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Ciências

Generalized Trichotomies: robustness and global and local invariant manifolds

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To my parents

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Resumo

Resumo:

Num espaço de Banach, dada uma equação diferencial $v'(t) = A(t)v(t)$, sujeita a uma condição inicial $v(s) = v_s$ e que admite uma tricotomia generalizada, estudámos o tipo de condições a impor às perturbações lineares B de modo que a equação $v'(t) = [A(t) + B(t)]\, v(t)$ ainda admita uma tricotomia generalizada, ou seja, estudámos a robustez das tricotomias generalizadas. Da mesma forma, foi também objecto deste trabalho, o estudo de uma equação diferencial com outro tipo de perturbações não lineares, $v'(t) = A(t)v(t) + f(t, v)$. Procurámos condições necessárias a impor à função f por forma a que a nova equação perturbada admitisse uma variedade invariante Lipschitz global, bem como as condições necessárias para a existência de variedades invariantes Lipschitz locais.

Palavras-Chave:

Equações diferenciais ordinárias não-autonomoas, tricotomias generalizadas, robustez, variedades invariantes, perturbações Lipschitz

Resumo Alargado

Este trabalho foi realizado no âmbito do doutoramento em Matemática e Aplicações e resulta essencialmente do estudo de vários artigos na área de sistemas dinâmicos resultantes de equações diferenciais ordinárias. A maior inspiração foi obtida pela análise cuidada de artigos de Barreira e Valls [11, 5, 3], bem como de Bento e Silva [17, 16, 20, 19], entre outros.

Sejam X um espaço de Banach, $B(X)$ a álgebra de Banach dos operadores lineares limitados que actuam em X e $A\colon \mathbb{R} \to B(X)$ uma aplicação contínua. Consideremos a equação diferencial ordinária não autónoma

$$v'(t) = A(t)v(t), \tag{$*$}$$

sujeita a uma condição inicial $v(s) = v_s$ e suponhamos que esta equação tem uma solução global. Nestas condições, pretendemos estudar as equações perturbadas

$$v'(t) = [A(t) + B(t)]\, v(t)$$

e

$$v'(t) = A(t)v(t) + f(t, v)$$

onde $B\colon \mathbb{R} \to B(X)$ e $f\colon \mathbb{R} \times X \to X$ são funções contínuas. É claro que a resposta vai depender do tipo de perturbações e das condições impostas a essas perturbações e das hipóteses que assumimos sobre a equação linear $(*)$.

A hipótese usada recentemente pelos autores referidos, bem como por outros, passa por assumir que a equação $(*)$ admite dicotomias ou tricotomias. Neste tra-

balho só consideramos tricotomias definidas de forma o mais geral possível. Assim, na secção inicial do primeiro capítulo explicamos este conceito – tricotomias generalizadas – de uma forma muito abrangente. Considerando o operador de evolução $T_{t,s}$ associado à equação diferencial $(*)$, i.e., $T_{t,s}v(s) = v(t)$ para quaisquer $t, s \in \mathbb{R}$, dizemos que esta admite uma decomposição invariante se, para todo $t \in \mathbb{R}$, existem projecções $P_t,\ Q_t^+,\ Q_t^-\ \in B(X)$ tais que

(S1) $P_t + Q_t^+ + Q_t^- = \mathrm{Id}$ para todo $t \in \mathbb{R}$;

(S2) $P_t Q_t^+ = 0$ para todo $t \in \mathbb{R}$;

(S3) $P_t T_{t,s} = T_{t,s} P_s$ para todo $t, s \in \mathbb{R}$;

(S4) $Q_t^+ T_{t,s} = T_{t,s} Q_s^+$ para todo $t, s \in \mathbb{R}$.

Definindo os subespaços lineares $E_t = P_t(X)$, $F_t^+ = Q_t^+(X)$ e $F_t^- = Q_t^-(X)$ e dadas funções $\alpha\colon \mathbb{R}^2 \to \mathbb{R}^+$, $\beta^+\colon \mathbb{R}^2_\geqslant \to \mathbb{R}^+$ e $\beta^-\colon \mathbb{R}^2_\leqslant \to \mathbb{R}^+$, onde

$$\mathbb{R}^2_\leqslant = \left\{(t,s) \in \mathbb{R}^2 \colon t \leqslant s\right\} \qquad \text{e} \qquad \mathbb{R}^2_\geqslant = \left\{(t,s) \in \mathbb{R}^2 \colon t \geqslant s\right\},$$

denotanto $\alpha(t,s)$, $\beta^+(t,s)$ e $\beta^-(t,s)$ por $\alpha_{t,s}$, $\beta^+_{t,s}$ e $\beta^-_{t,s}$, dizemos que a equação diferencial $v'(t) = A(t)v(t)$ admite uma tricotomia generalizada com majorantes $\alpha = (\alpha_{t,s})_{(t,s)\in\mathbb{R}^2}$, $\beta^+ = \left(\beta^+_{t,s}\right)_{(t,s)\in\mathbb{R}^2_\geqslant}$ e $\beta^- = \left(\beta^-_{t,s}\right)_{(t,s)\in\mathbb{R}^2_\leqslant}$, ou simplesmente com majorantes $\alpha_{t,s}$, $\beta^+_{t,s}$ e $\beta^-_{t,s}$, se admite uma decomposição invariante tal que

(D1) $\|T_{t,s}P_s\| \leqslant \alpha_{t,s}$ para todo $(t,s) \in \mathbb{R}^2$;

(D2) $\|T_{t,s}Q_s^+\| \leqslant \beta^+_{t,s}$ para todo $(t,s) \in \mathbb{R}^2_\geqslant$;

(D3) $\|T_{t,s}Q_s^-\| \leqslant \beta^-_{t,s}$ para todo $(t,s) \in \mathbb{R}^2_\leqslant$.

Ainda no primeiro capítulo construímos, em $\mathbb{R}^4$, um exemplo de uma equação diferencial ordinária que admite uma tricotomia generalizada que denominámos de tricotomia$-(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d})$ não uniforme, tricotomia essa que explorámos em todos os capítulos seguintes. Apresentamos ainda vários casos particulares deste tipo de tricotomias. As tricotomias $\rho-$exponenciais não uniformes e exponenciais não uniformes são casos particulares do anterior e vão ao encontro dos exemplos apresentados por outros autores, nomeadamente, por Barreira e Valls em [11], [12] e [2]. Além disso, apresentamos mais exemplos que segundo julgamos saber, são inovadores, nomeadamente as tricotomias às quais demos o nome de tricotomias $\mu-$polinomiais não uniformes e polinomiais não uniformes.

No Capítulo 2, estudamos perturbações lineares da equação diferencial $(*)$ da forma

$$v'(t) = [A(t) + B(t)]v(t),$$

onde $B\colon \mathbb{R} \to B(X)$ é uma aplicação contínua. Este problema designa-se usualmente por problema da robustez. Supondo que a equação diferencial $(*)$ admite uma tricotomia generalizada que verifica algumas hipóteses adicionais, provamos que a equação perturbada irá também admitir um comportamento tricotómico generalizado, desde que os operadores $B(t)$ tenham norma suficientemente pequena.

Denotando $B(t)$ por B_t e definindo as constantes, λ, λ^+ e λ^- à custa de $\alpha_{t,s}$, $\beta^+_{t,s}$, $\beta^-_{t,s}$ e de $\|B_t\|$ da seguinte maneira:

$$\lambda := \sup_{(t,s)\in\mathbb{R}^2} \frac{\lambda_{t,s}}{\alpha_{t,s}}, \quad \lambda^+ := \sup_{(t,s)\in\mathbb{R}^2_{\geqslant}} \frac{\lambda^+_{t,s}}{\beta^+_{t,s}} \quad \text{e} \quad \lambda^- := \sup_{(t,s)\in\mathbb{R}^2_{\leqslant}} \frac{\lambda^-_{t,s}}{\beta^-_{t,s}},$$

com $\lambda_{t,s}$ dado por

$$\begin{aligned}\lambda_{t,s} = &\int_{-\infty}^{s} \alpha_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \left|\int_s^t \alpha_{t,r}\|B_r\|\alpha_{r,s}\,dr\right| + \int_t^{+\infty} \beta^-_{t,r}\|B_r\|\alpha_{r,s}\,dr\\ &+ \int_{-\infty}^{t} \beta^+_{t,r}\|B_r\|\alpha_{r,s}\,dr + \int_s^{+\infty} \alpha_{t,r}\|B_r\|\beta^+_{r,s}\,dr,\end{aligned}$$

$\lambda^+_{t,s}$ definido por

$$\begin{aligned}\lambda^+_{t,s} = &\int_{-\infty}^{s} \beta^+_{t,r}\|B_r\|\alpha_{r,s}\,dr + \int_s^t \beta^+_{t,r}\|B_r\|\beta^+_{r,s}\,dr + \int_t^{+\infty} \alpha_{t,r}\|B_r\|\beta^+_{r,s}\,dr\\ &+ \int_{-\infty}^{s} \beta^+_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_t^{+\infty} \beta^-_{t,r}\|B_r\|\beta^+_{r,s}\,dr\end{aligned}$$

e $\lambda^-_{t,s}$ por

$$\begin{aligned}\lambda^-_{t,s} = &\int_{-\infty}^{t} \alpha_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_t^s \beta^-_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_s^{+\infty} \beta^-_{t,r}\|B_r\|\alpha_{r,s}\,dr\\ &+ \int_{-\infty}^{t} \beta^+_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_s^{+\infty} \beta^-_{t,r}\|B_r\|\beta^+_{r,s}\,dr,\end{aligned}$$

podemos enunciar o teorema que se segue.

Teorema 2.1.1 *Seja X um espaço de Banach. Suponhamos que a equação diferencial $v'(t) = A(t)v(t)$ admite uma tricotomia generalizada com majorantes $\alpha_{t,s}$, $\beta^+_{t,s}$ e $\beta^-_{t,s}$ tal que*

$$\sup_{t\in\mathbb{R}} \frac{\alpha_{t,s}}{\alpha_{t,\ell}} < +\infty \qquad \textit{para todo } (\ell,s)\in\mathbb{R}^2,$$
$$\sup_{t\geqslant\ell} \frac{\beta^+_{t,s}}{\beta^+_{t,\ell}} < +\infty \qquad \textit{para todo } (\ell,s)\in\mathbb{R}^2_{\geqslant},$$
$$\sup_{t\leqslant\ell} \frac{\beta^-_{t,s}}{\beta^-_{t,\ell}} < +\infty \qquad \textit{para todo } (\ell,s)\in\mathbb{R}^2_{\leqslant}.$$

Seja $B\colon \mathbb{R}\to B(X)$ uma função contínua. Se

$$\max\left\{\lambda,\lambda^+,\lambda^-\right\} < 1$$

onde λ, λ^+ and λ^- estão definidos anteriormente, então a equação perturbada

$$v'(t) = [A(t)+B(t)]v(t)$$

admite uma tricotomia generalizada com majorantes $\sigma\alpha_{t,s}$, $\sigma\beta^+_{t,s}$ e $\sigma\beta^-_{t,s}$ e onde σ é dado por

$$\sigma := \frac{1}{1-\max\{\lambda,\lambda^+,\lambda^-\}}.$$

Consideramos este resultado, o Teorema 2.1.1, o resultado principal deste capítulo, sendo enunciado no início do capítulo, mas sendo somente demonstrado na última secção deste. De seguida apresentamos casos particulares do teorema principal do capítulo. Começamos por mostrar no Teorema 2.2.1 que se a tricotomia exibida pela equação diferencial inicial for uma tricotomia$-(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d})$ não uniforme com algumas condições adicionais impostas, e se a perturbação B também obedecer a certos requisitos, então todas as condições do Teorema 2.1.1 são verificadas. Os dois exemplos seguintes, para tricotomias $\rho-$exponencial não uniforme e exponencial não uniforme, são apresentados como casos particulares do anterior, e é mostrado que os resultados obtidos no caso da tricotomia exponencial são menos exigentes do que os obtidos por Barreira e Valls em [10]. Os dois últimos teoremas deste capítulo exibem as condições que as tricotomias $\mu-$polinomias não uniformes e polinomiais não uniformes devem obedecer bem como as condições que devemos impor às perturbações por forma a serem verificadas todas as condições do Teorema 2.1.1. Terminamos este capítulo com a demonstração do referido teorema.

No terceiro capítulo estudamos outro tipo de problema. A equação diferencial inicial é sujeita agora a uma perturbação da forma

$$v'(t) = A(t)v(t) + f(t, v)$$

onde $f\colon \mathbb{R} \times X \to X$ é uma função contínua tal que $f(t, 0) = 0$ e $f_t\colon X \to X$, definida por $f_t(x) = f(t, x)$, é uma função Lipschitz para todo o $t \in \mathbb{R}$. Para podermos enunciar o principal resultado deste capítulo temos de introduzir alguma notação. Para cada $\tau \in \mathbb{R}$, o fluxo da equação perturbada é definido por

$$\Psi_\tau(s, v_s) = \big(s+\tau, x(s+\tau, s, v_s), y^+(s+\tau, s, v_s), y^-(s+\tau, s, v_s)\big),$$

com $s \in \mathbb{R}$, $v_s = (\xi, \eta^+, \eta^-) \in E_s \times F_s^+ \times F_s^-$ e onde

$$\big(x(t, s, v_s), y^+(t, s, v_s), y^-(t, s, v_s)\big) \in E_t \times F_t^+ \times F_t^-$$

denota a solução da equação perturbada. Considerando o conjunto

$$G = \bigcup_{t \in \mathbb{R}} \{t\} \times E_t,$$

e uma constante positiva N, denotamos por $\mathcal{A}_N$ o espaço das funções contínuas $\varphi\colon G \to X$ tais que

$$\varphi(t, 0) = 0 \quad \text{para todo } t \in \mathbb{R};$$

$$\varphi(t, \xi) \in F_t^+ \oplus F_t^- \quad \text{para todo } (t, \xi) \in G;$$

$$\sup\left\{\frac{\left\|\varphi(t,\xi) - \varphi(t,\bar\xi)\right\|}{\left\|\xi - \bar\xi\right\|} \colon (t,\xi), (t,\bar\xi) \in G,\ \xi \neq \bar\xi\right\} \leqslant N;$$

e por $\mathcal{V}_\varphi$ o gráfico de φ, ou seja, o conjunto

$$\mathcal{V}_\varphi = \{(s, \xi, \varphi(s, \xi)) : (s, \xi) \in G\} \subseteq \mathbb{R} \times X.$$

Definimos ainda as quantidades σ e ω por

$$\sigma := \sup_{(t,s)\in\mathbb{R}^2} \left| \int_s^t \frac{\alpha_{t,r} \operatorname{Lip}(f_r)\, \alpha_{r,s}}{\alpha_{t,s}}\, dr \right|$$

e

$$\omega := \sup_{s\in\mathbb{R}} \left[\int_{-\infty}^s \beta^+_{s,r} \operatorname{Lip}(f_r)\, \alpha_{r,s}\, dr + \int_s^{+\infty} \beta^-_{s,r} \operatorname{Lip}(f_r)\, \alpha_{r,s}\, dr \right].$$

Teorema 3.1.3 *Seja X um espaço de Banach. Suponhamos que $v'(t) = A(t)v(t)$ admite uma tricotomia generalizada com majorantes $\alpha_{t,s}$, $\beta^+_{t,s}$ e $\beta^-_{t,s}$ e seja $f\colon \mathbb{R} \times X \to X$ uma função contínua nas condições descritas. Se*

$$\lim_{r\to+\infty} \beta^-_{s,r}\alpha_{r,s} = \lim_{r\to-\infty} \beta^+_{s,r}\alpha_{r,s} = 0 \quad \textit{para todo } s \in \mathbb{R}$$

e

$$2\sigma + 2\omega < 1,$$

então existe $N \in\,]0, 1[$ e uma única função $\varphi \in \mathcal{A}_N$ tal que

$$\Psi_\tau(\mathcal{V}_\varphi) \subset \mathcal{V}_\varphi$$

para todo $\tau \in \mathbb{R}$ onde Ψ_τ e $\mathcal{V}_\varphi$ foram anteriormente definidos. Além disso,

$$\left\|\Psi_{t-s}(s, \xi, \varphi(s, \xi)) - \Psi_{t-s}(s, \bar{\xi}, \varphi(s, \bar{\xi}))\right\| \leqslant \frac{N}{\omega}\, \alpha_{t,s} \left\|\xi - \bar{\xi}\right\|$$

para todo $(t, s) \in \mathbb{R}^2$ e todo $\xi, \bar{\xi} \in E_s$.

O resultado global que se obteve, o Teorema 3.1.3, é feito para tricotomias generalizadas e, como tal, podemos aplicá-lo a todos os exemplos de tricotomias apresentados no primeiro capítulo: tricotomias$-(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d})$ não uniformes, tricotomias $\rho-$exponenciais não uniformes e exponenciais não uniformes (que são casos particulares do anterior) e tricotomias $\mu-$polinomias não uniformes e polinomiais não uniformes. A demonstração deste resultado também é apresentada no fim do capítulo.

Terminamos este trabalho com a apresentação, no Capítulo 4, de um resultado semelhante ao anterior mas em que as variedades invariantes são locais. Aqui as perturbações $f\colon \mathbb{R}\times X \to X$ são funções contínuas tais que $f(t,0)=0$, $f_t(\cdot) := f(t,\cdot)$ é Lipschitz na bola

$$\mathfrak{B}(R(t)) = \{x \in X\colon \|x\| \leqslant R(t)\}\,,$$

para todo o $t \in \mathbb{R}$ e onde $R\colon \mathbb{R} \to]0,+\infty[$. Denotanto a constante de Lipschitz de f_t na bola $\mathfrak{B}(R(t))$ por $\operatorname{Lip}\left(f_t|_{\mathfrak{B}(R(t))}\right)$, definindo $\widetilde{\sigma}$ e $\widetilde{\omega}$, por

$$\widetilde{\sigma} := \sup_{(t,s)\in\mathbb{R}^2} \left| \int_s^t \frac{\alpha_{t,r} \operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right) \alpha_{r,s}}{\alpha_{t,s}}\, dr \right|$$

e

$$\widetilde{\omega} := \sup_{s\in\mathbb{R}} \left[\int_{-\infty}^s \beta^+_{s,r} \operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right) \alpha_{r,s}\, dr + \int_s^{+\infty} \beta^-_{s,r} \operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right) \alpha_{r,s}\, dr \right]$$

e denotanto o gráfico de φ nas bolas $\mathfrak{B}(R(t))$ por

$$\mathcal{V}^*_{\varphi,R} = \{(s,\xi,\varphi(s,\xi)) \in \mathcal{V}_\varphi\colon \ \|\xi\| \leqslant R(s)\}$$

estamos em condições de enunciar o referido resultado.

Teorema 4.1.2 *Seja X um espaço de Banach. Suponhamos que $v'(t) = A(t)v(t)$ admite uma tricotomia generalizada com majorantes $\alpha_{t,s}$, $\beta^+_{t,s}$ e $\beta^-_{t,s}$ e seja $f\colon \mathbb{R}\times X \to X$ uma função contínua com as condições locais descritas. Se*

$$\lim_{r\to+\infty} \beta^-_{s,r}\alpha_{r,s} = \lim_{r\to-\infty} \beta^+_{s,r}\alpha_{r,s} = 0 \quad \textit{para todo } s\in\mathbb{R},$$

$$4\widetilde{\sigma} + 4\widetilde{\omega} < 1$$

e

$$\sup_{t\in\mathbb{R}} \frac{\alpha_{t,s}}{R(t)} < +\infty \quad \textit{para todo } s\in\mathbb{R},$$

então existe $N \in]0,1[$ e uma função $\varphi \in \mathcal{A}_N$ tal que para todo $\tau \in \mathbb{R}$ se tem

$$\Psi_\tau(\mathcal{V}^*_{\varphi,\overline{R}}) \subseteq \mathcal{V}^*_{\varphi,R}\,,$$

onde $\overline{R}$ *denota a função* $\overline{R}\colon \mathbb{R} \to \mathbb{R}^+$ *dada por*

$$\overline{R}(s) = \frac{\widetilde{\omega}}{N \sup\limits_{t\in\mathbb{R}} \left[\alpha_{t,s}/R(t)\right]}.$$

Além disso, tem-se

$$\left\|\Psi_{t-s}(s,\xi,\varphi(s,\xi)) - \Psi_{t-s}(s,\bar{\xi},\varphi(s,\bar{\xi}))\right\| \leqslant \frac{N}{\widetilde{\omega}}\alpha_{t,s}\left\|\xi-\bar{\xi}\right\|$$

para todo $(t,s) \in \mathbb{R}^2$ *e todo* $\xi, \bar{\xi} \in \mathfrak{B}(\overline{R}(s)) \cap E_s$.

Nas duas secções seguintes trabalhámos com dois tipos de perturbações f. Uma família de funções que verificava para todo o $t \in \mathbb{R}$ e $u, v \in X$, com $q > 0$

$$\|f(t,u) - f(t,v)\| \leqslant k(t)\,\|u - v\|\,(\|u\| + \|v\|)^q\,,$$

onde $k\colon \mathbb{R} \to]0, +\infty[$, e outra família que obedecia a

$$\|f(t,u) - f(t,v)\| \leqslant k\,\|u - v\|\,(\|u\| + \|v\|)^q$$

com k constante positiva. Para cada tipo de funções foram apresentados em subsecções, teoremas e corolários para as diversas tricotomias consideradas neste trabalho e, em alguns casos, foi possível exibir o raio $\overline{R}$, invocado no teorema local atrás apresentado, de uma forma bastante simplificada. Por fim, é feita a demonstração deste teorema local recorrendo ao teorema global, o Teorema 3.1.3.

Abstract

Abstract:

In a Banach space, given a differential equation $v'(t) = A(t)v(t)$, with an initial condition $v(s) = v_s$ and that admits a generalized trichotomy, we studied which type of conditions we need to impose to the linear perturbations B so that $v'(t) = [A(t) + B(t)]\, v(t)$ continues to admit a generalized trichotomy, that is, we studied the robustness of generalized trichotomies. In the same way, it was also the aim of our work the study of a differential equation with another type of nonlinear perturbations, $v'(t) = A(t)v(t) + f(t, v)$. We sought conditions to impose on the function f so that the new perturbed equation would admit a global Lipschitz invariant manifold as well as the necessary conditions for the existence of local Lipschitz invariant manifolds.

Keywords:

Nonautonomous ordinary differential equations, generalized trichotomies, robustness, invariant manifolds, Lipschitz perturbations

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Contents

Introduction

A major problem in the study of dynamical systems is to understand which properties are preserved when a dynamical system is perturbed. In particular, given a Banach space X and a continuous function A, defined in $\mathbb{R}$ and with values in the Banach algebra $B(X)$ of the bounded linear operators acting on X, what are the hypotheses that we have to assume about the solutions of the linear ordinary differential equation

$$v'(t) = A(t)v(t)$$

and about a perturbation $f\colon \mathbb{R} \times X \to X$, in order to be able to study the solutions of the perturbed differential equation

$$v'(t) = A(t)v(t) + f(t, v)\,?$$

In this context, the concept of exponential dichotomy is a very fruitful tool.

The concept of uniform exponential dichotomy goes back to 1929/30 with the work of Perron [48, 49]. Since then, many authors have studied the role played by uniform exponential dichotomies in dynamical systems, namely in linear ordinary differential equations and in particularly in the study of the existence of stable and unstable manifolds. However, the concept of uniform exponential dichotomy is very demanding and it was convenient to consider weaker definitions.

Thus, a notion of nonuniform exponential dichotomy was used by Preda and Megan [56] in 1983 and by Megan, Sasu and Sasu [43] in 2002 to study evolution operators. In 2006, inspired by the notion of nonuniform hyperbolic trajectories of Pesin [50], which allowed him to obtain invariant stable manifolds for diffeomorfisms defined on finite dimensional manifolds, Barreira and Valls [4] introduced the notion of nonuniform exponential dichotomy for linear ordinary differential equations in Banach spaces (see also [7]).

On the other hand, in 1994, appeared uniform dichotomies with nonexponential growth rates presented by Pinto [51]. In the same year, Naulin and Pinto in [45] introduced the concept of $(h,k)-$dichotomies for nonlinear differential systems. Subsequently, in 2010, Pötzsche introduced in [55], for nonautonomous discrete equations, nonexponential growth rates but expressed by general exponential functions.

Hence, it is natural to consider dichotomies that are simultaneous nonuniform and nonexponential. Bento and Silva [15], in 2009, obtained stable manifolds for difference equations that admit nonuniform polynomial dichotomies where the growth rates are more restrictive in the nonuniform part, but the uniform part obeys a polynomial law instead of an exponential (more restrictive) law. Also in 2009 and independently, Barreira and Valls introduced in [8] another type of nonuniform polynomial dichotomy.

In 2012 and in 2013, to allow the notion of nonexponential growth and nonuniform behavior simultaneously and with different growth rates in the uniform and nonuniform parts, a new nonuniform dichotomy, the nonuniform $(\mu,\nu)-$dichotomy, was proposed by Bento and Silva in [16] for difference equations and in [18] for differential equation. This notion includes the traditional uniform exponential dichotomies, the nonuniform exponential dichotomies, the uniform polynomial dichotomies and the nonuniform polynomial dichotomies which greatly enlarged the range of applications of uniform and nonuniform dichotomies. In these papers, Bento and Silva also established the existence of stable local manifolds.

In 2014 and 2016, Bento and Silva considered more general growth rates for dichotomies with nonuniform behavior for the continuous case in [20] and for the discrete case in [21].

In the theory of linear perturbations it is also important to study the so-called problem of robustness for these dichotomies, that is, the problem of finding the necessary conditions that an additive linear perturbation, say B, so that the new differential equation

$$v' = [A(t) + B(t)]v$$

admits the same type of dichotomy of equation

$$v' = A(t)v.$$

The problem of robustness has a long history. In 1958 Massera and Schäffer [42] addressed this theme based on Perron's work [49], as did Copel [30] in 1974 and Palmer [47] in 1984 in a finite dimension. Other authors that studied the problem of robustness are Daleckii and Krein [31], Naulin and Pinto [46], Chow and Leiva [25], Pliss and Sell [53] and Popescu [54]. It should be pointed out that all these papers consider only uniform exponential behaviors.

In 2008, Barreira and Valls [6] obtained a robustness result for nonuniform exponential dichotomies and in 2009 did the same for $\rho-$nonuniform exponential dichotomies in [9]. Robustness has also been studied, among others, by Chang, Zhang and Qin for nonuniform $(\mu, \nu)-$dichotomies in Banach spaces in [23] in the continuous case. Bento and Silva [19] also discussed this problem of robustness for difference equations.

Another important concept associated with the concept of dichotomy is the notion of trichotomy. Trichotomies play an important role in the study of the asymptotic behavior of dynamical systems, namely, when a dynamic linear system has no unstable directions, the stability of the system is completely determined by the behavior in the center manifold! Center manifolds are also useful in the study of bifurcations because it might allow the reduction of the dimension of the state space; for more details we recommend the books by Carr [22], Henry [36], Guckenheimer and Holmes [33], Hale and Koçak [34] and Haragus and Iooss [35].

The study of center manifolds started with Pliss [52] and Kelley [39, 38] in the

60's. After that, many authors studied the problem and proved results about center manifolds. A good expository paper for the case of autonomous differential equations in finite dimension was written by Vanderbauwhede [59] (see also Vanderbauwhede and Gils [61]) and for the case of autonomous differential equations in infinite dimension see Vanderbauwhede and Iooss [60]. For more details in the finite dimensional case see Chow, Liu and Yi [27, 26] and for the infinite dimensional case see Sijbrand [58], Mielke [44], Chow and Lu [28, 29] and Chicone and Latushkin [24].

For nonautonomous differential equations the concept of exponential trichotomy is an important tool to obtain center manifolds theorems. This notion goes back to Sacker and Sell [57], Aulbach [1] and Elaydi and Hajek [32] and is inspired by the notion of exponential dichotomy that can be traced back to the work of Perron in [48, 49]. However, as in the case of exponential dichotomies, the notion of exponential trichotomy is very demanding and several generalizations have appeared in the literature. Essentially we can find two ways of generalization: on one hand replace the exponential growth rates by nonexponential growth rates and on the other hand consider exponential trichotomies that also depend on the initial time and hence are nonuniform. Trichotomies with nonexponential growth rates have been introduced by Fenner and Pinto in [40] where the authors study the so called $(h, k)-$trichotomies and the nonuniform exponential trichotomies have been consider by Barreira and Valls in [2, 3].

Hence, it is natural to consider trichotomies that are both nonuniform and nonexponential. This was done by Barreira and Valls in [11, 12] where have been introduced the so-called $\rho-$nonuniform exponential trichotomies, but these trichotomies do not include as a particular case the $(h, k)-$trichotomies of Fenner and Pinto.

The robustness problem was also studied for trichotomies in 2009 by Barreira and Valls [10] who proved the robustness of nonuniform exponential trichotomies, and by Jiang [37] in 2012, who obtained robustness for nonuniform $(\mu, \nu)-$trichotomies.

The aim of this work is to define more general types of trichotomies in order to study, for linear and nonlinear perturbations, the solutions of an equation of the type

$$v' = A(t)v + f(t, v),$$

supposing that

$$v' = A(t)v$$

admits this type of general trichotomy.

Thus, we studied the robustness of trichotomies with a general situation, included a wide range of examples of nonuniform behavior and improved some of the existing results.

It is also our goal the study of the differential problem

$$v'(t) = A(t)v + f(t, v), \quad v(s) = v_s$$

where nonlinear perturbations f are continuous functions with certain properties, namely being Lipschitz functions in the second variable. Note that for dichotomies this has already been done by Bento and Silva in [20] and in [21] for differential and for difference equations, respectively. Here we present results both in a global and in a local form. Initially we obtained a result on the existence of global center Lipschitz invariant manifolds and, from this, we obtained a similar result in the existence of local Lipschitz invariant manifolds.

Now we present the structure of this thesis. In the first chapter, we give the basic notions and the necessary preliminaries of the generalized trichotomies for the forward work. We also include, in a subsection, some examples of generalized trichotomies.

In the second chapter we study the robustness problem for generalized trichotomies, i.e, we find necessary conditions that the linear perturbation should exhibit in order that the trichotomy is preserved. That is the main result of the chapter. Then we present a section with several particular cases that generalize some results already existing in the literature. The proof of the main robustness result is given in the last section of the chapter.

In the following chapter we are going to consider another type of perturbation of the differential equations $v' = A(t)v$. Here, our main goal is to establish the existence of global Lipschitz invariant manifolds when the linear differential equation admits a generalized trichotomy and is submitted to a nonlinear perturbation f, $v'(t) = A(t)v + f(t, v)$, satisfying some conditions. We present in the first section the main result, in the second section particular cases of the main result and in the last section the proof of the main result.

In the last chapter we prove, for the differential problem $v'(t) = A(t)v + f(t, v)$, the existence of local Lipschitz invariant manifolds. This result is stated in the first section and proved in the last one. In the middle sections, we have considered two types of nonlinear perturbations f and for each type of perturbation we present the usual particular cases.

Chapter 1

Generalized Trichotomies

In this chapter we consider, on a Banach space, the notion of generalized trichotomy for linear ordinary differential equations which include as particular cases some notions of tricotomies that already exist in the literature, namely in Barreira and Valls [2, 11, 12].

§1.1 Notation and Preliminaries

Let X be a Banach space, let $B(X)$ be the space of bounded linear operators in X and let $A\colon \mathbb{R} \to B(X)$ be a continuous map. Consider the linear differential equation

$$v' = A(t)v, \qquad v(s) = v_s \tag{1.1}$$

with $s \in \mathbb{R}$, $v_s \in X$. We are going to assume that (1.1) has a global solution and denote by $T_{t,s}$ the linear evolution operator associated to equation (1.1), i.e.,

$$v(t) = T_{t,s}v(s)$$

for every $t, s \in \mathbb{R}$.

Definition 1.1.1. *We say that* (1.1) *admits an* ***invariant splitting*** *if, for every* $t \in \mathbb{R}$*, there exist bounded projections* P_t*,* Q_t^+*,* $Q_t^- \in B(X)$ *such that*

(**S1**) $P_t + Q_t^+ + Q_t^- = \mathrm{Id}$ *for every* $t \in \mathbb{R}$*;*

(**S2**) $P_t Q_t^+ = 0$ *for every* $t \in \mathbb{R}$*;*

(**S3**) $P_t T_{t,s} = T_{t,s} P_s$ *for every* $t, s \in \mathbb{R}$*;*

(**S4**) $Q_t^+ T_{t,s} = T_{t,s} Q_s^+$ *for every* $t, s \in \mathbb{R}$*.*

From (**S1**) and (**S2**) we have

(**S5**) $P_t Q_t^- = Q_t^+ P_t = Q_t^- P_t = Q_t^+ Q_t^- = Q_t^- Q_t^+ = 0$ for every $t \in \mathbb{R}$

and from (**S1**), (**S3**) and (**S4**) it follows immediately that

(**S6**) $Q_t^- T_{t,s} = T_{t,s} Q_s^-$ for every $t, s \in \mathbb{R}$.

For each $t \in \mathbb{R}$, we define the linear subspaces $E_t = P_t(X)$, $F_t^+ = Q_t^+(X)$ and $F_t^- = Q_t^-(X)$, and, as usual, we identify $E_t \times F_t^+ \times F_t^-$ and $E_t \oplus F_t^+ \oplus F_t^- = X$ as the same vector space.

Now we give the definition of generalized trichotomy that is fundamental for our work.

Definition 1.1.2. *Let* $\alpha \colon \mathbb{R}^2 \to \mathbb{R}^+$*,* $\beta^+ \colon \mathbb{R}^2_{\geqslant} \to \mathbb{R}^+$ *and* $\beta^- \colon \mathbb{R}^2_{\leqslant} \to \mathbb{R}^+$*, where*

$$\mathbb{R}^2_{\leqslant} = \left\{(t,s) \in \mathbb{R}^2 \colon t \leqslant s\right\} \qquad \textit{and} \qquad \mathbb{R}^2_{\geqslant} = \left\{(t,s) \in \mathbb{R}^2 \colon t \geqslant s\right\},$$

and denote $\alpha(t,s)$*,* $\beta^+(t,s)$ *and* $\beta^-(t,s)$ *by* $\alpha_{t,s}$*,* $\beta^+_{t,s}$ *and* $\beta^-_{t,s}$*, respectively.*

We say that equation (1.1) *admits a* ***generalized trichotomy*** *with bounds* $\alpha = (\alpha_{t,s})_{(t,s)\in\mathbb{R}^2}$*,* $\beta^+ = \left(\beta^+_{t,s}\right)_{(t,s)\in\mathbb{R}^2_{\geqslant}}$ *and* $\beta^- = \left(\beta^-_{t,s}\right)_{(t,s)\in\mathbb{R}^2_{\leqslant}}$*, or simply with bounds* $\alpha_{t,s}$*,* $\beta^+_{t,s}$ *and* $\beta^-_{t,s}$*, if it admits an invariant splitting such that*

(**D1**) $\|T_{t,s} P_s\| \leqslant \alpha_{t,s}$ *for every* $(t,s) \in \mathbb{R}^2$*;*

(**D2**) $\|T_{t,s} Q_s^+\| \leqslant \beta^+_{t,s}$ *for every* $(t,s) \in \mathbb{R}^2_{\geqslant}$*;*

(**D3**) $\|T_{t,s} Q_s^-\| \leqslant \beta^-_{t,s}$ *for every* $(t,s) \in \mathbb{R}^2_{\leqslant}$*.*

§1.2 Examples of generalized trichotomies

§1.2.1 Nonuniform $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d})-$trichotomies

Now we present, in $\mathbb{R}^4$, an example of a differential equation that under certain conditions admits a generalized trichotomy with some bounds of a special type. This is a new and more general definition when compared to what has been done until now and is inspired in the notions of $(h, k)-$ dichotomy and $(h, k)-$trichotomy introduced by Pinto [51], Naulin and Pinto [45] and Fenner and Pinto [40].

