Extended anisotropic models in noncompact Kaluza-Klein theory

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In this paper, new exact solutions for locally rotational symmetric (LRS) space-times are obtained within the modified Brans-Dicke theory (MBDT) [1]. Specifically, extended five-dimensional (5D) versions of Kantowski-Sachs, LRS Bianchi type I and Bianchi type III are investigated in the context of the standard Brans-Dicke theory. We subsequently extract their corresponding dynamics on a 4D hypersurface. Our results are discussed regarding others obtained in the standard Brans-Dicke theory, induced-matter theory and general relativity. Moreover, we comment on the evolution of the scale factor of the extra spatial dimension, which is of interest in Kaluza-Klein frameworks.

I. INTRODUCTION

Several significant results have motivated researchers to consider scalar-tensor theories (instead of general relativity) in noncompact Kaluza-Klein frameworks, e.g., space-time matter theory [2–5], in order to establish modified theories [1, 6–10]. In induced-matter theory, by starting from pure geometrical $5D$ field equations in vacuum, induced (effective) matter is obtained on a $4D$ space time, as a direct consequence of the extra dimensions, in accordance with embedding theorems [11–14]. Moreover, there are corresponding reduced cosmologies which can bear a reasonable agreement with specific cosmological observables [15–18]. Different cosmological models are present in the literature [1, 9, 10, 19–22], whose (effective) matter sources emerge from the geometry of higher dimensions. Such contribution not only can yield implications similar to ordinary matter sources but it can also play the role of dark energy or dark matter in the universe [9, 10, 23, 24].

Within MBDT [1] (as well as other noncompact Kaluza-Klein models [25]) a significant advantage is achieved. For instance, to obtain an accelerating scale factor, a scalar potential does not need to be added to the action by hand [26], but instead, an induced potential is dictated from the intrinsic geometry [1, 10, 25] (recently, it has been shown that appropriate kinetic inflation can be obtained in the absence of a cosmological constant, ordinary matter and scalar potential [27–31]). An alternative approach was proposed in [32]. Concretely, the dynamical space-time theory [33] has been applied to a particular case of Kaluza-Klein cosmology, associated with a torus space (for a detailed study of this model, see [34]), and a mechanism of inflation has been analyzed therein.

In this paper, we are interested in applying the MBDT in obtaining anisotropic cosmological solutions on a hypersurface. Let us note that without any restricting conditions (for instance, a cylinder condition on the extra coordinate and/or a higher dimensional matter hypothesis [7, 8]), it has been shown that the MBDT [1] possesses four sets of field equations established on a D-dimensional hypersurface orthogonal to the extra dimension: two sets correspond to those of the standard Brans-Dicke (BD) theory [35], including a self interacting scalar potential; one set can be considered as an extended conservation law introduced in the induced-matter theory [2]; the other one has no counterpart in either standard BD theory or in the induced-matter theory. Furthermore, the effective matter as well as the induced scalar potential introduced in [1] emerge entirely from the geometry of the extra dimension.

A generalized Friedmann-Robertson-Walker (FRW) universe for the three values of the spatial curvature index has been investigated in [23]. However, our universe at very early times may not have been so completely uniform. In this respect, a generalized Bianchi type I anisotropic universe has also been studied in the context of the induced-matter

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theory and MBDT [36–39]. Therefore, the purpose of our herein work is to obtain new exact solutions for the field equations of the BD cosmology, in which the universe is described by either of three different anisotropic space times. More precisely, we will consider the extended versions of Kantowski-Sachs [40, 41], LRS Bianchi type I and Bianchi type III line elements in vacuum. Subsequently, by applying the MBDT, the corresponding reduced cosmology will be analysed on a $4D$ hypersuface. Our new solutions will be compared with some produced from standard scalar-tensor theories as well as from general relativity.

Our research also conveys another innovative feature to obtain exact cosmological solutions (which play a pivotal role in cosmology), providing insights into the quantitative as well as qualitative behaviour of our universe. Specifically, one of the advantages of the modified induced-matter models (based on the scalar-tensor theories [1, 10, 37]) is that the methodology may assist in solving the field equations more easier. Let us be more precise. In the herein framework, we benefit from solving the field equations in the bulk in the absence of an energy momentum tensor. Subsequently, we construct the corresponding dynamics on the hypersurface. Moreover, in contrast to phenomenological frameworks, we do not assume a scalar potential (on the hypersurface) for obtaining the favorable consequences. Instead, an induced scalar potential is dictated from a geometrical reduction procedure.

Our paper is organized as follows. In Section II, we present a brief review of the MBDT. In Section III, we consider a $5D$ vacuum universe, which is described by Kantowski-Sachs, Bianchi type I and Bianchi type III line-elements. By defining a new time coordinate, we find new exact cosmological solutions for the field equations. Moreover, we obtain a set of constraints for the parameters of the models and we investigate particular cases of these solutions. In section IV, we derive the effective energy momentum tensor, induced scalar potential and other physical quantities associated with our herein anisotropic models. Then, we study a corresponding reduced cosmology in greater detail. We show that the conservation law for the resulted induced matter is satisfied identically for all the models. In section V, when it is possible, the solutions are represented in terms of the cosmic time. For the models whose solutions are not feasible to represent in terms of the cosmic time, we restrict ourselves to analyse the consequences in terms of the conformal time. Finally, we present a summary and conclusions in Section VI.

II. MODIFIED BRANS-DICKE THEORY IN FOUR DIMENSIONS

In this section, let us provide a brief review of the MBDT [1]. The $5D$ action of the BD theory in the Jordan frame can be given by [1, 23]

$$\mathcal{S}^{(5)} = \int d^5x \sqrt{\left|\mathcal{G}^{(5)}\right|} \left[\phi R^{(5)} - \frac{\omega}{\phi}\, \mathcal{G}^{ab}\,(\nabla_a\phi)(\nabla_b\phi) + 16\pi\, L_{\text{matt}}^{(5)}\right], \tag{2.1}$$

where ω and ϕ are the BD coupling parameter and BD scalar field, respectively; the Latin indices take values from 0 to 4 and $L_{\text{matt}}^{(5)}$ is the Lagrangian density of the ordinary matter (in five dimensions). The determinant of the $5D$ metric $\mathcal{G}_{ab}$ is denoted by $\mathcal{G}^{(5)}$; $R^{(5)}$ is the Ricci curvature scalar and ∇_a represents the covariant derivative in the $5D$ space-time. Throughout this paper, we use Planck units.

The field equations extracted from the action (2.1) can be written as

$$G_{ab}^{(5)} = \frac{8\pi}{\phi}\, T_{ab}^{(5)} + \frac{\omega}{\phi^2}\left[(\nabla_a\phi)(\nabla_b\phi) - \frac{1}{2}\mathcal{G}_{ab}(\nabla^c\phi)(\nabla_c\phi)\right] + \frac{1}{\phi}\left(\nabla_a\nabla_b\phi - \mathcal{G}_{ab}\nabla^2\phi\right) \tag{2.2}$$

and

$$\nabla^2\phi = \frac{8\pi T^{(5)}}{3\omega + 4}, \tag{2.3}$$

where $\nabla^2 \equiv \nabla_a\nabla^a$ and $T^{(5)} = \mathcal{G}^{ab}T_{ab}^{(5)}$ is the trace of the energy-momentum tensor $T_{ab}^{(5)}$ associated with the ordinary matter fields in a $5D$ space-time.

The field equations of MBDT convey the dynamics on the $4D$ hypersurface [1]. More specifically, we take the line-element [42]

$$dS^2 = \mathcal{G}_{ab}(x^c)dx^a dx^b = g_{\mu\nu}(x^\alpha, l)dx^\mu dx^\nu + \epsilon\psi^2\,(x^\alpha, l)\,dl^2. \tag{2.4}$$

We use the notation $x^\alpha = (x^0, x^1, x^2, x^3)$ for the coordinates in $4D$ space-time and l is the non-compact coordinate associated with the fifth dimension. Moreover, we have $\epsilon = \pm 1$ to indicate if the extra dimension is either time-like or space-like. We are also assuming a specific hypersurface $\Sigma_0(l = l_0 = \text{constant})$, which is orthogonal to the unit

vector $n^a = \delta_4^a/\psi$ (where $n_a n^a = \epsilon$). Among four sets of modified field equations associated with the MBDT, only the following two sets are of interest for our objectives in this paper:

$$G_{\mu\nu}^{(4)} = \frac{8\pi}{\phi}\left(S_{\mu\nu} + T_{\mu\nu}^{[\mathrm{BD}]}\right) + \frac{\omega}{\phi^2}\left[(\mathcal{D}_\mu\phi)(\mathcal{D}_\nu\phi) - \frac{1}{2}g_{\mu\nu}(\mathcal{D}_\alpha\phi)(\mathcal{D}^\alpha\phi)\right]$$

$$+\frac{1}{\phi}\left(\mathcal{D}_\mu\mathcal{D}_\nu\phi - g_{\mu\nu}\mathcal{D}^2\phi\right) - g_{\mu\nu}\frac{V(\phi)}{2\phi} \tag{2.5}$$

and

$$\mathcal{D}^2\phi = \frac{8\pi}{2\omega+3}\left(S + T^{[\mathrm{BD}]}\right) + \frac{1}{2\omega+3}\left[\phi\frac{dV(\phi)}{d\phi} - 2V(\phi)\right]. \tag{2.6}$$

