB theory. In our herein procedure, the five-dimensional SB field equations (2.2) and (2.3), by taking the metric (2.5), produce four sets of equations (2.15), (2.17), (2.22) and (2.27), such that equations (2.17) and (2.22) reproduce the standard SB field equations, including a scalar potential on a four-dimensional space-time, with an effective energy-momentum source. Moreover, they can equivalently be retrieved from standard SB action including a scalar potential. Such a correspondence is asserted by the extended versions of the CM theorem [26–28]. However, we should note that equation (2.15) has no SB analog, and the set of equations (2.27) is an extended version of a conservation law acquired in the IMT procedure.

As a case study, we have considered a five-dimensional bulk geometry and then constructed the reduced cosmology on the four-dimensional hypersurface. We started from a generalized FLRW line-element and the standard SB theory in a five-dimensional vacuum space-time. We have shown that there are two constants of motion, which properly assisted us to find exact solutions for the field equations. Depending on the integration constants, we have obtained two classes of exact solutions: in the first class (Case I), both the scale factors $a(t)$ and $\psi(t)$ have power-law behaviors, whereas for the second class (Case II), they constitutes exponential functions of the cosmic time. We should note that for both the classes, when the scale factors are written concerning the cosmic time, they do not depend on n , whilst, the scalar field does. For $n \neq -2$, it is in the form of the logarithmic and power-law functions of the cosmic time for Case I and II, respectively. While for $n = -2$, it is a power-law and exponential functions (of cosmic time) for the Cases I and II, respectively.

Subsequently, as an application of the MSBT setting (constructed in section II), we have used the MSBT field equations to deal with the induced (effective) quantities onto the four-dimensional hypersurface. We should note that these quantities have interesting properties, which are presented in what follows. We found that, for both classes of the solutions, the induced scalar potential [which is produced from the geometry (as a fundamental concept) rather than adding it by *ad hoc* assumptions to the action] as well as the components of the induced matter, are in power-law forms (of the cosmic time) and are independent of n . However, the expressions associated to the scalar field depend also on n and are not always power-law functions of the cosmic time. We have shown that the induced matter onto the hypersurface obeys a conventional conservation law (similar to ordinary matter in the standard SB theory and

GR).

For case I, we observed that the formula for the effective matter yield a barotropic equation of state, which is associated with a perfect fluid. We then investigate dust, vacuum, radiation, stiff fluid and false vacuum cosmologies. In what follows, we would present a brief review of the results:

- *Dust solutions:* For a matter-dominated universe, we have shown that there are two disconnected branches occur for the time ranges $t < t_i$ and $t > t_i$; such that for the former range, the scale factor contracts with cosmic time, while for the latter one, it expands. However, the extra dimension does behave differently. For both of the branches, we have studied the time behavior of the quantities with different values of the integration constants and n .

We then focused on the branch II in which $t > t_i$. We have shown that the scale factor of the universe decelerates with cosmic time, which is inapplicable for the present epoch of the universe. Whilst, the induced energy density, the extra dimension and the effective scalar potential contract with the cosmic time. However, to describe the time behavior of the scalar field, we should specify not only the integration constants but also n . We should note that for the case $n \neq -2$, the allowed range of the cosmic time is $t \geq t_i > 0$; namely, our herein model does not have a big bang singularity. Whilst, for $n = -2$ there is a big bang singularity.

- *Vacuum solutions:* For this case, the only possibility is to set $\beta = 0$, i.e., considering $\psi = \text{constant}$, which means the components of the induced matter and the induced scalar potential must vanish. We showed that when $\gamma \mapsto -\tilde{\beta}/3$, features of the scalar field associated to this case reduce to those obtained for the dust case. Namely, by taking suitable initial values, we get similar time behaviors for the corresponding quantities.

We have then studied the quantities correspond to the branch II in which $t > t_i$. This solution, for all values of n , yields a decelerating expansion for the universe. Despite the scale factor, the scalar field depends not only on the integration constants but also on n . We have plotted the behavior of the scalar field concerning cosmic time for different values of n and the integration constants. We have shown that, for the case $n \neq -2$, there is no big bang singularity for the universe, but for $n = -2$, it can start to decelerate from a vanishing size. Disregarding the behavior of the scalar field and considering the big bang singularity, the scale factor behaves similar to the corresponding case in the context of the BD theory when the BD coupling goes to infinity.

- *Radiation solutions:* We found that our results correspond to the particular class of the Nariai solution associated to the radiative fluid for a spatially flat FLRW metric in the context of the BD theory. In this case, the scale factor decelerates, the induced scalar potential and Ricci scalar vanish, the scalar field takes constant values and the induced energy density, pressure and the extra dimension decrease with the cosmic time.
- *Stiff fluid and false vacuum:* For the case I, we found that the only acceptable solution for the stiff fluid is to set $\beta = 0$, which corresponds to the vacuum case already discussed. Moreover, for a false vacuum, the quantity $3\gamma + \beta$ must vanish, which is not permitted in the case I, but, it corresponds to the case II.

For the case II, we found that the scale factor and the components of the induced matter take constant values at an initial time $t = t_i = \text{constant}$. In order to investigate the behavior of the physical quantities at subsequent times, we have discussed the corresponding integration constants. Therefore, when the cosmic time increases, the scale factor of the universe accelerates exponentially, while the induced energy density decreases linearly such that it takes very small values at late times. These results have been obtained by assuming the geometrized units where $\chi = 8\pi$. While increasing the cosmic time, the induced scalar potential increases linearly, while the behavior of the scalar field for $n = -2$ decreases exponentially. Moreover, for $n \neq -2$, it is straightforward to show that it decreases and increases for $n < -2$ and $n > -2$, respectively. However, the fifth dimension always contracts with the cosmic time. Finally, we should note that although the solutions for case II can be considered as de Sitter-like, in our MSBT setting this behavior emerges instead from non-vanishing induced matter where $\rho_{\text{SB}} + p_{\text{SB}} = \frac{4}{3\chi} \left(\frac{c_1 \tilde{\beta}}{a_0^3 \psi_0} \right)^2 \neq 0$.

Finally, we would like to mention a few points regarding our herein MSBT framework and its applications. (i) In this paper, we have constructed the MSBT setting by going onto a four-dimensional hypersurface $\Sigma_l : l = l_0 = \text{constant}$ which is orthogonal to the fifth imension. We can generalize the embedding procedure by taking a dynamical hypersurface, see, e.g., [75]. (ii) Instead of five-dimensional embedding space-time, we can start from an arbitrary dimensional space-time, see, e.g., [3, 76]. Therefore, it will be possible to discuss concerning lower-dimensional gravity settings. (iii) As the equations (2.15) and (2.27) lie outside the conventional SB theory, studying them may produce interesting consequences. (iv) We have seen that the sign of the four-dimensional Ricci curvature [see Equation (2.16)] as well as all the induced quantities on the hypersurface depend on the signature of the five-dimensional line-element which can also be taken as two-time metric. We should note that applying the modified versions of the IMT-scalar-tensor settings (for instance, MBDT as well as MSBT) for two-time metrics may produce more interesting and generalized results than those obtained in the context of the IMT framework.

Acknowledgments

We thank anonymous referees for their beneficial points regarding the embedding theorems and the two-time physics. SMMR appreciates the support of the grant SFRH/BPD/82479/2011 by the Portuguese Agency Fundação para a Ciência e Tecnologia. This research is supported by the grant PEst-OE/MAT/UI0212/2014.

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