Example 1.2.1. *Let*

$$\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \colon \mathbb{R} \to \,]0, +\infty[$$

be C^1 functions and let

$$\varepsilon_{\mathfrak{a}}, \varepsilon_{\mathfrak{b}}, \varepsilon_{\mathfrak{c}}, \varepsilon_{\mathfrak{d}} \colon \mathbb{R} \to [1, +\infty[$$

be C^1 functions in $\mathbb{R} \setminus \{0\}$ and with derivatives from the left and from the right at $t = 0$. In $\mathbb{R}^4$, equipped with the maximum norm, consider the differential equation

$$\begin{cases} u' = \left[-\dfrac{\mathfrak{a}'(t)}{\mathfrak{a}(t)} + \varepsilon^*_{\mathfrak{a}}(t) \right] u, \\ v' = \left[\dfrac{\mathfrak{c}'(t)}{\mathfrak{c}(t)} + \varepsilon^*_{\mathfrak{c}}(t) \right] v, \\ w' = \left[-\dfrac{\mathfrak{d}'(t)}{\mathfrak{d}(t)} + \varepsilon^*_{\mathfrak{d}}(t) \right] w, \\ z' = \left[\dfrac{\mathfrak{b}'(t)}{\mathfrak{b}(t)} + \varepsilon^*_{\mathfrak{b}}(t) \right] z, \end{cases} \tag{1.2}$$

where

$$\varepsilon^*_i(t) = \begin{cases} \dfrac{\varepsilon'_i(t)}{\varepsilon_i(t)} \dfrac{\cos t - 1}{2} - \ln(\varepsilon_i(t)) \dfrac{\sin t}{2} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0, \end{cases}$$

for $i = \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$. Taking into account that

$$\frac{u'(t)}{u(t)} = -\frac{\mathfrak{a}'(t)}{\mathfrak{a}(t)} + \varepsilon^*_{\mathfrak{a}}(t)$$

we have

$$u(t) = \frac{\varepsilon_{\mathfrak{a}}(t)^{(\cos t-1)/2}}{\mathfrak{a}(t)}.$$

In a similar way we get

$$v(t) = \mathfrak{c}(t)\varepsilon_{\mathfrak{c}}(t)^{(\cos t-1)/2}, \quad w(t) = \frac{\varepsilon_{\mathfrak{d}}(t)^{(\cos t-1)/2}}{\mathfrak{d}(t)} \quad \text{and} \quad z(t) = \mathfrak{b}(t)\varepsilon_{\mathfrak{b}}(t)^{(\cos t-1)/2}.$$

The evolution operator of this equation is given by

$$T_{t,s}(u,v,w,z) = \big(U_{t,s}(u,v), V_{t,s}^{+}w, V_{t,s}^{-}z\big)$$

where $U_{t,s}\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is defined by*

$$U_{t,s}(u,v) = \left(\frac{\mathfrak{a}(s)}{\mathfrak{a}(t)}\,\frac{\varepsilon_{\mathfrak{a}}(t)^{(\cos t-1)/2}}{\varepsilon_{\mathfrak{a}}(s)^{(\cos s-1)/2}}u, \frac{\mathfrak{c}(t)}{\mathfrak{c}(s)}\,\frac{\varepsilon_{\mathfrak{c}}(t)^{(\cos t-1)/2}}{\varepsilon_{\mathfrak{c}}(s)^{(\cos s-1)/2}}v\right)$$

and $V_{t,s}^{+}, V_{t,s}^{-}\colon \mathbb{R} \to \mathbb{R}$ *are defined by*

$$V_{t,s}^{+}w = \frac{\mathfrak{d}(s)}{\mathfrak{d}(t)}\,\frac{\varepsilon_{\mathfrak{d}}(t)^{(\cos t-1)/2}}{\varepsilon_{\mathfrak{d}}(s)^{(\cos s-1)/2}}\,w \qquad \text{and} \qquad V_{t,s}^{-}z = \frac{\mathfrak{b}(t)}{\mathfrak{b}(s)}\,\frac{\varepsilon_{\mathfrak{b}}(t)^{(\cos t-1)/2}}{\varepsilon_{\mathfrak{b}}(s)^{(\cos s-1)/2}}\,z.$$

Using the projections $P_s, Q_s^{+}, Q_s^{-}\colon \mathbb{R}^4 \to \mathbb{R}^4$ *given by*

$$\begin{aligned} P_s(u,v,w,z) &= (u,v,0,0), \\ Q_s^{+}(u,v,w,z) &= (0,0,w,0), \\ Q_s^{-}(u,v,w,z) &= (0,0,0,z), \end{aligned}$$

we will have, for every $(t,s) \in \mathbb{R}^2$,

$$\big\|T_{t,s}Q_s^{+}\big\| = \big\|V_{t,s}^{+}\big\| = \frac{\mathfrak{d}(s)}{\mathfrak{d}(t)}\,\frac{\varepsilon_{\mathfrak{d}}(t)^{(\cos t-1)/2}}{\varepsilon_{\mathfrak{d}}(s)^{(\cos s-1)/2}} \leqslant \frac{\mathfrak{d}(s)}{\mathfrak{d}(t)}\,\varepsilon_{\mathfrak{d}}(s)$$

and

$$\big\|T_{t,s}Q_s^{-}\big\| = \big\|V_{t,s}^{-}\big\| = \frac{\mathfrak{b}(t)}{\mathfrak{b}(s)}\,\frac{\varepsilon_{\mathfrak{b}}(t)^{(\cos t-1)/2}}{\varepsilon_{\mathfrak{b}}(s)^{(\cos s-1)/2}} \leqslant \frac{\mathfrak{b}(t)}{\mathfrak{b}(s)}\,\varepsilon_{\mathfrak{b}}(s),$$

because $\varepsilon_{\mathfrak{d}}(s) \geqslant 1$ *and* $\varepsilon_{\mathfrak{b}}(s) \geqslant 1$. *Morover, assuming that*

$$\frac{\mathfrak{a}(s)\mathfrak{c}(s)}{\mathfrak{a}(t)\mathfrak{c}(t)}\left(\frac{\varepsilon_{\mathfrak{a}}(t)}{\varepsilon_{\mathfrak{c}}(t)}\right)^{(\cos t-1)/2}\left(\frac{\varepsilon_{\mathfrak{c}}(s)}{\varepsilon_{\mathfrak{a}}(s)}\right)^{(\cos s-1)/2} \geqslant 1 \quad \text{for every } (t,s) \in \mathbb{R}_{\geqslant}^2, \tag{1.3}$$

which is equivalent to

$$\frac{\mathfrak{a}(s)}{\mathfrak{a}(t)}\frac{\varepsilon_{\mathfrak{a}}(t)^{((\cos t-1)/2)}}{\varepsilon_{\mathfrak{a}}(s)^{(\cos s-1)/2}} \geqslant \frac{\mathfrak{c}(t)}{\mathfrak{c}(s)}\frac{\varepsilon_{\mathfrak{c}}(t)^{(\cos t-1)/2}}{\varepsilon_{\mathfrak{c}}(s)^{(\cos s-1)/2}} \quad \text{for every } (t,s)\in\mathbb{R}^2_{\geqslant}$$

and

$$\frac{\mathfrak{a}(s)}{\mathfrak{a}(t)}\frac{\varepsilon_{\mathfrak{a}}(t)^{((\cos t-1)/2)}}{\varepsilon_{\mathfrak{a}}(s)^{(\cos s-1)/2}} \leqslant \frac{\mathfrak{c}(t)}{\mathfrak{c}(s)}\frac{\varepsilon_{\mathfrak{c}}(t)^{(\cos t-1)/2}}{\varepsilon_{\mathfrak{c}}(s)^{(\cos s-1)/2}} \quad \text{for every } (t,s)\in\mathbb{R}^2_{\leqslant},$$

we have

$$\begin{aligned}\|T_{t,s}P_s\| &= \begin{cases}\dfrac{\mathfrak{a}(s)}{\mathfrak{a}(t)}\,\dfrac{\varepsilon_{\mathfrak{a}}(t)^{(\cos t-1)/2}}{\varepsilon_{\mathfrak{a}}(s)^{(\cos s-1)/2}} & \text{for all } (t,s)\in\mathbb{R}^2_{\geqslant},\\ \dfrac{\mathfrak{c}(t)}{\mathfrak{c}(s)}\,\dfrac{\varepsilon_{\mathfrak{c}}(t)^{(\cos t-1)/2}}{\varepsilon_{\mathfrak{c}}(s)^{(\cos s-1)/2}} & \text{for all } (t,s)\in\mathbb{R}^2_{\leqslant},\end{cases}\\ &\leqslant \begin{cases}\dfrac{\mathfrak{a}(s)}{\mathfrak{a}(t)}\,\varepsilon_{\mathfrak{a}}(s) & \text{for all } (t,s)\in\mathbb{R}^2_{\geqslant},\\ \dfrac{\mathfrak{c}(t)}{\mathfrak{c}(s)}\,\varepsilon_{\mathfrak{c}}(s) & \text{for all } (t,s)\in\mathbb{R}^2_{\leqslant}.\end{cases}\end{aligned}$$

Therefore, if (1.3) *is satisfied, equation* (1.2) *has a generalized trichotomy with bounds*

$$\begin{aligned}\alpha_{t,s} &= \begin{cases}\dfrac{\mathfrak{a}(s)}{\mathfrak{a}(t)}\,\varepsilon_{\mathfrak{a}}(s) & \text{for all } (t,s)\in\mathbb{R}^2_{\geqslant} \text{ with } t\neq s,\\ \min\{\varepsilon_{\mathfrak{a}}(s),\varepsilon_{\mathfrak{c}}(s)\} & \text{for all } (t,s)\in\mathbb{R}^2 \text{ with } t=s,\\ \dfrac{\mathfrak{c}(t)}{\mathfrak{c}(s)}\,\varepsilon_{\mathfrak{c}}(s) & \text{for all } (t,s)\in\mathbb{R}^2_{\leqslant} \text{ with } t\neq s,\end{cases}\\ \beta^+_{t,s} &= \frac{\mathfrak{d}(s)}{\mathfrak{d}(t)}\,\varepsilon_{\mathfrak{d}}(s) \qquad \text{for all } (t,s)\in\mathbb{R}^2_{\geqslant},\\ \beta^-_{t,s} &= \frac{\mathfrak{b}(t)}{\mathfrak{b}(s)}\,\varepsilon_{\mathfrak{b}}(s) \qquad \text{for all } (t,s)\in\mathbb{R}^2_{\leqslant}.\end{aligned} \tag{1.4}$$

We are going to call the trichotomies with this type of bounds by ***nonuniform*** $(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d})-$***trichotomies***.

§1.2.2 $\rho-$nonuniform exponential trichotomies

In this subsection we present particular cases of the nonuniform $(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d})-$tricotomies.

Example 1.2.2. *Let $\rho\colon \mathbb{R} \to \mathbb{R}$ an odd increasing differentiable function such that*

$$\lim_{t\to+\infty} \rho(t) = +\infty.$$

In (1.4), *making*

$$\mathfrak{a}(t) = \mathrm{e}^{-a\rho(t)}, \quad \mathfrak{c}(t) = \mathrm{e}^{-c\rho(t)}, \quad \mathfrak{b}(t) = \mathrm{e}^{-b\rho(t)}, \quad \mathfrak{d}(t) = \mathrm{e}^{-d\rho(t)}$$

and

$$\varepsilon_{\mathfrak{a}}(t) = \varepsilon_{\mathfrak{b}}(t) = \varepsilon_{\mathfrak{c}}(t) = \varepsilon_{\mathfrak{d}}(t) = D\,\mathrm{e}^{\varepsilon|\rho(t)|},$$

with

$$a, b, c, d, D, \varepsilon \in \mathbb{R} \text{ such that } D \geqslant 1 \text{ and } \varepsilon \geqslant 0,$$

we get

$$\begin{aligned} \alpha_{t,s} &= \begin{cases} D\,\mathrm{e}^{a[\rho(t)-\rho(s)]+\varepsilon|\rho(s)|} & \text{for all } (t,s) \in \mathbb{R}^2_{\geqslant}, \\ D\,\mathrm{e}^{c[\rho(s)-\rho(t)]+\varepsilon|\rho(s)|} & \text{for all } (t,s) \in \mathbb{R}^2_{\leqslant}, \end{cases} \\ \beta^+_{t,s} &= D\,\mathrm{e}^{d[\rho(t)-\rho(s)]+\varepsilon|\rho(s)|} \qquad \text{for all } (t,s) \in \mathbb{R}^2_{\geqslant}, \\ \beta^-_{t,s} &= D\,\mathrm{e}^{b[\rho(s)-\rho(t)]+\varepsilon|\rho(s)|} \qquad \text{for all } (t,s) \in \mathbb{R}^2_{\leqslant}. \end{aligned} \tag{1.5}$$

*This kind of bounds for the trichotomy, called the $\rho-$**nonuniform exponential trichotomy**, were considered by Barreira and Valls* [12, 11]*. Note that in this case condition* (1.3) *is equivalent to $a + c \geqslant 0$.*

When $\rho(t) = t$ we obtain the trichotomies considered by Barreira and Valls in [2]*, the **nonuniform exponential trichotomies**, with the bounds of the form*

$$\begin{aligned} \alpha_{t,s} &= \begin{cases} D\,\mathrm{e}^{a(t-s)+\varepsilon|s|} & \text{for all } (t,s) \in \mathbb{R}^2_{\geqslant}, \\ D\,\mathrm{e}^{c(s-t)+\varepsilon|s|} & \text{for all } (t,s) \in \mathbb{R}^2_{\leqslant}, \end{cases} \\ \beta^+_{t,s} &= D\,\mathrm{e}^{d(t-s)+\varepsilon|s|} \qquad \text{for all } (t,s) \in \mathbb{R}^2_{\geqslant}, \\ \beta^-_{t,s} &= D\,\mathrm{e}^{b(s-t)+\varepsilon|s|} \qquad \text{for all } (t,s) \in \mathbb{R}^2_{\leqslant}. \end{aligned}$$

Another example of $\rho-$nonuniform exponential trichotomies that we are going to consider is when the function ρ is given by

$$\rho(t) = \operatorname{sgn}(t)\ln(1+|t|) = \ln\left([1+|t|]^{\operatorname{sgn}(t)}\right). \tag{1.6}$$

It is clear that (1.6) *is an odd differentiable function with*

$$\rho'(t) = \frac{1}{1+|t|}$$

always positive. For this choice of ρ *in* (1.5) *we have*

$$\alpha_{t,s} = \begin{cases} D\left[\dfrac{(1+|t|)^{\operatorname{sgn}(t)}}{(1+|s|)^{\operatorname{sgn}(s)}}\right]^a (1+|s|)^\varepsilon & \text{for } (t,s) \in \mathbb{R}^2_\geqslant, \\ D\left[\dfrac{(1+|s|)^{\operatorname{sgn}(s)}}{(1+|t|)^{\operatorname{sgn}(t)}}\right]^c (1+|s|)^\varepsilon & \text{for } (t,s) \in \mathbb{R}^2_\leqslant, \end{cases}$$

$$\beta^+_{t,s} = D\left[\frac{(1+|t|)^{\operatorname{sgn}(t)}}{(1+|s|)^{\operatorname{sgn}(s)}}\right]^d (1+|s|)^\varepsilon \quad \text{for } (t,s) \in \mathbb{R}^2_\geqslant,$$

$$\beta^-_{t,s} = D\left[\frac{(1+|s|)^{\operatorname{sgn}(s)}}{(1+|t|)^{\operatorname{sgn}(t)}}\right]^b (1+|s|)^\varepsilon \quad \text{for } (t,s) \in \mathbb{R}^2_\leqslant.$$

§1.2.3 $\mu-$nonuniform polynomial trichotomies

Here we present another type of bounds for the trichotomy.

Example 1.2.3. *Let* $\mu\colon \mathbb{R} \to \mathbb{R}$ *be an odd, differentiable function with positive derivative such that* $\lim\limits_{t\to+\infty} \mu(t) = +\infty$*. Obviously* $\mu(0) = 0$*. Consider* $a, b, c, d, D, \varepsilon$ *real constants with* $D \geqslant 1$ *and* $\varepsilon \geqslant 0$*. Suppose that* (1.1) *admits bounds such as*

$$\alpha_{t,s} = \begin{cases} D(\mu(t)-\mu(s)+1)^a(|\mu(s)|+1)^\varepsilon & \text{for all } (t,s) \in \mathbb{R}^2_\geqslant, \\ D(\mu(s)-\mu(t)+1)^c(|\mu(s)|+1)^\varepsilon & \text{for all } (t,s) \in \mathbb{R}^2_\leqslant, \end{cases}$$

$$\beta^+_{t,s} = D(\mu(t)-\mu(s)+1)^d(|\mu(s)|+1)^\varepsilon \quad \text{for all } (t,s) \in \mathbb{R}^2_\geqslant,$$

$$\beta^-_{t,s} = D(\mu(s)-\mu(t)+1)^b(|\mu(s)|+1)^\varepsilon \quad \text{for all } (t,s) \in \mathbb{R}^2_\leqslant.$$

These trichotomies are called $\mu-$***nonuniform polynomial trichotomy****.*

When $\mu(t) = t$ *we name the trichotomy by* ***nonuniform polynomial trichot-***

omy:

$$\begin{aligned}
\alpha_{t,s} &= \begin{cases} D(t-s+1)^a(|s|+1)^\varepsilon & \textit{for all } (t,s) \in \mathbb{R}^2_{\geqslant}, \\ D(s-t+1)^c(|s|+1)^\varepsilon & \textit{for all } (t,s) \in \mathbb{R}^2_{\leqslant}, \end{cases} \\
\beta^+_{t,s} &= D(t-s+1)^d(|s|+1)^\varepsilon \qquad \textit{for all } (t,s) \in \mathbb{R}^2_{\geqslant}, \\
\beta^-_{t,s} &= D(s-t+1)^b(|s|+1)^\varepsilon \qquad \textit{for all } (t,s) \in \mathbb{R}^2_{\leqslant}.
\end{aligned} \tag{1.7}$$

Chapter 2

Robustness

The purpose of this chapter is the study of the robustness problem for equation (1.1), where $A\colon \mathbb{R} \to B(X)$ is a continuous map. Supposing that (1.1) admits a generalized trichotomy with bounds $\alpha_{t,s}$, $\beta^+_{t,s}$ and $\beta^-_{t,s}$, we are going to prove that equation

$$v'(t) = [A(t) + B(t)]v(t) \tag{2.1}$$

also admits a generalized trichotomy when $B\colon \mathbb{R} \to B(X)$ is a continuous function such that $B(t)$ has sufficiently small norm.

In Section 2.1 we state the main result of this chapter, in Section 2.2 we present several particular cases of the main theorem and in the last section we prove the main result. Our main goal in this chapter is to unify the several settings in the literature considering a general situation that includes a wide range of nonuniform behaviors. Moreover, it was also our goal to improve some existing results in the literature, namely the ones achieved by Barreira and Valls in [10].

It should be pointed out that the proof of Theorem 2.1.1 that we give in this work is different from the proofs given by Barreira and Valls [10] for nonuniform exponential trichotomies and by Jiang [37] for nonuniform $(\mu, \nu)-$trichotomies. In [10] and [37] the proof of robustness is made in terms of the robustness of the corresponding dichotomies. In our work we give a direct proof without using the corresponding robustness for the dichotomies. As far as we are aware, this proof is a new one for trichotomies.

The results of this chapter are from Bento and Costa [14].

§2.1 Main Theorem

First we need to introduce some notation. Denoting the perturbation function $B(t)$ by B_t, we define the constants λ, λ^+ and λ^- by

$$\lambda := \sup_{(t,s)\in\mathbb{R}^2} \frac{\lambda_{t,s}}{\alpha_{t,s}}, \quad \lambda^+ := \sup_{(t,s)\in\mathbb{R}^2_\geqslant} \frac{\lambda^+_{t,s}}{\beta^+_{t,s}} \quad \text{and} \quad \lambda^- := \sup_{(t,s)\in\mathbb{R}^2_\leqslant} \frac{\lambda^-_{t,s}}{\beta^-_{t,s}} \tag{2.2}$$

where $\lambda_{t,s}$ is given by

$$\begin{aligned}\lambda_{t,s} = &\int_{-\infty}^{s} \alpha_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \left|\int_s^t \alpha_{t,r}\|B_r\|\alpha_{r,s}\,dr\right| + \int_t^{+\infty} \beta^-_{t,r}\|B_r\|\alpha_{r,s}\,dr\\ &+ \int_{-\infty}^{t} \beta^+_{t,r}\|B_r\|\alpha_{r,s}\,dr + \int_s^{+\infty} \alpha_{t,r}\|B_r\|\beta^+_{r,s}\,dr,\end{aligned} \tag{2.3}$$

$\lambda^+_{t,s}$ is defined by

$$\begin{aligned}\lambda^+_{t,s} = &\int_{-\infty}^{s} \beta^+_{t,r}\|B_r\|\alpha_{r,s}\,dr + \int_s^t \beta^+_{t,r}\|B_r\|\beta^+_{r,s}\,dr + \int_t^{+\infty} \alpha_{t,r}\|B_r\|\beta^+_{r,s}\,dr\\ &+ \int_{-\infty}^{s} \beta^+_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_t^{+\infty} \beta^-_{t,r}\|B_r\|\beta^+_{r,s}\,dr\end{aligned} \tag{2.4}$$

and $\lambda^-_{t,s}$ is given by

$$\begin{aligned}\lambda^-_{t,s} = &\int_{-\infty}^{t} \alpha_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_t^s \beta^-_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_s^{+\infty} \beta^-_{t,r}\|B_r\|\alpha_{r,s}\,dr\\ &+ \int_{-\infty}^{t} \beta^+_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_s^{+\infty} \beta^-_{t,r}\|B_r\|\beta^+_{r,s}\,dr.\end{aligned} \tag{2.5}$$

Now we state the main theorem of this chapter which says that, under certain conditions, we can guarantee that the perturbed equation (2.1) admits a generalized trichotomy, when (1.1) admits the same type of trichotomy.

Theorem 2.1.1. *Suppose that equation* (1.1) *admits a generalized trichotomy with bounds* $\alpha_{t,s}$*,* $\beta^-_{t,s}$ *and* $\beta^+_{t,s}$ *such that*

$$\sup_{t\in\mathbb{R}} \frac{\alpha_{t,s}}{\alpha_{t,\ell}} < +\infty \qquad \text{for every } (\ell,s)\in\mathbb{R}^2, \tag{2.6}$$

$$\sup_{t\geqslant\ell} \frac{\beta^+_{t,s}}{\beta^+_{t,\ell}} < +\infty \qquad \text{for every } (\ell,s)\in\mathbb{R}^2_{\geqslant}, \tag{2.7}$$

$$\sup_{t\leqslant\ell} \frac{\beta^-_{t,s}}{\beta^-_{t,\ell}} < +\infty \qquad \text{for every } (\ell,s)\in\mathbb{R}^2_{\leqslant}. \tag{2.8}$$

Let $B\colon \mathbb{R}\to B(X)$ *continuous. If*

$$\max\left\{\lambda,\lambda^+,\lambda^-\right\} < 1 \tag{2.9}$$

where λ*,* λ^+ *and* λ^- *are defined by* (2.2)*, then equation* (2.1) *admits a generalized trichotomy with bounds* $\sigma\alpha_{t,s}$*,* $\sigma\beta^+_{t,s}$ *and* $\sigma\beta^-_{t,s}$ *with* σ *given by*

$$\sigma := \frac{1}{1-\max\{\lambda,\lambda^+,\lambda^-\}}.$$

The proof of this theorem will be given in the last section of the chapter.

§2.2 Examples

In this section we apply Theorem 2.1.1 to nonuniform $(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d})-$trichotomies, $\rho-$nonuniform exponential trichotomies and $\mu-$nonuniform polynomial trichotomies.

§2.2.1 Nonuniform $(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d})-$trichotomies

We begin with the nonuniform $(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d})-$trichotomies.

Theorem 2.2.1. *Suppose that equation* (1.1) *admits a nonuniform* $(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d})-$*trichotomy. Let* $B\colon\mathbb{R}\to B(X)$ *be a perturbation function such that*

$$\|B_t\| \leqslant \delta\frac{\gamma(t)}{\max\{\varepsilon_{\mathfrak{a}}(t),\varepsilon_{\mathfrak{b}}(t),\varepsilon_{\mathfrak{c}}(t),\varepsilon_{\mathfrak{d}}(t)\}}$$

where $\delta > 0$ *and* $\gamma\colon \mathbb{R} \to \mathbb{R}$ *is a function such that* $\displaystyle\int_{-\infty}^{+\infty} \gamma(r)\,dr < +\infty$. *If*

$$\frac{\mathfrak{a}(r)\ \mathfrak{d}(s)\,\varepsilon_{\mathfrak{d}}(s)}{\mathfrak{a}(s)\ \mathfrak{d}(r)\,\varepsilon_{\mathfrak{a}}(s)} \leqslant M \quad \text{for } (r,s) \in \mathbb{R}^2_{\geqslant}, \tag{2.10}$$

$$\frac{\mathfrak{c}(s)\,\mathfrak{b}(r)\,\varepsilon_{\mathfrak{b}}(s)}{\mathfrak{c}(r)\,\mathfrak{b}(s)\,\varepsilon_{\mathfrak{c}}(s)} \leqslant M \quad \text{for } (r,s) \in \mathbb{R}^2_{\leqslant}, \tag{2.11}$$

$$\frac{\mathfrak{c}(s)\,\mathfrak{d}(s)\,\varepsilon_{\mathfrak{c}}(r)\,\varepsilon_{\mathfrak{d}}(s)}{\mathfrak{c}(r)\,\mathfrak{d}(r)} \leqslant M \min\{\varepsilon_{\mathfrak{c}}(r)\,\varepsilon_{\mathfrak{d}}(s), \varepsilon_{\mathfrak{c}}(s)\,\varepsilon_{\mathfrak{d}}(r)\} \quad \text{for } (r,s) \in \mathbb{R}^2_{\geqslant}, \tag{2.12}$$

$$\frac{\mathfrak{a}(r)\,\mathfrak{b}(r)\,\varepsilon_{\mathfrak{a}}(r)\,\varepsilon_{\mathfrak{b}}(s)}{\mathfrak{a}(s)\,\mathfrak{b}(s)} \leqslant M \min\{\varepsilon_{\mathfrak{a}}(r)\,\varepsilon_{\mathfrak{b}}(s), \varepsilon_{\mathfrak{a}}(s)\,\varepsilon_{\mathfrak{b}}(r)\} \quad \text{for } (r,s) \in \mathbb{R}^2_{\leqslant} \tag{2.13}$$

for some $M \geqslant 1$ *such that*

$$\delta(M^2 + M + 1)\int_{-\infty}^{+\infty} \gamma(r)\,dr < 1, \tag{2.14}$$

then equation (2.1) *admits a nonuniform trichotomy with bounds of the form* $\sigma\alpha_{t,s}$, $\sigma\beta^{+}_{t,s}$ *and* $\sigma\beta^{-}_{t,s}$.

Proof: For this type of bounds it is clear that (2.6), (2.7) and (2.8) are verified. Clearly, conditions (2.10), (2.11), (2.12) and (2.13) are equivalent to

$$\beta^{+}_{r,s} \leqslant M\alpha_{r,s} \quad \text{for} \quad (r,s) \in \mathbb{R}^2_{\geqslant}, \tag{2.15}$$

$$\beta^{-}_{r,s} \leqslant M\alpha_{r,s} \quad \text{for} \quad (r,s) \in \mathbb{R}^2_{\leqslant}, \tag{2.16}$$

$$\alpha_{s,r}\beta^{+}_{r,s} \leqslant M \min\{\varepsilon_{\mathfrak{c}}(r)\,\varepsilon_{\mathfrak{d}}(s), \varepsilon_{\mathfrak{d}}(r)\,\varepsilon_{\mathfrak{c}}(s)\} \quad \text{for} \quad (r,s) \in \mathbb{R}^2_{\geqslant}, \tag{2.17}$$

$$\alpha_{s,r}\beta^{-}_{r,s} \leqslant M \min\{\varepsilon_{\mathfrak{a}}(r)\,\varepsilon_{\mathfrak{b}}(s), \varepsilon_{\mathfrak{a}}(s)\,\varepsilon_{\mathfrak{b}}(r)\} \quad \text{for} \quad (r,s) \in \mathbb{R}^2_{\leqslant}, \tag{2.18}$$

respectively. We must prove that the integrals present in the formulas (2.3), (2.4) and (2.5) are finite and that (2.9) is verified. By definition we have

$$\alpha_{t,r}\alpha_{r,s} = \begin{cases} \alpha_{t,s}\,\varepsilon_{\mathfrak{a}}(r) & \text{if } t > r > s \\ \alpha_{t,s}\,\varepsilon_{\mathfrak{c}}(r) & \text{if } s > r > t, \end{cases} \tag{2.19}$$

$$\beta^{+}_{t,r}\beta^{+}_{r,s} = \beta^{+}_{t,s}\varepsilon_{\mathfrak{d}}(r) \text{ if } t \geqslant r \geqslant s, \tag{2.20}$$

$$\beta^{-}_{t,r}\beta^{-}_{r,s} = \beta^{-}_{t,s}\varepsilon_{\mathfrak{b}}(r) \text{ if } s \geqslant r \geqslant t. \tag{2.21}$$

Therefore, using (2.15), (2.17), (2.18), (2.16) and the last three equalities we have

$$\alpha_{t,r}\beta^+_{r,s} \leqslant \begin{cases} \alpha_{t,s}\alpha_{s,r}\beta^+_{r,s}/\varepsilon_{\mathfrak{c}}(s) & \text{if } r \geqslant s \geqslant t \\ M\alpha_{t,r}\alpha_{r,s} & \text{if } t \geqslant r \geqslant s \\ \alpha_{t,r}\beta^+_{r,t}\beta^+_{t,s}/\varepsilon_{\mathfrak{d}}(t) & \text{if } r \geqslant t \geqslant s \end{cases} \leqslant \begin{cases} M\alpha_{t,s}\varepsilon_{\mathfrak{d}}(r) & \text{if } r \geqslant s \geqslant t \\ M\alpha_{t,s}\varepsilon_{\mathfrak{a}}(r) & \text{if } t \geqslant r \geqslant s \\ M\beta^+_{t,s}\varepsilon_{\mathfrak{c}}(r) & \text{if } r \geqslant t \geqslant s \end{cases} \tag{2.22}$$

$$\alpha_{t,r}\beta^-_{r,s} \leqslant \begin{cases} \alpha_{t,s}\alpha_{s,r}\beta^-_{r,s}/\varepsilon_{\mathfrak{a}}(s) & \text{if } t \geqslant s \geqslant r \\ M\alpha_{t,r}\alpha_{r,s} & \text{if } s \geqslant r \geqslant t \\ \alpha_{t,r}\beta^-_{r,t}\beta^-_{t,s}/\varepsilon_{\mathfrak{b}}(t) & \text{if } s \geqslant t \geqslant r \end{cases} \leqslant \begin{cases} M\alpha_{t,s}\varepsilon_{\mathfrak{b}}(r) & \text{if } t \geqslant s \geqslant r \\ M\alpha_{t,s}\varepsilon_{\mathfrak{c}}(r) & \text{if } s \geqslant r \geqslant t \\ M\beta^-_{t,s}\varepsilon_{\mathfrak{a}}(r) & \text{if } s \geqslant t \geqslant r \end{cases} \tag{2.23}$$

$$\beta^+_{t,r}\alpha_{r,s} \leqslant \begin{cases} \beta^+_{t,r}\alpha_{r,t}\alpha_{t,s}/\varepsilon_{\mathfrak{c}}(t) & \text{if } s \geqslant t \geqslant r \\ M\alpha_{t,r}\alpha_{r,s} & \text{if } t \geqslant r \geqslant s \\ \beta^+_{t,s}\beta^+_{s,r}\alpha_{r,s}/\varepsilon_{\mathfrak{d}}(s) & \text{if } t \geqslant s \geqslant r \end{cases} \leqslant \begin{cases} M\alpha_{t,s}\varepsilon_{\mathfrak{d}}(r) & \text{if } s \geqslant t \geqslant r \\ M\alpha_{t,s}\varepsilon_{\mathfrak{a}}(r) & \text{if } t \geqslant r \geqslant s \\ M\beta^+_{t,s}\varepsilon_{\mathfrak{c}}(r) & \text{if } t \geqslant s \geqslant r \end{cases} \tag{2.24}$$

$$\beta^-_{t,r}\alpha_{r,s} \leqslant \begin{cases} \beta^-_{t,s}\beta^-_{s,r}\alpha_{r,s}/\varepsilon_{\mathfrak{b}}(s) & \text{if } r \geqslant s \geqslant t \\ \beta^-_{t,r}\alpha_{r,t}\alpha_{t,s}/\varepsilon_{\mathfrak{a}}(t) & \text{if } r \geqslant t \geqslant s \\ M\alpha_{t,r}\alpha_{r,s} & \text{if } s \geqslant r \geqslant t \end{cases} \leqslant \begin{cases} M\beta^-_{t,s}\varepsilon_{\mathfrak{a}}(r) & \text{if } r \geqslant s \geqslant t \\ M\alpha_{t,s}\varepsilon_{\mathfrak{b}}(r) & \text{if } r \geqslant t \geqslant s \\ M\alpha_{t,s}\varepsilon_{\mathfrak{c}}(r) & \text{if } s \geqslant r \geqslant t \end{cases} \tag{2.25}$$

$$\begin{aligned} \beta^+_{t,r}\beta^-_{r,s} &= \begin{cases} \dfrac{\beta^+_{t,s}\beta^+_{s,r}\beta^-_{r,s}}{\varepsilon_{\mathfrak{d}}(s)} & \text{if } t \geqslant s \geqslant r \\ \dfrac{\beta^+_{t,r}\beta^-_{r,t}\beta^-_{t,s}}{\varepsilon_{\mathfrak{b}}(t)} & \text{if } s \geqslant t \geqslant r \end{cases} \\ &\leqslant \begin{cases} M\dfrac{\beta^+_{t,s}\beta^+_{s,r}\alpha_{r,s}}{\varepsilon_{\mathfrak{d}}(s)} & \text{if } t \geqslant s \geqslant r \\ M\dfrac{\alpha_{t,r}\beta^-_{r,t}\beta^-_{t,s}}{\varepsilon_{\mathfrak{b}}(t)} & \text{if } s \geqslant t \geqslant r \end{cases} \\ &\leqslant \begin{cases} M^2\beta^+_{t,s}\varepsilon_{\mathfrak{c}}(r) & \text{if } t \geqslant s \geqslant r \\ M^2\beta^-_{t,s}\varepsilon_{\mathfrak{a}}(r) & \text{if } s \geqslant t \geqslant r \end{cases} \end{aligned} \tag{2.26}$$

and finally

$$\begin{aligned}
\beta^-_{t,r}\beta^+_{r,s} &= \begin{cases} \beta^-_{t,s}\beta^-_{s,r}\beta^+_{r,s}/\varepsilon_{\mathfrak{b}}(s) & \text{if } r \geqslant s \geqslant t \\ \beta^-_{t,r}\beta^+_{r,t}\beta^+_{t,s}/\varepsilon_{\mathfrak{d}}(t) & \text{if } r \geqslant t \geqslant s \end{cases} \\
&\leqslant \begin{cases} M\beta^-_{t,s}\beta^-_{s,r}\alpha_{r,s}/\varepsilon_{\mathfrak{b}}(s) & \text{if } r \geqslant s \geqslant t \\ M\alpha_{t,r}\beta^+_{r,t}\beta^+_{t,s}/\varepsilon_{\mathfrak{d}}(t) & \text{if } r \geqslant t \geqslant s \end{cases} \\
&\leqslant \begin{cases} M^2\beta^-_{t,s}\varepsilon_{\mathfrak{a}}(r) & \text{if } r \geqslant s \geqslant t \\ M^2\beta^+_{t,s}\varepsilon_{\mathfrak{c}}(r) & \text{if } r \geqslant t \geqslant s. \end{cases}
\end{aligned} \tag{2.27}$$

We are now in conditions of prove that λ, λ^+ and λ^- are finite. So for every $(t,s) \in \mathbb{R}^2_\geqslant$ using (2.3) and inequalities (2.19), (2.22), (2.23), (2.24) and (2.25) we have

$$\begin{aligned}
\lambda_{t,s} &= \int_{-\infty}^{s} \alpha_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_{s}^{t} \alpha_{t,r}\|B_r\|\alpha_{r,s}\,dr + \int_{t}^{+\infty} \beta^-_{t,r}\|B_r\|\alpha_{r,s}\,dr + \\
&\quad + \int_{-\infty}^{t} \beta^+_{t,r}\|B_r\|\alpha_{r,s}\,dr + \int_{s}^{+\infty} \alpha_{t,r}\|B_r\|\beta^+_{r,s}\,dr \\
&\leqslant \left(M^2+M+1\right)\alpha_{t,s}\int_{-\infty}^{+\infty} \max\left\{\varepsilon_{\mathfrak{a}}(r),\varepsilon_{\mathfrak{b}}(r),\varepsilon_{\mathfrak{c}}(r),\varepsilon_{\mathfrak{d}}(r)\right\}\|B_r\|\,dr \\
&\leqslant \delta\left(M^2+M+1\right)\alpha_{t,s}\int_{-\infty}^{+\infty}\gamma(r)\,dr.
\end{aligned}$$

On the other hand, for every $(t,s) \in \mathbb{R}^2_\leqslant$

$$\begin{aligned}
\lambda_{t,s} &\leqslant \left(M^2+M+1\right)\alpha_{t,s}\int_{-\infty}^{+\infty} \max\left\{\varepsilon_{\mathfrak{a}}(r),\varepsilon_{\mathfrak{b}}(r),\varepsilon_{\mathfrak{c}}(r),\varepsilon_{\mathfrak{d}}(r)\right\}\|B_r\|\,dr \\
&\leqslant \delta\left(M^2+M+1\right)\alpha_{t,s}\int_{-\infty}^{+\infty}\gamma(r)\,dr
\end{aligned}$$

and therefore

$$\lambda = \sup_{(t,s)\in\mathbb{R}^2} \frac{\lambda_{t,s}}{\alpha_{t,s}} \leqslant \delta(M^2+M+1)\int_{-\infty}^{+\infty}\gamma(r)\,dr < 1$$

which implies that $\lambda < +\infty$.