In what follows let us briefly explain the symbols and quantities that appear in equations (2.5) and (2.6); for detailed review of the MBDT, see [1]. In these equations, $\mathcal{D}_\alpha$ denotes the covariant derivative on a $4D$ hypersurface and $\mathcal{D}^2 \equiv \mathcal{D}_\alpha\mathcal{D}^\alpha$. In (2.5) we have defined

$$S_{\mu\nu} \equiv T_{\mu\nu}^{(5)} - g_{\mu\nu}\left[\frac{(\omega+1)T^{(5)}}{3\omega+4} - \frac{\epsilon T_{44}^{(5)}}{\psi^2}\right], \tag{2.7}$$

constituting the effective part of ordinary matter that can be assumed in $5D$ bulk. In addition, we also introduced

$$T_{\mu\nu}^{[\mathrm{BD}]} \equiv T_{\mu\nu}^{[\mathrm{IMT}]} + T_{\mu\nu}^{[\phi]} + \frac{1}{16\pi}g_{\mu\nu}V(\phi), \tag{2.8}$$

which is an induced geometrically energy momentum tensor, as is composed of the following parts

$$\frac{8\pi}{\phi}T_{\mu\nu}^{[\mathrm{IMT}]} \equiv \frac{\mathcal{D}_\mu\mathcal{D}_\nu\psi}{\psi} - \frac{\epsilon}{2\psi^2}\left(\frac{\overset{*}{\psi}\overset{*}{g}_{\mu\nu}}{\psi} - \overset{**}{g}_{\mu\nu} + g^{\lambda\alpha}\overset{*}{g}_{\mu\lambda}\overset{*}{g}_{\nu\alpha} - \frac{1}{2}g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\overset{*}{g}_{\mu\nu}\right)$$

$$-\frac{\epsilon g_{\mu\nu}}{8\psi^2}\left[\overset{*}{g}{}^{\alpha\beta}\overset{*}{g}_{\alpha\beta} + \left(g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right)^2\right], \tag{2.9}$$

$$\frac{8\pi}{\phi}T_{\mu\nu}^{[\phi]} \equiv \frac{\epsilon\overset{*}{\phi}}{2\psi^2\phi}\left[\overset{*}{g}_{\mu\nu} + g_{\mu\nu}\left(\frac{\omega\overset{*}{\phi}}{\phi} - g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right)\right]. \tag{2.10}$$

Moreover, the notation $\overset{*}{A}$ has been used to denote the derivative of a quantity A with respect to the fifth coordinate, l. Finally, the induced scalar potential $V(\phi)$ is derived from the following differential equation (see Ref. [1])

$$\phi\frac{dV(\phi)}{d\phi} \equiv -2(\omega+1)\left[\frac{(\mathcal{D}_\alpha\psi)(\mathcal{D}^\alpha\phi)}{\psi} + \frac{\epsilon}{\psi^2}\left(\overset{*}{\phi} - \frac{\overset{*}{\psi}\overset{*}{\phi}}{\psi}\right)\right] - \frac{2\epsilon\omega\overset{*}{\phi}}{2\psi^2}\left(\frac{\overset{*}{\phi}}{\phi} + g^{\mu\nu}\overset{*}{g}_{\mu\nu}\right) \tag{2.11}$$

$$+\frac{\epsilon\phi}{4\psi^2}\left[\overset{*}{g}{}^{\alpha\beta}\overset{*}{g}_{\alpha\beta} + \left(g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right)^2\right] + 16\pi\left[\frac{(\omega+1)T^{(5)}}{3\omega+4} - \frac{\epsilon T_{44}^{(5)}}{\psi^2}\right].$$

Let us close this section by mentioning a few points: (i) in general, the BD scalar field is not a constant and therefore $T_{\mu\nu}^{[\mathrm{IMT}]}$ is a generalized version of the corresponding quantity introduced in [2, 3]; (ii) $T_{\mu\nu}^{[\phi]}$ is constructed from ϕ and its derivative with respect to l and it has no analogue in the induced-matter theory; (iii) the scalar potential emerges solely from the geometry of the fifth dimension, rather than considering any ad hoc phenomenological assumptions [26]. (iv) equations (2.5) and (2.6) can be derived from the following action: $\mathcal{S}^{(4)} = \int d^4x\sqrt{-g}\left[\phi R^{(4)} - \frac{\omega}{\phi}g^{\alpha\beta}(\mathcal{D}_\alpha\phi)(\mathcal{D}_\beta\phi) - V(\phi) + 16\pi L_{\mathrm{matt}}^{(4)}\right]$, where $\sqrt{-g}\left(S_{\alpha\beta} + T_{\alpha\beta}^{[\mathrm{BD}]}\right) \equiv 2\delta\left(\sqrt{-g}L_{\mathrm{matt}}^{(4)}\right)/\delta g^{\alpha\beta}$ and $T_{\alpha\beta}^{[\mathrm{BD}]}$ and $S_{\alpha\beta}$ are covariantly conserved.

In section IV, we will employ the above geometrical description to establish the reduced cosmological models associated with the Kantowski-Sachs, LRS Bianchi type I and Bianchi type III metrics. Moreover, we will compare our herein results with others obtained instead in the context of the standard BD theory, induced-matter theory and general relativity.

III. EXACT BRANS-DICKE ANISOTROPIC VACUUM SOLUTIONS IN A FIVE-DIMENSIONAL SPACE-TIME

In this section, we will use the $5D$ field equations to analyse a vacuum (i.e., $T_{ab}^{(5)} = 0$) universe described by the extended versions of the spatially homogeneous and anisotropic Kantowski-Sachs, LRS Bianchi type I and Bianchi type III space times. We solve the equations analytically and obtain exact solutions in the bulk. We assume the line element as [40, 41]

$$dS^2 = -dt^2 + a^2(t)dr^2 + b^2(t)d\Omega_\zeta^2 + \epsilon\psi^2(t)dl^2, \tag{3.1}$$

where the angular metric is given by

$$d\Omega_\zeta^2 = d\theta^2 + f^2(\theta)d\phi^2,$$

$$f(\theta) = \begin{cases} sin\theta, & \zeta = +1 & (\text{Kantowski} - \text{Sachs}), \\\\ \theta, & \zeta = 0 & (\text{LRS Bianchi type I}), \\\\ sinh\theta, & \zeta = -1 & (\text{LRS Bianchi type III}), \end{cases} \tag{3.2}$$

whit t being the cosmic time; $a(t)$, $b(t)$ and $\psi(t)$ are cosmological scale factors.

From (2.2), (2.3) and the line-element (3.1), we obtain the equations of motion associated with all the three curvatures as

$$\frac{\ddot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi}\left(\frac{\dot{a}}{a} + \frac{2\dot{b}}{b} + \frac{\dot{\psi}}{\psi}\right) = 0, \tag{3.3}$$

$$\frac{\dot{b}}{b}\left(\frac{2\dot{a}}{a} + \frac{\dot{b}}{b}\right) + \frac{\dot{\phi}}{\phi}\left[\frac{\dot{a}}{a} + \frac{2\dot{b}}{b} - \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right) + \frac{\dot{\psi}}{\psi}\right] + \frac{\dot{\psi}}{\psi}\left(\frac{\dot{a}}{a} + \frac{2\dot{b}}{b}\right) + \frac{\zeta}{b^2} = 0, \tag{3.4}$$

$$\frac{\ddot{b}}{b} + \frac{\dot{b}}{b}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{\phi}}{\phi}\right) + \frac{1}{2}\left[\frac{\ddot{\psi}}{\psi} + \frac{\dot{\psi}}{\psi}\left(\frac{\dot{a}}{a} + \frac{4\dot{b}}{b} + \frac{\dot{\phi}}{\phi}\right)\right] + \frac{\zeta}{b^2} = 0, \tag{3.5}$$

$$\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\left(\frac{2\dot{b}}{b} + \frac{\dot{\phi}}{\phi}\right) + \frac{1}{2}\left[\frac{\ddot{\psi}}{\psi} + \frac{\dot{\psi}}{\psi}\left(\frac{3\dot{a}}{a} + \frac{2\dot{b}}{b} + \frac{\dot{\phi}}{\phi}\right)\right] = 0, \tag{3.6}$$

$$\frac{\ddot{a}}{a} + \frac{2\ddot{b}}{b} + \frac{\dot{b}}{b}\left(\frac{2\dot{a}}{a} + \frac{\dot{b}}{b}\right) + \frac{\dot{\phi}}{\phi}\left[\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right) - \frac{\dot{\psi}}{\psi}\right] + \frac{\zeta}{b^2} = 0, \tag{3.7}$$

where an overdot represents a derivative with respect to the cosmic time. Since the metrics are spatially homogeneous, we have taken the BD scalar field depending only on the cosmic time.

Concerning field equations (3.3)-(3.7), we should note that there are four unknowns a, b, ϕ and ψ, with five coupled non-linear field equations which are not independent. To obtain exact solutions we introduce a new time coordinate,η, which is related to the cosmic time t as (notwithstanding the specific power-law assumption taken for the Bianchi type I model in [39], we will show that using the following transformation yields more generalized set of solutions; such a coordinate transformation has also been used in [43])

$$dt = bd\eta. \tag{3.8}$$

Consequently, equations (3.3)-(3.7) in terms of the new time coordinate η can be rewritten as

$$ab\psi\phi' = c_1, \tag{3.9}$$

$$Z'' + \zeta Z = 0, \qquad \text{where} \qquad Z \equiv ab\phi\psi, \tag{3.10}$$

$$(XZ)' + 2\zeta Z = 0, \qquad \text{where} \qquad X \equiv [\ln(ab^2)]', \tag{3.11}$$

$$[\ln(YZ)]' = 0, \qquad \text{where} \qquad Y \equiv [\ln(a\psi^{\frac{1}{2}})]', \tag{3.12}$$

$$\frac{b'}{b}\left(\frac{2a'}{a} + \frac{b'}{b}\right) + \frac{\phi'}{\phi}\left[\frac{\psi'}{\psi} - \frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)\right] + \left(\frac{a'}{a} + \frac{2b'}{b}\right)\left(\frac{\phi'}{\phi} + \frac{\psi'}{\psi}\right) + \zeta = 0, \tag{3.13}$$

with c_1 being an integration constant and a prime represents $d/d\eta$.

From (3.10), we get $Z(\eta) = Z_0 f(\eta + \eta_0)$ where $Z_0 \neq 0$ is an integration constant. Without loss of generality, we can set $\eta_0 = 0$. It is straightforward to retrieve the exact solutions for the field equations:

- $\zeta = -1$:

$$a(\eta) = a_0\left[tanh\left(\frac{\eta}{2}\right)\right]^{m_1}, \qquad b(\eta) = b_0 sinh\eta\left[tanh\left(\frac{\eta}{2}\right)\right]^{m_2}, \tag{3.14}$$

$$\phi(\eta) = \phi_0\left[tanh\left(\frac{\eta}{2}\right)\right]^{m_3}, \qquad \psi(\eta) = \psi_0\left[tanh\left(\frac{\eta}{2}\right)\right]^{m_4}, \tag{3.15}$$

- $\zeta = +1$:

$$a(\eta) = a_0\left[tan\left(\frac{\eta}{2}\right)\right]^{m_1}, \qquad b(\eta) = b_0 sin\eta\left[tan\left(\frac{\eta}{2}\right)\right]^{m_2}, \tag{3.16}$$

$$\phi(\eta) = \phi_0\left[tan\left(\frac{\eta}{2}\right)\right]^{m_3}, \qquad \psi(\eta) = \psi_0\left[tan\left(\frac{\eta}{2}\right)\right]^{m_4}, \tag{3.17}$$

- $\zeta = 0$:

$$a(\eta) = a_0\eta^{n_1}, \qquad b(\eta) = b_0\eta^{n_2}, \qquad \phi(\eta) = \phi_0\eta^{n_3}, \qquad \psi(\eta) = \psi_0\eta^{n_4}, \tag{3.18}$$

where m_i and n_i $(i = 1, 2, 3, 4)$ are given by

$$m_1 \equiv \frac{2}{3}(2\alpha + \beta), \qquad m_2 \equiv -\frac{1}{3}(2\alpha + \beta), \tag{3.19}$$

$$m_3 \equiv \beta, \qquad m_4 \equiv -\frac{2}{3}(\alpha + 2\beta), \tag{3.20}$$

$$n_1 \equiv \frac{2}{3}\left[2\alpha + \beta - \frac{1}{2}(\gamma + 3)\right], \qquad n_2 \equiv -\frac{1}{3}(2\alpha + \beta + \gamma), \tag{3.21}$$

$$n_3 \equiv \beta, \qquad n_4 \equiv -\frac{2}{3}[\alpha + 2\beta - (\gamma + 3)], \tag{3.22}$$

with a_0, b_0, ϕ_0, ψ_0, α, γ and $\beta \equiv \frac{c}{Z_0}$ constituting integration constants or parameters. We should note that the integration constants are related as $a_0 b_0 \phi_0 \psi_0 = Z_0$, which is valid for the models (3.2). However, relating the parameters α, β (and γ) for $\zeta = 0$ does differ from the other two models. More concretely, for $\zeta = \pm 1$, from equation (3.13), we get

$$4\alpha^2 + 6\alpha\beta + \left(\frac{3\omega}{2} + 5\right)\beta^2 - 3 = 0, \tag{3.23}$$
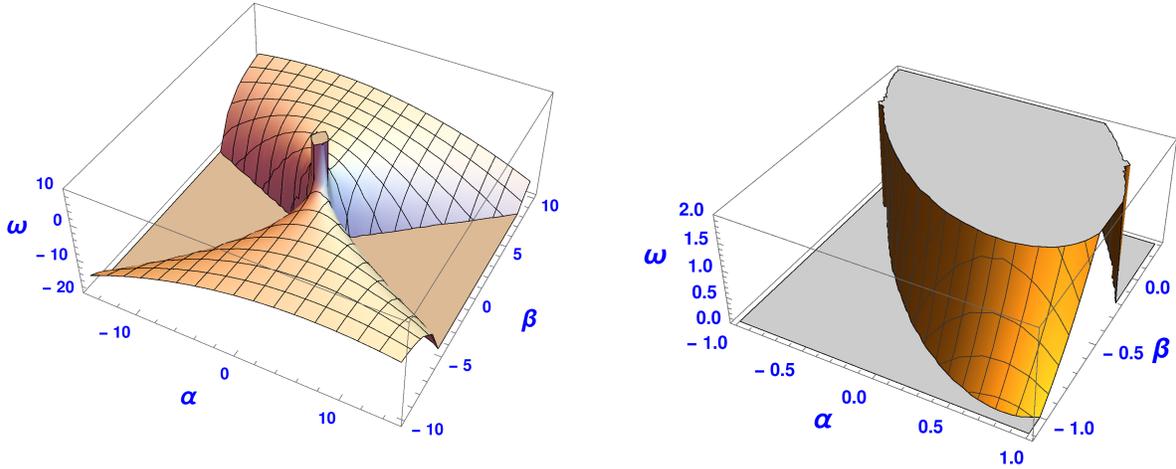
Figure 1: The allowed range of ω (associated with $\zeta = \pm 1$) in terms of α and β. The right panel shows that for very restricted ranges of α and β, the BD coupling parameter can take positive values.

or

$$\omega = -\frac{2(4\alpha^2 + 6\alpha\beta + 5\beta^2 - 3)}{3\beta^2}. \tag{3.24}$$

Whereas, for $\zeta = 0$, it is instead

$$-8\alpha^2 + 4\alpha(3 - 3\beta + \gamma) + 6\beta(3 + \gamma) - 2[6 + \gamma(6 + \gamma)] - \beta^2(3\omega + 10) = 0, \tag{3.25}$$

which can be rewritten as

$$\omega = \frac{-2}{3\beta^2}\left[4\alpha^2 + 2\alpha(3\beta - \gamma - 3) + 5\beta^2 - 3\beta(\gamma + 3) + \gamma^2 + 6\gamma + 6\right], \tag{3.26}$$

where we made care of equation (3.13).

Furthermore, in analogy with the Kasner relations in general relativity, it is straightforward to show that there are constraints which relate the powers present in (3.14)-(3.18)

$$\sum_{i=1}^{4} m_i = 0, \qquad \sum_{i=1}^{4} m_i^2 = 2 - \omega m_3^2, \tag{3.27}$$

$$\sum_{i=1}^{4} n_i = 1, \qquad \sum_{i=1}^{4} n_i^2 = 1 + 2n_2 - \omega n_3^2, \tag{3.28}$$

where we have imported (3.24) and (3.26).

We see that for the solutions associated with $\zeta = \pm 1$, we have just two independent parameters and the third one is constrained by relation (3.23). It is worthwhile to plot ω in terms of α and β for $\zeta = \pm 1$, see figure 1. As seen from the right panel, for restricted intervals of α and β, it is possible to get positive values for ω. Concerning the allowed range of the BD coupling parameter for the Bianchi type I, we will discuss it with the deparametrized solutions in section V.

We should mention a few particular cases: (i) For all three models, when β goes to zero, then $|\omega|$ tends to infinity and consequently we get $\phi = \phi_0 =$ constant; therefore the solutions (3.14)-(3.17) may reduce to those derived in a $5D$ vacuum space time in the context of general relativity (we should mention that when ω tends to infinity, the BD solutions reduce (but not always [44–46]) to the corresponding ones in general relativity). (ii) By assuming $\alpha = -2\beta$ and $\alpha + 2\beta = \gamma + 3$, associated with $\zeta = \pm 1$ and $\zeta = 0$, respectively, then $\psi(\eta)$ takes constant values and our solutions may reduce to those obtained in the context of the BD cosmology in a $4D$ vacuum space time. (iii) For $\zeta = 0$ and $\zeta = +1$, when $a = b$, the solutions may reduce to the corresponding ones obtained for the spatially flat and closed FRW universes, respectively, in the context of the BD theory.

IV. EFFECTIVE BRANS-DICKE COSMOLOGIES ON A FOUR DIMENSIONAL HYPERSURFACE

In the present section, by means of the framework reviewed in section II, we will obtain the components of the effective energy momentum tensor and the induced scalar potential. These will then assist us to retrieve the dynamics on the $4D$ hypersuface.

Substituting the components of the metric (3.1) to (2.8), it is straightforward to show that the non-vanishing components, in terms of the comoving time, on the hypersuface, are given by

$$\frac{8\pi}{\phi}T^{0[\text{BD}]}_{0} = -\frac{\ddot{\psi}}{\psi} + \frac{V(\phi)}{2\phi}, \tag{4.1}$$

$$\frac{8\pi}{\phi}T^{1[\text{BD}]}_{1} = -\frac{\dot{a}\dot{\psi}}{a\psi} + \frac{V(\phi)}{2\phi}, \tag{4.2}$$

where the induced scalar potential $V(\phi)$ is obtained from (2.11). Note that when we replace a by b in relation (4.2), then we get $\frac{8\pi}{\phi}T^{2[\text{BD}]}_{2}$ (which is equal to $\frac{8\pi}{\phi}T^{3[\text{BD}]}_{3}$) for any of the models in (3.2).