In a similar way we can prove that λ^+ is finite. By (2.4) and the inequalities above (2.24), (2.20), (2.22), (2.26) and (2.27) we have for every $(t,s) \in \mathbb{R}^2_{\geqslant}$

$$\begin{aligned}
\lambda^+_{t,s} &= \int_{-\infty}^{s} \beta^+_{t,r}\|B_r\|\alpha_{r,s}\,dr + \int_{s}^{t} \beta^+_{t,r}\|B_r\|\beta^+_{r,s}\,dr + \int_{t}^{+\infty} \alpha_{t,r}\|B_r\|\beta^+_{r,s}\,dr +\\
&\quad + \int_{-\infty}^{s} \beta^+_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_{t}^{+\infty} \beta^-_{t,r}\|B_r\|\beta^+_{r,s}\,dr\\
&\leqslant \int_{-\infty}^{s} M\beta^+_{t,s}\varepsilon_{\mathfrak{c}}(r)\|B_r\|\,dr + \int_{s}^{t} \beta^+_{t,s}\varepsilon_{\mathfrak{d}}(r)\|B_r\|\,dr + \int_{t}^{+\infty} M\beta^+_{t,s}\varepsilon_{\mathfrak{c}}(r)\|B_r\|\,dr +\\
&\quad + \int_{-\infty}^{s} M^2\beta^+_{t,s}\varepsilon_{\mathfrak{c}}(r)\|B_r\|\,dr + \int_{t}^{+\infty} M^2\beta^+_{t,s}\varepsilon_{\mathfrak{c}}(r)\|B_r\|\,dr\\
&\leqslant \delta\left(M^2+M\right)\beta^+_{t,s}\int_{-\infty}^{+\infty}\gamma(r)\,dr.
\end{aligned}$$

Finally, using (2.5) and the inequalities (2.23), (2.21), (2.25), (2.26) and (2.27) we get, for every $(t,s) \in \mathbb{R}^2_{\leqslant}$

$$\begin{aligned}
\lambda^-_{t,s} &= \int_{-\infty}^{t} \alpha_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_{t}^{s} \beta^-_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_{s}^{+\infty} \beta^-_{t,r}\|B_r\|\alpha_{r,s}\,dr\\
&\quad + \int_{-\infty}^{t} \beta^+_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_{s}^{+\infty} \beta^-_{t,r}\|B_r\|\beta^+_{r,s}\,dr\\
&\leqslant \int_{-\infty}^{t} M\beta^-_{t,s}\varepsilon_{\mathfrak{a}}(r)\|B_r\|\,dr + \int_{t}^{s} \beta^-_{t,s}\varepsilon_{\mathfrak{b}}(r)\|B_r\|\,dr + \int_{s}^{+\infty} M\beta^-_{t,s}\varepsilon_{\mathfrak{a}}(r)\|B_r\|\,dr\\
&\quad + \int_{-\infty}^{t} M^2\beta^-_{t,s}\varepsilon_{\mathfrak{a}}(r)\|B_r\|\,dr + \int_{s}^{+\infty} M^2\beta^-_{t,s}\varepsilon_{\mathfrak{a}}(r)\|B_r\|\,dr\\
&\leqslant \delta\left(M^2+M\right)\beta^-_{t,s}\int_{-\infty}^{+\infty}\gamma(r)\,dr,
\end{aligned}$$

which implies that $\lambda^- < +\infty$.

Hence, for

$$\delta < \frac{1}{(M^2+M+1)\displaystyle\int_{-\infty}^{+\infty}\gamma(r)\,dr}$$

the condition (2.9), $\max\{\lambda,\lambda^+,\lambda^-\} < 1$, is verified and so all the conditions required by the theorem are satisfied. This completes the proof. □

§2.2.2 $\rho-$nonuniform exponential trichotomies

Now, we are going to apply the last result to $\rho-$nonuniform exponential trichotomies.

Theorem 2.2.2. *Suppose that* (1.1) *admits a* $\rho-$*nonuniform exponential trichotomy. Let* $B\colon \mathbb{R} \to B(X)$ *be a continuous perturbation function such that*

$$\|B_t\| \leqslant \frac{\delta\rho'(t)\,\mathrm{e}^{-\gamma|\rho(t)|}}{D\,\mathrm{e}^{\varepsilon|\rho(t)|}} = \frac{\delta}{D}\rho'(t)\,\mathrm{e}^{-(\gamma+\varepsilon)|\rho(t)|}$$

for some $\delta, \gamma > 0$ *such that* $\delta < \frac{\gamma}{6}$. *If*

$$b \leqslant c, \quad d \leqslant a, \quad c+d \leqslant 0, \quad a+b \leqslant 0 \quad \text{and} \quad \varepsilon - \gamma < 0,$$

then equation (2.1) *admits a* $\rho-$*nonuniform exponential trichotomy with bounds of the form* $\sigma\alpha_{t,s}$, $\sigma\beta^+_{t,s}$ *and* $\sigma\beta^-_{t,s}$.

Proof: The bounds of the form (1.5) are a particular case of the bounds of the previous example if we consider

$$\mathfrak{a}(t) = \mathrm{e}^{-a\rho(t)}, \quad \mathfrak{c}(t) = \mathrm{e}^{-c\rho(t)}, \quad \mathfrak{b}(t) = \mathrm{e}^{-b\rho(t)}, \quad \mathfrak{d}(t) = \mathrm{e}^{-d\rho(t)}$$

and

$$\varepsilon_{\mathfrak{a}}(t) = \varepsilon_{\mathfrak{b}}(t) = \varepsilon_{\mathfrak{c}}(t) = \varepsilon_{\mathfrak{d}}(t) = D\,\mathrm{e}^{\varepsilon|\rho(t)|},$$

where $D \geqslant 1$, $\varepsilon \geqslant 0$ and $a, b, c, d \in \mathbb{R}$. Then the conditions required (2.16), (2.15), (2.17) and (2.18) become, respectively,

$$b \leqslant c, \quad d \leqslant a, \quad c+d \leqslant 0 \quad \text{and} \quad a+b \leqslant 0,$$

with $M = 1$. Moreover, since $\gamma(t) = \rho'(t)\,\mathrm{e}^{-\gamma|\rho(t)|}$ and $\gamma > \varepsilon$ then

$$\int_{-\infty}^{+\infty} \gamma(r)\,dr = \frac{2}{\gamma} < +\infty$$

and so (2.14) becomes $\delta < \frac{\gamma}{6}$. □

Corollary 2.2.3. *Suppose that* (1.1) *admits a nonuniform exponential trichotomy and* $B\colon \mathbb{R} \to B(X)$ *a continuous perturbation function such that*

$$\|B_t\| \leqslant \frac{\delta}{D}\,\mathrm{e}^{-(\varepsilon+\gamma)|t|},$$

with $\gamma \in \mathbb{R}$ *and for some* $0 < \delta < \frac{\gamma}{6}$. *It is obvious that then equation* (2.1) *admits a nonuniform exponential trichotomy with bounds of the form* $\sigma\alpha_{t,s}$, $\sigma\beta^+_{t,s}$ *and* $\sigma\beta^-_{t,s}$ *if*

$$b \leqslant c, \; d \leqslant a, \; c+d \leqslant 0, \; a+b \leqslant 0 \quad \text{and} \quad \varepsilon - \gamma < 0.$$

Here we accomplished a better result than Barreira and Valls had previously achieved in [10]. Note that in [10] is used, in our notation, the condition

$$\varepsilon < \min\left\{\frac{-d-c}{2}, \frac{-b-a}{2}\right\}$$

which is more restricted than our hypotheses.

§2.2.3 $\mu-$nonuniform polynomial trichotomies

Now we will consider $\mu-$nonuniform polynomial trichotomies.

Theorem 2.2.4. *Suppose that equation* (1.1) *admits a $\mu-$nonuniform polynomial trichotomy. Let $B\colon \mathbb{R} \to B(X)$ be a continuous perturbation function such that*

$$\|B_t\| \leqslant \delta\mu'(t)(|\mu(t)|+1)^{-\gamma},$$

for some $\gamma \in \mathbb{R}$. If $0 < \delta < \dfrac{|\varepsilon-\gamma+1|}{6D}$,

$$b \leqslant c \leqslant 0, \quad d \leqslant a \leqslant 0 \qquad \textit{and} \qquad \varepsilon - \gamma + 1 < 0,$$

then equation (2.1) *admits a $\mu-$nonuniform polynomial trichotomy with bounds of the form $\sigma\alpha_{t,s}$, $\sigma\beta^+_{t,s}$ and $\sigma\beta^-_{t,s}$.*

Proof: To prove that the conditions of Theorem 2.1.1 are verified we need to compute the integrals in formulas (2.3), (2.4) and (2.5).
It follows,
for every $t \leqslant r \leqslant s$,

$$\alpha_{t,r}\alpha_{r,s} \leqslant D\alpha_{t,s}(|\mu(r)|+1)^{\varepsilon} \text{ because } c \leqslant 0,$$

$$\alpha_{t,r}\beta^-_{r,s} \leqslant D\alpha_{t,s}(|\mu(r)|+1)^{\varepsilon} \text{ because } b \leqslant c \leqslant 0,$$

$$\beta^-_{t,r}\alpha_{r,s} \leqslant \alpha_{t,r}\alpha_{r,s} \leqslant D\alpha_{t,s}(|\mu(r)|+1)^{\varepsilon} \text{ because } b \leqslant c \leqslant 0,$$

for every $r \leqslant t \leqslant s$,

$$\alpha_{t,r}\beta^-_{r,s} \leqslant D\alpha_{t,s}(|\mu(r)|+1)^{\varepsilon} \text{ because } b \leqslant c \leqslant 0 \text{ and } a \leqslant 0,$$

$$\beta^+_{t,r}\alpha_{r,s} \leqslant \beta^+_{t,r}\alpha_{t,s} \leqslant D\alpha_{t,s}(|\mu(r)|+1)^{\varepsilon} \text{ because } c \leqslant 0 \text{ and } d \leqslant 0,$$

for every $t \leqslant s \leqslant r$,

$$\alpha_{t,r}\beta^+_{r,s} \leqslant D\alpha_{t,s}(|\mu(r)|+1)^\varepsilon \text{ because } c \leqslant 0 \text{ and } d \leqslant 0,$$

$$\beta^-_{t,r}\alpha_{r,s} \leqslant D\beta^-_{t,r}(|\mu(s)|+1)^\varepsilon \leqslant D\alpha_{t,s}(|\mu(r)|+1)^\varepsilon \text{ because } a \leqslant 0 \text{ and } b \leqslant c \leqslant 0.$$

We can also write,

for every $r \leqslant s \leqslant t$,

$$\beta^+_{t,r}\alpha_{r,s} \leqslant \alpha_{t,r}\alpha_{r,s} \leqslant D\alpha_{t,s}(|\mu(r)|+1)^\varepsilon \text{ because } c \leqslant 0 \text{ and } d \leqslant a,$$

$$\alpha_{t,r}\beta^-_{r,s} \leqslant D\alpha_{t,s}(|\mu(r)|+1)^\varepsilon \text{ because } a \leqslant 0 \text{ and } b \leqslant 0,$$

for every $s \leqslant r \leqslant t$,

$$\alpha_{t,r}\alpha_{r,s} \leqslant D\alpha_{t,s}(|\mu(r)|+1)^\varepsilon \text{ because } a \leqslant 0,$$

$$\beta^+_{t,r}\alpha_{r,s} \leqslant \alpha_{t,r}\alpha_{r,s} \leqslant D\alpha_{t,s}(|\mu(r)|+1)^\varepsilon \text{ because } d \leqslant a \leqslant 0,$$

$$\alpha_{t,r}\beta^+_{r,s} \leqslant \alpha_{t,r}\alpha_{r,s} \leqslant D\alpha_{t,s}(|\mu(r)|+1)^\varepsilon \text{ because } d \leqslant a \leqslant 0,$$

and for every $s \leqslant t \leqslant r$,

$$\alpha_{t,r}\beta^+_{r,s} \leqslant \alpha_{t,r}\alpha_{t,s} \leqslant D\alpha_{t,s}(|\mu(r)|+1)^\varepsilon \text{ because } d \leqslant a \text{ and } c \leqslant 0,$$

$$\beta^-_{t,r}\alpha_{r,s} \leqslant \beta^-_{t,r}\alpha_{t,s} \leqslant D\alpha_{t,s}(|\mu(r)|+1)^\varepsilon \text{ because } a \leqslant 0 \text{ and } b \leqslant 0.$$

Therefore, the integrals we need to compute $\lambda_{t,s}$ become,

for every $(t,s) \in \mathbb{R}^2_\leqslant$, if $b \leqslant c \leqslant 0$, $a \leqslant 0$ and $d \leqslant 0$

$$\int_{-\infty}^t \alpha_{t,r}\|B_r\|\beta^-_{r,s}\,dr \leqslant D\delta\alpha_{t,s}\int_{-\infty}^t \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr, \tag{2.28}$$

$$\int_t^s \alpha_{t,r}\|B_r\|\beta^-_{r,s}\,dr \leqslant D\delta\alpha_{t,s}\int_t^s \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr, \tag{2.29}$$

$$\int_t^s \alpha_{t,r}\|B_r\|\alpha_{r,s}\,dr \leqslant D\delta\alpha_{t,s}\int_t^s \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr, \tag{2.30}$$

$$\int_t^s \beta^-_{t,r}\|B_r\|\alpha_{r,s}\,dr \leqslant D\delta\alpha_{t,s}\int_t^s \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr, \tag{2.31}$$

$$\int_s^{+\infty} \beta^-_{t,r}\|B_r\|\alpha_{r,s}\,dr \leqslant D\delta\alpha_{t,s}\int_s^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr, \tag{2.32}$$

$$\int_{-\infty}^t \beta^+_{t,r}\|B_r\|\alpha_{r,s}\,dr \leqslant D\delta\alpha_{t,s}\int_{-\infty}^t \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr, \tag{2.33}$$

$$\int_s^{+\infty} \alpha_{t,r}\|B_r\|\beta^+_{r,s}\,dr \leqslant D\delta\alpha_{t,s}\int_s^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr. \tag{2.34}$$

At last, for every $(t,s) \in \mathbb{R}^2_{\geqslant}$, if we consider $d \leqslant a \leqslant 0$, $b \leqslant 0$ and $c \leqslant 0$ then

$$\int_{-\infty}^{s} \alpha_{t,r} \|B_r\| \beta^{-}_{r,s} \, dr \leqslant D\delta\alpha_{t,s} \int_{-\infty}^{s} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma} \, dr, \tag{2.35}$$

$$\int_{s}^{t} \alpha_{t,r} \|B_r\| \alpha_{r,s} \, dr \leqslant D\delta\alpha_{t,s} \int_{s}^{t} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma} \, dr, \tag{2.36}$$

$$\int_{t}^{+\infty} \beta^{-}_{t,r} \|B_r\| \alpha_{r,s} \, dr \leqslant D\delta\alpha_{t,s} \int_{t}^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma} \, dr, \tag{2.37}$$

$$\int_{-\infty}^{s} \beta^{+}_{t,r} \|B_r\| \alpha_{r,s} \, dr \leqslant D\delta\alpha_{t,s} \int_{-\infty}^{s} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma} \, dr, \tag{2.38}$$

$$\int_{s}^{t} \beta^{+}_{t,r} \|B_r\| \alpha_{r,s} \, dr \leqslant D\delta\alpha_{t,s} \int_{s}^{t} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma} \, dr, \tag{2.39}$$

$$\int_{s}^{t} \alpha_{t,r} \|B_r\| \beta^{+}_{r,s} \, dr \leqslant D\delta\alpha_{t,s} \int_{s}^{t} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma} \, dr \tag{2.40}$$

$$\int_{t}^{+\infty} \alpha_{t,r} \|B_r\| \beta^{+}_{r,s} \, dr \leqslant D\delta\alpha_{t,s} \int_{t}^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma} \, dr. \tag{2.41}$$

Now if we consider

for every $r \leqslant s \leqslant t$ we have

$$\beta^{+}_{t,r}\alpha_{r,s} \leqslant D\beta^{+}_{t,s}(|\mu(r)|+1)^{\varepsilon} \text{ because } c \leqslant 0 \text{ and } d \leqslant 0,$$

for every $s \leqslant r \leqslant t$,

$$\beta^{+}_{t,r}\beta^{+}_{r,s} \leqslant D\beta^{+}_{t,s}(|\mu(r)|+1)^{\varepsilon} \text{ because } d \leqslant 0,$$

for every $s \leqslant t \leqslant r$,

$$\alpha_{t,r}\beta^{+}_{r,s} \leqslant \alpha_{t,r}\beta^{+}_{t,s} \leqslant D\beta^{+}_{t,s}(|\mu(r)|+1)^{\varepsilon} \text{ because } c \leqslant 0 \text{ and } d \leqslant 0,$$

for every $r \leqslant s \leqslant t$,

$$\beta^{+}_{t,r}\beta^{-}_{r,s} \leqslant D\beta^{+}_{t,s}(|\mu(r)|+1)^{\varepsilon} \text{ because } b \leqslant 0 \text{ and } d \leqslant 0,$$

and for every $s \leqslant t \leqslant r$,

$$\beta^{-}_{t,r}\beta^{+}_{r,s} \leqslant \beta^{-}_{t,r}\beta^{+}_{t,s} \leqslant D\beta^{+}_{t,s}(|\mu(r)|+1)^{\varepsilon} \text{ because } c \leqslant 0 \text{ and } d \leqslant 0.$$

Therefore if $b \leqslant 0$, $c \leqslant 0$ and $d \leqslant 0$ we get for every $(t,s) \in \mathbb{R}^2_{\geqslant}$

$$\int_{-\infty}^{s} \beta^+_{t,r}\|B_r\|\alpha_{r,s}\,dr \leqslant D\delta\beta^+_{t,s}\int_{-\infty}^{s}\mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr, \tag{2.42}$$

$$\int_{s}^{t} \beta^+_{t,r}\|B_r\|\beta^+_{r,s}\,dr \leqslant D\delta\beta^+_{t,s}\int_{s}^{t}\mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr, \tag{2.43}$$

$$\int_{t}^{+\infty} \alpha_{t,r}\|B_r\|\beta^+_{r,s}\,dr \leqslant D\delta\beta^+_{t,s}\int_{t}^{+\infty}\mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr, \tag{2.44}$$

$$\int_{-\infty}^{s} \beta^+_{t,r}\|B_r\|\beta^-_{r,s}\,dr \leqslant D\delta\beta^+_{t,s}\int_{-\infty}^{s}\mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr \tag{2.45}$$

$$\int_{t}^{+\infty} \beta^-_{t,r}\|B_r\|\beta^+_{r,s}\,dr \leqslant D\delta\beta^+_{t,s}\int_{t}^{+\infty}\mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr. \tag{2.46}$$

Finally,

for every $r \leqslant t \leqslant s$,

$$\alpha_{t,r}\beta^-_{r,s} \leqslant \alpha_{t,r}\beta^-_{t,s} \leqslant D\beta^-_{t,s}(|\mu(r)|+1)^{\varepsilon} \text{ if } a \leqslant 0 \text{ and } b \leqslant 0,$$
$$\beta^+_{t,r}\beta^-_{r,s} \leqslant \beta^+_{t,r}\beta^-_{t,s} \leqslant D\beta^-_{t,s}(|\mu(r)|+1)^{\varepsilon} \text{ if } b \leqslant 0 \text{ and } d \leqslant 0,$$

for every $t \leqslant r \leqslant s$,

$$\beta^-_{t,r}\beta^-_{r,s} \leqslant D\beta^-_{t,s}(|\mu(r)|+1)^{\varepsilon} \text{ if } b \leqslant 0,$$

and for every $t \leqslant s \leqslant r$,

$$\beta^-_{t,r}\alpha_{r,s} \leqslant \beta^-_{t,s}\alpha_{r,s} \leqslant D\beta^-_{t,s}(|\mu(r)|+1)^{\varepsilon} \text{ if } a \leqslant 0 \text{ and } b \leqslant 0,$$
$$\beta^-_{t,r}\beta^+_{r,s} \leqslant D\beta^-_{t,s}(|\mu(r)|+1)^{\varepsilon} \text{ if } b \leqslant 0 \text{ and } d \leqslant 0.$$

Therefore if $a, b, d \leqslant 0$ we get for every $(t,s) \in \mathbb{R}^2_{\leqslant}$

$$\int_{-\infty}^{t} \alpha_{t,r}\|B_r\|\beta^-_{r,s}\,dr \leqslant D\delta\beta^-_{t,s}\int_{-\infty}^{t}\mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr, \tag{2.47}$$

$$\int_{t}^{s} \beta^-_{t,r}\|B_r\|\beta^-_{r,s}\,dr \leqslant D\delta\beta^-_{t,s}\int_{t}^{s}\mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr, \tag{2.48}$$

$$\int_{s}^{+\infty} \beta^-_{t,r}\|B_r\|\alpha_{r,s}\,dr \leqslant D\delta\beta^-_{t,s}\int_{s}^{+\infty}\mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr, \tag{2.49}$$

$$\int_{-\infty}^{t} \beta^+_{t,r}\|B_r\|\beta^-_{r,s}\,dr \leqslant D\delta\beta^-_{t,s}\int_{-\infty}^{t}\mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr \tag{2.50}$$

$$\int_{s}^{+\infty} \beta^-_{t,r}\|B_r\|\beta^+_{r,s}\,dr \leqslant D\delta\beta^-_{t,s}\int_{s}^{+\infty}\mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr. \tag{2.51}$$

Now we are in conditions to estimate λ, λ^+ and λ^-.

So for every $(t,s) \in \mathbb{R}^2_{\geqslant}$ by (2.3), (2.35), (2.36), (2.37), (2.38), (2.39), (2.40) and (2.41) we can put

$$\begin{aligned}
\lambda_{t,s} &= \int_{-\infty}^{s} \alpha_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_s^t \alpha_{t,r}\|B_r\|\alpha_{r,s}\,dr + \int_t^{+\infty} \beta^-_{t,r}\|B_r\|\alpha_{r,s}\,dr \\
&\quad + \int_{-\infty}^{t} \beta^+_{t,r}\|B_r\|\alpha_{r,s}\,dr + \int_s^{+\infty} \alpha_{t,r}\|B_r\|\beta^+_{r,s}\,dr \\
&\leqslant D\delta\ \alpha_{t,s}\left(\int_{-\infty}^{s} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr + \int_s^t \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr\right. \\
&\quad + \int_t^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr + \int_{-\infty}^{t} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr \\
&\quad \left. + \int_s^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr\right) \\
&\leqslant D\delta\ \alpha_{t,s}\left(2\int_{-\infty}^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr + \int_s^t \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr\right) \\
&\leqslant 6D\delta\ \alpha_{t,s}\int_0^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr
\end{aligned}$$

and if we require that $\varepsilon-\gamma+1<0$, $d \leqslant a \leqslant 0$ and $b, c \leqslant 0$ we can write

$$\lambda_{t,s} \leqslant 6D\delta\alpha_{t,s}\frac{1}{|\varepsilon-\gamma+1|}, \quad \text{for every } (t,s) \in \mathbb{R}^2_{\geqslant}.$$

On the other hand, for every $(t,s) \in \mathbb{R}^2_{\leqslant}$ and by (2.28), (2.29), (2.30), (2.31), (2.32), (2.33) and (2.34) we have

$$\begin{aligned}
\lambda_{t,s} &\leqslant D\delta\ \alpha_{t,s}\left(\int_{-\infty}^{s} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr + \int_t^s \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr\right. \\
&\quad + \int_t^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr + \int_{-\infty}^{t} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr \\
&\quad \left. + \int_s^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr\right) \\
&\leqslant D\delta\ \alpha_{t,s}\left(2\int_{-\infty}^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr + \int_t^s \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr\right) \\
&\leqslant 6D\delta\ \alpha_{t,s}\int_0^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr
\end{aligned}$$

and therefore if we require that $\varepsilon-\gamma+1<0$, $b \leqslant c \leqslant 0$ and $a, d \leqslant 0$ we can write

$$\lambda_{t,s} \leqslant 6D\delta\ \alpha_{t,s}\frac{1}{|\varepsilon-\gamma+1|}, \text{ for every } (t,s) \in \mathbb{R}^2_{\leqslant}$$

and so

$$\lambda_{t,s} \leqslant 6D\delta \ \alpha_{t,s}\frac{1}{|\varepsilon-\gamma+1|}, \text{ for every } (t,s) \in \mathbb{R}^2,$$

which implies that by (2.2) , $\lambda < +\infty$. In a similar way we can prove that λ^+ is finite. Since by (2.42), (2.43), (2.44), (2.45) and (2.46) we can say, for every $(t,s) \in \mathbb{R}^2_\geqslant$ that

$$\begin{aligned}
\lambda^+_{t,s} &= \int_{-\infty}^{s} \beta^+_{t,r}\|B_r\|\alpha_{r,s}\,dr + \int_s^t \beta^+_{t,r}\|B_r\|\beta^+_{r,s}\,dr + \int_t^{+\infty} \alpha_{t,r}\|B_r\|\beta^+_{r,s}\,dr \\
&\quad + \int_{-\infty}^{s} \beta^+_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_t^{+\infty} \beta^-_{t,r}\|B_r\|\beta^+_{r,s}\,dr \\
&\leqslant D\delta\beta^+_{t,s}\bigg(\int_{-\infty}^{s} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr + \int_s^t \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr \\
&\quad + \int_t^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr + \int_{-\infty}^{s} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr \\
&\quad + \int_t^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr\bigg) \\
&= D\delta\beta^+_{t,s}\bigg(\int_{-\infty}^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr + \int_{-\infty}^{s} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr \\
&\quad + \int_t^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr\bigg) \\
&\leqslant 2D\delta\beta^+_{t,s}\int_{-\infty}^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr
\end{aligned}$$

where, considering $\varepsilon-\gamma+1<0$ and $b,c,d \leqslant 0$ we get

$$\lambda^+_{t,s} \leqslant 4D\delta\beta^+_{t,s}\frac{1}{|\varepsilon-\gamma+1|}, \text{ for every } (t,s) \in \mathbb{R}^2_\geqslant$$

which implies that, by (2.2), $\lambda^+ < +\infty$.

Finally we must prove that λ^- is finite.

By (2.47), (2.48), (2.49), (2.50) and (2.51) we get

$$\begin{aligned}
\lambda^-_{t,s} &= \int_{-\infty}^{t} \alpha_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_t^s \beta^-_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_s^{+\infty} \beta^-_{t,r}\|B_r\|\alpha_{r,s}\,dr \\
&\quad + \int_{-\infty}^{t} \beta^+_{t,r}\|B_r\|\beta^-_{r,s}\,dr + \int_s^{+\infty} \beta^-_{t,r}\|B_r\|\beta^+_{r,s}\,dr \\
&\leqslant D\;\delta\beta^-_{t,s}\left(\int_{-\infty}^{t} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr + \int_t^s \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr\right. \\
&\quad + \int_s^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr + \int_{-\infty}^{t} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr \\
&\quad \left. + \int_s^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr\right) \\
&= D\;\delta\beta^-_{t,s}\left[\int_{-\infty}^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr + \int_{-\infty}^{t} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr\right. \\
&\quad \left. + \int_s^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr\right] \\
&\leqslant 2D\;\delta\beta^-_{t,s}\int_{-\infty}^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr,
\end{aligned}$$

and if we suppose $\varepsilon-\gamma+1<0$ and $a, b, d \leqslant 0$ it follows

$$\lambda^-_{t,s} \leqslant 4D\;\delta\beta^-_{t,s}\frac{1}{|\varepsilon-\gamma+1|}, \text{ for every } (t,s) \in \mathbb{R}^2_{\leqslant}$$

which implies that, by (2.2), $\lambda^- < +\infty$. Hence, if $\delta < \dfrac{|\varepsilon-\gamma+1|}{6D}$ then condition (2.9), $\max\{\lambda, \lambda^+, \lambda^-\} < 1$, is verified and so all the conditions required by the theorem are satisfied. This completes the proof. □

In the next corollary we will consider nonuniform polynomial trichotomies, i.e., we make $\mu(t) = t$ in the last theorem and the result follows immediately.

Corollary 2.2.5. *Suppose that equation* (1.1) *admits a nonuniform polynomial trichotomy. Let* $B\colon \mathbb{R} \to B(X)$ *be a continuous perturbation function such that*

$$\|B_t\| \leqslant \delta(|t|+1)^{-\gamma},$$

for some $\gamma \in \mathbb{R}$. *If* $\delta < \dfrac{|\varepsilon-\gamma+1|}{6D}$

$$b \leqslant c \leqslant 0, \quad d \leqslant a \leqslant 0 \qquad \text{and} \qquad \varepsilon-\gamma+1<0,$$

then equation (2.1) *admits a nonuniform polynomial trichotomy with bounds of the form* $\sigma\alpha_{t,s}$, $\sigma\beta^+_{t,s}$ *and* $\sigma\beta^-_{t,s}$.

§2.3 Proof of Theorem 2.1.1

Before proving this theorem we need to introduce more notation and state several lemmas that are indispensable to accomplish the proof.

§2.3.1 Auxiliary Lemmas

For every $s \in \mathbb{R}$, let

$$\Omega^0_s = \left\{U = (U_{t,s})_{t\in\mathbb{R}} : U_{t,s} \in B(X),\ t \mapsto U_{t,s} \text{ is continuous, } \sup_{t\in\mathbb{R}} \frac{\|U_{t,s}\|}{\alpha_{t,s}} < +\infty\right\},$$
$$\Omega^+_s = \left\{V^+ = \left(V^+_{t,s}\right)_{t\geqslant s} : V^+_{t,s} \in B(X),\ t \mapsto V^+_{t,s} \text{ is continuous, } \sup_{t\geqslant s} \frac{\|V^+_{t,s}\|}{\beta^+_{t,s}} < +\infty\right\},$$
$$\Omega^-_s = \left\{V^- = \left(V^-_{t,s}\right)_{t\leqslant s} : V^-_{t,s} \in B(X),\ t \mapsto V^-_{t,s} \text{ is continuous, } \sup_{t\leqslant s} \frac{\|V^-_{t,s}\|}{\beta^-_{t,s}} < +\infty\right\}.$$

The pairs $(\Omega^0_s, \|\cdot\|^0_s)$, $(\Omega^+_s, \|\cdot\|^+_s)$ and $(\Omega^-_s, \|\cdot\|^-_s)$ are Banach spaces, where

$$\|U\|^0_s := \sup_{t\in\mathbb{R}} \frac{\|U_{t,s}\|}{\alpha_{t,s}}, \quad \|V^+\|^+_s := \sup_{t\geqslant s} \frac{\|V^+_{t,s}\|}{\beta^+_{t,s}} \quad \text{and} \quad \|V^-\|^-_s := \sup_{t\leqslant s} \frac{\|V^-_{t,s}\|}{\beta^-_{t,s}}.$$

For each $s \in \mathbb{R}$, let Ω_s be the Banach space $\Omega_s = \Omega^0_s \times \Omega^+_s \times \Omega^-_s$ equipped with the norm

$$\|(U, V^+, V^-)\|_s = \max\left\{\|U\|^0_s, \|V^+\|^+_s, \|V^-\|^-_s\right\}.$$

Letting, for every $t, s \in \mathbb{R}$

$$C_{t,s} = T_{t,s}P_sB_s, \quad D^+_{t,s} = T_{t,s}Q^+_sB_s \quad \text{and} \quad D^-_{t,s} = T_{t,s}Q^-_sB_s \tag{2.52}$$

from (**D1**), (**D2**) and (**D3**) we have

$$\begin{aligned}
&\|C_{t,s}\| \leqslant \|T_{t,s}P_s\|\ \|B_s\| \leqslant \alpha_{t,s}\|B_s\| && \text{for every } (t,s) \in \mathbb{R}^2,\\
&\|D^+_{t,s}\| \leqslant \|T_{t,s}Q^+_s\|\ \|B_s\| \leqslant \beta^+_{t,s}\|B_s\| && \text{for every } (t,s) \in \mathbb{R}^2_\geqslant,\\
&\|D^-_{t,s}\| \leqslant \|T_{t,s}Q^-_s\|\ \|B_s\| \leqslant \beta^-_{t,s}\|B_s\| && \text{for every } (t,s) \in \mathbb{R}^2_\leqslant.
\end{aligned} \tag{2.53}$$

Lemma 2.3.1. *Let $s \in \mathbb{R}$. For every $(U, V^+, V^-) \in \Omega_s$, define*

$$J_s(U, V^+, V^-) = \big(J_{t,s}(U, V^+, V^-)\big)_{t\in\mathbb{R}} \tag{2.54}$$

where

$$\begin{aligned} J_{t,s}(U, V^+, V^-) = &-\int_{-\infty}^{s} C_{t,r} V^-_{r,s}\, dr + \int_s^t C_{t,r} U_{r,s}\, dr - \int_t^{+\infty} D^-_{t,r} U_{r,s}\, dr \\ &+ \int_{-\infty}^{t} D^+_{t,r} U_{r,s}\, dr + \int_s^{+\infty} C_{t,r} V^+_{r,s}\, dr. \end{aligned} \tag{2.55}$$

Then J_s is a bounded linear operator from Ω_s into Ω_s^0 and

$$\|J_s\| \leqslant \lambda \tag{2.56}$$

where λ is given by (2.2).

Proof: From (2.2), (2.3), (2.55) and the definitions (2.52), (2.52) and (2.52) we obtain

$$\begin{aligned}
&\|J_{t,s}(U, V^+, V^-)\| \\
&\leqslant \int_{-\infty}^{s} \|C_{t,r}\|\ \|V^-_{r,s}\|\, dr + \left|\int_s^t \|C_{t,r}\|\ \|U_{r,s}\|\, dr\right| + \int_t^{+\infty} \|D^-_{t,r}\|\ \|U_{r,s}\|\, dr \\
&\quad + \int_{-\infty}^{t} \|D^+_{t,r}\|\ \|U_{r,s}\|\, dr + \int_s^{+\infty} \|C_{t,r}\|\ \|V^+_{r,s}\|\, dr \\
&\leqslant \int_{-\infty}^{s} \alpha_{t,r}\|B_r\|\beta^-_{r,s}\ \|V^-\|^-_s\, dr + \left|\int_s^t \alpha_{t,r}\|B_r\|\ \alpha_{r,s}\|U\|^0_s\, dr\right| \\
&\quad + \int_t^{+\infty} \beta^-_{t,r}\|B_r\|\ \alpha_{r,s}\|U\|^0_s\, dr + \int_{-\infty}^{t} \beta^+_{t,r}\|B_r\|\ \alpha_{r,s}\|U\|^0_s\, dr \\
&\quad + \int_s^{+\infty} \alpha_{t,r}\|B_r\|\ \beta^+_{r,s}\ \|V^+\|^+_s\, dr \\
&\leqslant \Bigg(\int_{-\infty}^{s} \alpha_{t,r}\|B_r\|\beta^-_{r,s}\, dr + \left|\int_s^t \alpha_{t,r}\|B_r\|\alpha_{r,s}\, dr\right| + \int_t^{+\infty} \beta^-_{t,r}\|B_r\|\alpha_{r,s}\, dr \\
&\quad + \int_{-\infty}^{t} \beta^+_{t,r}\|B_r\|\alpha_{r,s}\, dr + \int_s^{+\infty} \alpha_{t,r}\|B_r\|\beta^+_{r,s}\, dr\Bigg)\ \|(U, V^+, V^-)\|_s \\
&= \lambda_{t,s}\, \|(U, V^+, V^-)\|_s \\
&\leqslant \lambda\, \alpha_{t,s}\, \|(U, V^+, V^-)\|_s\ .
\end{aligned}$$

Therefore, $J_{t,s}$ is a bounded linear operator from Ω_s into $B(X)$ and

$$\|J_{t,s}\| \leqslant \lambda\, \alpha_{t,s} \quad \text{for all } t \in \mathbb{R}.$$

This proves that J_s is a linear bounded operator from Ω_s into Ω_s^0 and verifies condition (2.56). $\square$

Lemma 2.3.2. *Let $s \in \mathbb{R}$. For every $(U, V^+, V^-) \in \Omega_s$, define*

$$L_s^+(U, V^+, V^-) = \left(L_{t,s}^+(U, V^+, V^-)\right)_{t \geqslant s}, \tag{2.57}$$

where

$$\begin{aligned} L_{t,s}^+(U, V^+, V^-) = &- \int_{-\infty}^{s} D_{t,r}^+ U_{r,s}\, dr + \int_s^t D_{t,r}^+ V_{r,s}^+\, dr - \int_t^{+\infty} C_{t,r} V_{r,s}^+\, dr \\ &- \int_{-\infty}^{s} D_{t,r}^+ V_{r,s}^-\, dr - \int_t^{+\infty} D_{t,r}^- V_{r,s}^+\, dr. \end{aligned} \tag{2.58}$$

Then L_s^+ is a bounded linear operator from Ω_s into Ω_s^+ and

$$\|L_s^+\| \leqslant \lambda^+ \tag{2.59}$$

where λ^+ is given by (2.2).