In terms of new time coordinate η, the components of the induced matter are:

$$\rho(\eta) \equiv -T^{0[\text{BD}]}_{0} = \frac{\phi(\eta)}{8\pi b^2(\eta)}\left(\frac{\psi''}{\psi} - \frac{b'\psi'}{b\psi}\right) - \frac{V(\eta)}{16\pi}, \tag{4.3}$$

$$P_1(\eta) \equiv T^{1[\text{BD}]}_{1} = -\frac{\phi(\eta)}{8\pi b^2(\eta)}\frac{a'}{a}\frac{\psi'}{\psi} + \frac{V(\eta)}{16\pi}, \tag{4.4}$$

$$P_2(\eta) \equiv T^{2[\text{BD}]}_{2} = P_3(\eta) \equiv T^{3[\text{BD}]}_{3} = -\frac{\phi(\eta)}{8\pi b^2(\eta)}\frac{b'}{b}\frac{\psi'}{\psi} + \frac{V(\eta)}{16\pi}. \tag{4.5}$$

Moreover, equation (2.11) becomes

$$\frac{dV(\phi)}{d\phi} = \frac{2(1+\omega)}{b^2(\eta)}\left(\frac{\phi'}{\phi}\right)\left(\frac{\psi'}{\psi}\right). \tag{4.6}$$

A. Effective cosmologies for $\zeta = \pm 1$

In order to obtain the components of the induced energy momentum tensor, let us first calculate the expression for the induced scalar potential. Substituting solutions (3.14)-(3.17) to the equation (2.11) yields the form of the potential:

$$V(\eta) = \frac{V_0}{2}\int du \left(1 + \zeta u^2\right)^4 u^m, \tag{4.7}$$

where

$$m \equiv \frac{1}{3}(4\alpha + 5\beta - 15), \qquad V_0 \equiv -\frac{(1+\omega)(\alpha + 2\beta)\beta^2\phi_0}{12b_0^2}, \tag{4.8}$$

$$u(\eta) \equiv \begin{cases} tanh\left(\frac{\eta}{2}\right) & \text{for} \quad \zeta = -1, \\ tan\left(\frac{\eta}{2}\right) & \text{for} \quad \zeta = +1. \end{cases} \tag{4.9}$$

Integrating the right hand side of (4.7) yields

$$V(\eta) = V_0 u^m \sum_{n=0}^{4}\left\{\zeta^n \binom{4}{n}\frac{u^{2n+1}}{m + (2n+1)}\right\}, \tag{4.10}$$

where, without loss of generality, we have set the integration constant equal to zero.

We should note that, in some particular cases, $V(\eta)$ vanishes: (i) $\alpha = -2\beta$, in which $\psi(\eta)$ takes constant values and therefore all the solutions for $\zeta = \pm 1$ in the previous section as well as this section reduce to their counterparts obtained in the context of the standard BD theory in $4D$ space-time; (ii) $\omega = -1$: for this particular case, there are similarities between the scalar-tensor theories and supergravity [47] (it has been believed that, for this particular value of the BD coupling parameter, the standard BD theory can be considered as a low energy limit of the bosonic string theory [47, 48]); and (iii) $\beta = 0$: in this case, the BD scalar field takes constant values, therefore, our solutions in the previous section reduce to the corresponding Kantowski-Sachs and LRS Bianchi type III cosmological models in a $5D$ space-time obtained in general relativity, and consequently, the solutions of the present section describe the behavior of the quantities for the Kantowski-Sachs and LRS Bianchi type III cosmological models in the context of the induced-matter theory. We will further investigate the case (iii) in this paper.

Finally, substituting exact solutions (3.14)-(3.17) and the induced scalar potential (4.10) to relations (4.3)-(4.5), we obtain the components of the induced matter in terms of η

$$
\begin{aligned}
\rho(\eta) &= \left[(\beta + 2) + (\beta - 2)\zeta u^2\right] T_0 u^m \left(1 + \zeta u^2\right)^3 \\
&+ T_0(1+\omega)\beta^2 u^m \sum_{n=0}^{4} \zeta^n \binom{4}{n} \frac{u^{2n+1}}{m + (2n+1)},
\end{aligned}
\tag{4.11}
$$

$$
P_1(\eta) = T_0 \left\{ \left[\frac{2}{3}(2\alpha + \beta)\right] u \left(1 + \zeta u^2\right)^4 - (1+\omega)\beta^2 \sum_{n=0}^{4} \zeta^n \binom{4}{n} \frac{u^{2n+1}}{m + (2n+1)} \right\} u^m,
\tag{4.12}
$$

$$
\begin{aligned}
P_2(\eta) &= \frac{T_0}{3} \left[(3 - 2\alpha - \beta) - (3 + 2\alpha + \beta)\zeta u^2\right] u^{m+1} \left(1 + \zeta u^2\right)^3 \\
&- T_0(1+\omega)\beta^2 u^m \sum_{n=0}^{4} \zeta^n \binom{4}{n} \frac{u^{2n+1}}{m + (2n+1)},
\end{aligned}
\tag{4.13}
$$

where

$$
T_0 \equiv \frac{\phi_0(\alpha + 2\beta)}{192\pi b_0^2}.
\tag{4.14}
$$

In general, we see that the different components of $T_k^{k[\text{BD}]}$ (where $k = 1, 2$, with no sum) are not equal, therefore, the induced matter cannot be considered as a perfect fluid.

In the MBDT, the induced energy momentum tensor should also be conserved. Its conservation for three cases, in terms of the cosmic time, can be written as

$$
\dot{\rho} + \sum_{i=1}^{3} (\rho + P_i) H_i = 0,
\tag{4.15}
$$

where $H_1 = \dot{a}/a$ and $H_2 = H_3 = \dot{b}/b$ are the directional Hubble parameters. From (3.8), (3.14)-(3.17), it is easy to show that (4.15), in terms of the new time coordinate, is given by

$$
u\rho'(\eta) + \frac{1}{3}(2\alpha + \beta)\left(1 + \zeta u^2\right)\left[P_1(\eta) - P_2(\eta)\right] + \left(1 - \zeta u^2\right)\left[\rho(\eta) + P_2(\eta)\right] = 0,
\tag{4.16}
$$

where $u(\eta)$ is given by (4.9). Substituting (4.11)-(4.13) to equation (4.16) and then using (3.23), it is straightforward to show that the above equality is satisfied for both Kantowski-Sachs and LRS Bianchi type III models.

It is worthwhile to investigate the properties of a few physical quantities, which are important in observational cosmology. For instance, let us define the spatial volume V_s, average Hubble parameter H, mean anisotropy parameter A_h, the deceleration parameter q and the expansions for scalar expansion θ and the shear scalar σ^2, in terms of the

cosmic time, as

$$V_s = A^3(t) = a(t)b^2(t), \quad \theta = 3H = \left(\frac{\dot{a}}{a} + \frac{2\dot{b}}{b}\right),$$

$$A_h = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\Delta H_i}{H}\right)^2, \quad \text{where} \quad \Delta H_i = H_i - H,$$

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = -\frac{A\ddot{A}}{\dot{A}^2},$$

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{3}\left(H_1^2 + H_2^2 - 2H_1H_2\right), \tag{4.17}$$

where $i, j = 1, 2, 3$ and A(t) is mean scale factor of the universe.

Substituting solutions (3.14)-(3.17) in the corresponding relations (4.17), we obtain the above quantities in terms of η:

$$V_s(\eta) = A^3(\eta) = a_0 b_0^2 \left(\frac{2u}{1 + \zeta u^2}\right)^2,$$

$$\theta(\eta) = 3H(\eta) = \left(\frac{1 - u^4}{2b_0 u^2}\right) u^{\frac{1}{3}(2\alpha+\beta)},$$

$$A_h(\eta) = (2\alpha + \beta)\left(\frac{1 + \zeta u^2}{1 - \zeta u^2}\right)\left[\left(\frac{2\alpha + \beta}{2}\right)\left(\frac{1 + \zeta u^2}{1 - \zeta u^2}\right) - 1\right] + \frac{1}{2},$$

$$q(\eta) = \frac{(4 + 2\alpha + \beta)u^4 + 4(1 + \zeta u^2) - (2\alpha + \beta)}{2(1 - \zeta u^2)^2},$$

$$\sigma^2(\eta) = \frac{1}{3b_0^2}\left(\frac{1 + \zeta u^2}{2u}\right)^4\left[(2\alpha + \beta) - \left(\frac{1 - \zeta u^2}{1 + \zeta u^2}\right)\right]^2 u^{\frac{2}{3}(2\alpha+\beta)}. \tag{4.18}$$

As mentioned, when the BD coupling parameter goes to infinity, the BD solutions may reduce to the corresponding counterparts in general relativity. Let us close this subsection by studying this particular case. From (3.24), we see that when β goes to zero, then $|\omega|$ tends to infinity, and consequently, the BD scalar field takes constant values. In this limit, from (3.23), we get

$$\alpha = \pm\frac{\sqrt{3}}{2}. \tag{4.19}$$

Therefore, letting $\beta = 0$ and substituting the values of α from the above relation to the solutions (3.15), (3.16) and (3.17), we obtain

- $\zeta = -1$:

$$a(\eta) = a_0 u^{\pm\frac{2\sqrt{3}}{3}}, \quad b(\eta) = b_0 sinh\eta \, u^{\mp\frac{\sqrt{3}}{3}}, \quad \psi(\eta) = \psi_0 \, u^{\mp\frac{\sqrt{3}}{3}}, \tag{4.20}$$

- $\zeta = +1$:

$$a(\eta) = a_0 u^{\pm\frac{2\sqrt{3}}{3}}, \quad b(\eta) = b_0 sin\eta \, u^{\mp\frac{\sqrt{3}}{3}}, \quad \psi(\eta) = \psi_0 u^{\mp\frac{\sqrt{3}}{3}}, \tag{4.21}$$

where $u = u(\eta)$ was introduced by relations (4.9). It is straightforward to show that solutions (4.20) and (4.21) satisfy equations (3.10)-(3.13). Moreover, for this particular case, assuming $c_1 = 0$, (3.9) yields an identity, $0 = 0$.