Proof: Let $s \in \mathbb{R}$. From (2.58), (2.53) and (2.2) we have

$$\begin{aligned} &\|L_{t,s}^+(U, V^+, V^-)\| \\ &\leqslant \int_{-\infty}^{s} \|D_{t,r}^+\|\, \|U_{r,s}\|\, dr + \int_s^t \|D_{t,r}^+\|\, \|V_{r,s}^+\|\, dr + \int_t^{+\infty} \|C_{t,r}\|\, \|V_{r,s}^+\|\, dr \\ &\quad + \int_{-\infty}^{s} \|D_{t,r}^+\|\, \|V_{r,s}^-\|\, dr + \int_t^{+\infty} \|D_{t,r}^-\|\, \|V_{r,s}^+\|\, dr \\ &\leqslant \left(\int_{-\infty}^{s} \beta_{t,r}^+ \|B_r\| \alpha_{r,s}\, dr + \int_s^t \beta_{t,r}^+ \|B_r\| \beta_{r,s}^+\, dr + \int_t^{+\infty} \alpha_{t,r} \|B_r\| \beta_{r,s}^+\, dr \right. \\ &\quad \left. + \int_{-\infty}^{s} \beta_{t,r}^+ \|B_r\| \beta_{r,s}^-\, dr + \int_t^{+\infty} \beta_{t,r}^- \|B_r\| \beta_{r,s}^+\, dr \right) \|(U, V^+, V^-)\|_s \\ &= \lambda_{t,s}^+ \|(U, V^+, V^-)\|_s \\ &\leqslant \lambda^+ \beta_{t,s}^+ \|(U, V^+, V^-)\|_s\,. \end{aligned}$$

Thus, $L_{t,s}^+$ is a bounded linear operator from Ω_s into $B(X)$ such that

$$\|L_{t,s}^+\| \leqslant \lambda^+ \beta_{t,s}^+ \quad \text{for all} \quad t \geqslant s.$$

Therefore L_s^+ is a linear operator from Ω_s into Ω_s^+ and verifies the condition (2.59). $\square$

Lemma 2.3.3. *Let $s \in \mathbb{R}$. For every $(U, V^+, V^-) \in \Omega_s$, define*

$$L_s^-(U, V^+, V^-) = \left(L_{t,s}^-(U, V^+, V^-)\right)_{t \leqslant s}, \tag{2.60}$$

where

$$\begin{aligned} L_{t,s}^-(U, V^+, V^-) = \int_{-\infty}^{t} C_{t,r} V_{r,s}^- \, dr - \int_t^s D_{t,r}^- V_{r,s}^- \, dr + \int_s^{+\infty} D_{t,r}^- U_{r,s} \, dr \\ + \int_{-\infty}^{t} D_{t,r}^+ V_{r,s}^- \, dr + \int_s^{+\infty} D_{t,r}^- V_{r,s}^+ \, dr. \end{aligned} \tag{2.61}$$

Then L_s^- is a bounded linear operator from Ω_s into Ω_s^- and

$$\|L_s^-\| \leqslant \lambda^- \tag{2.62}$$

where λ^- is given by (2.2)

Proof: From (2.61), (2.53) and (2.2) we have

$$\begin{aligned} &\|L_{t,s}^-(U, V^+, V^-)\| \\ &\leqslant \int_{-\infty}^{t} \|C_{t,r}\| \; \|V_{r,s}^-\| \, dr + \int_t^s \|D_{t,r}^-\| \;\; \|V_{r,s}^-\| \, dr + \int_s^{+\infty} \|D_{t,r}^-\| \;\; \|U_{r,s}\| \, dr \\ &\quad + \int_{-\infty}^{t} \|D_{t,r}^+\| \; \|V_{r,s}^-\| \, dr + \int_s^{+\infty} \|D_{t,r}^-\| \; \|V_{r,s}^+\| \, dr \\ &\leqslant \left(\int_{-\infty}^{t} \alpha_{t,r} \|B_r\| \beta_{r,s}^- \, dr + \int_t^s \beta_{t,r}^- \|B_r\| \beta_{r,s}^- \, dr + \int_s^{+\infty} \beta_{t,r}^- \|B_r\| \alpha_{r,s} \; dr \right. \\ &\quad \left. + \int_{-\infty}^{t} \beta_{t,r}^+ \|B_r\| \beta_{r,s}^- \; dr + \int_s^{+\infty} \beta_{t,r}^- \|B_r\| \beta_{r,s}^+ \; dr \right) \|(U, V^+, V^-)\|_s \\ &= \lambda_{t,s}^- \; \|(U, V^+, V^-)\|_s \\ &\leqslant \; \lambda^- \; \beta_{t,s}^- \; \|(U, V^+, V^-)\|_s \; . \end{aligned}$$

Therefore, $L_{t,s}^-$ is a bounded linear operator from Ω_s into $B(X)$ and

$$\|L_{t,s}\| \leqslant \lambda^- \, \beta_{t,s}^-,$$

and this proves that L_s^- is a linear operator from Ω_s into Ω_s^- that verifies (2.62). □

Lemma 2.3.4. *Let $s \in \mathbb{R}$. For all $(U, V^+, V^-) \in \Omega_s$, let*

$$T_s(U, V^+, V^-) = \left(J_s(U, V^+, V^-), L_s^+(U, V^+, V^-), L_s^-(U, V^+, V^-)\right)$$

where J_s, L_s^+ *and* L_s^- *are defined by* (2.54), (2.57) *and* (2.60) *respectively. Then* T_s *is a linear operator from* Ω_s *into* Ω_s *such that*

$$\|T_s\| \leqslant \max\left\{\lambda, \lambda^+, \lambda^-\right\} < 1.$$

Proof: It is obvious considering Lemmas 2.3.1, 2.3.2 and 2.3.3. □

Lemma 2.3.5. *Let* $s \in \mathbb{R}$. *Then there exists a unique* $(U, V^+, V^-) \in \Omega_s$ *such that*

$$U_{t,s} = T_{t,s}P_s + J_{t,s}(U, V^+, V^-) \quad \textit{for all } (t,s) \in \mathbb{R}^2, \tag{2.63}$$

$$V_{t,s}^+ = T_{t,s}Q_s^+ + L_{t,s}^+(U, V^+, V^-) \ \textit{for all } (t,s) \in \mathbb{R}^2_{\geqslant}, \tag{2.64}$$

$$V_{t,s}^- = T_{t,s}Q_s^- + L_{t,s}^-(U, V^+, V^-) \ \textit{for all } (t,s) \in \mathbb{R}^2_{\leqslant}. \tag{2.65}$$

Moreover,

$$\|U_{t,s}\| \leqslant \sigma \ \alpha_{t,s} \quad \textit{for every } (t,s) \in \mathbb{R}^2,$$

$$\|V_{t,s}^+\| \leqslant \sigma \ \beta_{t,s}^+ \quad \textit{for every } (t,s) \in \mathbb{R}^2_{\geqslant}$$

and

$$\|V_{t,s}^-\| \leqslant \sigma \ \beta_{t,s}^- \quad \textit{for every } (t,s) \in \mathbb{R}^2_{\leqslant},$$

where $\sigma = \dfrac{1}{1 - \max\{\lambda, \lambda^-, \lambda^+\}}$.

Proof: Let $s \in \mathbb{R}$ and define Γ_s by

$$\Gamma_s = \left((T_{t,s}P_s)_{t\in\mathbb{R}},\ (T_{t,s}Q_s^+)_{t\geqslant s},\ (T_{t,s}Q_s^-)_{t\leqslant s}\right).$$

Then, considering (**D1**), (**D2**) and (**D3**), we can say that $\Gamma_s \in \Omega_s$ and $\|\Gamma_s\|_s \leqslant 1$. Let $\Upsilon_s \colon \Omega_s \to \Omega_s$ be the operator defined by

$$\Upsilon_s = \Gamma_s \ + \ T_s.$$

Since T_s is a linear contraction with Lipschitz constant $\max\{\lambda, \lambda^+, \lambda^-\}$, Υ_s is also a contraction with the same Lipschitz constant. Therefore, since Ω_s is a Banach space, by the Banach fixed point Theorem, Υ_s has a unique fixe point, that we will call (U, V^+, V^-) and obviously verifies (2.63), (2.64) and (2.65).

Also from the proof of the Banach fixed point Theorem, we have

$$\|(U, V^+, V^-) - (0,0,0)\|_s \leqslant \sigma\|\Upsilon_s(0,0,0) - (0,0,0)\|_s = \sigma\|\Gamma_s\|_s \leqslant \sigma.$$

Considering the definition of $\|\cdot\|_s$, we have $\|U_{t,s}\| \leqslant \sigma\alpha_{t,s}$ for every $(t,s) \in \mathbb{R}^2$, $\|V^+_{t,s}\| \leqslant \sigma\beta^+_{t,s}$ for every $(t,s) \in \mathbb{R}^2_{\geqslant}$ and $\|V^-_{t,s}\| \leqslant \sigma\beta^-_{t,s}$ for every $(t,s) \in \mathbb{R}^2_{\leqslant}$. □

Lemma 2.3.6. *Let $s \in \mathbb{R}$. The point $(U, V^+, V^-) \in \Omega_s$ is a solution of the equation* (2.1).

Proof: From (2.63) and (2.55), for all $(t,s) \in \mathbb{R}^2$, we have

$$\begin{aligned} U_{t,s} = T_{t,s}P_s &- \int_{-\infty}^{s} C_{t,r}V^-_{r,s}\,dr + \int_s^t C_{t,r}U_{r,s}\,dr - \int_t^{+\infty} D^-_{t,r}U_{r,s}\,dr \\ &+ \int_{-\infty}^{t} D^+_{t,r}U_{r,s}\,dr + \int_s^{+\infty} C_{t,r}V^+_{r,s}\,dr. \end{aligned}$$

Noting that the right-hand of (2.63) is differentiable in t, then

$$\begin{aligned} \frac{\partial U_{t,s}}{\partial t} &= v'(t)P_s - v'(t)\int_{-\infty}^{s} P_rB_rV^-_{r,s}\,dr + v'(t)\int_s^t P_rB_rU_{r,s}\,dr \\ &\quad + P_tB_tU_{t,s} - v'(t)\int_t^{+\infty} Q^-_rB_rU_{r,s}\,dr + Q^-_tB_tU_{t,s} \\ &\quad + v'(t)\int_{-\infty}^{t} Q^+_rB_rU_{r,s}\,dr + Q^+_tB_tU_{t,s} + v'(t)\int_s^{+\infty} P_rB_rV^+_{r,s}\,dr \\ &= A_tT_{t,s}P_s - A_t\int_{-\infty}^{s} C_{t,r}V^-_{r,s}\,dr + A_t\int_s^t C_{t,r}U_{r,s}\,dr \\ &\quad + P_tB_tU_{t,s} - A_t\int_t^{+\infty} D^-_{t,r}U_{r,s}\,dr + Q^-_tB_tU_{t,s} \\ &\quad + A_t\int_{-\infty}^{t} D^+_{t,r}U_{r,s}\,dr + Q^+_tB_tU_{t,s} + A_t\int_s^{+\infty} C_{t,r}V^+_{r,s}\,dr \\ &= A_tU_{t,s} + P_tB_tU_{t,s} + Q^-_tB_tU_{t,s} + Q^+_tB_tU_{t,s} \\ &= A_tU_{t,s} + B_tU_{t,s} \\ &= (A_t + B_t)\,U_{t,s}. \end{aligned}$$

In a similar way, from (2.64) and (2.58) we have for all $(t,s) \in \mathbb{R}^2_{\geqslant}$,

$$\begin{aligned} V^+_{t,s} &= T_{t,s}Q^+_s - \int_{-\infty}^{s} D^+_{t,r}U_{r,s}\,dr + \int_s^t D^+_{t,r}V^+_{r,s}\,dr - \int_t^{+\infty} C_{t,r}V^+_{r,s}\,dr \\ &\quad - \int_{-\infty}^{s} D^+_{t,r}V^-_{r,s}\,dr - \int_t^{+\infty} D^-_{t,r}V^+_{r,s}\,dr \end{aligned}$$

and

$$\begin{aligned} \frac{\partial V^+_{t,s}}{\partial t} &= A_tT_{t,s}Q^+_s - v'(t)\int_{-\infty}^{s} Q^+_rB_rU_{r,s}\,dr + v'(t)\int_s^t Q^+_rB_rV^+_{r,s}\,dr \\ &\quad + Q^+_tB_tV^+_{t,s} - v'(t)\int_t^{+\infty} P_rB_rV^+_{r,s}\,dr + P_tB_tV^+_{t,s} \\ &\quad - v'(t)\int_{-\infty}^{s} Q^+_rB_rV^-_{r,s}\,dr - v'(t)\int_t^{+\infty} Q^-_rB_rV^+_{r,s}\,dr + Q^-_tB_tV^+_{t,s} \\ &= (A_t + B_t)\,V^+_{t,s}. \end{aligned}$$

Finally, from (2.65) and (2.61) we have that

$$\begin{aligned} V^-_{t,s} &= T_{t,s}Q^-_s + \int_{-\infty}^{t} C_{t,r}V^-_{r,s}\,dr - \int_t^s D^-_{t,r}V^-_{r,s}\,dr + \int_s^{+\infty} D^-_{t,r}U_{r,s}\,dr \\ &\quad + \int_{-\infty}^{t} D^+_{t,r}V^-_{r,s}\,dr + \int_s^{+\infty} D^-_{t,r}V^+_{r,s}\,dr \end{aligned}$$

for all $(t,s) \in \mathbb{R}^2_{\leqslant}$ and so

$$\begin{aligned} \frac{\partial V^-_{t,s}}{\partial t} &= A_tT_{t,s}Q^-_s + v'(t)\int_{-\infty}^{t} P_rB_rV^-_{r,s}\,dr + P_tB_tV^-_{t,s} \\ &\quad - v'(t)\int_t^s Q^-_rB_rV^-_{r,s}\,dr + Q^-_tB_tV^-_{t,s} + v'(t)\int_s^{+\infty} Q^-_rB_rU_{r,s}\,dr \\ &\quad + v'(t)\int_{-\infty}^{t} Q^+_rB_rV^-_{r,s}\,dr + Q^+_tB_tV^-_{t,s} + v'(t)\int_s^{+\infty} Q^-_rB_rV^+_{r,s}\,dr \\ &= A_tV^-_{t,s} + B_tV^-_{t,s} \\ &= (A_t + B_t)\,V^-_{t,s} \end{aligned}$$

and this completes the proof. □

Lemma 2.3.7. *Let $(t,s) \in \mathbb{R}^2$. Then we have*

$$\begin{cases} U_{\ell,s} = U_{\ell,t}U_{t,s} & \text{for every} \quad (\ell,t) \in \mathbb{R}^2, \\ V^+_{\ell,t}U_{t,s} = 0 & \text{for every} \quad (\ell,t) \in \mathbb{R}^2_{\geqslant}, \\ V^-_{\ell,t}U_{t,s} = 0 & \text{for every} \quad (\ell,t) \in \mathbb{R}^2_{\leqslant}. \end{cases}$$

Proof: Let $(t,s)\in\mathbb{R}^2$ and define

$$\begin{cases} W_{\ell,t} = U_{\ell,t}U_{t,s} - U_{\ell,s}\ , & (\ell,t)\in\mathbb{R}^2 \\ Z^+_{\ell,t} = V^+_{\ell,t}U_{t,s}, & (\ell,t)\in\mathbb{R}^2_{\geqslant} \\ Z^-_{\ell,t} = V^-_{\ell,t}U_{t,s}\ , & (\ell,t)\in\mathbb{R}^2_{\leqslant}. \end{cases}$$

We must prove that

$$W = (W_{\ell,t})_{\ell\in\mathbb{R}} \in \Omega^0_t, \quad Z^+ = \left(Z^+_{\ell,t}\right)_{\ell\geqslant t} \in \Omega^+_t \quad \text{and} \quad Z^- = \left(Z^-_{\ell,t}\right)_{\ell\leqslant t} \in \Omega^-_t.$$

From

$$\|W_{\ell,t}\| \leqslant \|U_{\ell,t}\|\|U_{t,s}\| + \|U_{\ell,s}\| \leqslant \sigma\ \alpha_{\ell,t}\|U_{t,s}\| + \sigma\ \alpha_{\ell,s} \leqslant \sigma\ \alpha_{\ell,t}\left(\|U_{t,s}\| + \frac{\alpha_{\ell,s}}{\alpha_{\ell,t}}\right)$$

and using (2.6) it follows that $W\in\Omega^0_t$. On the other hand, for every $(\ell,t)\in\mathbb{R}^2_{\geqslant}$ we have

$$\|Z^+_{\ell,t}\| \leqslant \|V^+_{\ell,t}\|\ \|U_{t,s}\| \leqslant \sigma\ \beta^+_{\ell,t}\|U_{t,s}\|.$$

Clearly we have $Z^+\in\Omega^+_t$. Finally, since for every $(\ell,t)\in\mathbb{R}^2_{\leqslant}$

$$\|Z^-_{\ell,t}\| \leqslant \|V^-_{\ell,t}\|\ \|U_{t,s}\| \leqslant \sigma\ \beta^-_{\ell,t}\|U_{t,s}\|,$$

then $Z^-\in\Omega^-_t$. Hence $(W,Z^+,Z^-)\in\Omega_t$. Now we have to prove that (W,Z^+,Z^-) is a fixed point of T_t. From (2.63), for every $(\ell,t)\in\mathbb{R}^2$, we have

$$W_{\ell,t} = U_{\ell,t}U_{t,s} - U_{\ell,s} = T_{\ell,t}P_tU_{t,s} + J_{\ell,t}(U,V^+,V^-)U_{t,s} - U_{\ell,s}$$

and since

$$T_{\ell,t}P_tU_{t,s} = T_{\ell,s}P_s - \int_{-\infty}^{s} C_{\ell,r}V^-_{r,s}\,dr + \int_s^t C_{\ell,r}U_{r,s}\,dr + \int_s^{+\infty} C_{\ell,r}V^+_{r,s}\,dr$$

we have

$$T_{\ell,t}P_tU_{t,s} - U_{\ell,s} = -\int_t^{\ell} C_{\ell,r}U_{r,s}\,dr + \int_{\ell}^{+\infty} D^-_{\ell,r}U_{r,s}\,dr - \int_{-\infty}^{\ell} D^+_{\ell,r}U_{r,s}\,dr$$

and this implies

$$W_{\ell,t} = J_{\ell,t}(W, Z^+, Z^-).$$

On the other hand, from (2.64)

$$Z^+_{\ell,t} = V^+_{\ell,t}U_{t,s} = T_{\ell,t}Q^+_t U_{t,s} + L^+_{\ell,t}(U, V^+, V^-)U_{t,s}$$

and because

$$T_{\ell,t}Q^+_t U_{t,s} = \int_{-\infty}^{t} D^+_{\ell,r}U_{r,s}\,dr,$$

we have

$$Z^+_{\ell,t} = L^+_{\ell,t}(W, Z^+, Z^-).$$

Finally, using (2.65)

$$Z^-_{\ell,t} = V^-_{\ell,t}U_{t,s} = T_{\ell,t}Q^-_t U_{t,s} + L^-_{\ell,t}(U, V^+, V^-)U_{t,s}$$

and because

$$T_{\ell,t}Q^-_t U_{t,s} = -\int_{t}^{+\infty} D^-_{\ell,r}U_{r,s}\,dr,$$

we obtain

$$Z^-_{\ell,t} = L^-_{\ell,t}(W, Z^+, Z^-).$$

Therefore (W, Z^+, Z^-) is a fixed point of the linear contraction $T_t = (J_t, L^+_t, L^-_t)$ and since T_t has a unique fixed point, it must be the zero of Ω_t. So, we must have

$$\begin{cases} U_{\ell,t}U_{t,s} = U_{\ell,s} & \text{for every } (\ell,t) \in \mathbb{R}^2, \\ V^+_{\ell,t}U_{t,s} = 0 & \text{for every } (\ell,t) \in \mathbb{R}^2_{\geqslant}, \\ V^-_{\ell,t}U_{t,s} = 0 & \text{for every } (\ell,t) \in \mathbb{R}^2_{\leqslant}. \end{cases}$$

□

Lemma 2.3.8. *Let $(t,s) \in \mathbb{R}^2_{\leqslant}$. Then we have*

$$\begin{cases} U_{\ell,t}V^-_{t,s} = 0 & \textit{for every } (\ell,t) \in \mathbb{R}^2, \\ V^+_{\ell,t}V^-_{t,s} = 0 & \textit{for every } (\ell,t) \in \mathbb{R}^2_{\geqslant}, \\ V^-_{\ell,s} = V^-_{\ell,t}V^-_{t,s} & \textit{for every } (\ell,t) \in \mathbb{R}^2_{\leqslant}. \end{cases}$$

Proof: Let $(t,s) \in \mathbb{R}^2_{\leqslant}$ and define

$$\begin{cases} W_{\ell,t} = U_{\ell,t} V^-_{t,s} & \text{for every } (\ell,t) \in \mathbb{R}^2, \\ Z^+_{\ell,t} = V^+_{\ell,t} V^-_{t,s} & \text{for every } (\ell,t) \in \mathbb{R}^2_{\geqslant}, \\ Z^-_{\ell,t} = V^-_{\ell,t} V^-_{t,s} - V^-_{\ell,s} & \text{for every } (\ell,t) \in \mathbb{R}^2_{\leqslant}. \end{cases}$$

Since

$$\|W_{\ell,t}\| \leqslant \|U_{\ell,t}\| \; \|V^-_{t,s}\| \leqslant \sigma \alpha_{\ell,t} \|V^-_{t,s}\|$$

we can say that $W \in \Omega^0_t$. From

$$\|Z^+_{\ell,t}\| \leqslant \|V^+_{\ell,t}\| \; \|V^-_{t,s}\| \leqslant \sigma \beta^+_{\ell,t} \|V^-_{t,s}\|$$

it follows $Z^+ \in \Omega^+_t$. At last,

$$\|Z^-_{\ell,t}\| \leqslant \|V^-_{\ell,t}\| \; \|V^-_{t,s}\| + \|V^-_{\ell,s}\| \leqslant \sigma \beta^-_{\ell,t} \|V^-_{t,s}\| + \sigma \beta^-_{\ell,s} \leqslant \sigma \beta^-_{\ell,t} \left(\|V^-_{t,s}\| + \frac{\beta^-_{\ell,s}}{\beta^-_{\ell,t}} \right)$$

and from (2.8) it follows that $Z^- \in \Omega^-_t$.

Therefore $(W, Z^+, Z^-) \in \Omega_t$. From (2.63) we have

$$W_{\ell,t} = U_{\ell,t} V^-_{t,s} = T_{\ell,t} P_t V^-_{t,s} + J_{\ell,t}(U, V^+, V^-) V^-_{t,s}$$

and since

$$T_{\ell,t} P_t V^-_{t,s} = \int_{-\infty}^{t} C_{\ell,r} V^-_{r,s} \, dr$$

we have

$$W_{\ell,t} = J_{\ell,t}(W, Z^+, Z^-).$$

On the other hand, we have

$$Z^+_{\ell,t} = V^+_{\ell,t} V^-_{t,s} = T_{\ell,t} Q^+_t V^-_{t,s} + L^+_{\ell,t}(U, V^+, V^-) V^-_{t,s}.$$

But

$$T_{\ell,t} Q^+_t V^-_{t,s} = \int_{-\infty}^{t} D^+_{\ell,r} V^-_{r,s} \, dr$$

and so we can write

$$Z^+_{\ell,t} = L^+_{\ell,t}(W, Z^+, Z^-).$$

Finally, for $(\ell, t) \in \mathbb{R}^2_{\geqslant}$ we have

$$\begin{aligned} Z^-_{\ell,t} &= V^-_{\ell,t}V^-_{t,s} - V^-_{\ell,s} \\ &= T_{\ell,t}Q^-_t V^-_{t,s} + L^-_{\ell,t}(U, V^+, V^-)V^-_{t,s} - \ V^-_{\ell,s} \end{aligned}$$

and because

$$T_{\ell,t}Q^-_t V^-_{t,s} = T_{\ell,s}Q^-_s - \int_t^s D^-_{\ell,r}V^-_{r,s}\,dr + \int_s^{+\infty} D^-_{\ell,r}U_{r,s}\,dr + \int_s^{+\infty} D^-_{\ell,r}V^+_{r,s}\,dr,$$

we have

$$T_{\ell,t}Q^-_t V^-_{t,s} - V^-_{\ell,s} = -\int_{-\infty}^{\ell} C_{\ell,r}V^-_{r,s}\,dr + \int_\ell^t D^-_{\ell,r}V^-_{r,s}\,dr - \int_{-\infty}^{\ell} D^+_{\ell,r}V^-_{r,s}\,dr$$

and this implies

$$Z^-_{\ell,t} = L^-_{\ell,t}(W, Z^+, Z^-).$$

Once more, (W, Z^+, Z^-) is a fixed point of the linear contraction and since T_t has a unique fixed point, it must be the zero of Ω_t. So, it follows that

$$\begin{cases} U_{\ell,t}V^-_{t,s} = 0 & \text{for every } (\ell,t) \in \mathbb{R}^2, \\ V^+_{\ell,t}V^-_{t,s} = 0 & \text{for every } (\ell,t) \in \mathbb{R}^2_{\geqslant}, \\ V^-_{\ell,s} = V^-_{\ell,t}V^-_{t,s} & \text{for every } (\ell,t) \in \mathbb{R}^2_{\leqslant}. \end{cases}$$

□

Lemma 2.3.9. *Let $(t, s) \in \mathbb{R}^2_{\geqslant}$. Then we have*

$$\begin{cases} U_{\ell,t}V^+_{t,s} = 0 & \textit{for every } (\ell,t) \in \mathbb{R}^2, \\ V^+_{\ell,s} = V^+_{\ell,t}V^+_{t,s} & \textit{for every } (\ell,t) \in \mathbb{R}^2_{\geqslant}, \\ V^-_{\ell,t}V^+_{\ell,t} = 0 & \textit{for every } (\ell,t) \in \mathbb{R}^2_{\leqslant}. \end{cases}$$

Proof: Let $(t,s) \in \mathbb{R}^2_{\geqslant}$ and define

$$\begin{cases} W_{\ell,t} = U_{\ell,t} V^+_{t,s} & \text{for every } (\ell,t) \in \mathbb{R}^2, \\ Z^+_{\ell,t} = V^+_{\ell,t} V^+_{t,s} - V^+_{\ell,s} & \text{for every } (\ell,t) \in \mathbb{R}^2_{\geqslant}, \\ Z^-_{\ell,t} = V^-_{\ell,t} V^+_{t,s} & \text{for every } (\ell,t) \in \mathbb{R}^2_{\leqslant}. \end{cases}$$

Since

$$\|W_{\ell,t}\| \leqslant \|U_{\ell,t}\| \; \|V^+_{t,s}\| \leqslant \sigma \alpha_{\ell,t} \|V^+_{t,s}\|$$

it follows that $W \in \Omega^0_t$. From

$$\|Z^+_{\ell,t}\| \leqslant \|V^+_{\ell,t}\| \; \|V^+_{t,s}\| + \|V^+_{\ell,s}\| \leqslant \sigma \beta^+_{\ell,t} \|V^+_{t,s}\| + \sigma \beta^+_{\ell,s} \leqslant \sigma \beta^+_{\ell,t} \left(\|V^+_{t,s}\| + \frac{\beta^+_{\ell,s}}{\beta^+_{\ell,t}} \right)$$

and (2.7) it follows that $Z^+ \in \Omega^+_t$. At last,

$$\|Z^-_{\ell,t}\| \leqslant \|V^-_{\ell,t}\| \; \|V^+_{t,s}\| \leqslant \sigma \beta^-_{\ell,t} \|V^+_{t,s}\|$$

we have $Z^- \in \Omega^-_t$. Therefore $(W, Z^+, Z^-) \in \Omega_t$. From (2.63) we have

$$W_{\ell,t} = U_{\ell,t} V^+_{t,s} = T_{\ell,t} P_t V^+_{t,s} + J_{\ell,t}(U, V^+, V^-) V^+_{t,s}$$

and since

$$T_{\ell,t} P_t V^+_{t,s} = \int_t^{+\infty} C_{\ell,r} V^+_{r,s} \, dr$$

we have

$$W_{\ell,t} = J_{\ell,t}(W, Z^+, Z^-).$$

On the other hand, for $(\ell,t) \in \mathbb{R}^2_{\geqslant}$

$$Z^+_{\ell,t} = V^+_{\ell,t} V^+_{t,s} - \; V^+_{\ell,s} = T_{\ell,t} Q^+_t V^+_{t,s} + L^+_{\ell,t}(U, V^+, V^-) V^+_{t,s} - \; V^+_{\ell,s}.$$

But

$$T_{\ell,t} Q^+_t V^+_{t,s} = T_{\ell,s} Q^+_s - \int_{-\infty}^{s} D^+_{\ell,r} U_{r,s} \, dr + \int_s^t D^+_{\ell,r} V^+_{r,s} \, dr - \int_{-\infty}^{s} D^+_{\ell,r} V^-_{r,s} \, dr,$$

therefore

$$T_{\ell,t}Q_t^+V_{t,s}^+ - V_{\ell,s}^+ = -\int_t^\ell D_{\ell,r}^+V_{r,s}^+\,dr - \int_\ell^{+\infty} C_{\ell,r}V_{r,s}^+\,dr - \int_\ell^{+\infty} D_{\ell,r}^-V_{r,s}^+\,dr,$$

and so we can write

$$Z_{\ell,t}^+ = L_{\ell,t}^+(W, Z^+, Z^-).$$

Finally, from (2.65) we get

$$Z_{\ell,t}^- = V_{\ell,t}^-V_{t,s}^+ \; = T_{\ell,t}Q_t^-V_{t,s}^+ + L_{\ell,t}^-(U, V^+, V^-)V_{t,s}^+$$

and because

$$T_{\ell,t}Q_t^-V_{t,s}^+ = -\int_t^{+\infty} D_{\ell,r}^-V_{r,s}^+\,dr,$$

we have

$$Z_{\ell,t}^- = L_{\ell,t}^-(W, Z^+, Z^-).$$

Then (W, Z^+, Z^-) is a fixed point of the linear contraction $T_t = (J_t, L_t^+, L_t^-)$ and it follows that $U_{\ell,t}V_{t,s}^+ = 0$ for every $(\ell, t) \in \mathbb{R}^2$, $V_{\ell,s}^+ = V_{\ell,t}^+V_{t,s}^+$ for every $(\ell, t) \in \mathbb{R}^2_\geqslant$ and $V_{\ell,t}^-V_{\ell,t}^+ = 0$ for every $(\ell, t) \in \mathbb{R}^2_\leqslant$. □

Now we denote by $\widehat{T}_{t,s}$ the linear evolution operator associated to equation (2.1),

$$\widehat{T}_{t,s} = T_{t,s} + \int_s^t T_{t,r}B_r\widehat{T}_{r,s}\,dr \qquad \text{for every} \quad (t, s) \in \mathbb{R}^2. \tag{2.66}$$

Lemma 2.3.10. *Let $(\ell, s) \in \mathbb{R}^2_\geqslant$. Then*

$$U_{t,\ell}\widehat{T}_{\ell,s}V_{s,s}^- = 0 \quad \textit{for every } (t, \ell) \in \mathbb{R}^2,$$

$$V_{t,\ell}^+\widehat{T}_{\ell,s}V_{s,s}^- = 0 \quad \textit{for every } (t, \ell) \in \mathbb{R}^2_\geqslant,$$

$$V_{t,\ell}^-\widehat{T}_{\ell,s}V_{s,s}^- = \begin{cases} \widehat{T}_{t,s}V_{s,s}^- & \textit{if } \; s \leqslant t \\ V_{t,s}^- & \textit{if } \; t \leqslant s \end{cases} \qquad \textit{for every } (t, \ell) \in \mathbb{R}^2_\leqslant.$$

Proof: Let $(\ell, s) \in \mathbb{R}^2_\geqslant$ and define the operators

$$W_{t,\ell} = U_{t,\ell}\widehat{T}_{\ell,s}V_{s,s}^- \;\text{ for every } (t, \ell) \in \mathbb{R}^2,$$

$$Z_{t,\ell}^+ = V_{t,\ell}^+\widehat{T}_{\ell,s}V_{s,s}^- \;\text{ for every}(t, \ell) \in \mathbb{R}^2_\geqslant,$$

$$Z_{t,\ell}^- = \begin{cases} V_{t,\ell}^-\widehat{T}_{\ell,s}V_{s,s}^- - \widehat{T}_{t,s}V_{s,s}^- & \text{if } s \leqslant t \\ V_{t,\ell}^-\widehat{T}_{\ell,s}V_{s,s}^- - V_{t,s}^- & \text{if } t \leqslant s \end{cases} \qquad \text{for every } (t, \ell) \in \mathbb{R}^2_\leqslant.$$

It is clear that $(W, Z^+, Z^-) \in \Omega_\ell$. For every $(t,\ell) \in \mathbb{R}^2$ we can write from (2.63)

$$W_{t,\ell} = T_{t,\ell}P_\ell \widehat{T}_{\ell,s}V^-_{s,s} + J_{t,\ell}(U, V^+, V^-)\widehat{T}_{\ell,s}V^-_{s,s}.$$

But from the definition of $\widehat{T}$, (2.66), (2.65) and (2.52) we have

$$\begin{aligned} T_{t,\ell}P_\ell \widehat{T}_{\ell,s}V^-_{s,s} &= T_{t,\ell}P_\ell T_{\ell,s}V^-_{s,s} + T_{t,\ell}P_\ell \int_s^\ell T_{\ell,r}B_r\widehat{T}_{r,s}\,drV^-_{s,s} \\ &= T_{t,s}P_sV^-_{s,s} + \int_s^\ell T_{t,r}P_r\ B_r\widehat{T}_{r,s}V^-_{s,s}\,dr \\ &= T_{t,s}P_s\int_{-\infty}^s C_{s,r}V^-_{r,s}\,dr + \int_s^\ell C_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr \\ &= \int_{-\infty}^s C_{t,r}V^-_{r,s}\,dr + \int_s^\ell C_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr. \end{aligned} \tag{2.67}$$

Since, from (2.55) we have, for every $(t,\ell) \in \mathbb{R}^2$

$$\begin{aligned} J_{t,\ell}(U,V^+,V^-)\widehat{T}_{\ell,s}V^-_{s,s} &= -\int_{-\infty}^\ell C_{t,r}V^-_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr + \int_\ell^t C_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr \\ -\int_t^{+\infty} D^-_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr &+ \int_{-\infty}^t D^+_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr + \int_\ell^{+\infty} C_{t,r}V^+_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr, \end{aligned} \tag{2.68}$$

then, from (2.68) and (2.67) we get

$$\begin{aligned} W_{t,\ell} &= U_{t,\ell}\widehat{T}_{\ell,s}V^-_{s,s} \\ &= -\int_{-\infty}^\ell C_{t,r}(V^-_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s} - \widehat{T}_{r,s}V^-_{s,s})\,dr - \int_s^\ell C_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr + \int_s^\ell C_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr \\ &\quad + \int_\ell^t C_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr - \int_t^{+\infty} D^-_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr + \int_{-\infty}^t D^+_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr \\ &\quad + \int_\ell^{+\infty} C_{t,r}V^+_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr \\ &= -\int_{-\infty}^\ell C_{t,r}Z^-_{r,\ell}\,dr + \int_\ell^t C_{t,r}W_{r,\ell}\,dr - \int_t^{+\infty} D^-_{t,r}W_{r,\ell}\,dr \\ &\quad + \int_{-\infty}^t D^+_{t,r}W_{r,\ell}\,dr + \int_\ell^{+\infty} C_{t,r}Z^+_{r,\ell}\,dr \\ &= J_{t,\ell}(W, Z^+, Z^-). \end{aligned}$$

For every $(t,\ell) \in \mathbb{R}^2_\geqslant$, using (2.64) we can write

$$Z^+_{t,\ell} = V^+_{t,\ell}\widehat{T}_{\ell,s}V^-_{s,s} = T_{t,\ell}Q^+_\ell\widehat{T}_{\ell,s}V^-_{s,s} + L^+_{t,\ell}(U,V^+,V^-)\widehat{T}_{\ell,s}V^-_{s,s}$$

and by (2.66)

$$\begin{aligned}
T_{t,\ell}Q_\ell^+\widehat{T}_{\ell,s}V_{s,s}^- &= T_{t,\ell}Q_\ell^+T_{\ell,s}V_{s,s}^- + T_{t,\ell}Q_\ell^+\int_s^\ell T_{\ell,r}B_r\widehat{T}_{r,s}V_{s,s}^-\,dr\\
&= T_{t,s}Q_s^+V_{s,s}^- + \int_s^\ell T_{t,r}Q_r^+B_r\widehat{T}_{r,s}V_{s,s}^-\,dr\\
&= T_{t,s}Q_s^+V_{s,s}^- + \int_s^\ell D_{t,r}^+\widehat{T}_{r,s}V_{s,s}^-\,dr\\
&= \int_{-\infty}^s D_{t,r}^+V_{r,s}^-\,dr + \int_s^\ell D_{t,r}^+\widehat{T}_{r,s}V_{s,s}^-\,dr.
\end{aligned}$$

From (2.58) we get

$$\begin{aligned}
L_{t,\ell}^+(U,V^+,V^-)\widehat{T}_{\ell,s}V_{s,s}^- = &-\int_{-\infty}^\ell D_{t,r}^+U_{r,\ell}\widehat{T}_{\ell,s}V_{s,s}^-\,dr + \int_\ell^t D_{t,r}^+V_{r,\ell}^+\widehat{T}_{\ell,s}V_{s,s}^-\,dr\\
&-\int_t^{+\infty} C_{t,r}V_{r,\ell}^+\widehat{T}_{\ell,s}V_{s,s}^-\,dr - \int_{-\infty}^\ell D_{t,r}^+V_{r,\ell}^-\widehat{T}_{\ell,s}V_{s,s}^-\,dr\\
&-\int_t^{+\infty} D_{t,r}^-V_{r,\ell}^+\widehat{T}_{\ell,s}V_{s,s}^-\,dr
\end{aligned}$$

and so

$$\begin{aligned}
Z_{t,\ell}^+ &= T_{t,\ell}Q_\ell^+\widehat{T}_{\ell,s}V_{s,s}^- + L_{t,\ell}^+(U,V^+,V^-)\widehat{T}_{\ell,s}V_{s,s}^-\\
&= \int_{-\infty}^s D_{t,r}^+V_{r,s}^-\,dr + \int_s^\ell D_{t,r}^+\widehat{T}_{r,s}V_{s,s}^-\,dr - \int_{-\infty}^\ell D_{t,r}^+U_{r,\ell}\widehat{T}_{\ell,s}V_{s,s}^-\,dr\\
&\quad+\int_\ell^t D_{t,r}^+V_{r,\ell}^+\widehat{T}_{\ell,s}V_{s,s}^-\,dr - \int_t^{+\infty} C_{t,r}V_{r,\ell}^+\widehat{T}_{\ell,s}V_{s,s}^-\,dr\\
&\quad-\int_{-\infty}^\ell D_{t,r}^+V_{r,\ell}^-\widehat{T}_{\ell,s}V_{s,s}^-\,dr - \int_t^{+\infty} D_{t,r}^-V_{r,\ell}^+\widehat{T}_{\ell,s}V_{s,s}^-\,dr\\
&= -\int_{-\infty}^\ell D_{t,r}^+W_{r,\ell}\,dr + \int_\ell^t D_{t,r}^+Z_{r,\ell}^+\,dr - \int_t^{+\infty} C_{t,r}Z_{r,\ell}^+\,dr\\
&\quad-\int_{-\infty}^\ell D_{t,r}^+Z_{r,\ell}^-\,dr - \int_{-\infty}^\ell D_{t,r}^+\widehat{T}_{r,s}V_{s,s}^-\,dr - \int_t^{+\infty} D_{t,r}^-Z_{r,\ell}^+\,dr\\
&\quad+\int_{-\infty}^s D_{t,r}^+V_{r,s}^-\,dr + \int_s^\ell D_{t,r}^+\widehat{T}_{r,s}V_{s,s}^-\,dr\\
&= L_{t,\ell}^+(W,Z^+,Z^-)
\end{aligned}$$

For every $(t,\ell)\in\mathbb{R}^2_\leqslant$, considering that $t\leqslant s$ and from (2.65) we have

$$Z_{t,\ell}^- = V_{t,\ell}^-\widehat{T}_{\ell,s}V_{s,s}^- - V_{t,s}^- = T_{t,\ell}Q_\ell^-\widehat{T}_{\ell,s}V_{s,s}^- + L_{t,\ell}^-(U,V^+,V^-)\widehat{T}_{\ell,s}V_{s,s}^- - V_{t,s}^-,$$

from (2.66) it follows that

$$\begin{aligned}T_{t,\ell}Q^-_\ell \widehat{T}_{\ell,s}V^-_{s,s} &= T_{t,\ell}Q^-_\ell T_{\ell,s}V^-_{s,s} + T_{t,\ell}Q^-_\ell \int_s^\ell T_{\ell,r}B_r\widehat{T}_{r,s}V^-_{s,s}\,dr\\ &= T_{t,s}Q^-_s V^-_{s,s} + \int_s^\ell D^-_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr\end{aligned}$$

and by (2.61) we have

$$\begin{aligned}&L^-_{t,\ell}(U,V^+,V^-)\widehat{T}_{\ell,s}V^-_{s,s} =\\ &= \int_{-\infty}^t C_{t,r}V^-_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr - \int_t^\ell D^-_{t,r}V^-_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr + \int_\ell^{+\infty} D^-_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr\\ &\quad + \int_{-\infty}^t D^+_{t,r}V^-_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr + \int_\ell^{+\infty} D^-_{t,r}V^+_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr.\end{aligned} \tag{2.69}$$

From (2.66) and the definition (2.65) for every $s \in \mathbb{R}$ we can write

$$\begin{aligned}&T_{t,\ell}Q^-_\ell \widehat{T}_{\ell,s}V^-_{s,s} - V^-_{t,s}\\ &= T_{t,s}Q^-_s V^-_{s,s} + \int_s^\ell D^-_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr - T_{t,s}Q^-_s - \int_{-\infty}^t C_{t,r}V^-_{r,s}\,dr\\ &\quad + \int_t^s D^-_{t,r}V^-_{r,s}\,dr - \int_s^{+\infty} D^-_{t,r}U_{r,s}\,dr - \int_{-\infty}^t D^+_{t,r}V^-_{r,s}\,dr - \int_s^{+\infty} D^-_{t,r}V^+_{r,s}\,dr\\ &= T_{t,s}Q^-_s + \int_s^{+\infty} T_{t,s}Q^-_s D^-_{s,r}U_{r,s}\,dr + \int_s^{+\infty} T_{t,s}Q^-_s D^-_{s,r}V^+_{r,s}\,dr\\ &\quad + \int_s^\ell D^-_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr - T_{t,s}Q^-_s - \int_{-\infty}^t C_{t,r}V^-_{r,s}\,dr + \int_t^s D^-_{t,r}V^-_{r,s}\,dr\\ &\quad - \int_s^{+\infty} D^-_{t,r}U_{r,s}\,dr - \int_{-\infty}^t D^+_{t,r}V^-_{r,s}\,dr - \int_s^{+\infty} D^-_{t,r}V^+_{r,s}\,dr\\ &= \int_s^{+\infty} D^-_{t,r}U_{r,s}\,dr + \int_s^{+\infty} D^-_{t,r}V^+_{r,s}\,dr + \int_s^\ell D^-_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr - \int_{-\infty}^t C_{t,r}V^-_{r,s}\,dr\\ &\quad + \int_t^s D^-_{t,r}V^-_{r,s}\,dr - \int_s^{+\infty} D^-_{t,r}U_{r,s}\,dr - \int_{-\infty}^t D^+_{t,r}V^-_{r,s}\,dr - \int_s^{+\infty} D^-_{t,r}V^+_{r,s}\,dr\\ &= \int_s^\ell D^-_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr - \int_{-\infty}^t C_{t,r}V^-_{r,s}\,dr + \int_t^s D^-_{t,r}V^-_{r,s}\,dr - \int_{-\infty}^t D^+_{t,r}V^-_{r,s}\,dr\end{aligned} \tag{2.70}$$

Therefore from (2.69) and (2.70) we have, for every $t \leqslant s \leqslant \ell$,

$$\begin{aligned}
&Z^-_{t,\ell}\\
&= V^-_{t,\ell}\widehat{T}_{\ell,s}V^-_{s,s} - V^-_{t,s}\\
&= \int_{-\infty}^{t} C_{t,r}\left(V^-_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s} - \widehat{T}_{r,s}V^-_{s,s}\right) dr - \int_{t}^{\ell} D^-_{t,r}\left(V^-_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s} - \widehat{T}_{r,s}V^-_{s,s}\right) dr\\
&\quad + \int_{\ell}^{s} D^-_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr + \int_{\ell}^{+\infty} D^-_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr + \int_{-\infty}^{t} D^+_{t,r}\left(V^-_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s} - V^-_{r,s}\right) dr\\
&\quad + \int_{\ell}^{+\infty} D^-_{t,r}V^+_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr + \int_{s}^{\ell} D^-_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr\\
&= \int_{-\infty}^{t} C_{t,r}Z^-_{r,\ell}\,dr - \int_{t}^{\ell} D^-_{t,r}Z^-_{r,\ell}\,dr + \int_{\ell}^{+\infty} D^-_{t,r}W_{r,\ell}\,dr + \int_{-\infty}^{t} D^+_{t,r}Z^-_{r,\ell}\,dr\\
&\quad + \int_{\ell}^{+\infty} D^-_{t,r}Z^+_{r,\ell}\,dr\\
&= L^-_{t,\ell}(W, Z^+, Z^-).
\end{aligned}$$

Now for every $(t,\ell) \in \mathbb{R}^2_{\leqslant}$, considering $s \leqslant t$ we have

$$Z^-_{t,\ell} = V^-_{t,\ell}\widehat{T}_{\ell,s}V^-_{s,s} - \widehat{T}_{t,s}V^-_{s,s} = T_{t,\ell}Q^-_{\ell}\widehat{T}_{\ell,s}V^-_{s,s} + L^-_{t,\ell}(U, V^+, V^-)\widehat{T}_{\ell,s}V^-_{s,s} - \widehat{T}_{t,s}V^-_{s,s},$$

from (2.66) and (2.65) it follows that, for every $s \in \mathbb{R}$,

$$
\begin{aligned}
&T_{t,\ell}Q_\ell^- \widehat{T}_{\ell,s}V_{s,s}^- - \widehat{T}_{t,s}V_{s,s}^- = \\
&= \left(T_{t,\ell}Q_\ell^- T_{\ell,s} + T_{t,\ell}Q_\ell^- \int_s^\ell T_{\ell,r}B_r\widehat{T}_{r,s}\,dr - T_{t,s} - \int_s^t T_{t,r}B_r\widehat{T}_{r,s}\,dr\right)V_{s,s}^- \\
&= \left(T_{t,s}Q_s^- + \int_s^\ell D_{t,r}^-\widehat{T}_{r,s}\,dr - T_{t,s} - \int_s^t T_{t,r}B_r\widehat{T}_{r,s}\,dr\right)V_{s,s}^- \\
&= \left(-T_{t,s}P_s - T_{t,s}Q_s^+ + \int_s^\ell D_{t,r}^-\widehat{T}_{r,s}\,dr - \int_s^t T_{t,r}B_r\widehat{T}_{r,s}\,dr\right)V_{s,s}^- \\
&= -T_{t,s}P_sV_{s,s}^- - T_{t,s}Q_s^+V_{s,s}^- + \int_s^\ell D_{t,r}^-\widehat{T}_{r,s}V_{s,s}^-\,dr - \int_s^t T_{t,r}B_r\widehat{T}_{r,s}V_{s,s}^-\,dr \\
&= -\int_{-\infty}^s T_{t,s}P_sC_{s,r}V_{r,s}^-\,dr - \int_{-\infty}^s T_{t,s}Q_s^+D_{s,r}^+V_{r,s}^-\,dr + \int_s^\ell D_{t,r}^-\widehat{T}_{r,s}V_{s,s}^-\,dr \\
&\quad - \int_s^t T_{t,r}B_r\widehat{T}_{r,s}V_{s,s}^-\,dr \\
&= -\int_{-\infty}^s C_{t,r}V_{r,s}^-\,dr - \int_{-\infty}^s D_{t,r}^+V_{r,s}^-\,dr + \int_s^\ell D_{t,r}^-\widehat{T}_{r,s}V_{s,s}^-\,dr \\
&\quad - \int_s^t T_{t,r}B_r\widehat{T}_{r,s}V_{s,s}^-\,dr
\end{aligned}
$$

and by (2.61) we also have (2.69). Therefore from (2.69) and (2.70) we have, for every $s \leqslant t \leqslant \ell$,

$$\begin{aligned}
Z^-_{t,\ell} &= V^-_{t,\ell}\widehat{T}_{\ell,s}V^-_{s,s} - \widehat{T}_{t,s}V^-_{s,s} \\
&= \int_{-\infty}^{t} C_{t,r}\left(V^-_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s} - \widehat{T}_{r,s}V^-_{s,s}\right)dr + \int_{-\infty}^{t} C_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr \\
&\quad - \int_{t}^{\ell} D^-_{t,r}\left(V^-_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s} - \widehat{T}_{r,s}V^-_{s,s}\right)dr + \int_{s}^{t} D^-_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr \\
&\quad + \int_{\ell}^{+\infty} D^-_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr + \int_{-\infty}^{t} D^+_{t,r}\left(V^-_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s} - \widehat{T}_{r,s}V^-_{s,s}\right)dr \\
&\quad + \int_{-\infty}^{t} D^+_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr + \int_{\ell}^{+\infty} D^-_{t,r}V^+_{r,\ell}\widehat{T}_{\ell,s}V^-_{s,s}\,dr - \int_{-\infty}^{s} C_{t,r}V^-_{r,s}\,dr \\
&\quad - \int_{-\infty}^{s} D^+_{t,r}V^-_{r,s}\,dr - \int_{s}^{t} T_{t,r}B_r\widehat{T}_{r,s}V^-_{s,s}\,dr \\
&= \int_{-\infty}^{t} C_{t,r}Z^-_{r,\ell}\,dr - \int_{t}^{\ell} D^-_{t,r}Z^-_{r,\ell}\,dr + \int_{\ell}^{+\infty} D^-_{t,r}W_{r,\ell}\,dr + \int_{-\infty}^{t} D^+_{t,r}Z^-_{r,\ell}\,dr \\
&\quad + \int_{\ell}^{+\infty} D^-_{t,r}Z^+_{r,\ell}\,dr + \int_{-\infty}^{t} C_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr - \int_{-\infty}^{s} C_{t,r}V^-_{r,s}\,dr \\
&\quad + \int_{s}^{t} D^-_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr - \int_{-\infty}^{s} D^+_{t,r}V^-_{r,s}\,dr + \int_{-\infty}^{t} D^+_{t,r}\widehat{T}_{r,s}V^-_{s,s}\,dr \\
&\quad - \int_{s}^{t} T_{t,r}(P_r + Q^+_r + Q^-_r)B_r\widehat{T}_{r,s}V^-_{s,s}\,dr \\
&= L^-_{t,\ell}(W, Z^+, Z^-).
\end{aligned}$$

Therefore, (W, Z^+, Z^-) is a fixed point of $T_\ell = (J_\ell, L^+_\ell, L^-_\ell)$, a linear operator. It follows $W_{t,\ell} = 0$ for every $(t,\ell) \in \mathbb{R}^2$, $Z^-_{t,\ell} = 0$ for every $(t,\ell) \in \mathbb{R}^2_\leqslant$ and $Z^+_{t,\ell} = 0$ for every $(t,\ell) \in \mathbb{R}^2_\geqslant$. □

Lemma 2.3.11. *Let $(\ell, s) \in \mathbb{R}^2_\leqslant$. Then*

$$U_{t,\ell}\widehat{T}_{\ell,s}V^+_{s,s} = 0 \quad \textit{for every } (t,\ell) \in \mathbb{R}^2,$$

$$V^-_{t,\ell}\widehat{T}_{\ell,s}V^+_{s,s} = 0 \quad \textit{for every } (t,\ell) \in \mathbb{R}^2_\leqslant,$$

$$V^+_{t,\ell}\widehat{T}_{\ell,s}V^+_{s,s} = \begin{cases} \widehat{T}_{t,s}V^+_{s,s} & \textit{if } t \leqslant s \\ V^+_{t,s} & \textit{if } t \geqslant s \end{cases} \qquad \textit{for every } (t,\ell) \in \mathbb{R}^2_\geqslant.$$

Proof: Let $(\ell, s) \in \mathbb{R}^2_{\leqslant}$ and define the operators

$$W_{t,\ell} = U_{t,\ell}\widehat{T}_{\ell,s}V^+_{s,s} \quad \text{for every } (t,\ell) \in \mathbb{R}^2,$$

$$Z^-_{t,\ell} = V^-_{t,\ell}\widehat{T}_{\ell,s}V^+_{s,s} \quad \text{for every } (t,\ell) \in \mathbb{R}^2_{\leqslant},$$

$$Z^+_{t,\ell} = \begin{cases} V^+_{t,\ell}\widehat{T}_{\ell,s}V^+_{s,s} - \widehat{T}_{t,s}V^+_{s,s} & \text{if } t \leqslant s \\ V^+_{t,\ell}\widehat{T}_{\ell,s}V^+_{s,s} - V^+_{t,s} & \text{if } t \geqslant s \end{cases} \qquad \text{for every } (t,\ell) \in \mathbb{R}^2_{\geqslant}.$$

Obviously $(W, Z^+, Z^-) \in \Omega_\ell$. For every $(t,\ell) \in \mathbb{R}^2$ we can write from (2.63)

$$W_{t,\ell} = T_{t,\ell}P_\ell\widehat{T}_{\ell,s}V^+_{s,s} + J_{t,\ell}(U, V^+, V^-)\widehat{T}_{\ell,s}V^+_{s,s}.$$

But from the definition of $\widehat{T}$, (2.66), (2.64) and (2.52) we have

$$\begin{aligned} T_{t,\ell}P_\ell\widehat{T}_{\ell,s}V^+_{s,s} &= T_{t,\ell}P_\ell T_{\ell,s}V^+_{s,s} + T_{t,\ell}P_\ell \int_s^\ell T_{\ell,r}B_r\widehat{T}_{r,s}\,dr V^+_{s,s} \\ &= T_{t,s}P_sV^+_{s,s} + \int_s^\ell T_{t,r}P_r\ B_r\widehat{T}_{r,s}V^+_{s,s}\,dr \\ &= -T_{t,s}P_s\int_s^{+\infty} C_{s,r}V^+_{r,s}\,dr + \int_s^\ell C_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr \\ &= -\int_s^{+\infty} C_{t,r}V^+_{r,s}\,dr + \int_s^\ell C_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr. \end{aligned} \tag{2.71}$$

Since, from (2.55) we have, for every $(t,\ell) \in \mathbb{R}^2$

$$\begin{aligned} &J_{t,\ell}(U, V^+, V^-)\widehat{T}_{\ell,s}V^+_{s,s} = \\ &= -\int_{-\infty}^\ell C_{t,r}V^-_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr + \int_\ell^t C_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr - \int_t^{+\infty} D^-_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr \\ &\quad + \int_{-\infty}^t D^+_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr + \int_\ell^{+\infty} C_{t,r}V^+_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr, \end{aligned} \tag{2.72}$$

then, from (2.72) and (2.71) we get

$$\begin{aligned}
W_{t,\ell} &= U_{t,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\\
&= -\int_{-\infty}^{\ell} C_{t,r}V^-_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr + \int_{\ell}^{t} C_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr - \int_{t}^{+\infty} D^-_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr\\
&\quad + \int_{-\infty}^{t} D^+_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr + \int_{\ell}^{+\infty} C_{t,r}(V^+_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s} - \widehat{T}_{r,s}V^+_{s,s})\,dr\\
&\quad - \int_{s}^{\ell} C_{t,r}V^+_{r,s}\,dr + \int_{s}^{\ell} C_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr\\
&= -\int_{-\infty}^{\ell} C_{t,r}Z^-_{r,\ell}\,dr + \int_{\ell}^{t} C_{t,r}W_{r,\ell}\,dr - \int_{t}^{+\infty} D^-_{t,r}W_{r,\ell}\,dr\\
&\quad + \int_{-\infty}^{t} D^+_{t,r}W_{r,\ell}\,dr + \int_{\ell}^{+\infty} C_{t,r}Z^+_{r,\ell}\,dr\\
&= J_{t,\ell}(W, Z^+, Z^-).
\end{aligned}$$

For every $(t,\ell) \in \mathbb{R}^2_{\leqslant}$ and from (2.65) we have

$$Z^-_{t,\ell} = V^-_{t,\ell}\widehat{T}_{\ell,s}V^+_{s,s} = T_{t,\ell}Q^-_{\ell}\widehat{T}_{\ell,s}V^+_{s,s} + L^-_{t,\ell}(U, V^+, V^-)\widehat{T}_{\ell,s}V^+_{s,s},$$

from (2.66) and (2.52) it follows that

$$\begin{aligned}
T_{t,\ell}Q^-_{\ell}\widehat{T}_{\ell,s}V^+_{s,s} &= T_{t,\ell}Q^-_{\ell}T_{\ell,s}V^+_{s,s} + T_{t,\ell}Q^-_{\ell}\int_{s}^{\ell} T_{\ell,r}B_r\widehat{T}_{r,s}V^+_{s,s}\,dr\\
&= T_{t,s}Q^-_{s}V^+_{s,s} + \int_{s}^{\ell} D^-_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr \qquad (2.73)\\
&= -\int_{s}^{+\infty} D^-_{t,r}V^+_{r,s}\,dr + \int_{s}^{\ell} D^-_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr
\end{aligned}$$

and by (2.61) we have

$$\begin{aligned}
&L^-_{t,\ell}(U, V^+, V^-)\widehat{T}_{\ell,s}V^+_{s,s} =\\
&= \int_{-\infty}^{t} C_{t,r}V^-_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr - \int_{t}^{\ell} D^-_{t,r}V^-_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr + \int_{\ell}^{+\infty} D^-_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr \qquad (2.74)\\
&\quad + \int_{-\infty}^{t} D^+_{t,r}V^-_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr + \int_{\ell}^{+\infty} D^-_{t,r}V^+_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr.
\end{aligned}$$

Therefore, from (2.73) and (2.74) we have, for every $t \leqslant \ell$,

$$\begin{aligned}
Z^-_{t,\ell} &= V^-_{t,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\\
&= -\int_s^{+\infty} D^-_{t,r}V^+_{r,s}\,dr + \int_{-\infty}^t C_{t,r}V^-_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr - \int_t^\ell D^-_{t,r}V^-_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr\\
&\quad + \int_\ell^{+\infty} D^-_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr + \int_{-\infty}^t D^+_{t,r}V^-_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr\\
&\quad + \int_\ell^{+\infty} D^-_{t,r}(V^+_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s} - \widehat{T}_{r,s}V^+_{s,s})\,dr + \int_\ell^{+\infty} D^-_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr\\
&\quad + \int_s^\ell D^-_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr\\
&= -\int_s^{+\infty} D^-_{t,r}V^+_{r,s}\,dr + \int_{-\infty}^t C_{t,r}Z^-_{r,\ell}\,dr - \int_t^\ell D^-_{t,r}Z^-_{r,\ell}\,dr + \int_\ell^{+\infty} D^-_{t,r}W_{r,\ell}\,dr\\
&\quad + \int_{-\infty}^t D^+_{t,r}Z^-_{r,\ell}\,dr + \int_\ell^{+\infty} D^-_{t,r}Z^+_{r,\ell}\,dr + \int_\ell^{+\infty} D^-_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr\\
&\quad + \int_s^\ell D^-_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr\\
&= L^-_{t,\ell}(W, Z^+, Z^-).
\end{aligned}$$

At last, for every $(t,\ell) \in \mathbb{R}^2_\geqslant$, supposing $t \geqslant s$ and using (2.64) we can write

$$Z^+_{t,\ell} = V^+_{t,\ell}\widehat{T}_{\ell,s}V^+_{s,s} - V^+_{t,s} = T_{t,\ell}Q^+_\ell\widehat{T}_{\ell,s}V^+_{s,s} + L^+_{t,\ell}(U, V^+, V^-)\widehat{T}_{\ell,s}V^+_{s,s} - V^+_{t,s}$$

and by (2.66) and (2.64) we have

$$\begin{aligned}
T_{t,\ell}Q^+_\ell\widehat{T}_{\ell,s}V^+_{s,s} &= T_{t,\ell}Q^+_\ell T_{\ell,s}V^+_{s,s} + T_{t,\ell}Q^+_\ell \int_s^\ell T_{\ell,r}B_r\widehat{T}_{r,s}V^+_{s,s}\,dr\\
&= T_{t,s}Q^+_s V^+_{s,s} + \int_s^\ell T_{t,r}Q^+_r B_r\widehat{T}_{r,s}V^+_{s,s}\,dr\\
&= T_{t,s}Q^+_s V^+_{s,s} + \int_s^\ell D^+_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr\\
&= T_{t,s}Q^+_s - \int_{-\infty}^s D^+_{t,r}U_{r,s}\,dr - \int_{-\infty}^s D^+_{t,r}V^-_{r,s}\,dr + \int_s^\ell D^+_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr.
\end{aligned}$$

Therefore we can put

$$
\begin{aligned}
&T_{t,\ell}Q_\ell^+\widehat{T}_{\ell,s}V_{s,s}^+ - V_{t,s}^+ = \\
&= T_{t,s}Q_s^+ - \int_{-\infty}^s D_{t,r}^+U_{r,s}\,dr - \int_{-\infty}^s D_{t,r}^+V_{r,s}^-\,dr + \int_s^\ell D_{t,r}^+\widehat{T}_{r,s}V_{s,s}^+\,dr - T_{t,s}Q_s^+ \\
&\quad + \int_{-\infty}^s D_{t,r}^+U_{r,s}\,dr - \int_s^t D_{t,r}^+V_{r,s}^+\,dr + \int_t^{+\infty} C_{t,r}V_{r,s}^+\,dr + \int_{-\infty}^s D_{t,r}^+V_{r,s}^-\,dr \\
&\quad + \int_t^{+\infty} D_{t,r}^-V_{r,s}^+\,dr \\
&= \int_s^\ell D_{t,r}^+\widehat{T}_{r,s}V_{s,s}^+\,dr - \int_s^t D_{t,r}^+V_{r,s}^+\,dr + \int_t^{+\infty} C_{t,r}V_{r,s}^+\,dr + \int_t^{+\infty} D_{t,r}^-V_{r,s}^+\,dr.
\end{aligned}
$$

From (2.58) we get

$$
\begin{aligned}
&L_{t,\ell}^+(U,V^+,V^-)\widehat{T}_{\ell,s}V_{s,s}^+ = \\
&= -\int_{-\infty}^\ell D_{t,r}^+U_{r,\ell}\widehat{T}_{\ell,s}V_{s,s}^+\,dr + \int_\ell^t D_{t,r}^+V_{r,\ell}^+\widehat{T}_{\ell,s}V_{s,s}^+\,dr - \int_t^{+\infty} C_{t,r}V_{r,\ell}^+\widehat{T}_{\ell,s}V_{s,s}^+\,dr \\
&\quad - \int_{-\infty}^\ell D_{t,r}^+V_{r,\ell}^-\widehat{T}_{\ell,s}V_{s,s}^+\,dr - \int_t^{+\infty} D_{t,r}^-V_{r,\ell}^+\widehat{T}_{\ell,s}V_{s,s}^+\,dr
\end{aligned}
$$

and therefore

$$
\begin{aligned}
Z_{t,\ell}^+ &= \int_s^\ell D_{t,r}^+\widehat{T}_{r,s}V_{s,s}^+\,dr - \int_s^t D_{t,r}^+V_{r,s}^+\,dr + \int_t^{+\infty} C_{t,r}V_{r,s}^+\,dr \\
&\quad + \int_t^{+\infty} D_{t,r}^-V_{r,s}^+\,dr - \int_{-\infty}^\ell D_{t,r}^+U_{r,\ell}\widehat{T}_{\ell,s}V_{s,s}^+\,dr + \int_\ell^t D_{t,r}^+V_{r,\ell}^+\widehat{T}_{\ell,s}V_{s,s}^+\,dr \\
&\quad - \int_t^{+\infty} C_{t,r}V_{r,\ell}^+\widehat{T}_{\ell,s}V_{s,s}^+\,dr - \int_{-\infty}^\ell D_{t,r}^+V_{r,\ell}^-\widehat{T}_{\ell,s}V_{s,s}^+\,dr \\
&\quad - \int_t^{+\infty} D_{t,r}^-V_{r,\ell}^+\widehat{T}_{\ell,s}V_{s,s}^+\,dr \\
&= -\int_{-\infty}^\ell D_{t,r}^+W_{r,\ell}\,dr + \int_\ell^t D_{t,r}^+(V_{r,\ell}^+\widehat{T}_{\ell,s}V_{s,s}^+ - \widehat{T}_{r,s}V_{s,s}^+)\,dr + \int_\ell^s D_{t,r}^+\widehat{T}_{r,s}V_{s,s}^+\,dr \\
&\quad - \int_t^{+\infty} C_{t,r}Z_{r,\ell}^+\,dr - \int_{-\infty}^\ell D_{t,r}^+Z_{r,\ell}^-\,dr - \int_t^{+\infty} D_{t,r}^-Z_{r,\ell}^+\,dr \\
&\quad + \int_s^\ell D_{t,r}^+\widehat{T}_{r,s}V_{s,s}^+\,dr \\
&= L_{t,\ell}^+(W,Z^+,Z^-).
\end{aligned}
$$

Finally, for every $(t,\ell) \in \mathbb{R}^2_{\geqslant}$, supposing $t \leqslant s$ and using (2.64) we can write

$$Z^+_{t,\ell} = V^+_{t,\ell}\widehat{T}_{\ell,s}V^+_{s,s} - \widehat{T}_{t,s}V^+_{s,s} = T_{t,\ell}Q^+_\ell\widehat{T}_{\ell,s}V^+_{s,s} + L^+_{t,\ell}(U,V^+,V^-)\widehat{T}_{\ell,s}V^+_{s,s} - \widehat{T}_{t,s}V^+_{s,s}$$

and by (2.66) and (2.64) we have

$$\begin{aligned}
&T_{t,\ell}Q^+_\ell\widehat{T}_{\ell,s}V^+_{s,s} - \widehat{T}_{t,s}V^+_{s,s} = \\
&= T_{t,\ell}Q^+_\ell T_{\ell,s}V^+_{s,s} + T_{t,\ell}Q^+_\ell\int_s^\ell T_{\ell,r}B_r\widehat{T}_{r,s}V^+_{s,s}\,dr - T_{t,s}V^+_{s,s} - \int_s^t T_{t,r}B_r\widehat{T}_{r,s}V^+_{s,s}\,dr \\
&= T_{t,s}Q^+_s V^+_{s,s} + \int_s^\ell T_{t,r}Q^+_r B_r\widehat{T}_{r,s}V^+_{s,s}\,dr - T_{t,s}V^+_{s,s} - \int_s^t T_{t,r}B_r\widehat{T}_{r,s}V^+_{s,s}\,dr \\
&= (-T_{t,s}P_s - T_{t,s}Q^-_s)V^+_{s,s} + \int_s^\ell D^+_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr - \int_s^t T_{t,r}B_r\widehat{T}_{r,s}V^+_{s,s}\,dr \\
&= \int_s^{+\infty} T_{t,s}P_sC_{s,r}V^+_{r,s}\,dr + \int_s^{+\infty} T_{t,s}Q^-_s D^-_{s,r}V^+_{r,s}\,dr + \int_s^\ell D^+_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr \\
&\quad - \int_s^t T_{t,r}\,\mathrm{Id}\,B_r\widehat{T}_{r,s}V^+_{s,s}\,dr \\
&= \int_s^{+\infty} C_{t,r}V^+_{r,s}\,dr + \int_s^{+\infty} D^-_{t,r}V^+_{r,s}\,dr + \int_s^\ell D^+_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr \\
&\quad - \int_s^t T_{t,r}P_rB_r\widehat{T}_{r,s}V^+_{s,s}\,dr - \int_s^t T_{t,r}Q^+_r B_r\widehat{T}_{r,s}V^+_{s,s}\,dr - \int_s^t T_{t,r}Q^-_r B_r\widehat{T}_{r,s}V^+_{s,s}\,dr \\
&= \int_s^{+\infty} C_{t,r}V^+_{r,s}\,dr + \int_s^{+\infty} D^-_{t,r}V^+_{r,s}\,dr + \int_t^\ell D^+_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr \\
&\quad - \int_s^t C_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr - \int_s^t D^-_{t,r}\widehat{T}_{r,s}V^+_{s,s}\,dr.
\end{aligned}$$

From (2.58) we get

$$\begin{aligned}
&L^+_{t,\ell}(U,V^+,V^-)\widehat{T}_{\ell,s}V^+_{s,s} = \\
&= -\int_{-\infty}^\ell D^+_{t,r}U_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr + \int_\ell^t D^+_{t,r}V^+_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr - \int_t^{+\infty} C_{t,r}V^+_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr \\
&\quad - \int_{-\infty}^\ell D^+_{t,r}V^-_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr - \int_t^{+\infty} D^-_{t,r}V^+_{r,\ell}\widehat{T}_{\ell,s}V^+_{s,s}\,dr
\end{aligned}$$

and therefore

$$\begin{aligned}
Z^+_{t,\ell} = &\int_s^{+\infty} C_{t,r} V^+_{r,s}\, dr + \int_s^{+\infty} D^-_{t,r} V^+_{r,s}\, dr + \int_t^{\ell} D^+_{t,r} \widehat{T}_{r,s} V^+_{s,s}\, dr \\
&- \int_s^t C_{t,r} \widehat{T}_{r,s} V^+_{s,s}\, dr - \int_s^t D^-_{t,r} \widehat{T}_{r,s} V^+_{s,s}\, dr - \int_{-\infty}^{\ell} D^+_{t,r} U_{r,\ell} \widehat{T}_{\ell,s} V^+_{s,s}\, dr \\
&+ \int_\ell^t D^+_{t,r} V^+_{r,\ell} \widehat{T}_{\ell,s} V^+_{s,s}\, dr - \int_t^{+\infty} C_{t,r} V^+_{r,\ell} \widehat{T}_{\ell,s} V^+_{s,s}\, dr \\
&- \int_{-\infty}^{\ell} D^+_{t,r} V^-_{r,\ell} \widehat{T}_{\ell,s} V^+_{s,s}\, dr - \int_t^{+\infty} D^-_{t,r} V^+_{r,\ell} \widehat{T}_{\ell,s} V^+_{s,s}\, dr \\
= &- \int_{-\infty}^{\ell} D^+_{t,r} U_{r,\ell} \widehat{T}_{\ell,s} V^+_{s,s}\, dr + \int_\ell^t D^+_{t,r} (V^+_{r,\ell} \widehat{T}_{\ell,s} V^+_{s,s} - \widehat{T}_{r,s} V^+_{s,s})\, dr \\
&- \int_t^{+\infty} C_{t,r} (V^+_{r,\ell} \widehat{T}_{\ell,s} V^+_{s,s} - \widehat{T}_{r,s} V^+_{s,s})\, dr - \int_t^{+\infty} C_{t,r} \widehat{T}_{r,s} V^+_{s,s}\, dr \\
&- \int_{-\infty}^{\ell} D^+_{t,r} V^-_{r,\ell} \widehat{T}_{\ell,s} V^+_{s,s}\, dr - \int_t^{+\infty} D^-_{t,r} (V^+_{r,\ell} \widehat{T}_{\ell,s} V^+_{s,s} - \widehat{T}_{r,s} V^+_{s,s})\, dr \\
&- \int_t^{+\infty} D^-_{t,r} \widehat{T}_{r,s} V^+_{s,s}\, dr + \int_s^{+\infty} C_{t,r} V^+_{r,s}\, dr + \int_s^{+\infty} D^-_{t,r} V^+_{r,s}\, dr \\
&- \int_s^t C_{t,r} \widehat{T}_{r,s} V^+_{s,s}\, dr - \int_s^t D^-_{t,r} \widehat{T}_{r,s} V^+_{s,s}\, dr \\
= &\, L^+_{t,\ell}(W, Z^+, Z^-).
\end{aligned}$$

Hence, (W, Z^+, Z^-) is a fixed point of the linear operator $T_\ell = \big(J_\ell, L^+_\ell, L^-_\ell\big)$. It follows $W_{t,\ell} = 0$ for every $(t,\ell) \in \mathbb{R}^2$, $Z^+_{t,\ell} = 0$ for every $(t,\ell) \in \mathbb{R}^2_\geqslant$ and $Z^-_{t,\ell} = 0$ for every $(t,\ell) \in \mathbb{R}^2_\leqslant$. □

Now we are going to define, for each $s \in \mathbb{R}$, the linear subspaces $\widehat{E}_s = U_{s,s}(X)$, $\widehat{F}^+_s = V^+_{s,s}(X)$ and $\widehat{F}^-_s = V^-_{s,s}(X)$ and state the following lemma.