In addition, from (4.7), we find that the induced scalar potential vanishes, and therefore, from (4.11)-(4.13), the

components of the induced energy momentum tensor reduce to

$$\rho(\eta) = 2T_0 \left(1 - \zeta u^2\right) \left(1 + \zeta u^2\right)^3 u^{m+1}, \tag{4.22}$$

$$P_1(\eta) = \pm \frac{2\sqrt{3}}{3} T_0 \left(1 + \zeta u^2\right)^4 u^{m+1}, \tag{4.23}$$

$$P_2(\eta) = \frac{T_0}{3} \left[(3 \mp \sqrt{3}) - 3(3 \pm \sqrt{3})\zeta u^2\right] \left(1 + \zeta u^2\right)^3 u^{m+1}, \tag{4.24}$$

where

$$T_0 \equiv \frac{\pm\sqrt{3}\phi_0}{384\pi b_0^2}, \qquad m = \frac{1}{3}(\pm 2\sqrt{3} - 15). \tag{4.25}$$

B. Effective cosmologies for $\zeta = 0$

In what follows, we obtain the corresponding induced scalar potential, the effective matter and the other physical quantities. Using relations (3.18), equation (4.6) can be written as

$$\frac{dV(\phi)}{d\phi} = -\frac{4\beta(1+\omega)(\alpha + 2\beta - \gamma - 3)}{3b_0^2} \left(\frac{\phi}{\phi_0}\right)^{\frac{2\xi}{3\beta}}, \tag{4.26}$$

where, for convenience, we introduced a new parameter

$$\xi \equiv 2\alpha + \beta + \gamma - 3. \tag{4.27}$$

Setting the integration constant equal to zero, (4.26) gives

$$V(\phi) = \begin{cases} -\frac{4\phi_0\beta^2(1+\omega)(\alpha+2\beta-\gamma-3)}{b_0^2(2\xi+3\beta)} \left(\frac{\phi}{\phi_0}\right)^{\frac{2\xi}{3\beta}} & \text{for} \qquad 2\xi + 3\beta \neq 0, \\ \\ -\frac{4\phi_0\beta(1+\omega)(\alpha+2\beta-\gamma-3)}{3b_0^2}\ln\phi & \text{for} \qquad 2\xi + 3\beta = 0. \end{cases} \tag{4.28}$$

In addition, by substituting the BD scalar field from (3.18) to the above relation, the induced scalar potential is written in terms of η

$$V(\eta) = \begin{cases} -\frac{4\phi_0\beta^2(1+\omega)(\alpha+2\beta-\gamma-3)}{b_0^2(2\xi+3\beta)}\eta^{\frac{1}{3}(2\xi+3\beta)} & \text{for} \qquad 2\xi + 3\beta \neq 0, \\ \\ -\frac{4\phi_0\beta(1+\omega)(\alpha+2\beta-\gamma-3)}{3b_0^2}\ln(\phi_0\eta^\beta) & \text{for} \qquad 2\xi + 3\beta = 0. \end{cases} \tag{4.29}$$

As the logarithmic branch of the induced scalar potential leads to an effective matter which is complicated to discuss regarding the energy conditions, let us just investigate the power law solutions. However, for the power-law branch, substituting relations (3.18) and (4.29) to (4.3)-(4.5), the components of the induced energy momentum tensor for $\zeta = 0$ are given by

$$\rho(\eta) = \frac{\phi_0(\alpha + 2\beta - \gamma - 3)}{4\pi b_0^2} \left[\frac{\beta - \gamma - 1}{3} + \frac{\beta^2(1+\omega)}{2\xi + 3\beta}\right] \eta^{\frac{1}{3}(2\xi+3\beta)}, \tag{4.30}$$

$$P_1(\eta) = \frac{\phi_0(\alpha + 2\beta - \gamma - 3)}{4\pi b_0^2} \left[\frac{2\xi + 3(1-\gamma)}{9} - \frac{\beta^2(1+\omega)}{2\xi + 3\beta}\right] \eta^{\frac{1}{3}(2\xi+3\beta)}, \tag{4.31}$$

$$P_2(\eta) = -\frac{\phi_0(\alpha + 2\beta - \gamma - 3)}{4\pi b_0^2} \left[\frac{\xi + 3}{9} + \frac{\beta^2(1+\omega)}{2\xi + 3\beta}\right] \eta^{\frac{1}{3}(2\xi+3\beta)}. \tag{4.32}$$

In order to check the conservation of the induced energy momentum tensor, let us proceed as follows: using (3.18) and the above relations, equation (4.15) is rewritten in terms of the new time coordinate as

$$\eta\rho'(\eta) - (\gamma+1)\Big[\rho(\eta) + P_1(\eta)\Big] + \frac{2}{3}\,(\xi+3)\,\Big[P_1(\eta) - P_2(\eta)\Big] = 0. \tag{4.33}$$

From (4.30)-(4.32) and the constraint (3.25), it is easy to show that (4.33) is satisfied.

For $\zeta = 0$, it is straightforward to show that the physical quantities (4.17), in terms of the new time coordinate are given by

$$V_s(\eta) = a_0 b_0^2 \eta^{-(1+\gamma)},$$

$$\theta(\eta) = 3H(\eta) = -\frac{\gamma+1}{b_0}\eta^{\frac{\xi}{3}},$$

$$A_h(\eta) = 2\left(\frac{2\alpha+\beta-1}{1+\gamma}\right)^2,$$

$$q(\eta) = \frac{2\alpha+\beta-4}{\gamma+1},$$

$$\sigma^2(\eta) = \frac{(2\alpha+\beta-1)^2}{3b_0^2}\eta^{\frac{2\xi}{3}}. \tag{4.34}$$

V. ANALYTIC SOLUTIONS AND COSMIC TIME

As seen from (3.14) and (3.16), for the cases $\zeta = \pm 1$, $b(\eta)$ is a complicated function of η, and thus finding analytical solutions in terms of the cosmic time, t, requires calculating complicated integrals, which even for the special cases are almost impossible. Hence, we shall restrict ourselves to depict the behaviour of quantities in terms of the parametric time, which is related to the cosmic time by equation (3.8). For instance, figure 2 shows the behavior of the scale factors a and b in terms of η. Moreover, we should note that the plot of $a(\eta)$ also depicts the behavior of ϕ and ψ (in terms of η) by setting $2(2\alpha+\beta)/3 \to \beta$ and $2(2\alpha+\beta)/3 \to -2(\alpha+2\beta)/3$, respectively. Furthermore, figure 2 also implies that the behaviors of the scale factors (in terms of η) can be stated in terms of only two parameters; for instance, herein, we took parameters α and β, and remove the BD coupling constant, because the latter is constrained by relation (3.24).

However, for the solutions associated with the LRS Bianchi type I, the de-parametrization procedure is feasible and we will show that our solutions can be considered as extended versions of those obtained, by assuming specific ansatzes, in the context of either the standard BD theory or even the MBDT, see for instance [39]. In what follows, let us investigate the case $\zeta = 0$. Substituting $b(\eta)$ from (3.18) to (3.8) and then integrating both sides of it, we obtain

$$t(\eta) = \begin{cases} -\frac{3b_0}{\xi}\left(\eta^{-\frac{\xi}{3}} - \eta_0^{-\frac{\xi}{3}}\right) & \text{for} \quad \xi \neq 0, \\[2ex] b_0\ln(\frac{\eta}{\eta_0}) & \text{for} \quad \xi = 0, \end{cases} \tag{5.1}$$

where η_0 is an integration constant. Without loss of generality, we can set the integration constant η_0 equal to zero. We also should note that as we have not been investigating the logarithmic induced scalar potential, thus, according to (4.29), the constraint $4\alpha + 5\beta + 2\gamma - 6 = 2\xi + 3\beta \neq 0$ must hold for all solutions.

Therefore, from relations (5.1), we get all the solutions in terms of the cosmic time. In the following subsections, we will show that there are two types of solutions, namely, the power-law and exponential-law solutions. Subsequently, we will investigate the physical properties of each case.

Let us introduce new parameters. Concretely,

$$w_i = \frac{P_i(t)}{\rho(t)}, \tag{5.2}$$

where $\rho(t)$ and $P_i(t)$ are the components of the induced energy momentum tensor in terms of the cosmic time and w_i are the directional equation of state parameters along the axes. Moreover, we set w as the deviation-free equation of state parameter associated with the induced matter. In order to parameterize the deviation from the isotropy, we set $w = w_1$ and then we introduce the skewness parameters as $\delta_j = w_j - w$ (where $j = 2,3$), which indicate the deviation from w along the other two directions. As in our model $P_2 = P_3$, we therefore obtain $\delta_2 = \delta_3 \equiv \delta$.

Using relations (3.18) and (5.1), we obtain power-law and exponential-law solutions in terms of the cosmic time. Let us investigate them in separated subsections.
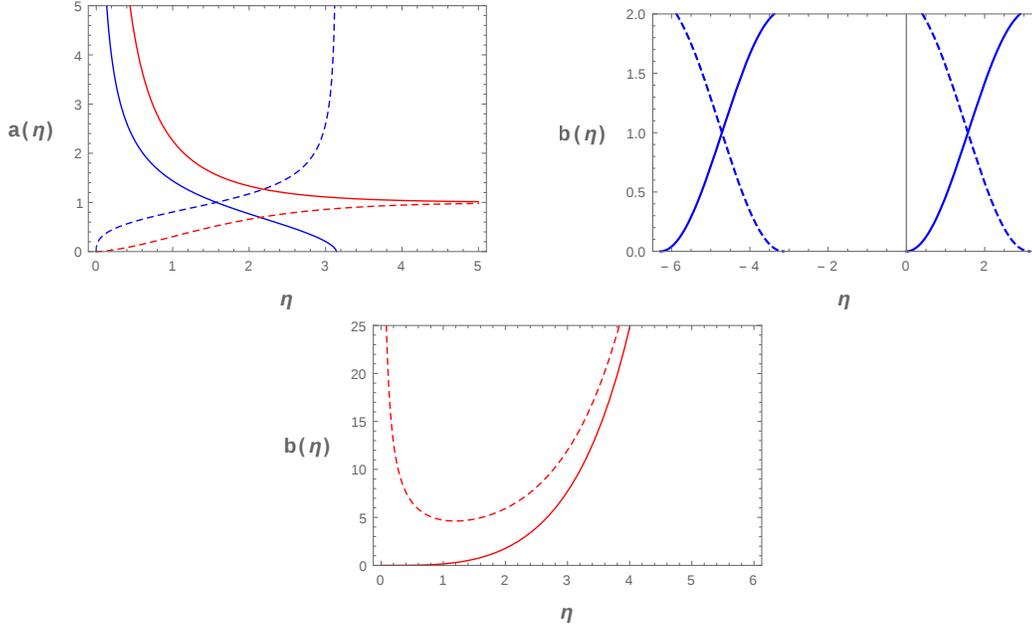
Figure 2: The behavior of the scale factors a and b associated with $\zeta = -1$ (red curves) and $\zeta = +1$ (blue curves) in terms of the parametric time η (which is related to the cosmic time by equation (3.8)). We have set $a_0 = 1 = b_0$, and the dashed and solid curves are for $2\alpha + \beta > 0$ and $2\alpha + \beta < 0$, respectively.