Lemma 2.3.12. *For every* $(t,s) \in \mathbb{R}^2$ *we have*

$$\widehat{T}_{t,s}(\widehat{E}_s) \subseteq \widehat{E}_t, \quad \widehat{T}_{t,s}(\widehat{F}^+_s) \subseteq \widehat{F}^+_t \quad \text{and} \quad \widehat{T}_{t,s}(\widehat{F}^-_s) \subseteq \widehat{F}^-_t.$$

Proof: Let $x \in \widehat{E}_s$. Then $x = U_{s,s} y$ for some $y \in X$. Since $U_{s,s} U_{s,s} = U_{s,s}$ then $x = U_{s,s} x$. By Lemmas 2.3.6 and 2.3.7

$$\widehat{T}_{t,s} x = \widehat{T}_{t,s} U_{s,s} x = U_{t,s} x = U_{t,t} U_{t,s} x$$

and so $\widehat{T}_{t,s}x \in \widehat{E}_t$. To prove the second inclusion, let $x \in \widehat{F}_s^+$. In a similar way, we have $x = V_{s,s}^+ x$. Using Lemmas 2.3.6 and 2.3.9 it follows that

$$\widehat{T}_{t,s}x = \widehat{T}_{t,s}V_{s,s}^+ x = V_{t,s}^+ x = V_{t,t}^+ V_{t,s}^+ x$$

and so $\widehat{T}_{t,s}x \in \widehat{F}_t^+$. At last, let $x \in \widehat{F}_s^-$. Then $x = V_{s,s}^- x$. By Lemmas 2.3.6 and 2.3.8 we have

$$\widehat{T}_{t,s}x = \widehat{T}_{t,s}V_{s,s}^- x = V_{t,s}^- x = V_{t,t}^- V_{t,s}^- x$$

and so $\widehat{T}_{t,s}x \in \widehat{F}_t^-$. □

§2.3.2 Proof of Theorem 2.1.1

Setting $\widehat{P}_s = U_{s,s}$, $\widehat{Q}_s^+ = V_{s,s}^+$ and $\widehat{Q}_s^- = V_{s,s}^-$, we can now prove Theorem 2.1.1l. First of all we need to prove that $\widehat{P}_s$, $\widehat{Q}_s^+$ and $\widehat{Q}_s^-$ are projections. Attending to (2.63), (2.64) and (2.65) we have

$$\begin{aligned}
&\widehat{P}_s + \widehat{Q}_s^+ + \widehat{Q}_s^- \\
&= P_s + J_{s,s}(U, V^+, V^-) + Q_s^+ + L_{s,s}^+(U, V^+, V^-) + Q_s^- + L_{s,s}^-(U, V^+, V^-) \\
&= P_s - \int_{-\infty}^{s} C_{s,r}V_{r,s}^- \, dr - \int_{s}^{+\infty} D_{s,r}^- U_{r,s} \, dr + \int_{-\infty}^{s} D_{s,r}^+ U_{r,s} \, dr \\
&\quad + \int_{s}^{+\infty} C_{s,r}V_{r,s}^+ \, dr + Q_s^+ - \int_{-\infty}^{s} D_{s,r}^+ U_{r,s} \, dr - \int_{s}^{+\infty} C_{s,r}V_{r,s}^+ \, dr \\
&\quad - \int_{-\infty}^{s} D_{s,r}^+ V_{r,s}^- \, dr - \int_{s}^{+\infty} D_{s,r}^- V_{r,s}^+ \, dr + Q_s^- + \int_{-\infty}^{s} C_{s,r}V_{r,s}^- \, dr \\
&\quad + \int_{s}^{+\infty} D_{s,r}^- U_{r,s} \, dr + \int_{-\infty}^{s} D_{s,r}^+ V_{r,s}^- \, dr + \int_{s}^{+\infty} D_{s,r}^- V_{r,s}^+ \, dr \\
&= P_s + Q_s^+ + Q_s^- \\
&= \mathrm{Id}
\end{aligned}$$

and by Lemmas 2.3.7, 2.3.8 and 2.3.9 we can say $\widehat{P}_s\widehat{P}_s = \widehat{P}_s$, $\widehat{Q}_s^+\widehat{Q}_s^+ = \widehat{Q}_s^+$ and $\widehat{Q}_s^-\widehat{Q}_s^- = \widehat{Q}_s^-$. Therefore $\widehat{P}_s$, $\widehat{Q}_s^+$ and $\widehat{Q}_s^-$ are projections and by Lemma 2.3.9 setting

$t = \ell = s$ we have $\widehat{P}_s\widehat{Q}_s^+ = 0$. By Lemmas 2.3.10 and 2.3.11 we have, for every $t, s \in \mathbb{R}$

$$\begin{aligned}
\widehat{T}_{t,s}\widehat{P}_s &= \widehat{P}_t\widehat{T}_{t,s}\widehat{P}_s + \widehat{Q}_t^+\widehat{T}_{t,s}\widehat{P}_s + \widehat{Q}_t^-\widehat{T}_{t,s}\widehat{P}_s \\
&= \widehat{P}_t\widehat{T}_{t,s}\left(\mathrm{Id} - \widehat{Q}_s^+ - \widehat{Q}_s^-\right) + \widehat{Q}_t^+\widehat{T}_{t,s}\widehat{P}_s + \widehat{Q}_t^-\widehat{T}_{t,s}\widehat{P}_s \\
&= \widehat{P}_t\widehat{T}_{t,s} - \widehat{P}_t\widehat{T}_{t,s}\widehat{Q}_s^+ - \widehat{P}_t\widehat{T}_{t,s}\widehat{Q}_s^- + \widehat{Q}_t^+\widehat{T}_{t,s}\widehat{P}_s + \widehat{Q}_t^-\widehat{T}_{t,s}\widehat{P}_s \\
&= \widehat{P}_t\widehat{T}_{t,s}.
\end{aligned}$$

In a similar way we can say

$$\widehat{T}_{t,s}\widehat{Q}_s^+ = \widehat{Q}_t^+\widehat{T}_{t,s} \quad \text{and} \quad \widehat{T}_{t,s}\widehat{Q}_s^- = \widehat{Q}_t^-\widehat{T}_{t,s} \quad \text{for every} \quad t, s \in \mathbb{R},$$

i.e., (**S1**), (**S2**), (**S3**) and (**S4**) are verified which means that (2.1) has an invariant splitting. At last, from Lemmas 2.3.5 and 2.3.6 we can write

$$\begin{aligned}
\|\widehat{T}_{t,s}\widehat{P}_s\| &= \|U_{t,s}\| \leqslant \sigma\alpha_{t,s} && \text{for every } (t, s) \in \mathbb{R}^2, \\
\|\widehat{T}_{t,s}\widehat{Q}_s^+\| &= \|V_{t,s}^+\| \leqslant \sigma\beta_{t,s}^+ && \text{for every } (t, s) \in \mathbb{R}^2_\geqslant, \\
\|\widehat{T}_{t,s}\widehat{Q}_s^-\| &= \|V_{t,s}^-\| \leqslant \sigma\beta_{t,s}^- && \text{for every } (t, s) \in \mathbb{R}^2_\leqslant.
\end{aligned}$$

Therefore equation (2.1) admits a generalized trichotomy with bounds $(\sigma\alpha_{t,s})$, $(\sigma\beta_{t,s}^+)$ and $(\sigma\beta_{t,s}^-)$ and this completes the proof.

Chapter 3

Global Lipschitz invariant manifolds

This chapter is dedicated to the study of the existence of global Lipschitz invariant manifolds when equation (1.1) admits a generalized trichotomy and is submitted to a nonlinear perturbation f satisfying some conditions, namely a Lipschitz condition. Here, in section 3.1, we formulate a theorem concerning the existence of center invariant manifolds considering bounds of a more general form and to prove it we need to establish some lemmas. We also include, in section 3.2, the exhibition of some examples of invariant center manifolds. The first example is a new general case that includes some already existent in the literature, namely the ones done by Barreira and Valls in [11, 3]. In the last section of this chapter we give the proof of the theorem stated before in the first section.

The proof of the main theorem of this chapter is based in the so-called classical Lyapunov-Perron method (see [41, 49]) that consists in the following:

– the variation of constants formula that allows to relate the solutions of the linear equation with the solutions of the perturbed equation;

– the construction of a suitable space of functions that is a complete metric space;

– the construction of a suitable contraction on the complete metric space mentioned above;

– the application of Banach's fixed point theorem to the mentioned contraction gives a function that is the only fixed point of the contraction whose graph is the invariant manifold.

This method was used by many authors, namely by [3, 11, 20, 21]. However we have introduced a novelty in the application of the method. In [3, 11, 20, 21] uses two applications of the Banach's fixed point theorem, the first one to obtain the solutions of the perturbed equation along the stable/center direction and the other to obtain the solutions of the perturbed equation in the other directions. Here, with only one application of the Banach's fixed point theorem, we obtain the solutions of the perturbed equation in all the directions.

The results of this chapter are from the preprint Bento and Costa [13].

§3.1 Existence of global Lipschitz invariant manifolds

Suppose that equation (1.1) admits a generalized trichotomy. Consider the initial value problem

$$v' = A(t)v + f(t, v), \qquad v(s) = v_s \tag{3.1}$$

where $f : \mathbb{R} \times X \to X$ is a continuous function such that

$$f(t, 0) = 0 \text{ for every } t \in \mathbb{R} \tag{3.2}$$

and, for every $t \in \mathbb{R}$,

$$\operatorname{Lip}(f_t) := \sup \left\{ \frac{\|f(t, x) - f(t, y)\|}{\|x - y\|} : x, y \in X, \ x \neq y \right\} < +\infty, \tag{3.3}$$

i.e., the function $f_t \colon X \to X$ given by $f_t(x) = f(t, x)$ is a Lipschitz function (in x). Clearly

$$\|f(t, x) - f(t, y)\| \leqslant \operatorname{Lip}(f_t) \, \|x - y\| \tag{3.4}$$

for every $x, y \in X$ and every $t \in \mathbb{R}$ and taking $y = 0$ in the last inequality and by (3.2), we have

$$\|f(t,x)\| \leqslant \operatorname{Lip}(f_t)\,\|x\| \tag{3.5}$$

for every $x \in X$ and every $t \in \mathbb{R}$.

When (1.1) admits a generalized trichotomy, we can write the only solution of (3.1) in the form

$$\big(x(t,s,v_s), y^+(t,s,v_s), y^-(t,s,v_s)\big) \in E_t \times F_t^+ \times F_t^-$$

where $v_s = (\xi, \eta^+, \eta^-) \in E_s \times F_s^+ \times F_s^-$, then solving problem (3.1) is equivalent to solve the following problem

$$x(t) = T_{t,s}P_s\xi + \int_s^t T_{t,r}P_r f(r, x(r), y^+(r), y^-(r))\, dr \tag{3.6}$$

$$y^+(t) = T_{t,s}Q_s^+\eta^+ + \int_s^t T_{t,r}Q_r^+ f(r, x(r), y^+(r), y^-(r))\, dr \tag{3.7}$$

$$y^-(t) = T_{t,s}Q_s^-\eta^- + \int_s^t T_{t,r}Q_r^- f(r, x(r), y^+(r), y^-(r))\, dr \tag{3.8}$$

for every $t \in \mathbb{R}$.

Definition 3.1.1. *We define the **flow** of differential equation* (3.1) *as*

$$\Psi_\tau(s, v_s) = \big(s+\tau, x(s+\tau, s, v_s), y^+(s+\tau, s, v_s), y^-(s+\tau, s, v_s)\big) \tag{3.9}$$

for each $\tau \in \mathbb{R}$.

We are going to study the existence of invariant center manifolds for equation (3.1) when (1.1) admits a generalized trichotomy. The invariant center manifolds that we are going to obtain are given by the graph of a function belonging to a certain function space that we define now.

For

$$G = \{(s,\xi) : s \in \mathbb{R},\ \xi \in E_s\} \subset \mathbb{R} \times X$$

and $N \in]0,+\infty[$, we denote by $\mathcal{A}_N$ the space of continuous functions $\varphi\colon G \to X$ such that

$$\varphi(t,0) = 0 \quad \text{for all } t \in \mathbb{R}; \tag{3.10}$$

$$\varphi(t,\xi) \in F_t^+ \oplus F_t^- \quad \text{for all } (t,\xi) \in G; \tag{3.11}$$

$$\sup\left\{\frac{\left\|\varphi(t,\xi)-\varphi(t,\bar\xi)\right\|}{\left\|\xi-\bar\xi\right\|}\colon (t,\xi),(t,\bar\xi)\in G,\ \xi\neq\bar\xi\right\} \leqslant N. \tag{3.12}$$

Note that from (3.12) it follows immediately that

$$\left\|\varphi(t,\xi)-\varphi(t,\bar\xi)\right\| \leqslant N\left\|\xi-\bar\xi\right\| \text{ for all } (t,\xi),(t,\bar\xi)\in G, \tag{3.13}$$

and making $\bar\xi = 0$ in (3.13), we have

$$\|\varphi(t,\xi)\| \leqslant N\|\xi\| \text{ for every } (t,\xi)\in G. \tag{3.14}$$

By (3.11), and identifying $F_t^+ \oplus F_t^-$ and $F_t^+ \times F_t^-$ as the same space, we can write $\varphi = (\varphi^+,\varphi^-)$, where $\varphi^+(t,\xi) = Q_t^+\varphi(t,\xi)$ and $\varphi^-(t,\xi) = Q_t^-\varphi(t,\xi)$.

We also define the graph, for all the functions $\varphi \in \mathcal{A}_N$, as follows.

Definition 3.1.2. *Let $\varphi \in \mathcal{A}_N$. We define the* ***graph of*** *φ as*

$$\begin{aligned}\mathcal{V}_\varphi &= \{(s,\xi,\varphi(s,\xi)) : (s,\xi)\in G\}\\ &= \big\{\big(s,\xi,\varphi^+(s,\xi),\varphi^-(s,\xi)\big) : (s,\xi)\in G\big\}\\ &\subseteq \mathbb{R}\times X.\end{aligned} \tag{3.15}$$

Before state the main theorem we need to define the following quantities:

$$\sigma := \sup_{(t,s)\in\mathbb{R}^2}\left|\int_s^t \frac{\alpha_{t,r}\operatorname{Lip}(f_r)\,\alpha_{r,s}}{\alpha_{t,s}}\,dr\right| \tag{3.16}$$

and

$$\omega := \sup_{s\in\mathbb{R}}\left[\int_{-\infty}^s \beta_{s,r}^+\operatorname{Lip}(f_r)\,\alpha_{r,s}\,dr + \int_s^{+\infty}\beta_{s,r}^-\operatorname{Lip}(f_r)\,\alpha_{r,s}\,dr\right]. \tag{3.17}$$

Theorem 3.1.3. *Let X be a Banach space. Suppose that* (1.1) *admits a generalized trichotomy with bounds $\alpha_{t,s}$, $\beta^+_{t,s}$ and $\beta^-_{t,s}$ and let $f\colon \mathbb{R} \times X \to X$ be a continuous function such that* (3.2) *and* (3.3) *are satisfied. If*

$$\lim_{r \to +\infty} \beta^-_{s,r} \alpha_{r,s} = \lim_{r \to -\infty} \beta^+_{s,r} \alpha_{r,s} = 0 \quad \text{for every } s \in \mathbb{R} \tag{3.18}$$

and

$$2\sigma + 2\omega < 1,$$

where σ and ω are given by (3.16) *and* (3.17), *respectively, then there is $N \in \,]0,1[$ and a unique $\varphi \in \mathcal{A}_N$ such that*

$$\Psi_\tau(\mathcal{V}_\varphi) \subset \mathcal{V}_\varphi$$

for every $\tau \in \mathbb{R}$, where Ψ_τ is given by (3.9) *and $\mathcal{V}_\varphi$ is given by* (3.15). *Moreover,*

$$\left\|\Psi_{t-s}(s,\xi,\varphi(s,\xi)) - \Psi_{t-s}(s,\bar{\xi},\varphi(s,\bar{\xi}))\right\| \leqslant \frac{N}{\omega}\, \alpha_{t,s}\, \left\|\xi - \bar{\xi}\right\|$$

for all $(t,s) \in \mathbb{R}^2$ and all $\xi, \bar{\xi} \in E_s$.

The proof of last theorem will be given in Section 3.3.

§3.2 Examples of invariant center manifolds

In this section we will give particular cases of Theorem 3.1.3.

§3.2.1 Nonuniform $(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d})-$trichotomies

In this subsection, we will apply this last result to trichotomies with bounds of the form (1.4).

Theorem 3.2.1. *Let X be a Banach space. Suppose that equation* (1.1) *admits a nonuniform $(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d})-$trichotomy and let $f\colon \mathbb{R} \times X \to X$ be a continuous function* *such that* (3.2) *and* (3.3) *are satisfied and*

$$\operatorname{Lip}(f_r) \leqslant \delta \min\left\{ \frac{1}{\mathfrak{c}(r)\mathfrak{d}(r)\varepsilon_{\mathfrak{d}}(r)} \left[\frac{\mathfrak{c}(r)\mathfrak{d}(r)}{\varepsilon_{\mathfrak{c}}(r)}\right]', \frac{\mathfrak{a}(r)\mathfrak{b}(r)}{\varepsilon_{\mathfrak{b}}(r)} \left[-\frac{1}{\mathfrak{a}(r)\mathfrak{b}(r)\varepsilon_{\mathfrak{a}}(r)}\right]', \gamma(r) \right\} \tag{3.19}$$

for every $r \in \mathbb{R} \setminus \{0\}$, *where* $\delta < 1/6$ *and* $\gamma \colon \mathbb{R} \to]0, +\infty[$ *is a function such that*

$$\max\left\{ \int_{-\infty}^{+\infty} \varepsilon_{\mathfrak{a}}(r)\gamma(r)\, dr, \int_{-\infty}^{+\infty} \varepsilon_{\mathfrak{c}}(r)\gamma(r)\, dr \right\} \leqslant 1. \tag{3.20}$$

If

$$\lim_{r\to-\infty} \mathfrak{c}(r)\mathfrak{d}(r)\varepsilon_{\mathfrak{d}}(r) = \lim_{r\to+\infty} \frac{\varepsilon_{\mathfrak{b}}(r)}{\mathfrak{a}(r)\mathfrak{b}(r)} = 0 \tag{3.21}$$

then equation (3.1) *admits an invariant center manifold, i.e., there is* $N \in]0,1[$ *and a unique* $\varphi \in \mathcal{A}_N$ *such that*

$$\Psi_\tau(\mathcal{V}_\varphi) \subset \mathcal{V}_\varphi \quad \textit{for every } \tau \in \mathbb{R},$$

where Ψ_τ *is given by* (3.9) *and* $\mathcal{V}_\varphi$ *is given by* (3.15). *Furthermore,*

$$\left\|\Psi_{t-s}(p_{s,\xi}) - \Psi_{t-s}(p_{s,\bar\xi})\right\| \leqslant \begin{cases} \dfrac{N}{\omega}\,\dfrac{\mathfrak{a}(s)}{\mathfrak{a}(t)}\,\varepsilon_{\mathfrak{a}}(s)\,\left\|\xi - \bar\xi\right\| & \textit{if } t \geqslant s, \\ \dfrac{N}{\omega}\,\dfrac{\mathfrak{c}(t)}{\mathfrak{c}(s)}\,\varepsilon_{\mathfrak{c}}(s)\,\left\|\xi - \bar\xi\right\| & \textit{if } t \leqslant s, \end{cases}$$

for all $(s,\xi), (s,\bar\xi) \in G$ *and where* $p_{s,\xi} = (s, \xi, \varphi(s,\xi))$ *and* $p_{s,\bar\xi} = \big(s, \bar\xi, \varphi(s,\bar\xi)\big)$.

Proof: It is obvious that for this type of bounds (3.18) is equivalent to (3.21). Moreover, since

$$\frac{\alpha_{t,r}\alpha_{r,s}}{\alpha_{t,s}} = \begin{cases} \dfrac{\mathfrak{a}(r)/\mathfrak{a}(t)\,\varepsilon_{\mathfrak{a}}(r)\ \mathfrak{a}(s)/\mathfrak{a}(r)\,\varepsilon_{\mathfrak{a}}(s)}{\mathfrak{a}(s)/\mathfrak{a}(t)\,\varepsilon_{\mathfrak{a}}(s)} & \text{if } t \geqslant r \geqslant s, \\ \dfrac{\mathfrak{c}(t)/\mathfrak{c}(r)\,\varepsilon_{\mathfrak{c}}(r)\ \mathfrak{c}(r)/\mathfrak{c}(s)\,\varepsilon_{\mathfrak{c}}(s)}{\mathfrak{c}(t)/\mathfrak{c}(s)\,\varepsilon_{\mathfrak{c}}(s)} & \text{if } t \leqslant r \leqslant s, \end{cases} = \begin{cases} \varepsilon_{\mathfrak{a}}(r) & \text{if } t \geqslant r \geqslant s, \\ \varepsilon_{\mathfrak{c}}(r) & \text{if } t \leqslant r \leqslant s, \end{cases}$$

from (3.19) and (3.20) we have

$$\begin{aligned}
\sigma &= \sup_{(t,s)\in\mathbb{R}^2} \left| \int_s^t \frac{\alpha_{t,r}\operatorname{Lip}(f_r)\alpha_{r,s}}{\alpha_{t,s}}\,dr \right| \\
&= \max\left\{ \sup_{(t,s)\in\mathbb{R}^2_\geqslant} \int_s^t \varepsilon_{\mathfrak{a}}(r)\operatorname{Lip}(f_r)\,dr,\ \sup_{(t,s)\in\mathbb{R}^2_\leqslant} \int_t^s \varepsilon_{\mathfrak{c}}(r)\operatorname{Lip}(f_r)\,dr \right\} \\
&= \max\left\{ \int_{-\infty}^{+\infty} \varepsilon_{\mathfrak{a}}(r)\operatorname{Lip}(f_r)\,dr,\ \int_{-\infty}^{+\infty} \varepsilon_{\mathfrak{c}}(r)\operatorname{Lip}(f_r)\,dr \right\} \\
&\leqslant \delta \max\left\{ \int_{-\infty}^{+\infty} \varepsilon_{\mathfrak{a}}(r)\gamma(r)\,dr,\ \int_{-\infty}^{+\infty} \varepsilon_{\mathfrak{c}}(r)\gamma(r)\,dr \right\} \\
&\leqslant \delta.
\end{aligned}$$

From (3.19) and (3.21) it follows that

$$\begin{aligned}
\omega &= \sup_{s\in\mathbb{R}} \left[\int_{-\infty}^s \beta^+_{s,r}\operatorname{Lip}(f_r)\,\alpha_{r,s}\,dr + \int_s^{+\infty} \beta^-_{s,r}\operatorname{Lip}(f_r)\,\alpha_{r,s}\,dr \right] \\
&= \sup_{s\in\mathbb{R}} \left[\int_{-\infty}^s \frac{\mathfrak{d}(r)\varepsilon_{\mathfrak{d}}(r)\mathfrak{c}(r)\varepsilon_{\mathfrak{c}}(s)}{\mathfrak{d}(s)\mathfrak{c}(s)}\operatorname{Lip}(f_r)\,dr + \int_s^{+\infty} \frac{\mathfrak{b}(s)\varepsilon_{\mathfrak{b}}(r)\mathfrak{a}(s)\varepsilon_{\mathfrak{a}}(s)}{\mathfrak{b}(r)\mathfrak{a}(r)}\operatorname{Lip}(f_r)\,dr \right] \\
&\leqslant \delta \left[\frac{\varepsilon_{\mathfrak{c}}(s)}{\mathfrak{c}(s)\mathfrak{d}(s)} \int_{-\infty}^s \left(\frac{\mathfrak{c}(r)\mathfrak{d}(r)}{\varepsilon_{\mathfrak{c}}(r)} \right)' dr - \mathfrak{a}(s)\mathfrak{b}(s)\varepsilon_{\mathfrak{a}}(s) \int_s^{+\infty} \left(\frac{1}{\mathfrak{a}(r)\mathfrak{b}(r)\varepsilon_{\mathfrak{a}}(r)} \right)' dr \right] \\
&= 2\delta.
\end{aligned}$$

Hence, since $\delta < 1/6$ we have $2\sigma + 2\omega < 1$ and all hypothesis of Theorem 3.1.3 are satisfied and this finishes the proof. ☐

Remark 3.2.2. *Note that the function γ can be chosen such that*

$$\gamma(t) = \frac{\min\left\{ \dfrac{\varepsilon_{\mathfrak{c}}(s)}{2\mathfrak{c}(s)\mathfrak{d}(s)} \left[\dfrac{\mathfrak{c}(t)\mathfrak{d}(t)}{\varepsilon_{\mathfrak{c}}(t)} \right]', \dfrac{\mathfrak{a}(s)\mathfrak{b}(s)\varepsilon_{\mathfrak{a}}(s)}{2} \left[-\dfrac{1}{\mathfrak{a}(t)\mathfrak{b}(t)\varepsilon_{\mathfrak{a}}(t)} \right]' \right\}}{\max\{\varepsilon_{\mathfrak{a}}(t), \varepsilon_{\mathfrak{c}}(t)\}}$$

for $t \neq 0$ and where s is a fixed real number. In fact, by (3.21) *we have*

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \varepsilon_{\mathfrak{a}}(r)\gamma(r)\,dr \\
&= \int_{-\infty}^s \varepsilon_{\mathfrak{a}}(r)\gamma(r)\,dr + \int_s^{+\infty} \varepsilon_{\mathfrak{a}}(r)\gamma(r)\,dr \\
&\leqslant \int_{-\infty}^s \frac{\varepsilon_{\mathfrak{c}}(s)}{2\mathfrak{c}(s)\mathfrak{d}(s)} \left[\frac{\mathfrak{c}(r)\mathfrak{d}(r)}{\varepsilon_{\mathfrak{c}}(r)} \right]' dr + \int_s^{+\infty} \frac{\mathfrak{a}(s)\mathfrak{b}(s)\varepsilon_{\mathfrak{a}}(s)}{2} \left[\frac{-1}{\mathfrak{a}(r)\mathfrak{b}(r)\varepsilon_{\mathfrak{a}}(r)} \right]' dr \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \varepsilon_{\mathfrak{c}}(r)\gamma(r)\,dr \\
&= \int_{-\infty}^{s} \varepsilon_{\mathfrak{c}}(r)\gamma(r)\,dr + \int_{s}^{+\infty} \varepsilon_{\mathfrak{c}}(r)\gamma(r)\,dr \\
&\leqslant \int_{-\infty}^{s} \frac{\varepsilon_{\mathfrak{c}}(s)}{2\mathfrak{c}(s)\mathfrak{d}(s)} \left[\frac{\mathfrak{c}(r)\mathfrak{d}(r)}{\varepsilon_{\mathfrak{c}}(r)}\right]' dr + \int_{s}^{+\infty} \frac{\mathfrak{a}(s)\mathfrak{b}(s)\varepsilon_{\mathfrak{c}}(s)}{2} \left[\frac{-1}{\mathfrak{a}(r)\mathfrak{b}(r)\varepsilon_{\mathfrak{a}}(r)}\right]' dr \\
&= 1.
\end{aligned}$$

§3.2.2 $\rho-$nonuniform exponential trichotomies

Now we are going to apply last theorem of this chapter to the $\rho-$nonuniform exponential trichotomies.

Theorem 3.2.3. *Let X be a Banach space and assume that equation* (1.1) *admits a $\rho-$nonuniform exponential trichotomy. Suppose that $f\colon \mathbb{R}\times X \to X$ is a continuous function that satisfies* (3.2) *and* (3.3) *and*

$$\operatorname{Lip}(f_r) \leqslant \frac{\delta\rho'(r)}{D^2\,\mathrm{e}^{2\varepsilon|\rho(r)|}} \min\left\{-c-d-\varepsilon\operatorname{sgn}(r), -a-b+\varepsilon\operatorname{sgn}(r), \frac{D\gamma}{2}\,\mathrm{e}^{(\varepsilon-\gamma)|\rho(r)|}\right\}, \tag{3.22}$$

where $\delta < 1/6$ and

$$\gamma = \varepsilon \quad \text{if} \quad \varepsilon > 0 \qquad \text{and} \qquad 0 < \gamma < \frac{2}{D}\min\{-c-d, -a-b\} \quad \text{if} \quad \varepsilon = 0. \tag{3.23}$$

If

$$a+b+\varepsilon < 0 \qquad \text{and} \qquad c+d+\varepsilon < 0, \tag{3.24}$$

then (3.1) *admits an invariant center manifold, i.e., there is $N \in \,]0,1[$ and a unique $\varphi \in \mathcal{A}_N$ such that*

$$\Psi_\tau(\mathcal{V}_\varphi) \subset \mathcal{V}_\varphi \quad \text{for every } \tau \in \mathbb{R},$$

where Ψ_τ is given by (3.9) *and $\mathcal{V}_\varphi$ is given by* (3.15). *Furthermore,*

$$\left\|\Psi_{t-s}(p_{s,\xi}) - \Psi_{t-s}(p_{s,\bar\xi})\right\| \leqslant \begin{cases} \dfrac{DN}{\omega}\,\mathrm{e}^{a[\rho(t)-\rho(s)]+\varepsilon|\rho(s)|}\,\left\|\xi-\bar\xi\right\| & \text{if } t \geqslant s, \\[2ex] \dfrac{DN}{\omega}\,\mathrm{e}^{c[\rho(s)-\rho(t)]+\varepsilon|\rho(s)|}\,\left\|\xi-\bar\xi\right\| & \text{if } t \leqslant s, \end{cases}$$

for all $(s,\xi),(s,\bar{\xi}) \in G$ *and where* $p_{s,\xi} = (s,\xi,\varphi(s,\xi))$ *and* $p_{s,\bar{\xi}} = \big(s,\bar{\xi},\varphi(s,\bar{\xi})\big)$.

Proof: For this bounds condition (3.21) is equivalent to (3.24) since

$$\lim_{r\to-\infty} \mathfrak{c}(r)\mathfrak{d}(r)\varepsilon_{\mathfrak{d}}(r) = 0 \Leftrightarrow \lim_{r\to-\infty} D\,\mathrm{e}^{(-c-d)\rho(r)+\varepsilon|\rho(r)|} = 0 \Leftrightarrow \lim_{r\to-\infty} D\,\mathrm{e}^{(-c-d-\varepsilon)\rho(r)} = 0$$

and

$$\lim_{r\to+\infty} \frac{\varepsilon_{\mathfrak{b}}(r)}{\mathfrak{a}(r)\mathfrak{b}(r)} = 0 \Leftrightarrow \lim_{r\to+\infty} D\,\mathrm{e}^{(a+b)\rho(r)+\varepsilon|\rho(r)|} = 0 \Leftrightarrow \lim_{r\to+\infty} D\,\mathrm{e}^{(a+b+\varepsilon)\rho(r)} = 0.$$

Moreover, since for $r \neq 0$ we have

$$\begin{aligned}
\frac{1}{\mathfrak{c}(r)\mathfrak{d}(r)\varepsilon_{\mathfrak{d}}(r)} \left[\frac{\mathfrak{c}(r)\mathfrak{d}(r)}{\varepsilon_{\mathfrak{c}}(r)}\right]' &= \frac{1}{\varepsilon_{\mathfrak{c}}(r)\varepsilon_{\mathfrak{d}}(r)} \left(\frac{\mathfrak{c}'(r)}{\mathfrak{c}(r)} + \frac{\mathfrak{d}'(r)}{\mathfrak{d}(r)} - \frac{\varepsilon_{\mathfrak{c}}'(r)}{\varepsilon_{\mathfrak{c}}(r)}\right) \\
&= \frac{1}{D^2\,\mathrm{e}^{2\varepsilon|\rho(r)|}} \left(\frac{-c\rho'(r)\,\mathrm{e}^{-c\rho(r)}}{\mathrm{e}^{-c\rho(r)}} + \frac{-d\rho'(r)\,\mathrm{e}^{-d\rho(r)}}{\mathrm{e}^{-d\rho(r)}} - \frac{\varepsilon\,\mathrm{sgn}(r)\rho'(r)\,\mathrm{e}^{\varepsilon|\rho(r)|}}{\mathrm{e}^{\varepsilon|\rho(r)|}}\right) \\
&= \frac{(-c-d-\varepsilon\,\mathrm{sgn}(r))\rho'(r)}{D^2\,\mathrm{e}^{2\varepsilon|\rho(r)|}}
\end{aligned} \tag{3.25}$$

and in a similar way

$$\begin{aligned}
\frac{\mathfrak{a}(r)\mathfrak{b}(r)}{\varepsilon_{\mathfrak{b}}(r)} \left[-\frac{1}{\mathfrak{a}(r)\mathfrak{b}(r)\varepsilon_{\mathfrak{a}}(r)}\right]' &= \frac{1}{\varepsilon_{\mathfrak{a}}(r)\varepsilon_{\mathfrak{b}}(r)} \left(\frac{\mathfrak{a}'(r)}{\mathfrak{a}(r)} + \frac{\mathfrak{b}'(r)}{\mathfrak{b}(r)} + \frac{\varepsilon_{\mathfrak{a}}'(r)}{\varepsilon_{\mathfrak{a}}(r)}\right) \\
&= \frac{(-a-b+\varepsilon\,\mathrm{sgn}(r))\,\rho'(r)}{D^2\,\mathrm{e}^{2\varepsilon|\rho(r)|}}.
\end{aligned} \tag{3.26}$$

Making

$$\gamma(r) = \frac{\gamma}{2D}\rho'(r)\,\mathrm{e}^{-(\varepsilon+\gamma)|\rho(r)|}, \tag{3.27}$$

with γ given by (3.23), and observing that

$$\int_{-\infty}^{+\infty} D\,\mathrm{e}^{\varepsilon|\rho(r)|}\,\gamma(r)\,dr = \int_{-\infty}^{+\infty} \frac{\gamma}{2}\rho'(r)\,\mathrm{e}^{-\gamma|\rho(r)|}\,dr = 2\int_{0}^{+\infty} \frac{\gamma}{2}\rho'(r)\,\mathrm{e}^{-\gamma\rho(r)}\,dr = 1,$$

since $\rho'(t)$ is an even function, condition (3.19) becomes (3.22) and the result follows. □

Note that this improves the result by Barreira and Valls in [11] because we have a better asymptotic behavior for the solutions with initial conditions in the invariant manifold. In fact, in the exponent we have a where Barreira and Valls have $a + 2\delta D$.

Taking $\rho(t) = t$ in last theorem we have the following result.

Corollary 3.2.4. *Let X be a Banach space and assume that equation* (1.1) *admits a nonuniform exponential trichotomy. Suppose that $f\colon \mathbb{R}\times X \to X$ is a continuous function that satisfies* (3.2) *and* (3.3) *and*

$$\operatorname{Lip}(f_r) \leqslant \frac{\delta}{D^2\, \mathrm{e}^{2\varepsilon|r|}} \min\left\{-c-d-\varepsilon\operatorname{sgn}(r), -a-b+\varepsilon\operatorname{sgn}(r), \frac{D\gamma}{2}\,\mathrm{e}^{(\varepsilon-\gamma)|r|}\right\},$$

where γ is given by (3.23) *and $\delta < 1/6$. If* (3.24) *is satisfied then* (3.1) *admits an invariant center manifold, i.e., there is $N \in\,]0,1[$ and a unique $\varphi \in \mathcal{A}_N$ such that*

$$\Psi_\tau(\mathcal{V}_\varphi) \subset \mathcal{V}_\varphi \quad \textit{for every } \tau \in \mathbb{R},$$

where Ψ_τ is given by (3.9) *and $\mathcal{V}_\varphi$ is given by* (3.15). *Furthermore,*

$$\left\|\Psi_{t-s}(p_{s,\xi}) - \Psi_{t-s}(p_{s,\bar\xi})\right\| \leqslant \begin{cases} \dfrac{DN}{\omega}\,\mathrm{e}^{a(t-s)+\varepsilon|s|}\,\left\|\xi-\bar\xi\right\| & \textit{if } t \geqslant s,\\ \dfrac{DN}{\omega}\,\mathrm{e}^{c(s-t)+\varepsilon|s|}\,\left\|\xi-\bar\xi\right\| & \textit{if } t \leqslant s,\end{cases}$$

for all $(s,\xi),(s,\bar\xi) \in G$ and where $p_{s,\xi} = (s,\xi,\varphi(s,\xi))$ and $p_{s,\bar\xi} = \big(s,\bar\xi,\varphi(s,\bar\xi)\big)$.

Again, as in the last theorem, we improve the asymptotic behavior of the result obtained by Barreira and Valls in [3].

§3.2.3 $\mu-$nonuniform polynomial trichotomies

In what follows, we are going to assume that equation (1.1) admits a $\mu-$nonuniform polynomial trichotomy.

Theorem 3.2.5. *Let X be a Banach space. Suppose that equation* (1.1) *admits a $\mu-$nonuniform polynomial trichotomy and let $f\colon \mathbb{R}\times X \to X$ be a continuous function such that* (3.2) *and* (3.3) *are satisfied and*

$$\operatorname{Lip}(f_r) \leqslant \delta\mu'(r)(|\mu(r)|+1)^{-\gamma}$$

with $\gamma > 0$. If (3.24) *is satisfied, δ is sufficiently small and*

$$a, c \leqslant 0, \quad 2\varepsilon \leqslant \gamma \quad \textit{and} \quad \varepsilon-\gamma+1 < 0,$$

then (3.1) *admits an invariant center manifold, i.e., there is* $N \in]0,1[$ *and a unique* $\varphi \in \mathcal{A}_N$ *such that*

$$\Psi_\tau(\mathcal{V}_\varphi) \subset \mathcal{V}_\varphi \quad \text{for every } \tau \in \mathbb{R},$$

where Ψ_τ *is given by* (3.9) *and* $\mathcal{V}_\varphi$ *is given by* (3.15). *Furthermore,*

$$\left\|\Psi_{t-s}(p_{s,\xi}) - \Psi_{t-s}(p_{s,\bar{\xi}})\right\| \leqslant \begin{cases} \dfrac{DN}{\omega}(\mu(t)-\mu(s)+1)^a(|\mu(s)|+1)^\varepsilon \left\|\xi-\bar{\xi}\right\| & \text{if } t \geqslant s, \\ \dfrac{DN}{\omega}(\mu(s)-\mu(t)+1)^c(|\mu(s)|+1)^\varepsilon \left\|\xi-\bar{\xi}\right\| & \text{if } t \leqslant s, \end{cases}$$

for all $(s,\xi),(s,\bar{\xi}) \in G$ *and where* $p_{s,\xi} = (s,\xi,\varphi(s,\xi))$ *and* $p_{s,\bar{\xi}} = \big(s,\bar{\xi},\varphi(s,\bar{\xi})\big)$.

To prove this theorem we need the following lemma.