A.   Power law solutions ($\xi \neq 0$)

The solutions which are power-law forms of the cosmic time are given by (applying the specific assumptions, such as *effective pressure* and the particular law of variation of the Hubble parameter, in Ref. [39] yielded more different and more restricted solutions than those we will investigate in this paper)

$$a(t) = a_0 \left(-\frac{\xi t}{3b_0}\right)^{\frac{-2\xi+3(\gamma-1)}{\xi}}, \qquad b(t) = \left(-\frac{\xi t}{3b_0}\right)^{\frac{\xi+3}{\xi}}, \tag{5.3}$$

$$\phi(t) = \phi_0 \left(-\frac{\xi t}{3b_0}\right)^{-\frac{3\beta}{\xi}}, \qquad\qquad \psi(t) = \psi_0 \left(-\frac{\xi t}{3b_0}\right)^{\frac{\xi+3(\beta-\gamma-1)}{\xi}}. \tag{5.4}$$

Moreover, by using (5.1) in relations (4.29)-(4.32), it is easy to show that the induced scalar potential, the effective energy density, the (deviation-free) equation of state and skewness parameters associated with the power-law solutions are written as

$$V(t) = V_p \left(-\frac{\xi t}{3b_0}\right)^{-\frac{2\xi+3\beta}{\xi}}, \qquad V_p \equiv -\frac{4\phi_0\beta^2(1+\omega)(\alpha+2\beta-\gamma-3)}{b_0^2(2\xi+3\beta)}, \tag{5.5}$$

$$\rho(t) = \rho_p \left(-\frac{\xi t}{3b_0}\right)^{-\frac{2\xi+3\beta}{\xi}}, \qquad \rho_p \equiv \frac{\phi_0(\alpha+2\beta-\gamma-3)}{4\pi b_0^2}\left[\frac{\beta-\gamma-1}{3} + \frac{\beta^2(1+\omega)}{2\xi+3\beta}\right], \tag{5.6}$$

$$w_p = \frac{1}{3}\left[\frac{16\alpha^2 + 4\alpha(7\beta+\gamma-9) + \beta^2(1-9\omega) - \beta(\gamma+27) - 2\gamma^2 + 18}{4\alpha(\beta-\gamma-1) + \beta^2(3\omega+8) - \beta(3\gamma+11) - 2(\gamma-3)(\gamma+1)}\right], \tag{5.7}$$

$$\delta_p = -\frac{(2\alpha+\beta-1)(2\xi+3\beta)}{4\alpha(\beta-\gamma-1) + \beta^2(3\omega+8) - \beta(3\gamma+11) - 2(\gamma-3)(\gamma+1)}, \tag{5.8}$$

where w_p and δ_p are the (deviation-free) equation of state and skewness parameters associated with this power-law solution. The solutions (5.3)-(5.8) constitute a new Bianchi type I cosmological dynamics in MBDT.

Relations (5.3), (5.6)-(5.8) and the constraint (3.26) allow to extract the energy momentum tensor conservation law (4.15), identically.

Furthermore, re-employing relations (5.1), the other physical quantities, in terms of the cosmic time, are given by

$$V_s(t) = A(t)^3 = a_0 b_0^2 \left(-\frac{\xi t}{3 b_0} \right)^{\frac{3(1+\gamma)}{\xi}}, \qquad \theta(t) = 3H(t) = \frac{3(\gamma + 1)}{\xi t},$$

$$A_h = 2 \left[\frac{\xi + 3 - (\gamma + 1)}{\gamma + 1} \right]^2, \qquad q = \frac{\xi - (\gamma + 1)}{\gamma + 1},$$

$$\sigma^2 = 3 \left[\frac{\xi + 3 - (\gamma + 1)}{\xi t} \right]^2. \tag{5.9}$$

Let us explain the time behaviors of the physical quantities (by assuming $a_0 > 0$): (i) we observe that for $t = 0$, there is a singularity. For any arbitrary value of b_0, if $\xi(1 + \gamma) > 0$, the spatial volume expands, while for $\xi(1 + \gamma) < 0$, it always contracts; (ii) the Hubble parameter goes to zero when $t \to \infty$; (iii) the shear and expansion scalars diverge at $t = 0$ and they vanish when $t \to \infty$; (iv) from the relation associated with the deceleration parameter, we see that for $\frac{\xi}{\gamma+1} > 1$, the mean scale factor of the universe decelerates, while for $0 < \frac{\xi}{\gamma+1} < 1$, we get an accelerating mean scale factor, which is in accordance with the observational data [49–51]; (v) from relations (5.9), it is seen that $\frac{\sigma^2}{\theta^2} = \frac{1}{3} \frac{\xi+3}{\gamma+1} - 1 = $ constant, which indicates that the model does not approach isotropy when the cosmic time takes large values.

In the rest of this subsection, let us investigate the reduced isotropic cosmological model resulted from the power-law solutions.

We first remind that for all the solutions of this section, we have a constraint as $4\alpha + 5\beta + 2\gamma - 6 = 2\xi + 3\beta \neq 0$. In order to get an isotropic fluid, we should set $\delta_p = 0$, and from relation (5.8), we obtain either $\beta = -\frac{2}{5}(2\alpha + \gamma - 3)$ or $\beta = 1 - 2\alpha$. However, the former value is not acceptable because it is in contradiction with the mentioned constraint. For the latter value of β, from relations (5.3)-(5.9), we get $A_h = 0 = \sigma^2$, as expected. The set of resulted solutions are summarized as

$$ds^2 = -dt^2 + \left[\left(\frac{2 - \gamma}{3 b_0} \right) t \right]^{\frac{2(\gamma+1)}{\gamma-2}} \left(dr^2 + d\Omega^2_{\zeta=0} \right), \tag{5.10}$$

$$\phi(t) = \phi_0 \left[\left(\frac{2 - \gamma}{3 b_0} \right) t \right]^{\frac{6\alpha-3}{\gamma-2}}, \qquad \psi(t) = \psi_0 \left[\left(\frac{2 - \gamma}{3 b_0} \right) t \right]^{-\frac{2(3\alpha+\gamma+1)}{\gamma-2}}, \tag{5.11}$$

$$\rho = \rho_p \left[\left(\frac{2 - \gamma}{3 b_0} \right) t \right]^{\frac{6\alpha-2\gamma+1}{\gamma-2}}, \qquad \rho_p = -\frac{\phi_0(1 + \gamma)(3\alpha + \gamma + 1)(6\alpha + 4\gamma + 1)}{12\pi b_0^2(6\alpha - 2\gamma + 1)}, \tag{5.12}$$

$$w_p = -\frac{6\alpha + \gamma + 4}{3(\gamma + 1)}, \qquad q = -\frac{3}{1 + \gamma}, \tag{5.13}$$

$$\omega = -\frac{2 \left[12\alpha^2 + (4\gamma - 2)\alpha + \gamma^2 + 3\gamma + 2 \right]}{3(1 - 2\alpha)^2}, \tag{5.14}$$

where we have set $a_0 = 1$. It is seen that relation (5.13) is an equation of state of a barotropic fluid.

In a particular case where $\beta = 0$, then, $\alpha = 1/2$, and therefore the BD scalar field takes a constant value and the BD coupling parameter goes to infinity. In this case, in order to find the exact solutions, we should start from the field equations (3.3)-(3.7). It is straightforward to show that they are satisfied whether $\gamma = -1$ or $\gamma = -4$. The former case leads to a static universe which is not of interest in this paper. For $\gamma = -4$, we obtain the unique solution associated with the general relativistic field equations for a $5D$ spatially flat FRW universe in vacuum

$$ds^2 = -dt^2 + \left(\frac{2t}{b_0} \right) \left(dr^2 + d\Omega^2_{\zeta=0} \right),$$

$$\psi(t) = \psi_0 \left(\frac{2t}{b_0} \right)^{-\frac{1}{2}}, \qquad \phi(t) = \phi_0 = \text{constant}. \tag{5.15}$$

In this case, we get a decelerating scale factor for the universe. Moreover, the fifth dimension contracts when the cosmic time increases. Inserting the above relations in (4.1) and (4.2) (the induced scalar potential, without loss of generality, can be assumed equal to zero), we retrieve

$$\rho = \frac{3\phi_0}{32\pi}\frac{1}{t^2}, \qquad\qquad w_p = \frac{1}{3}, \qquad\qquad (5.16)$$

which corresponds to a radiative fluid. These results associated with the power-law solutions for $\beta = 0$ are similar to those obtained in [1]. Moreover, other cases associated with a spatially flat FRW universe, in which the effective matter can play the role of an extended quintessence, radiation and dust, have been widely investigated in MBDT [1, 23].

As another particular case of the isotropic solutions (5.10)-(5.14), we investigate the universe in which the stiff fluid is dominant. To the best of our knowledge this case has not been studied in the context of MBDT. By setting $w_p = 1$ in (5.13), we obtain a relation for α in terms of γ as $\alpha = -(4\gamma+7)/6$ and consequently, β can also be written in terms of γ as $\beta = 2(2\gamma + 5)/3$. Substituting these values of α and β in the relations associated with the isotropic fluid, i.e., (5.10)-(5.14), all the quantities can be written in terms of γ

$$ds^2 = -dt^2 + \left[\left(\frac{2-\gamma}{3b_0}\right)t\right]^{\frac{2(\gamma+1)}{\gamma-2}}\left(dr^2 + d\Omega_{\zeta=0}^2\right), \qquad\qquad (5.17)$$

$$\phi(t) = \phi_0\left[\left(\frac{2-\gamma}{3b_0}\right)t\right]^{-\frac{2(2\gamma+5)}{\gamma-2}}, \qquad\qquad \psi(t) = \psi_0\left[\left(\frac{2-\gamma}{3b_0}\right)t\right]^{\frac{2\gamma+5}{\gamma-2}}, \qquad\qquad (5.18)$$

$$\rho = \frac{\phi_0(2\gamma+5)}{24\pi b_0^2}\left[\left(\frac{2-\gamma}{3b_0}\right)t\right]^{\frac{6(\gamma+1)}{2-\gamma}}, \qquad\qquad (5.19)$$

$$\omega = -\frac{11\gamma^2 + 55\gamma + 62}{2(2\gamma+5)^2}, \qquad\qquad q = -\frac{3}{1+\gamma}. \qquad\qquad (5.20)$$

Let us now study three important particular cases, concerning the stiff fluid:

- When $\gamma = -5/2$ then ω goes to infinity, $\alpha = 1/2$, $\beta = 0$ and the BD scalar field takes constant values. However, as we discussed above equation (5.15), with such values of the parameters α, β and γ, equations (3.3)-(3.7) are not satisfied. This result can be as an obvious evidence to suggest that general relativity is not always recovered from the BD theory, when the BD coupling parameter goes to infinity [44–46].

- With $\omega = -1$, from (5.20) we get either $\gamma = -1$ or $\gamma = -4$. The former leads to a static universe which is not of interest in this work. However, for the latter, we obtain the following solutions

$$ds^2 = -dt^2 + \left(\frac{2t}{b_0}\right)\left(dr^2 + d\Omega_{\zeta=0}^2\right), \qquad\qquad \psi(t) = \psi_0\left(\frac{2t}{b_0}\right)^{\frac{1}{2}}, \qquad\qquad (5.21)$$

$$\phi(t) = \frac{\phi_0 b_0}{2t}, \qquad\qquad \rho(t) = -\frac{b_0\phi_0}{64\pi t^3}, \qquad\qquad (5.22)$$

and $q = 1$, which describes a decelerating universe.

- Letting $\omega = -4/3$, from (5.20), we obtain either $\gamma = 2$ (which is not acceptable) or $\gamma = -7$. The latter yields

$$ds^2 = -dt^2 + \left(\frac{3t}{b_0}\right)^{\frac{4}{3}}\left(dr^2 + d\Omega_{\zeta=0}^2\right), \qquad\qquad \psi(t) = \left(\frac{3\psi_0 t}{b_0}\right), \qquad\qquad (5.23)$$

$$\phi(t) = \frac{\phi_0 b_0^2}{9t^2}, \qquad\qquad \rho(t) = -\frac{\phi_0 b_0^2}{216\pi t^4}, \qquad\qquad (5.24)$$

and $q = 3/4$, which, again, describes a decelerating universe.

B. Exponential-law solutions ($\xi = 0$)

By focusing on the logarithmic branch in (5.1), from relations (3.18), we get new solutions

$$a(t) = a_e \exp\left[\left(\frac{1-\gamma}{b_0}\right)t\right], \qquad\qquad b(t) = b_e \exp\left(-\frac{t}{b_0}\right), \tag{5.25}$$

$$\phi(t) = \phi_e \exp\left(\frac{\beta t}{b_0}\right), \qquad\qquad \psi(t) = \psi_e \exp\left[\left(\frac{1+\gamma-\beta}{b_0}\right)t\right], \tag{5.26}$$

where $a_e \equiv a_0\eta_0^{1-\gamma}$, $b_e \equiv b_0/\eta_0$, $\phi_e \equiv \phi_0\eta_0^\beta$ and $\psi_e \equiv \psi_0\eta_0^{1-\beta+\gamma}$. The induced potential and the effective matter on a $4D$ hypersurface are

$$V(t) = V_e \exp\left(\frac{\beta t}{b_0}\right), \qquad V_e \equiv -\frac{2\phi_0\eta_0^\beta(\omega+1)\beta(\beta-\gamma-1)}{b_0^2}, \tag{5.27}$$

$$\rho(t) = \rho_e \exp\left(\frac{\beta t}{b_0}\right), \qquad \rho_e \equiv \frac{\phi_0\eta_0^\beta(\beta-\gamma-1)\left[\beta(\omega+2)-(\gamma+1)\right]}{8\pi b_0^2}, \tag{5.28}$$

$$w_e = -\frac{\beta(\omega+1)+(\gamma-1)}{\beta(\omega+2)-(\gamma+1)}, \tag{5.29}$$

$$\delta_e = \frac{\gamma-2}{\beta(\omega+2)-(\gamma+1)}. \tag{5.30}$$

Moreover, using the corresponding constraint associated with this case ($\xi = 0$) in (3.26), gives the following BD coupling

$$\omega = -\frac{2\left[\beta^2-\beta(\gamma+1)+\gamma^2+2\right]}{\beta^2}. \tag{5.31}$$

Applying relations (5.25), (5.28), (5.29) and the constraint (5.31), it is straightforward to show that the conservation law for the energy momentum tensor is satisfied identically.

Furthermore, relations (5.1) provide

$$V_s(t) = A(t)^3 = a_0 b_0^2 \exp\left[-\frac{(\gamma+1)t}{b_0}\right],$$

$$\theta = 3H(t) = -\frac{\gamma+1}{b_0}, \qquad\qquad A_h = 2\left(\frac{2-\gamma}{\gamma+1}\right)^2,$$

$$q = -1, \qquad\qquad \sigma^2 = 3\left(\frac{2-\gamma}{3b_0}\right)^2. \tag{5.32}$$

Let us start with focusing on the physical properties of the solutions when the effective matter is an anisotropic fluid. Subsequently, we will discuss on the cases where the induced matter is assumed to be an isotropic fluid.

In order to get an expanding universe, from the relation associated with V_s in (5.32), by assuming that $a_0, \phi_0 > 0$, we get two classes of solutions with (i) $b_0 < 0, \gamma > -1$ and (ii) $b_0 > 0$, $\gamma < -1$. Let us restrict ourselves to the physical solutions. More concretely, we would consider the following assumptions: as the weak energy condition must be satisfied, the induced energy density should decrease with cosmic time; the fifth dimension should be contracted when cosmic time grows [42]. Consequently, for the cases (i) and (ii), we get $0 < \beta < \gamma + 1$ and $\gamma + 1 < \beta < 0$, respectively. For both the cases (i) and (ii), the BD coupling parameter is restricted as $\omega < (\gamma+1-2\beta)/\beta$. According to relation (5.31) and assuming the obtained constraints on β and γ, our numerical analysis show that the BD coupling parameter always takes negative values, see figure 3. We should note that if we do not restrict ourselves to get a contracting fifth dimension, then there is no upper (lower) bound for β as $\gamma + 1$. Moreover, with this assumption, the constraint on the BD coupling constant is replaced by a more generalized one. Therefore, we can obtain a much wider set of (extended) solutions.
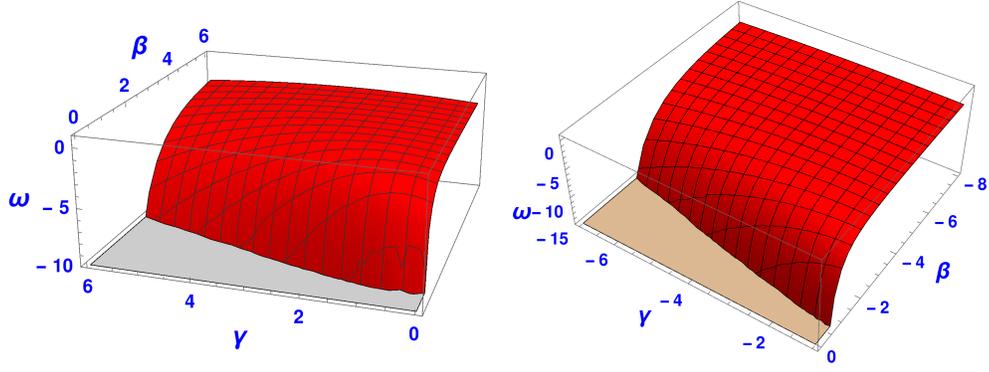
Figure 3:   The left and right panels demonstrate the allowed ranges of ω, associated with the exponential-law solution, in terms of γ and β for the cases (i) and (ii), respectively. These figures show that for the allowed ranges of γ and β, the BD coupling parameter always takes negative values.
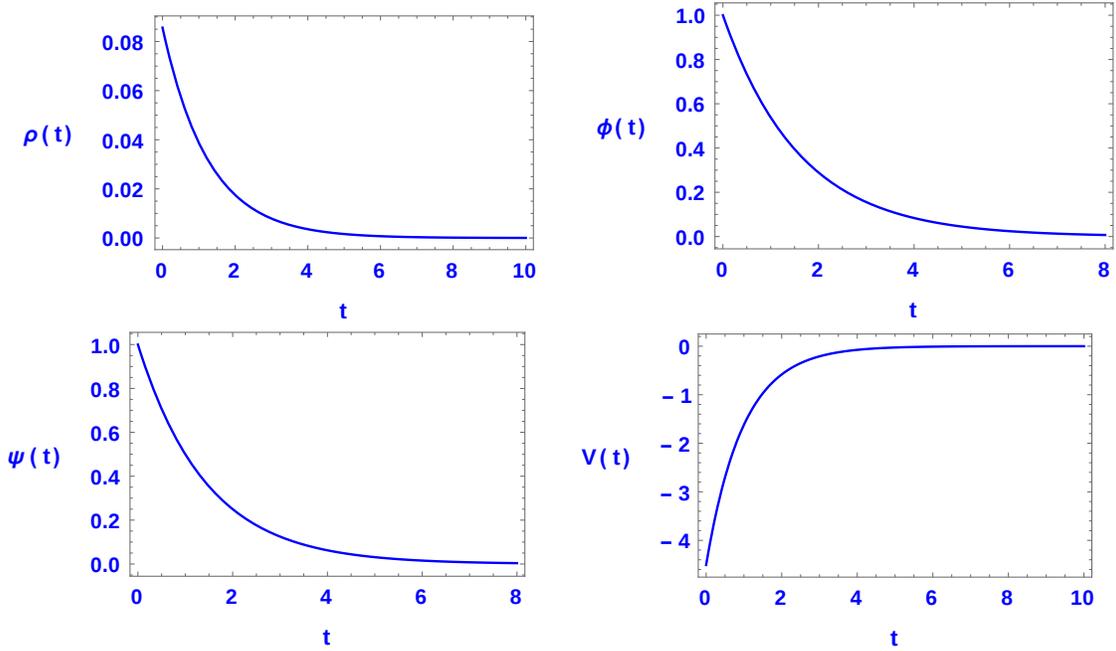
Figure 4:   The time behaviors of the induced energy density, BD scalar field, the scalar field associated with the fifth dimension and induced scalar potential associated with whether the case (i) or case (ii). Upper left panel: $b_0 = -1.43$, $\beta = 1.1362$ and $\gamma = 1.24$ are associated with case (i). Upper right panel: $b_0 = 3.78$, $\beta = -2.33$ are associated with case (ii). Lower left panel: $b_0 = 0.17$, $\beta = -1.15624$ and $\gamma = -2.4$ are associated with case (ii). Lower right panel: $b_0 = -1.8$, $\beta = 1.84204$ and $\gamma = 2.6$ are associated with case (i). We have set other constants, such as ϕ_0, η_0 and ψ_0, equal to one.