Lemma 3.2.6. *Let* λ, $\nu < 0$, $\varepsilon \geqslant 0$ *and* $p \in \mathbb{R}$. *If*

$$\lambda+\varepsilon+\nu+1<0, \quad \lambda+\varepsilon \leqslant 0 \quad \text{and} \quad \nu+\varepsilon \leqslant 0,$$

then

$$\int_0^{+\infty} (1+\tau)^\lambda \left(|\tau+p|+1\right)^\nu \left(|p|+1\right)^\varepsilon \, d\tau \leqslant \begin{cases} \dfrac{1}{|\lambda+\varepsilon+\nu+1|} & \text{if} \quad p \geqslant 0, \\ \dfrac{2^{\varepsilon+1}+1}{|\lambda+\varepsilon+\nu+1|} & \text{if} \quad p<0. \end{cases}$$

Proof: If $p \geqslant 0$, since $\varepsilon \geqslant 0$, $\nu+\varepsilon \leqslant 0$ and $\lambda+\varepsilon+\nu+1<0$, we have

$$\begin{aligned} \int_0^{+\infty} (\tau+1)^\lambda \left(|\tau+p|+1\right)^\nu \left(|p|+1\right)^\varepsilon \, d\tau &= \int_0^{+\infty} (\tau+1)^\lambda \left(\tau+p+1\right)^\nu \left(p+1\right)^\varepsilon \, d\tau \\ &\leqslant \int_0^{+\infty} (\tau+1)^\lambda \left(\tau+p+1\right)^{\nu+\varepsilon} \, d\tau \\ &\leqslant \int_0^{+\infty} (\tau+1)^{\lambda+\nu+\varepsilon} \, d\tau \\ &= \frac{1}{|\lambda+\nu+\varepsilon+1|}. \end{aligned}$$

If $p<0$, then

$$|\tau+p|+1 = \begin{cases} |p|-\tau+1 & \text{if} \quad 0 \leqslant \tau \leqslant |p|, \\ \tau-|p|+1 & \text{if} \quad \tau \geqslant |p|, \end{cases}$$

and this implies

$$\begin{aligned}
&\int_0^{|p|/2} (\tau+1)^{\lambda} (|p|-\tau+1)^{\nu} (|p|+1)^{\varepsilon} \, d\tau \\
&= \int_0^{|p|/2} (\tau+1)^{\lambda} (|p|-\tau+1)^{\nu+\varepsilon} \left(\frac{|p|+1}{|p|-\tau+1}\right)^{\varepsilon} d\tau \\
&\leqslant 2^{\varepsilon} \int_0^{|p|/2} (\tau+1)^{\lambda+\nu+\varepsilon} \, d\tau \\
&\leqslant \frac{2^{\varepsilon}}{|\lambda+\nu+\varepsilon+1|},
\end{aligned}$$

$$\begin{aligned}
&\int_{|p|/2}^{|p|} (\tau+1)^{\lambda} (|p|-\tau+1)^{\nu} (|p|+1)^{\varepsilon} \, d\tau \\
&= \int_{|p|/2}^{|p|} (\tau+1)^{\lambda+\varepsilon} (|p|-\tau+1)^{\nu} \left(\frac{|p|+1}{\tau+1}\right)^{\varepsilon} d\tau \\
&\leqslant 2^{\varepsilon} \int_{|p|/2}^{|p|} (|p|-\tau+1)^{\lambda+\nu+\varepsilon} \, d\tau \\
&\leqslant \frac{2^{\varepsilon}}{|\lambda+\nu+\varepsilon+1|}
\end{aligned}$$

and

$$\begin{aligned}
\int_{|p|}^{+\infty} (\tau+1)^{\lambda} (\tau-|p|+1)^{\nu} (|p|+1)^{\varepsilon} \, d\tau &\leqslant \int_{|p|}^{+\infty} (\tau+1)^{\lambda+\varepsilon} (\tau-|p|+1)^{\nu} \, d\tau \\
&\leqslant \int_{|p|}^{+\infty} (\tau-|p|+1)^{\lambda+\nu+\varepsilon} \, d\tau \\
&\leqslant \frac{1}{|\lambda+\nu+\varepsilon+1|}.
\end{aligned}$$

Hence, if $p<0$ we have

$$\int_0^{+\infty} (\tau+1)^{\lambda} (|\tau+p|+1)^{\nu} (|p|+1)^{\varepsilon} \, d\tau \leqslant \frac{2^{\varepsilon+1}+1}{|\lambda+\varepsilon+\nu+1|}.$$

□

Proof of Theorem 3.2.5. Since for $r \leqslant s$ we have

$$\beta^{+}_{s,r}\alpha_{r,s} = D^2 \left(\mu(s)-\mu(r)+1\right)^{c+d+\varepsilon} \left(\frac{|\mu(r)|+1}{\mu(s)-\mu(r)+1}\right)^{\varepsilon} \left(|\mu(s)|+1\right)^{\varepsilon}$$

and for $r \geqslant s$ we have

$$\beta^{-}_{s,r}\alpha_{r,s} = D^2 \left(\mu(r)-\mu(s)+1\right)^{a+b+\varepsilon} \left(\frac{|\mu(r)|+1}{\mu(r)-\mu(s)+1}\right)^{\varepsilon} \left(|\mu(s)|+1\right)^{\varepsilon},$$

it follows that if $c+d+\varepsilon<0$ and $a+b+\varepsilon<0$, then

$$\lim_{r\to-\infty}\beta^+_{s,r}\alpha_{r,s}=\lim_{r\to+\infty}\beta^-_{s,r}\alpha_{r,s}=0.$$

Since for every $t\geqslant r\geqslant s$ and every $t\leqslant r\leqslant s$ we have

$$\begin{aligned}[\mu(t)-\mu(r)+1]\,[\mu(r)-\mu(s)+1]&=[\mu(t)-\mu(r)]\,[\mu(r)-\mu(s)]+\mu(t)-\mu(s)+1\\&\geqslant\mu(t)-\mu(s)+1,\end{aligned}$$

and $a,c\leqslant 0$, it follows that for $t\geqslant r\geqslant s$

$$\frac{\alpha_{t,r}\alpha_{r,s}}{\alpha_{t,s}}=D\frac{(\mu(t)-\mu(r)+1)^a\,(\mu(r)-\mu(s)+1)^a}{(\mu(t)-\mu(s)+1)^a}\,(|\mu(r)|+1)^\varepsilon\leqslant D\,(|\mu(r)|+1)^\varepsilon \tag{3.28}$$

and for $t\leqslant r\leqslant s$ we also have

$$\frac{\alpha_{t,r}\alpha_{r,s}}{\alpha_{t,s}}=D\frac{(\mu(r)-\mu(t)+1)^c\,(\mu(s)-\mu(r)+1)^c}{(\mu(s)-\mu(t)+1)^c}\,(|\mu(r)|+1)^\varepsilon\leqslant D\,(|\mu(r)|+1)^\varepsilon. \tag{3.29}$$

Then, since $\gamma>\varepsilon+1$, it follows that

$$\begin{aligned}\sigma&=\sup_{(t,s)\in\mathbb{R}^2}\left|\int_s^t\frac{\alpha_{t,r}\operatorname{Lip}(f_r)\alpha_{r,s}}{\alpha_{t,s}}\,dr\right|\\&\leqslant D\delta\sup_{(t,s)\in\mathbb{R}^2}\left|\int_s^t\mu'(r)\,(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr\right|\\&=D\delta\int_{-\infty}^{+\infty}\mu'(r)\,(|\mu(r)|+1)^{\varepsilon-\gamma}\,dr\\&=D\delta\int_{-\infty}^{+\infty}(|\tau|+1)^{\varepsilon-\gamma}\,d\tau\\&=\frac{2D\delta}{|\varepsilon-\gamma+1|}.\end{aligned}$$

Here we made the substitution $\tau=\mu(r)$.

Making the substitution $\tau=\mu(s)-\mu(r)$ we have

$$\begin{aligned}&\int_{-\infty}^s\beta^+_{s,r}\operatorname{Lip}(f_r)\,\alpha_{r,s}\,dr\\&\leqslant D^2\delta\int_{-\infty}^s\mu'(r)\,(\mu(s)-\mu(r)+1)^{c+d}\,(|\mu(r)|+1)^{\varepsilon-\gamma}\,(|\mu(s)|+1)^\varepsilon\,dr\\&=D^2\delta\int_0^{+\infty}(\tau+1)^{c+d}\,(|\tau-\mu(s)|+1)^{\varepsilon-\gamma}\,(|\mu(s)|+1)^\varepsilon\,d\tau\end{aligned}$$

and with the substitution $\tau = \mu(r) - \mu(s)$ we obtain

$$\begin{aligned}
&\int_s^{+\infty} \beta^-_{s,r} \operatorname{Lip}(f_r)\, \alpha_{r,s}\, dr \\
&\leqslant D^2\delta \int_s^{+\infty} \mu'(r)\, (\mu(r) - \mu(s) + 1)^{a+b}\, (|\mu(r)| + 1)^{\varepsilon-\gamma}\, (|\mu(s)| + 1)^{\varepsilon}\, dr \\
&= D^2\delta \int_0^{+\infty} (\tau + 1)^{a+b}\, (|\tau + \mu(s)| + 1)^{\varepsilon-\gamma}\, (|\mu(s)| + 1)^{\varepsilon}\, d\tau.
\end{aligned}$$

Using Lemma 3.2.6 it follows that

$$\begin{aligned}
\omega &= \sup_{s\in\mathbb{R}} \left[\int_{-\infty}^{s} \beta^+_{s,r} \operatorname{Lip}(f_r)\, \alpha_{r,s}\, dr + \int_s^{+\infty} \beta^-_{s,r} \operatorname{Lip}(f_r)\, \alpha_{r,s}\, dr \right] \\
&\leqslant D^2\delta \left(\frac{2^{\varepsilon+1} + 1}{|\max\{a+b, c+d\} + 2\varepsilon - \gamma + 1|} + \frac{1}{|\min\{a+b, c+d\} + 2\varepsilon - \gamma + 1|} \right).
\end{aligned}$$

Hence, for δ sufficiently small we have $2\sigma + 2\omega < 1$ and the result follows. □

In the next corollary we will consider nonuniform polynomial trichotomy.

Corollary 3.2.7. *Let X be a Banach space. Suppose that equation* (1.1) *admits a trichotomy with bounds of the form* (1.7) *and let $f\colon \mathbb{R} \times X \to X$ be a continuous function such that* (3.2) *and* (3.3) *are satisfied and*

$$\operatorname{Lip}(f_r) \leqslant \delta(|r| + 1)^{-\gamma}$$

with $\gamma > 0$. If (3.24) *is satisfied, δ is sufficiently small and*

$$a, c \leqslant 0, \quad 2\varepsilon \leqslant \gamma \quad \textit{and} \quad \varepsilon - \gamma + 1 < 0,$$

then (3.1) *admits an invariant center manifold, i.e., there is $N \in\,]0,1[$ and a unique $\varphi \in \mathcal{A}_N$ such that*

$$\Psi_\tau(\mathcal{V}_\varphi) \subset \mathcal{V}_\varphi \quad \textit{for every } \tau \in \mathbb{R},$$

where Ψ_τ is given by (3.9) *and $\mathcal{V}_\varphi$ is given by* (3.15). *Furthermore,*

$$\left\| \Psi_{t-s}(p_{s,\xi}) - \Psi_{t-s}(p_{s,\bar\xi}) \right\| \leqslant \begin{cases} \dfrac{DN}{\omega} (t - s + 1)^a (|s| + 1)^{\varepsilon} \left\| \xi - \bar\xi \right\| & \textit{if } t \geqslant s, \\[2mm] \dfrac{DN}{\omega} (s - t + 1)^c (|s| + 1)^{\varepsilon} \left\| \xi - \bar\xi \right\| & \textit{if } t \leqslant s, \end{cases}$$

for all $(s,\xi), (s,\bar\xi) \in G$ and where $p_{s,\xi} = (s, \xi, \varphi(s,\xi))$ and $p_{s,\bar\xi} = \big(s, \bar\xi, \varphi(s,\bar\xi)\big)$.

§3.3 Proof of Theorem 3.1.3

Before doing the proof of the Theorem 3.1.3 we need to state and prove some lemmas that will be used in the proof of that result.

§3.3.1 Auxiliary Lemmas

Lemma 3.3.1. *If σ and ω are positive real numbers such that*

$$2\sigma + 2\omega < 1,$$

then there exist $M \in]1,2[$ and $N \in]0,1[$ such that

$$\sigma = \frac{M-1}{M(1+N)} \quad \text{and} \quad \omega = \frac{N}{M(1+N)}. \tag{3.30}$$

Proof: Clearly, equalities (3.30) are equivalent to

$$\frac{M-1}{\sigma} = \frac{N}{\omega} = M(1+N). \tag{3.31}$$

Hence making

$$M = \frac{1-\sigma+\omega-\sqrt{1-2\sigma-2\omega+(\sigma-\omega)^2}}{2\omega} \tag{3.32}$$

and

$$N = \frac{1-\sigma-\omega-\sqrt{1-2\sigma-2\omega+(\sigma-\omega)^2}}{2\sigma} \tag{3.33}$$

we obtain immediately the first equality in (3.31). Taking into account that

$$1-2\sigma-2\omega+(\sigma-\omega)^2 = (1-\sigma+\omega)^2 - 4\omega = (1-\sigma-\omega)^2 - 4\sigma\omega,$$

and $\sigma+\omega < 1-\sigma-\omega$, we have

$$\begin{aligned} M &= \frac{(1-\sigma+\omega)^2 - [1-2\sigma-2\omega+(\sigma-\omega)^2]}{2\omega\left[1-\sigma+\omega+\sqrt{1-2\sigma-2\omega+(\sigma-\omega)^2}\right]} \\ &= \frac{2}{1-\sigma+\omega+\sqrt{1-2\sigma-2\omega+(\sigma-\omega)^2}} \\ &< \frac{2}{1-\sigma+\omega+|\sigma-\omega|} \\ &\leqslant 2 \end{aligned}$$

and

$$\begin{aligned} N &= \frac{(1-\sigma-\omega)^2 - [1-2\sigma-2\omega+(\sigma-\omega)^2]}{2\sigma\left[(1-\sigma-\omega)+\sqrt{1-2\sigma-2\omega+(\sigma-\omega)^2}\right]} \\ &= \frac{2\omega}{1-\sigma-\omega+\sqrt{1-2\sigma-2\omega+(\sigma-\omega)^2}} \\ &< \frac{2\omega}{1-\sigma-\omega+|\sigma-\omega|} \\ &< \frac{2\omega}{\sigma+\omega+|\sigma-\omega|} \\ &\leqslant 1. \end{aligned}$$

Moreover, using the definition of N and M we can put

$$1+\frac{1}{N} = \frac{1-\sigma+\omega+\sqrt{1-2\sigma-2\omega+(\sigma-\omega)^2}}{2\omega} = \frac{1}{\omega M}$$

and this proves the second equality in (3.31). To finish the proof we note it is clear that $N > 0$ and since $M = 1 + \sigma N/\omega$ we have $M > 1$. □

In what follows M and N are those given by (3.32) and (3.33) in the proof of last lemma. Moreover, the constant N mentioned in Theorem 3.1.3 is also given by (3.33).

It is easy to see that $\mathcal{A}_N$ is a complete metric space with the metric

$$d(\varphi,\psi) = \sup\left\{\frac{\|\varphi(t,\xi)-\psi(t,\xi)\|}{\|\xi\|} : (t,\xi)\in G,\ \xi\neq 0\right\}, \tag{3.34}$$

for all $\varphi,\psi \in \mathcal{A}_N$.

Let

$$G' = \left\{(t,s,\xi) : (t,s)\in\mathbb{R}^2,\ \xi\in E_s\right\} \subset \mathbb{R}^2\times X.$$

Let $\mathcal{B}_M$ be the set of all continuous functions $x\colon G'\to X$ such that

$$x(t,s,0) = 0 \ \text{ for all } (t,s)\in\mathbb{R}^2, \tag{3.35}$$

$$x(s,s,\xi) = \xi \ \text{ for all } s\in\mathbb{R} \text{ and all } \xi\in E_s, \tag{3.36}$$

$$x(t,s,\xi)\in E_t \ \text{ for all } (t,s,\xi)\in G', \tag{3.37}$$

$$\sup\left\{\frac{\left\|x(t,s,\xi)-x(t,s,\bar\xi)\right\|}{\alpha_{t,s}\ \left\|\xi-\bar\xi\right\|} : (t,s,\xi),(t,s,\bar\xi)\in G',\ \xi\neq\bar\xi\right\} \leqslant M \tag{3.38}$$

where M is given by (3.30).

From (3.38), it follows that

$$\left\|x(t,s,\xi)-x(t,s,\bar{\xi})\right\| \leqslant M\alpha_{t,s}\left\|\xi-\bar{\xi}\right\| \quad \text{for all } (t,s,\xi),(t,s,\bar{\xi}) \in G' \tag{3.39}$$

and making $\bar{\xi}=0$ in (3.39), from (3.35) we have

$$\|x(t,s,\xi)\| \leqslant M\alpha_{t,s}\|\xi\| \quad \text{for all } (t,s,\xi) \in G'. \tag{3.40}$$

Defining

$$d'(x,y)=\sup\left\{\frac{\|x(t,s,\xi)-y(t,s,\xi)\|}{\alpha_{t,s}\;\|\xi\|}\colon (t,s,\xi)\in G',\ \xi\neq 0\right\} \tag{3.41}$$

taking into account (3.37), (3.36) and (3.38), it is easy to see that $(\mathcal{B}_M, d')$ is a complete metric space. Writing the only solution of (3.1) in the form

$$(x(t,s,\xi),\varphi(t,x(t,s,\xi))),\ \text{with } t\in\mathbb{R},$$

it is clear that solving (3.1) is the same as solving the equations

$$x(t,s,\xi)=T_{t,s}P_s\xi+\int_s^t T_{t,r}P_r f(r,x(r,s,\xi),\varphi(r,x(r,s,\xi)))\,dr, \tag{3.42}$$

$$\varphi^+(t,x(t,s,\xi))=T_{t,s}Q_s^+\varphi^+(s,\xi)+\int_s^t T_{t,r}Q_r^+ f(r,x(r,s,\xi),\varphi(r,x(r,s,\xi)))\,dr, \tag{3.43}$$

$$\varphi^-(t,x(t,s,\xi))=T_{t,s}Q_s^-\varphi^-(s,\xi)+\int_s^t T_{t,r}Q_r^- f(r,x(r,s,\xi),\varphi(r,x(r,s,\xi)))\,dr. \tag{3.44}$$

Let $\mathcal{C}_{M,N}=\mathcal{B}_M\times\mathcal{A}_N$. The space $\mathcal{C}_{M,N}$ with the metric defined by

$$d''\left((x,\varphi),(y,\psi)\right)=d'(x,y)+d(\varphi,\psi),\ \text{for all } (x,\varphi),(y,\psi)\in\mathcal{C}_{M,N}$$

is a complete metric space.

Next we state a lemma about the form of the solutions of (3.1).

Lemma 3.3.2. *Let* $(x,\varphi)\in\mathcal{C}_{M,N}$ *such that* (3.42) *is satisfied. The following properties are equivalent:*

a) for every $(t,s,\xi)\in G'$*, equations* (3.43) *and* (3.44) *hold;*

b) for every $(s,\xi)\in G$*,*

$$\varphi^+(s,\xi)=\int_{-\infty}^{s} T_{s,r}Q_r^+ f(r,x(r,s,\xi),\varphi(r,x(r,s,\xi)))\,dr \tag{3.45}$$

and

$$\varphi^-(s,\xi)=-\int_{s}^{+\infty} T_{s,r}Q_r^- f(r,x(r,s,\xi),\varphi(r,x(r,s,\xi)))\,dr. \tag{3.46}$$

Proof: First we must prove that the integrals in (3.45) and (3.46) are convergent. From (3.5), (3.14) and (3.40) we obtain

$$\begin{aligned}\|f(r,x(r,s,\xi),\varphi(r,x(r,s,\xi)))\| &\leqslant \operatorname{Lip}(f_r)\,\|x(r,s,\xi)+\varphi(r,x(r,s,\xi))\|\\ &\leqslant \operatorname{Lip}(f_r)\,(\|x(r,s,\xi)\|+\|\varphi(r,x(r,s,\xi))\|)\\ &\leqslant \operatorname{Lip}(f_r)\,(\|x(r,s,\xi)\|+N\,\|x(r,s,\xi)\|)\\ &\leqslant M(1+N)\operatorname{Lip}(f_r)\alpha_{r,s}\,\|\xi\|\end{aligned} \tag{3.47}$$

for every $r,s\in\mathbb{R}$ and using (**D2**), (3.47), (3.17) and (**D3**) we have

$$\begin{aligned}&\int_{-\infty}^{s}\left\|T_{s,r}Q_r^+ f(r,x(r,s,\xi),\varphi(r,x(r,s,\xi)))\right\|\,dr\\ &\leqslant \int_{-\infty}^{s}\left\|T_{s,r}Q_r^+\right\|\,\|f(r,x(r,s,\xi),\varphi(r,x(r,s,\xi)))\|\,dr\\ &\leqslant M(1+N)\,\|\xi\|\int_{-\infty}^{s}\beta_{s,r}^+\operatorname{Lip}(f_r)\alpha_{r,s}\,dr\\ &\leqslant M(1+N)\omega\,\|\xi\|\end{aligned}$$

and

$$\begin{aligned}&\int_{s}^{+\infty}\left\|T_{s,r}Q_r^- f(r,x(r,s,\xi),\varphi(r,x(r,s,\xi)))\right\|\,dr\\ &\leqslant \int_{s}^{+\infty}\left\|T_{s,r}Q_r^-\right\|\,\|f(r,x(r,s,\xi),\varphi(r,x(r,s,\xi)))\|\,dr\\ &\leqslant M(1+N)\,\|\xi\|\int_{s}^{+\infty}\beta_{s,r}^-\operatorname{Lip}(f_r)\alpha_{r,s}\,dr\\ &\leqslant M(1+N)\omega\,\|\xi\|\end{aligned}$$

for every $(s,\xi)\in G$. Thus the integrals (3.45) and (3.46) are convergent.

Now we prove that $a) \Rightarrow b)$. Suppose that (3.43) and (3.44) hold for every $(t, s, \xi) \in G'$. Then, from (3.43) we have

$$\begin{aligned}\varphi^+(s,\xi) &= T_{s,t}\varphi^+(t, x(t,s,\xi)) - \int_s^t T_{s,t}T_{t,r}Q_r^+ f(r, x(r,s,\xi), \varphi(r, x(r,s,\xi)))\, dr\\ &= T_{s,t}Q_t^+\varphi(t, x(t,s,\xi)) - \int_s^t T_{s,r}Q_r^+ f(r, x(r,s,\xi), \varphi(r, x(r,s,\xi)))\, dr.\end{aligned}$$

Since by (**D2**), (3.14) and (3.40) we have

$$\begin{aligned}\left\|T_{s,t}Q_t^+\varphi(t, x(t,s,\xi))\right\| &\leqslant \beta_{s,t}^+ \left\|\varphi(t, x(t,s,\xi))\right\|\\ &\leqslant N\beta_{s,t}^+ \left\|x(t,s,\xi)\right\|\\ &\leqslant MN\left\|\xi\right\| \beta_{s,t}^+\alpha_{t,s}\ ,\end{aligned}$$

by (3.18), making $t \to -\infty$, we conclude that

$$\lim_{t\to-\infty} T_{s,t}Q_t^+\varphi(t, x(t,s,\xi)) = 0$$

and this implies

$$\begin{aligned}\varphi^+(s,\xi) &= -\int_s^{-\infty} T_{s,r}Q_r^+ f(r, x(r,s,\xi), \varphi(r, x(r,s,\xi)))\, dr\\ &= \int_{-\infty}^s T_{s,r}Q_r^+ f(r, x(r,s,\xi), \varphi(r, x(r,s,\xi)))\, dr,\end{aligned}$$

i.e., (3.45) holds.

Similarly, from (3.44) we have

$$\begin{aligned}\varphi^-(s,\xi) &= T_{s,t}\varphi^-(t, x(t,s,\xi)) - \int_s^t T_{s,t}T_{t,r}Q_r^- f(r, x(r,s,\xi), \varphi(r, x(r,s,\xi)))\, dr\\ &= T_{s,t}Q_t^-\varphi(t, x(t,s,\xi)) - \int_s^t T_{s,t}T_{t,r}Q_r^- f(r, x(r,s,\xi), \varphi(r, x(r,s,\xi)))\, dr\end{aligned}$$

and since

$$\begin{aligned}\left\|T_{s,t}Q_t^-\varphi(t, x(t,s,\xi))\right\| &\leqslant \beta_{s,t}^- \left\|\varphi(t, x(t,s,\xi))\right\|\\ &\leqslant N\beta_{s,t}^- \left\|x(t,s,\xi)\right\|\\ &\leqslant MN\left\|\xi\right\| \beta_{s,t}^-\alpha_{t,s}\end{aligned}$$

from (**D3**), (3.14) and (3.40). Letting $t \to +\infty$ we have, by (3.18),

$$\lim_{t\to+\infty} T_{s,t}Q_t^- \varphi(t, x(t, s, \xi)) = 0$$

and we obtain

$$\varphi^-(s, \xi) = -\int_s^{+\infty} T_{s,r}Q_r^- f(r, x(r, s, \xi), \varphi(r, x(r, s, \xi)))\, dr$$

for every $(s, \xi) \in G$. Hence $a) \Rightarrow b)$.

Now we will prove that $b) \Rightarrow a)$. Assuming that for every $(s, \xi) \in G$ identities (3.45) and (3.46) hold, applying $T_{t,s}$ to both sides of equation (3.45) we have

$$T_{t,s}\varphi^+(s, \xi) = \int_{-\infty}^{s} T_{t,r}Q_r^+ f(r, x(r, s, \xi), \varphi(r, x(r, s, \xi)))\, dr$$

and this implies

$$\begin{aligned}
&T_{t,s}\varphi^+(s, \xi) + \int_s^t T_{t,r}Q_r^+ f(r, x(r, s, \xi), \varphi(r, x(r, s, \xi)))\, dr \\
&= \int_{-\infty}^{t} T_{t,r}Q_r^+ f(r, x(r, s, \xi), \varphi(r, x(r, s, \xi)))\, dr \\
&= \int_{-\infty}^{t} T_{t,r}Q_r^+ f(r, x(r, t, x(t, s, \xi)), \varphi(r, x(r, t, x(t, s, \xi))))\, dr \\
&= \varphi^+(t, x(t, s, \xi)),
\end{aligned}$$

for every $(t, s, \xi) \in G'$. In a similar way we have

$$T_{t,s}\varphi^-(s, \xi) = -\int_s^{+\infty} T_{t,s}T_{s,r}Q_r^- f(r, x(r, s, \xi), \varphi(r, x(r, s, \xi)))\, dr$$

and thus

$$\begin{aligned}
&T_{t,s}\varphi^-(s, \xi) + \int_s^t T_{t,r}Q_r^- f(r, x(r, s, \xi), \varphi(r, x(r, s, \xi)))\, dr \\
&= -\int_t^{+\infty} T_{t,r}Q_r^- f(r, x(r, s, \xi), \varphi(r, x(r, s, \xi)))\, dr \\
&= -\int_t^{+\infty} T_{t,r}Q_r^- f\left(r, x(r, t, x(t, s, \xi)), \varphi\left(r, x\left(r, t, x\left(t, s, \xi\right)\right)\right)\right)\, dr \\
&= \varphi^-(t, x(t, s, \xi)),
\end{aligned}$$

for every $(t, s, \xi) \in G'$. Therefore $b) \Rightarrow a)$ and this completes the proof of the lemma. □

Now we define in $\mathcal{C}_{M,N}$ an operator J as follows.

Consider in $\mathcal{C}_{M,N}$ the operator J such that, to each $(x,\varphi) \in \mathcal{C}_{M,N}$, assigns a function $J(x,\varphi)\colon G' \to X$ defined, for every $(t,s,\xi) \in G'$, by

$$[J(x,\varphi)]\,(t,s,\xi) = T_{t,s}P_s\xi + \int_s^t T_{t,r}P_r f(r, x(r,s,\xi), \varphi(r, x(r,s,\xi)))\,dr.$$

Lemma 3.3.3. *For every $(x,\varphi) \in \mathcal{C}_{M,N}$, we have*

$$J(x,\varphi) \in \mathcal{B}_M.$$

Proof: It is obvious that $J(x,\varphi)(t,s,0) = 0$ and $J(x,\varphi)(s,s,\xi) = \xi$ for all $(t,s) \in \mathbb{R}^2$ and every $(x,\varphi) \in \mathcal{C}_{M,N}$ and $\xi \in E_s$. Moreover, for all $(t,s) \in \mathbb{R}^2$ and using (**D1**)

$$\begin{aligned}\|J(x,\varphi)(t,s,\xi) - J(x,\varphi)(t,s,\bar\xi)\| &\leqslant \|T_{t,s}P_s\|\|\xi-\bar\xi\| + \left|\int_s^t \|T_{t,r}P_r\|\gamma_{r,s,\xi,\bar\xi}\,dr\right| \\ &\leqslant \alpha_{t,s}\|\xi-\bar\xi\| + \left|\int_s^t \alpha_{t,r}\gamma_{r,s,\xi,\bar\xi}\,dr\right|,\end{aligned}$$

where

$$\gamma_{r,s,\xi,\bar\xi} := \|f(r, x(r,s,\xi), \varphi(r, x(r,s,\xi))) - f(r, x(r,s,\bar\xi), \varphi(r, x(r,s,\bar\xi)))\|.$$

for every $(t,s,\xi), (t,s,\bar\xi) \in G'$. From (3.4), (3.39) and (3.13) we have

$$\begin{aligned}&\gamma_{r,s,\xi,\bar\xi} \\ &\leqslant \operatorname{Lip}(f_r)\left(\|x(r,s,\xi) - x(r,s,\bar\xi)\| + \|\varphi(r, x(r,s,\xi)) - \varphi(r,s,x(r,s,\bar\xi))\|\right) \\ &\leqslant \operatorname{Lip}(f_r)\left(M\alpha_{r,s}\|\xi-\bar\xi\| + N\|x(r,s,\xi)) - x(r,s,\bar\xi))\|\right) \\ &\leqslant \operatorname{Lip}(f_r)\left(M\alpha_{r,s}\|\xi-\bar\xi\| + MN\alpha_{r,s}\|\xi-\bar\xi\|\right) \\ &\leqslant M(1+N)\|\xi-\bar\xi\|\operatorname{Lip}(f_r)\alpha_{r,s}\end{aligned} \tag{3.48}$$

and so by (3.16) and using Lemma 3.3.1 it follows that

$$\begin{aligned}&\|J(x,\varphi)(t,s,\xi) - J(x,\varphi)(t,s,\bar\xi)\| \\ &\qquad\leqslant \alpha_{t,s}\|\xi-\bar\xi\| + M(1+N)\|\xi-\bar\xi\|\left|\int_s^t \alpha_{t,r}\operatorname{Lip}(f_r)\alpha_{r,s}\,dr\right| \\ &\qquad\leqslant (1+M(1+N)\sigma)\,\alpha_{t,s}\|\xi-\bar\xi\| \\ &\qquad= M\alpha_{t,s}\|\xi-\bar\xi\|,\end{aligned}$$

for every $(x,\varphi) \in \mathcal{C}_{M,N}$ and every $(t,s,\xi),(t,s,\bar{\xi}) \in G'$. Then considering $\xi \neq \bar{\xi}$ we have

$$\frac{\|J(x,\varphi)(t,s,\xi) - J(x,\varphi)(t,s,\bar{\xi})\|}{\alpha_{t,s}\|\xi - \bar{\xi}\|} \leqslant M$$

and we can say $J(\mathcal{B}_M \times \mathcal{A}_N) \subseteq \mathcal{B}_M$. □

In $\mathcal{C}_{M,N}$ define the operator L that assigns to every $(x,\varphi) \in \mathcal{C}_{M,N}$ a function

$$L(x,\varphi)\colon G \to X$$

defined, for every $(s,\xi) \in G$, by

$$[L(x,\varphi)]\,(s,\xi) = \left[L^+(x,\varphi)\right](s,\xi) + \left[L^-(x,\varphi)\right](s,\xi),$$

where

$$\left[L^+(x,\varphi)\right](s,\xi) = \int_{-\infty}^{s} T_{s,r}Q_r^+ f(r, x(r,s,\xi), \varphi(r, x(r,s,\xi)))\,dr$$

and

$$\left[L^-(x,\varphi)\right](s,\xi) = -\int_{s}^{+\infty} T_{s,r}Q_r^- f(r, x(r,s,\xi), \varphi(r, x(r,s,\xi)))\,dr.$$

Lemma 3.3.4. *For every $(x,\varphi) \in \mathcal{C}_{M,N}$, we have*

$$L(x,\varphi) \in \mathcal{A}_N.$$

Proof: From (3.35), (3.10) and (3.2) it follows that $[L(x,\varphi)]\,(t,0) = 0$. Moreover, by definition we have $[L(x,\varphi)]\,(t,\xi) \in F_t^+ \oplus F_t^-$.

From (**D2**), (**D3**) and (3.48) it follows for every $(s,\xi),(s,\bar{\xi}) \in G$ that

$$\begin{aligned}\|L^+(x,\varphi)(s,\xi) - L^+(x,\varphi)(s,\bar{\xi})\| &\leqslant \int_{-\infty}^{s} \|T_{s,r}Q_r^+\|\gamma_{r,s,\xi,\bar{\xi}}\,dr \\ &\leqslant M(1+N)\left\|\xi - \bar{\xi}\right\| \int_{-\infty}^{s} \beta_{s,r}^+ \operatorname{Lip}(f_r)\alpha_{r,s}\,dr\end{aligned}$$

and

$$\begin{aligned}\|L^-(x,\varphi)(s,\xi) - L^-(x,\varphi)(s,\bar{\xi})\| &\leqslant \int_{s}^{+\infty} \|T_{s,r}Q_r^-\|\gamma_{r,s,\xi,\bar{\xi}}\,dr \\ &\leqslant M(1+N)\left\|\xi - \bar{\xi}\right\| \int_{s}^{+\infty} \beta_{s,r}^- \operatorname{Lip}(f_r)\alpha_{r,s}\,dr\end{aligned}$$

and so, using (3.17) and Lemma 3.3.1 we have

$$\begin{aligned}
&\left\|[L(x,\varphi)](s,\xi)-[L(x,\varphi)](s,\bar{\xi})\right\| \\
&\leqslant \left\|\left[L^+(x,\varphi)\right](s,\xi)-\left[L^+(x,\varphi)\right](s,\bar{\xi})\right\|+\left\|\left[L^-(x,\varphi)\right](s,\xi)-\left[L^-(x,\varphi)\right](s,\bar{\xi})\right\| \\
&\leqslant M(1+N)\left\|\xi-\bar{\xi}\right\|\left(\int_{-\infty}^{s}\beta^+_{s,r}\operatorname{Lip}(f_r)\,\alpha_{r,s}\,dr+\int_{s}^{+\infty}\beta^-_{s,r}\operatorname{Lip}(f_r)\,\alpha_{r,s}\,dr\right) \\
&\leqslant M(1+N)\omega\left\|\xi-\bar{\xi}\right\| \\
&= N\left\|\xi-\bar{\xi}\right\|,
\end{aligned}$$

and the proof is complete. □

Lemma 3.3.5. *For every $(x,\varphi),(y,\psi)\in\mathcal{C}_{M,N}$ we have*

$$d'\left(J(x,\varphi),J(y,\psi)\right)\leqslant\sigma\left[(1+N)d'(x,y)+Md(\varphi,\psi)\right] \tag{3.49}$$

and

$$d\left(L(x,\varphi),L(y,\psi)\right)\leqslant\omega\left[(1+N)d'(x,y)+Md(\varphi,\psi)\right]. \tag{3.50}$$

Proof: For every $(r,s,\xi)\in G'$, putting

$$\overline{\gamma}_{r,s,\xi}:=\|f(r,x(r,s,\xi),\varphi(r,x(r,s,\xi)))-f(r,y(r,s,\xi),\psi(r,y(r,s,\xi)))\|,$$

and using (**D1**) we have

$$\|J(x,\varphi)(t,s,\xi)-J(y,\psi)(t,s,\xi)\|\leqslant\left|\int_s^t\|T_{t,r}P_r\|\,\overline{\gamma}_{r,s,\xi}\,dr\right|\leqslant\left|\int_s^t\alpha_{t,r}\,\overline{\gamma}_{r,s,\xi}\,dr\right|.$$

By (3.4), (3.13), (3.41), (3.34), and (3.40) we obtain

$$\begin{aligned}
\overline{\gamma}_{r,s,\xi}&=\|f(r,x(r,s,\xi),\varphi(r,x(r,s,\xi)))-f(r,y(r,s,\xi),\psi(r,y(r,s,\xi)))\| \\
&\leqslant\operatorname{Lip}(f_r)\left[\|x(r,s,\xi)-y(r,s,\xi)\|+\|\varphi(r,x(r,s,\xi))-\psi(r,y(r,s,\xi))\|\right] \\
&\leqslant\operatorname{Lip}(f_r)[\|x(r,s,\xi)-y(r,s,\xi)\|+\|\varphi(r,x(r,s,\xi))-\varphi(r,y(r,s,\xi))\| \\
&\qquad+\|\varphi(r,y(r,s,\xi))-\psi(r,y(r,s,\xi))\|] \\
&\leqslant\operatorname{Lip}(f_r)\left[(1+N)\|x(r,s,\xi)-y(r,s,\xi)\|+d(\varphi,\psi)\|y(r,s,\xi))\|\right] \\
&\leqslant\operatorname{Lip}(f_r)\left[d'(x,y)\alpha_{r,s}\|\xi\|(1+N)+d(\varphi,\psi)M\alpha_{r,s}\|\xi\|\right] \\
&\leqslant\operatorname{Lip}(f_r)\alpha_{r,s}\|\xi\|\left[(1+N)d'(x,y)+Md(\varphi,\psi)\right],
\end{aligned} \tag{3.51}$$

it follows by (3.16) that

$$\begin{aligned}&\|J(x,\varphi)(t,s,\xi)-J(y,\psi)(t,s,\xi)\|\\&\leqslant\left|\int_s^t \alpha_{t,r}\operatorname{Lip}(f_r)\alpha_{r,s}\,dr\right|\ \|\xi\|\,[(1+N)d'(x,y)+Md(\varphi,\psi)]\\&\leqslant \alpha_{t,s}\sigma\,\|\xi\|\,[(1+N)d'(x,y)+Md(\varphi,\psi)]\,,\end{aligned}$$

for every $(t,s,\xi)\in G'$. Thus from (3.41) we get (3.49).