Let us summarize the properties of the solutions. The above constraints on the parameters as well as integration constants, for both of the cases (i) and (ii), provide an exponentially expanding universe with the following properties. (i) From relations (5.32), we get $\frac{\sigma^2}{\theta^2} = \frac{1}{3}\left(\frac{2-\gamma}{\gamma+1}\right)^2 =$ constant, which indicates that the model, in general, does not near isotropy when the cosmic time takes large values. (ii) The average Hubble, mean anisotropy, deceleration, scalar and shear scalar parameters always take constant values. (iii) The volume V_s starts its exponential expansion from a nonzero constant. (iv) The induced energy density, the BD scalar field and and the fifth dimension decrease while the cosmic time increases. Moreover, they tend to zero when the cosmic time takes very large values. As a few examples, we have plotted their behaviors in terms of cosmic time in figure 4. (v) The induced scalar potential always increases with cosmic time, and it tends to zero when the cosmic time takes large values.

Regarding the isotropic fluid, by setting $\delta_e = 0$ in (5.30) and using (5.31), we get either $\gamma = 2$ or $\beta = 0$. In what follows, let us discuss their corresponding solutions.

<div align="center">1. $\beta = 0$</div>

In the limit $\beta \to 0$, we find that $|\omega| \to \infty$, and consequently, the BD scalar field takes a constant value. In this case, the field equations (3.3)-(3.7) are satisfied just for $\gamma = \pm i\sqrt{2}$ (where $i^2 = -1$), which is not of interest. This consequence may suggest that the general relativistic solutions are not always recovered from the corresponding BD solutions in the particular case where ω goes to infinity [44–46].

<div align="center">2. $\gamma = 2$</div>

By substituting this value for γ into the solutions associated with the exponential-law, we obtain

$$ds^2 = -dt^2 + \left(\frac{b_0}{\eta_0}\right)^2 \exp\left(2H_0 t\right)\left[dr^2 + d\Omega_{\zeta=0}^2\right], \tag{5.33}$$

$$V(t) = -\frac{2}{b_0^2}[\beta(\beta-3)(1+\omega)]\phi(t), \qquad\qquad \phi(t) = \phi_0\eta_0^\beta \exp\left(-\beta H_0 t\right), \tag{5.34}$$

$$\psi(t) = \psi_0\eta_0^{3-\beta}\exp\left[(\beta-3)H_0 t\right], \qquad\qquad \omega = -\frac{2\left(\beta^2 - 3\beta + 6\right)}{\beta^2}, \tag{5.35}$$

$$\rho(t) = \frac{\phi_0\eta_0^\beta}{8\pi b_0^2}(\beta-3)\left[\beta(\omega+2)-3\right]\exp\left(-\beta H_0 t\right), \qquad w = -\frac{\beta(\omega+1)+1}{\beta(\omega+2)-3}, \qquad\qquad \tag{5.36}$$

where $H_0 \equiv -1/b_0$, $A_h = 0 = \sigma$ and we have assumed $a_0 = b_0$. These results show a homogenous and isotropic spatially flat FRW universe in four dimensions. We should note that, in the context of the BD theory (with or without an ad hoc scalar potential), to the best of our knowledge, this set of solutions seems entirely novel and nobody has obtained them yet. It is seen that an exponentially accelerating universe can be obtained by assuming $\eta_0, b_0 < 0$. Concerning this case, let us focus on a particular case where $\omega = -4/3$ (for other interesting cases as $\omega = -1, 0$, our model does not yield appropriate physical solutions). From (5.36), we get $\beta = 3, 6$. Let us investigate the solutions associated with these particular cases.

- For $\beta = 3$, the fifth dimension takes constant value, the components of the effective matter as well as the induced scalar potential vanish. Therefore, we get a vacuum spatially flat FRW-BD universe

$$ds^2 = -dt^2 + \left(\frac{b_0}{\eta_0}\right)^2 \exp\left(2H_0 t\right)\left[dr^2 + d\Omega_{\zeta=0}^2\right], \qquad \phi(t) = \phi_0\eta_0^3\exp\left(-3H_0 t\right), \tag{5.37}$$

  which is exactly the Ohanlon-Tupper solution [47, 52–54] for $\omega = -4/3$ in the context of the standard BD theory. As claimed in [47], this is the only de Sitter solution associated with the vacuum spatially flat universe in the standard BD theory with vanishing scalar potential.

- For $\beta = 6$, we get $w = 1$ which corresponds to the stiff fluid. Moreover, from relations (5.33)-(5.36), we write

$$ds^2 = -dt^2 + \left(\frac{b_0}{\eta_0}\right)^2 \exp\left(2H_0 t\right)\left[dr^2 + d\Omega_{\zeta=0}^2\right], \tag{5.38}$$

$$V(t) = \frac{12\phi(t)}{b_0^2}, \qquad\qquad \phi(t) = \phi_0\eta_0^6 \exp\left(-6H_0 t\right), \tag{5.39}$$

$$\psi(t) = \frac{\psi_0}{\eta_0^3}\exp\left(3H_0 t\right), \qquad\qquad \rho(t) = \frac{3\phi_0\eta_0^6}{8\pi b_0^2}\exp\left(-6H_0 t\right). \tag{5.40}$$

  For $b_0 < 0$, we get an exponentially accelerating unverse. Moreover, by assuming $\phi_0 > 0$, the BD scalar field, the induced energy density and scalar potential decrease exponentially with cosmic time. However, by assuming $\frac{\psi_0}{\eta_0^3} > 0$, we see that the fifth dimension increases with cosmic time.

VI. CONCLUSIONS

In this paper, by considering an extended version of Kantowski-Sachs, LRS Bianchi type I and Bianchi type III models as a background space-time, we have investigated the $4D$ cosmologies that can be extracted within the MBDT [1]. In this respect, we first solved the extended equations of motion in a $5D$ bulk in vacuum. In order to solve these non-linear coupled differential equations, we defined a new time coordinate, which appropriately transforms the system of field equations into more easier counterparts. Consequently, we have obtained new exact solutions associated with each spatial curvature in the bulk. Moreover, for all solutions, a set of parameters and integration constants was extracted. We have shown that they are not independent, and found constraints which relate them to each other and to the BD coupling parameter. Subsequently, we analyzed the solutions by discussing the allowed values of the corresponding parameters. Moreover, by considering a few particular cases, we have compared our results with corresponding solutions obtained in the conventional BD theory and in general relativity.

The main objective of this paper was to discuss the reduced cosmologies on a $4D$ hypersurface, produced from applying the methodology of the MBDT. Therefore, we obtained expressions for induced physical quantities such as spatial volume, average Hubble parameter, mean anisotropy parameter, the deceleration parameter and the expansions for scalar expansion and the shear scalar. Moreover, we presented the properties and behaviors of these quantities, and discussing them. Concerning the solutions associated with LRS Bianchi type I model, we have also obtained all the induced physical quantities in terms of the cosmic time.

Concerning the Bianchi type I model, the scope of our solutions are more extended than those obtained in previous investigations, see for instance [39]. We have shown that for $\zeta = 0$, there are two general classes of anisotropic solutions. The first class is power-law in terms of the cosmic time, which have been also widely investigated in the context of general relativity, standard BD theory, MBDT as well as the generalized scalar tensor theories [37, 39, 43]. However, for the sake of completeness and comparison, we have analysed them briefly in this paper. Furthermore, we obtained new exact solutions for particular values of the BD coupling parameter as well as for the equation of state parameter. We have addressed the physical quantities which are important for both anisotropic and isotropic fluids. It should be noted that among the isotropic reduced cosmologies presented in this paper, the consequences associated with the stiff fluid are completely new and have not been investigated in the previous publications associated with the MBDT, see for instance, [1, 9, 23], and references therein. Moreover, the second class (which is exponential-law in terms of the cosmic time) to the best of our knowledge has not been obtained in the corresponding standard models as well as in the context of the MBDT, see, for instance, [39], and references therein. We investigated this class comprehensively in this work regarding the anisotropic and isotropic cosmologies. We have also presented solutions for particular well known values of the BD coupling parameter as well as equation of state parameter including stiff, radiative and false vacuum fluids. For each solution of the Bianchi type I model, we have also described the evolution of the extra dimension (in terms of the cosmic time). Let us also mention those pointed in [32], albeit obtained within another physical context.

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