On the other hand, using again (**D2**) and (3.51) we have

$$\begin{aligned}&\|L^+(x,\varphi)(s,\xi)-L^+(y,\psi)(s,\xi)\|\\&\leqslant\int_{-\infty}^s \|T_{s,r}Q_r^+\|\,\overline{\gamma}_{r,s,\xi}\,dr\\&\leqslant\|\xi\|\,[(1+N)d'(x,y)+Md(\varphi,\psi)]\int_{-\infty}^s \beta_{s,r}^+\operatorname{Lip}(f_r)\alpha_{r,s}\,dr\end{aligned}$$

and also from (**D3**) and (3.51) it follows that

$$\begin{aligned}&\|L^-(x,\varphi)(s,\xi)-L^-(s,\psi)(t,\xi)\|\\&\leqslant\int_s^{+\infty} \|T_{s,r}Q_r^-\|\,\overline{\gamma}_{r,s,\xi}\,dr\\&\leqslant\|\xi\|\,[(1+N)d'(x,y)+Md(\varphi,\psi)]\int_s^{+\infty} \beta_{s,r}^-\operatorname{Lip}(f_r)\alpha_{r,s}\,dr\end{aligned}$$

and thus from (3.17) we obtain

$$\|L(x,\varphi)(s,\xi)-L(s,\psi)(t,\xi)\|\leqslant\|\xi\|\,\omega\,[(1+N)d'(x,y)+Md(\varphi,\psi)]\,.$$

Therefore, from (3.34) we get (3.50). □

Below we define o new operator in $\mathcal{C}_{M,N}$ and we will prove that it is a contraction which will be essential in the proof of Theorem 3.1.3.

Define the operator $T\colon\mathcal{C}_{M,N}\to\mathcal{C}_{M,N}$ by

$$T(x,\varphi)=(J(x,\varphi),L(x,\varphi))=\big(J(x,\varphi),L^+(x,\varphi),L^-(x,\varphi)\big)\,.$$

Lemma 3.3.6. *The operator $T\colon\mathcal{C}_{M,N}\to\mathcal{C}_{M,N}$ is a contraction.*

Proof: Let $(x,\varphi),(y,\psi) \in \mathcal{C}_{M,N}$. Using last lemma, Lemma 3.3.1, and since $M < 2$ and $N < 1$ we have

$$\begin{aligned} d''\left(T(x,\varphi),T(y,\psi)\right) &= d''\left((J(x,\varphi),L(x,\varphi)),(J(y,\psi),L(y,\psi))\right) \\ &= d'\left(J(x,\varphi),J(y,\psi)\right) + d\left(L(x,\varphi),L(y,\psi)\right) \\ &\leqslant (\sigma+\omega)\left((1+N)d'(x,y) + Md(\varphi,\psi)\right) \\ &= \left(1 - \frac{1-N}{M}\right) d'(x,y) + \left(1 - \frac{2-M}{1+N}\right) d(\varphi,\psi) \\ &\leqslant \max\left\{1 - \frac{1-N}{M}, 1 - \frac{2-M}{1+N}\right\} d''((x,\varphi),(y,\psi)), \end{aligned}$$

and so T is a contraction in $\mathcal{C}_{M,N}$. □

§3.3.2 Proof of Theorem 3.1.3

Now we are going to prove Theorem 3.1.3. Since $\mathcal{C}_{M,N}$ is a complete metric space and by Lemma 3.3.6 the operator T is a contraction, by Banach Fixed Point Theorem, there is a unique point $(x,\varphi) \in \mathcal{C}_{M,N}$ such that

$$T(x,\varphi) = (x,\varphi)$$

and that verifies (3.42), (3.45) and (3.46). In Lemma 3.3.2 we proved that solve the last two equations was equivalent to solve another two, (3.43) and (3.44), if (3.42) holds. Therefore, by (3.6), (3.7) and (3.8), this establishes the existence of the invariant manifold, that is, the existence of a unique

$$\varphi = (\varphi^+,\varphi^-) \in \mathcal{A}_N \;\text{ such that }\; \Psi_\tau(\mathcal{V}_\varphi) \subseteq \mathcal{V}_\varphi$$

for every $\tau \in \mathbb{R}$. Moreover, for every $s, t \in \mathbb{R}$ and every $\xi, \bar{\xi} \in E_s$ we have

$$
\begin{aligned}
&\|\Psi_{t-s}(s,\xi,\varphi(s,\xi)) - \Psi_{t-s}(s,\bar{\xi},\varphi(s,\bar{\xi}))\| \\
&= \|(t, x(t,s,\xi), \varphi(t, x(t,s,\xi))) - (t, x(t,s,\bar{\xi}), \varphi(t, x(t,s,\bar{\xi})))\| \\
&\leqslant \|x(t,s,\xi) - x(t,s,\bar{\xi})\| + \|\varphi(t, x(t,s,\xi))) - \varphi(t, x(t,s,\bar{\xi})))\| \\
&\leqslant (1+N)\|x(t,s,\xi) - x(t,s,\bar{\xi})\| \\
&\leqslant M(1+N)\alpha_{t,s}\|\xi - \bar{\xi}\| \\
&= \frac{N}{\omega}\alpha_{t,s}\|\xi - \bar{\xi}\|
\end{aligned}
$$

and this completes the proof of the theorem.

Chapter 4

Local Lipschitz invariant manifolds

In this last chapter we prove the existence of local Lipschitz invariant manifolds. This theorem is stated in Section 4.1. In Sections 4.2 and 4.3 we have considered two types of nonlinear perturbations f in the differential equation

$$v' = A(t)v + f(t, v),$$

where f is a continuous locally Lipschtiz function. In the last section, we give the proof of the main result. The proof of the main theorem uses Theorem 3.1.3.

§4.1 Existence of local Lipschitz invariant manifolds

Now we are going to assume that equation (1.1) admits a generalized trichotomy with bounds $\alpha_{t,s}$, $\beta^+_{t,s}$ and $\beta^-_{t,s}$, that $f\colon \mathbb{R} \times X \to X$ is a continuous function such that (3.2) holds and, for each $t \in \mathbb{R}$, the function f_t is a Lipschitz function in

$$\mathfrak{B}(R(t)) = \{x \in X \colon \|x\| \leqslant R(t)\},$$

where $R\colon \mathbb{R} \to \mathbb{R}^+$, with Lipschitz constant $\operatorname{Lip}\left(f_t|_{\mathfrak{B}(R(t))}\right)$. Consider the initial value problem (3.1), i.e.,

$$v' = A(t)v + f(t, v), \qquad v(s) = v_s.$$

In these conditions we define the following constants:

$$\widetilde{\sigma} := \sup_{(t,s)\in\mathbb{R}^2} \left| \int_s^t \frac{\alpha_{t,r} \operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right) \alpha_{r,s}}{\alpha_{t,s}} \, dr \right|$$

and

$$\widetilde{\omega} := \sup_{s\in\mathbb{R}} \left[\int_{-\infty}^s \beta^+_{s,r} \operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right) \alpha_{r,s} \, dr + \int_s^{+\infty} \beta^-_{s,r} \operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right) \alpha_{r,s} \, dr \right].$$

Definition 4.1.1. *Given a function $R\colon \mathbb{R} \to \mathbb{R}^+$ and $\varphi \in \mathcal{A}_N$, we define the* ***graph of** φ **in** $\mathfrak{B}(R(s))$* *as*

$$\mathcal{V}^*_{\varphi,R} = \{(s,\xi,\varphi(s,\xi)) \in \mathcal{V}_\varphi \colon \|\xi\| \leqslant R(s)\}, \tag{4.1}$$

that will give us our ***local Lipschitz invariant manifolds****.*

Theorem 4.1.2. *Let X be a Banach space. Suppose that* (1.1) *admits a generalized trichotomy with bounds $\alpha_{t,s}$, $\beta^+_{t,s}$ and $\beta^-_{t,s}$. Let $f\colon \mathbb{R}\times X \to X$ be a continuous function such that* (3.2) *is verified and, for each $t\in\mathbb{R}$, the function f_t is a Lipschitz function in $\mathfrak{B}(R(t))$, where $R\colon\mathbb{R}\to\mathbb{R}^+$. If* (3.18) *holds,*

$$4\widetilde{\sigma} + 4\widetilde{\omega} < 1$$

and

$$\sup_{t\in\mathbb{R}} \frac{\alpha_{t,s}}{R(t)} < +\infty \quad \text{for every } s\in\mathbb{R}, \tag{4.2}$$

then there is $N \in \,]0,1[$ and $\varphi\in\mathcal{A}_N$ such that for every $\tau\in\mathbb{R}$ we have

$$\Psi_\tau(\mathcal{V}^*_{\varphi,\overline{R}}) \subseteq \mathcal{V}^*_{\varphi,R}\,,$$

where Ψ_τ is given by (3.9)*, $\overline{R}$ denotes the function $\overline{R}\colon\mathbb{R}\to\mathbb{R}^+$ given by*

$$\overline{R}(s) = \frac{\widetilde{\omega}}{N \sup_{t\in\mathbb{R}} [\alpha_{t,s}/R(t)]} \tag{4.3}$$

and $\mathcal{V}^*_{\varphi,\overline{R}}$ and $\mathcal{V}^*_{\varphi,R}$ are given by (4.1)*. Furthermore, we have*

$$\left\|\Psi_{t-s}(s,\xi,\varphi(s,\xi)) - \Psi_{t-s}(s,\bar{\xi},\varphi(s,\bar{\xi}))\right\| \leqslant \frac{N}{\widetilde{\omega}} \alpha_{t,s} \left\|\xi-\bar{\xi}\right\| \tag{4.4}$$

for every $(t,s)\in\mathbb{R}^2$ and every $\xi,\bar{\xi}\in\mathfrak{B}(\overline{R}(s))\cap E_s$.

The proof of this theorem will be given in Section 4.4.

§4.2 Examples of local invariant manifolds – first type of perturbations

In this section we are going to consider perturbations $f\colon \mathbb{R}\times X \to X$ such that

$$\|f(t,u)-f(t,v)\| \leqslant k(t)\,\|u-v\|\,(\|u\|+\|v\|)^q \tag{4.5}$$

for every $t\in\mathbb{R}$ and every $u,v\in X$ and where $k\colon \mathbb{R}\to]0,+\infty[$ and $q>0$. For every $R\colon \mathbb{R}\to]0,+\infty[$, the function f_t is Lipschitz in $\mathfrak{B}(R(t))$ and

$$\operatorname{Lip}\left(f_t|_{\mathfrak{B}(R(t))}\right) \leqslant 2^q k(t) R^q(t).$$

§4.2.1 Nonuniform $(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d})$−trichotomies

In this subsection, we will apply Theorem 4.1.2 to nonuniform $(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d})$−trichotomies.

Theorem 4.2.1. *Let X be a Banach space. Suppose that equation* (1.1) *admits a nonuniform* $(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d})$ *−trichotomy and that $f\colon \mathbb{R}\times X\to X$ is a continuous function that satisfies* (3.2) *and* (4.5) *with*

$$k(t)=\frac{\delta}{2^q\mathfrak{c}^q(t)}\min\left\{\frac{1}{\mathfrak{c}(t)\mathfrak{d}(t)\varepsilon_{\mathfrak{d}}(t)}\,\left[\frac{\mathfrak{c}(t)\mathfrak{d}(t)}{\varepsilon_{\mathfrak{c}}(t)}\right]',\frac{\mathfrak{a}(t)\mathfrak{b}(t)}{\varepsilon_{\mathfrak{b}}(t)}\,\left[-\frac{1}{\mathfrak{a}(t)\mathfrak{b}(t)\varepsilon_{\mathfrak{a}}(t)}\right]',\gamma(t)\right\}$$

and where $\gamma\colon \mathbb{R}\to]0,+\infty[$ is a function such that (3.20) *is satisfied. If* (3.21) *is fulfilled,* $0<\delta<1/12$ *and*

$$M(s):=\sup_{t\geqslant s}\frac{\mathfrak{a}(s)\mathfrak{c}(s)}{\mathfrak{a}(t)\mathfrak{c}(t)}<+\infty,$$

then there is $N\in]0,1[$ and $\varphi\in\mathcal{A}_N$ such that

$$\Psi_\tau\left(\mathcal{V}^*_{\varphi,\overline{R}}\right)\subset\mathcal{V}^*_{\varphi,R}$$

for every $\tau\in\mathbb{R}$, where Ψ_τ is given by (3.9)*, $R,\overline{R}\colon \mathbb{R}\to]0,\infty[$ are defined by*

$$R(s)=\mathfrak{c}(s)\quad\text{and}\quad \overline{R}(s)=\frac{\widetilde{\omega}}{N\max\left\{M(s)\varepsilon_{\mathfrak{a}}(s)/\mathfrak{c}(s),\varepsilon_{\mathfrak{c}}(s)/\mathfrak{c}(s)\right\}}$$

and $\mathcal{V}^*_{\varphi,\overline{R}}$ *and* $\mathcal{V}^*_{\varphi,R}$ *are given by* (4.1)*. Furthermore, we have*

$$\left\|\Psi_{t-s}(p_{s,\xi})-\Psi_{t-s}(p_{s,\bar\xi})\right\| \leqslant \begin{cases} \dfrac{N}{\widetilde{\omega}}\dfrac{\mathfrak{a}(s)}{\mathfrak{a}(t)}\varepsilon_{\mathfrak{a}}(s)\left\|\xi-\bar\xi\right\| & \text{for } t \geqslant s,\\ \dfrac{N}{\widetilde{\omega}}\dfrac{\mathfrak{c}(t)}{\mathfrak{c}(s)}\varepsilon_{\mathfrak{c}}(s)\left\|\xi-\bar\xi\right\| & \text{for } t \leqslant s,\end{cases}$$

for every $(s,\xi),(s,\bar\xi)\in G$ *and* $\xi,\bar\xi \in \mathfrak{B}(\overline{R}(s))$ *and where* $p_{s,\xi}=(s,\xi,\varphi(s,\xi))$ *and* $p_{s,\bar\xi}=\big(s,\bar\xi,\varphi(s,\bar\xi)\big)$.

Proof: With $R(t)=\mathfrak{c}(t)$, from

$$\sup_{t\geqslant s}\frac{\alpha_{t,s}}{R(t)}=\sup_{t\geqslant s}\frac{\mathfrak{a}(s)}{\mathfrak{a}(t)\mathfrak{c}(t)}\varepsilon_{\mathfrak{a}}(s)=\sup_{t\geqslant s}\frac{\mathfrak{a}(s)\mathfrak{c}(s)}{\mathfrak{a}(t)\mathfrak{c}(t)}\frac{\varepsilon_{\mathfrak{a}}(s)}{\mathfrak{c}(s)}=M(s)\frac{\varepsilon_{\mathfrak{a}}(s)}{\mathfrak{c}(s)}$$

and

$$\sup_{t\leqslant s}\frac{\alpha_{t,s}}{R(t)}=\sup_{t\leqslant s}\frac{\mathfrak{c}(t)}{\mathfrak{c}(s)\mathfrak{c}(t)}\varepsilon_{\mathfrak{c}}(s)=\frac{\varepsilon_{\mathfrak{c}}(s)}{\mathfrak{c}(s)}$$

we conclude that (4.2) is satisfied. The Lipschitz constants of f_t in $\mathfrak{B}(\mathfrak{c}(t))$ satisfy

$$\operatorname{Lip}\left(f_t|_{\mathfrak{B}(\mathfrak{c}(t))}\right)\leqslant 2^q k(t)\mathfrak{c}^q(t)$$

and therefore, using (2.19) and (3.20) we have

$$\begin{aligned}\widetilde{\sigma} &\leqslant \sup_{(t,s)\in\mathbb{R}^2}\left|\int_s^t \frac{\alpha_{t,r}\operatorname{Lip}\left(f_r|_{\mathfrak{B}(\mathfrak{c}(r))}\right)\alpha_{r,s}}{\alpha_{t,s}}\,dr\right|\\ &= 2^q\max\left\{\sup_{(t,s)\in\mathbb{R}^2_{\geqslant}}\int_s^t \varepsilon_{\mathfrak{a}}(r)k(r)\mathfrak{c}^q(r)\,dr,\ \sup_{(t,s)\in\mathbb{R}^2_{\leqslant}}\int_t^s \varepsilon_{\mathfrak{c}}(r)k(r)\mathfrak{c}^q(r)\,dr\right\}\\ &\leqslant \delta\max\left\{\int_{-\infty}^{+\infty}\varepsilon_{\mathfrak{a}}(r)\gamma(r)\,dr,\int_{-\infty}^{+\infty}\varepsilon_{\mathfrak{c}}(r)\gamma(r)\,dr\right\}\\ &\leqslant \delta.\end{aligned}$$

Moreover, using (3.21) we have

$$\begin{aligned}\int_{-\infty}^s \beta^+_{s,r}\operatorname{Lip}\left(f_r|_{\mathfrak{B}(\mathfrak{c}(r))}\right)\alpha_{r,s}\,dr &\leqslant 2^q\int_{-\infty}^s\frac{\mathfrak{d}(r)}{\mathfrak{d}(s)}\varepsilon_{\mathfrak{d}}(r)k(r)\mathfrak{c}^q(r)\frac{\mathfrak{c}(r)}{\mathfrak{c}(s)}\varepsilon_{\mathfrak{c}}(s)\,dr\\ &\leqslant \frac{\delta\varepsilon_{\mathfrak{c}}(s)}{\mathfrak{c}(s)\mathfrak{d}(s)}\int_{-\infty}^s\left(\frac{\mathfrak{c}(r)\mathfrak{d}(r)}{\varepsilon_{\mathfrak{c}}(r)}\right)' dr\\ &\leqslant \delta\end{aligned}$$

and

$$\begin{aligned}\int_s^{+\infty} \beta_{s,r}^{-} \operatorname{Lip}\left(f_r|_{\mathfrak{B}(\mathfrak{c}(r))}\right) \alpha_{r,s}\, dr &\leqslant 2^q \int_s^{+\infty} \frac{\mathfrak{b}(s)}{\mathfrak{b}(r)} \varepsilon_{\mathfrak{b}}(r) k(r) \mathfrak{c}^q(r) \frac{\mathfrak{a}(s)}{\mathfrak{a}(r)} \varepsilon_{\mathfrak{a}}(s)\, dr \\ &\leqslant \delta \mathfrak{a}(s)\mathfrak{b}(s)\varepsilon_{\mathfrak{a}}(s) \int_s^{+\infty} \left(\frac{1}{\mathfrak{a}(r)\mathfrak{b}(r)\varepsilon_{\mathfrak{a}}(r)}\right)' dr \\ &\leqslant \delta\end{aligned}$$

we also have $\widetilde{\omega} \leqslant 2\delta$. Hence, from $\delta < 1/12$ we have $4\widetilde{\sigma} + 4\widetilde{\omega} < 1$. Furthermore, for this type of bounds in the trichotomy condition (3.18) is equivalent to (3.21). Therefore all the hypothesis of Theorem 4.1.2 are satisfied and the result follows. □

§4.2.2 $\rho-$nonuniform exponential trichotomies

Now we will consider $\rho-$nonuniform exponential trichotomies.

Theorem 4.2.2. *Let X be a Banach space. Suppose that equation* (1.1) *admits a $\rho-$nonuniform exponential trichotomy and that $f\colon \mathbb{R} \times X \to X$ is a continuous function that verifies* (3.2) *and* (4.5) *with*

$$k(t) = \frac{\delta \rho'(t)\, \mathrm{e}^{-2\varepsilon|\rho(t)|}}{2^q D^2\, \mathrm{e}^{-cq\rho(t)}} \min\left\{-c-d-\varepsilon \operatorname{sgn}(t), -a-b+\varepsilon \operatorname{sgn}(t), \frac{D\gamma}{2}\, \mathrm{e}^{(\varepsilon-\gamma)|\rho(t)|}\right\}$$

and where γ is given by (3.23) *and $\delta > 0$. If* (3.24) *is satisfied, $a + c \leqslant 0$ and $\delta < 1/12$, then there is $N \in]0, 1[$ and $\varphi \in \mathcal{A}_N$ such that*

$$\Psi_\tau\left(\mathcal{V}^*_{\varphi,\overline{R}}\right) \subset \mathcal{V}^*_{\varphi,R}$$

for every $\tau \in \mathbb{R}$, where Ψ_τ is given by (3.9)*, the function $R, \overline{R}\colon \mathbb{R} \to]0, \infty[$ are defined by*

$$R(s) = \mathrm{e}^{-c\rho(s)} \qquad \text{and} \qquad \overline{R}(s) = \frac{\widetilde{\omega}}{DN}\, \mathrm{e}^{-c\rho(s)-\varepsilon|\rho(s)|},$$

and $\mathcal{V}^*_{\varphi,\overline{R}}$ and $\mathcal{V}^*_{\varphi,R}$ are given by (4.1)*. Furthermore, we have*

$$\left\|\Psi_{t-s}(p_{s,\xi}) - \Psi_{t-s}(p_{s,\bar{\xi}})\right\| \leqslant \begin{cases} \dfrac{DN}{\widetilde{\omega}}\, \mathrm{e}^{a[\rho(t)-\rho(s)]+\varepsilon|\rho(s)|} \left\|\xi - \bar{\xi}\right\| & \text{for } t \geqslant s, \\ \dfrac{DN}{\widetilde{\omega}}\, \mathrm{e}^{c[\rho(s)-\rho(t)]+\varepsilon|\rho(s)|} \left\|\xi - \bar{\xi}\right\| & \text{for } t \leqslant s, \end{cases}$$

for every $(s,\xi),(s,\bar\xi) \in G$ *and* $\xi,\bar\xi \in \mathfrak{B}(\overline{R}(s))$ *and where* $p_{s,\xi} = (s,\xi,\varphi(s,\xi))$ *and* $p_{s,\bar\xi} = \big(s,\bar\xi,\varphi(s,\bar\xi)\big)$.

Proof: From $a+c \leqslant 0$ it follows immediately that

$$M(s) = \sup_{t\geqslant s} \frac{\mathfrak{a}(s)\mathfrak{c}(s)}{\mathfrak{a}(t)\mathfrak{c}(t)} = \sup_{t\geqslant s} \mathrm{e}^{(a+c)(\rho(t)-\rho(s))} = 1.$$

Moreover, from (3.25) and (3.26) and (3.27) we have

$$\begin{aligned} k(t) &= \frac{\delta}{2^q\mathfrak{c}^q(t)} \min\left\{\frac{1}{\mathfrak{c}(t)\mathfrak{d}(t)\varepsilon_{\mathfrak{d}}(t)}\left[\frac{\mathfrak{c}(t)\mathfrak{d}(t)}{\varepsilon_{\mathfrak{c}}(t)}\right]', \frac{\mathfrak{a}(t)\mathfrak{b}(t)}{\varepsilon_{\mathfrak{b}}(t)}\left[-\frac{1}{\mathfrak{a}(t)\mathfrak{b}(t)\varepsilon_{\mathfrak{a}}(t)}\right]', \gamma(t)\right\} \\ &= \frac{\delta\rho'(t)}{2^q\,\mathrm{e}^{-cq\rho(t)}} \min\left\{\frac{(-c-d-\varepsilon\,\mathrm{sgn}(t))}{D^2\,\mathrm{e}^{2\varepsilon|\rho(t)|}}, \frac{(-a-b+\varepsilon\,\mathrm{sgn}(t))}{D^2\,\mathrm{e}^{2\varepsilon|\rho(t)|}}, \frac{\gamma}{2D}\,\mathrm{e}^{-(\varepsilon+\gamma)|\rho(t)|}\right\} \\ &= \frac{\delta\rho'(t)\,\mathrm{e}^{-2\varepsilon|\rho(t)|}}{2^qD^2\,\mathrm{e}^{-cq\rho(t)}} \min\left\{-c-d-\varepsilon\,\mathrm{sgn}(t), -a-b+\varepsilon\,\mathrm{sgn}(t), \frac{D\gamma}{2}\,\mathrm{e}^{(\varepsilon-\gamma)|\rho(t)|}\right\} \end{aligned}$$

and the result follows immediately from Theorem 4.2.1. □

Making $\rho(t)=t$ in the last theorem we have the next result.

Corollary 4.2.3. *Let* X *be a Banach space. Suppose that equation* (1.1) *admits a nonuniform exponential trichotomy. Assume that* $f\colon \mathbb{R}\times X \to X$ *is a continuous function that verifies* (3.2) *and* (4.5) *is satisfied with*

$$k(t) = \frac{\delta\,\mathrm{e}^{-2\varepsilon|t|}}{2^qD^2\,\mathrm{e}^{-cqt}} \min\left\{-c-d-\varepsilon\,\mathrm{sgn}(t), -a-b+\varepsilon\,\mathrm{sgn}(t), \frac{D\gamma}{2}\,\mathrm{e}^{(\varepsilon-\gamma)|t|}\right\}$$

and where γ *is given by* (3.23) *and* $\delta > 0$. *If* (3.24) *is satisfied,* $a+c \leqslant 0$ *and* $\delta < 1/12$ *then there is* $N \in]0,1[$ *and* $\varphi \in \mathcal{A}_N$ *such that*

$$\Psi_\tau\left(\mathcal{V}^*_{\varphi,\overline{R}}\right) \subset \mathcal{V}^*_{\varphi,R}$$

for every $\tau \in \mathbb{R}$, *where* Ψ_τ *is given by* (3.9), *the function* $R,\overline{R}\colon \mathbb{R} \to]0,\infty[$ *are defined by*

$$R(s) = \mathrm{e}^{-cs} \qquad \textit{and} \qquad \overline{R}(s) = \frac{\widetilde{\omega}}{DN}\,\mathrm{e}^{-cs+\varepsilon|s|},$$

and $\mathcal{V}^*_{\varphi,\overline{R}}$ *and* $\mathcal{V}^*_{\varphi,R}$ *are given by* (4.1). *Furthermore, we have*

$$\left\|\Psi_{t-s}(p_{s,\xi}) - \Psi_{t-s}(p_{s,\bar\xi})\right\| \leqslant \begin{cases} \dfrac{DN}{\widetilde{\omega}}\,\mathrm{e}^{a(t-s)+\varepsilon|s|}\left\|\xi-\bar\xi\right\| & \textit{for } t \geqslant s, \\[2ex] \dfrac{DN}{\widetilde{\omega}}\,\mathrm{e}^{c(s-t)+\varepsilon|s|}\left\|\xi-\bar\xi\right\| & \textit{for } t \leqslant s, \end{cases}$$

for every $(s,\xi),(s,\bar{\xi}) \in G$ *and* $\xi,\bar{\xi} \in \mathfrak{B}(\overline{R}(s))$ *and where* $p_{s,\xi} = (s,\xi,\varphi(s,\xi))$ *and* $p_{s,\bar{\xi}} = \big(s,\bar{\xi},\varphi(s,\bar{\xi})\big)$.

§4.2.3 $\mu-$nonuniform polynomial trichotomies

In this subsection we will apply Theorem 4.1.2 to $\mu-$nonuniform polynomial trichotomies.

Theorem 4.2.4. *Let* X *be a Banach space. Suppose that equation* (1.1) *admits a* $\mu-$*nonuniform polynomial trichotomy and that* $f\colon \mathbb{R}\times X \to X$ *is a continuous function that verifies* (3.2) *and that* (4.5) *is satisfied with*

$$k(t) = \frac{\delta}{2^q(|\mu(t)|+1)^{-cq}}$$

and $\delta > 0$. *Let* $R\colon \mathbb{R} \to]0,+\infty[$ *be the function defined by*

$$R(s) = \mu'(s)^{1/q}\ (|\mu(s)|+1)^{-\gamma}, \quad \gamma > 0.$$

If (3.24) *is fulfilled,* $a,c \leqslant 0$, $\varepsilon + (-\gamma+c)q + 1 < 0$, $2\varepsilon + (-\gamma+c)q \leqslant 0$ *and*

$$\max\left\{\sup_{t\geqslant s}\left[\frac{(\mu(t)-\mu(s)+1)^a}{\mu'(t)^{1/q}(|\mu(t)|+1)^{-\gamma}}\right],\ \sup_{t\leqslant s}\left[\frac{(\mu(s)-\mu(t)+1)^c}{\mu'(t)^{1/q}(|\mu(t)|+1)^{-\gamma}}\right]\right\} < +\infty, \qquad (4.6)$$

then there is $N \in]0,1[$ *and* $\varphi \in \mathcal{A}_N$ *such that*

$$\Psi_\tau\left(\mathcal{V}^*_{\varphi,\overline{R}}\right) \subset \mathcal{V}^*_{\varphi,R}$$

for every $\tau \in \mathbb{R}$, *where* Ψ_τ *is given by* (3.9), $\overline{R}\colon \mathbb{R} \to]0,\infty[$ *is defined by* (4.3) *and* $\mathcal{V}^*_{\varphi,\overline{R}}$ *and* $\mathcal{V}^*_{\varphi,R}$ *are given by* (4.1). *Furthermore, we have*

$$\left\|\Psi_{t-s}(p_{s,\xi}) - \Psi_{t-s}(p_{s,\bar{\xi}})\right\| \leqslant \begin{cases} \dfrac{DN}{\widetilde{\omega}}(\mu(t)-\mu(s)+1)^a(|\mu(s)|+1)^\varepsilon \left\|\xi-\bar{\xi}\right\| & \text{if } t \geqslant s, \\ \dfrac{DN}{\widetilde{\omega}}(\mu(s)-\mu(t)+1)^c(|\mu(s)|+1)^\varepsilon \left\|\xi-\bar{\xi}\right\| & \text{if } t \leqslant s, \end{cases}$$

for every $(s,\xi),(s,\bar{\xi}) \in G$ *and* $\xi,\bar{\xi} \in \mathfrak{B}(\overline{R}(s))$ *and where* $p_{s,\xi} = (s,\xi,\varphi(s,\xi))$ *and* $p_{s,\bar{\xi}} = \big(s,\bar{\xi},\varphi(s,\bar{\xi})\big)$.

Proof: It is clear that

$$\operatorname{Lip}\left(f_t|_{\mathfrak{B}(R(t))}\right) \leqslant \delta\mu'(t)(|\mu(t)|+1)^{(-\gamma+c)q}. \tag{4.7}$$

Using (3.28), (3.29), $\varepsilon+(-\gamma+c)q+1<0$ and making the substitution $\tau=\mu(r)$, we have

$$\begin{aligned}
\widetilde{\sigma} &= \sup_{(t,s)\in\mathbb{R}^2}\left|\int_s^t \frac{\alpha_{t,r}\operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right)\alpha_{r,s}}{\alpha_{t,s}}\,dr\right| \\
&\leqslant D\delta \sup_{(t,s)\in\mathbb{R}^2}\left|\int_s^t \mu'(r)(|\mu(r)|+1)^{\varepsilon+(-\gamma+c)q}\,dr\right| \\
&= D\delta\int_{-\infty}^{+\infty}\mu'(r)(|\mu(r)|+1)^{\varepsilon+(-\gamma+c)q}\,dr \\
&= 2D\delta\int_0^{+\infty}(\tau+1)^{\varepsilon+(-\gamma+c)q}\,d\tau \\
&= \frac{2D\delta}{|\varepsilon+(-\gamma+c)q+1|}.
\end{aligned}$$

By (4.7) and making the substitution $\tau=\mu(s)-\mu(r)$ we have

$$\begin{aligned}
&\int_{-\infty}^{s}\beta^{+}_{s,r}\operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right)\alpha_{r,s}\,dr \\
&\leqslant D^2\delta\int_{-\infty}^{s}\mu'(r)(\mu(s)-\mu(r)+1)^{c+d}(|\mu(r)|+1)^{\varepsilon+(-\gamma+c)q}(|\mu(s)|+1)^{\varepsilon}\,dr \\
&= D^2\delta\int_0^{+\infty}(\tau+1)^{c+d}(|\tau-\mu(s)|+1)^{\varepsilon+(-\gamma+c)q}(|\mu(s)|+1)^{\varepsilon}\,d\tau,
\end{aligned}$$

and using again (4.7) combined with the substitution $\tau=\mu(r)-\mu(s)$ it follows that

$$\begin{aligned}
&\int_{s}^{+\infty}\beta^{-}_{s,r}\operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right)\alpha_{r,s}\,dr \\
&\leqslant D^2\delta\int_{s}^{+\infty}\mu'(r)(\mu(r)-\mu(s)+1)^{a+b}(|\mu(r)|+1)^{\varepsilon+(-\gamma+c)q}(|\mu(s)|+1)^{\varepsilon}\,dr \\
&= D^2\delta\int_0^{+\infty}(\tau+1)^{a+b}(|\tau+\mu(s)|+1)^{\varepsilon+(-\gamma+c)q}(|\mu(s)|+1)^{\varepsilon}\,d\tau.
\end{aligned}$$

Hence, since $a+b+\varepsilon<0$, $c+d+\varepsilon<0$, $\varepsilon+(-\gamma+c)q+1<0$ and $2\varepsilon+(-\gamma+c)q\leqslant 0$, by Lemma 3.2.6 we get

$$\begin{aligned}\widetilde{\omega} &= \sup_{s\in R}\left[\int_{-\infty}^{s}\beta^{+}_{s,r}\,\mathrm{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right)\alpha_{r,s}\,dr + \int_{s}^{+\infty}\beta^{-}_{s,r}\,\mathrm{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right)\alpha_{r,s}\,dr\right]\\ &\leqslant \frac{D^2\delta(2^{\varepsilon+1}+1)}{|\max\{a+b,c+d\}+2\varepsilon+(c-\gamma)q+1|}+\\ &\qquad\qquad + \frac{D^2\delta}{|\min\{a+b,c+d\}+2\varepsilon+(c-\gamma)q+1|}.\end{aligned}$$

Moreover, since for every $s \in \mathbb{R}$, we have

$$\begin{aligned}\sup_{t\geqslant s}\frac{\alpha_{t,s}}{R(t)} &= \sup_{t\geqslant s}\left[\frac{D(\mu(t)-\mu(s)+1)^a(|\mu(s)|+1)^{\varepsilon}}{\mu'(t)^{1/q}(|\mu(t)|+1)^{-\gamma}}\right]\\ &= D(|\mu(s)|+1)^{\varepsilon}\ \sup_{t\geqslant s}\left[\frac{(\mu(t)-\mu(s)+1)^a}{\mu'(t)^{1/q}(|\mu(t)|+1)^{-\gamma}}\right]\end{aligned}$$

and

$$\begin{aligned}\sup_{t\leqslant s}\frac{\alpha_{t,s}}{R(t)} &= \sup_{t\leqslant s}\left[\frac{D(\mu(s)-\mu(t)+1)^c(|\mu(s)|+1)^{\varepsilon}}{\mu'(t)^{1/q}(|\mu(t)|+1)^{-\gamma}}\right]\\ &= D(|\mu(s)|+1)^{\varepsilon}\sup_{t\leqslant s}\left[\frac{(\mu(s)-\mu(t)+1)^c}{\mu'(t)^{1/q}(|\mu(t)|+1)^{-\gamma}}\right],\end{aligned}$$

condition (4.6) implies (4.2). Therefore choosing δ sufficiently small such that $4\widetilde{\sigma}+4\widetilde{\omega}<1$, it follows that all conditions of Theorem 4.1.2 are satisfied. □

Remarks 4.2.5.

a) *Note that if*

$$\vartheta(s) := \max\left\{\sup_{t\geqslant s}\left[\frac{(\mu(t)-\mu(s)+1)^a}{\mu'(t)^{1/q}(|\mu(t)|+1)^{-\gamma}}\right], \sup_{t\leqslant s}\left[\frac{(\mu(s)-\mu(t)+1)^c}{\mu'(t)^{1/q}(|\mu(t)|+1)^{-\gamma}}\right]\right\}, \tag{4.8}$$

then (4.3) *becomes*

$$\overline{R}(s) = \frac{\widetilde{\omega}}{DN\vartheta(s)}\ (|\mu(s)|+1)^{-\varepsilon}.$$

b) *Since*

$$|\mu(t)|+1 \leqslant [|\mu(t)-\mu(s)|+1]+|\mu(s)|,$$

it follows that

$$\begin{aligned}\sup_{t\geqslant s}\frac{\alpha_{t,s}}{R(t)} &= D(|\mu(s)|+1)^{\varepsilon}\ \sup_{t\geqslant s}\left[\frac{(\mu(t)-\mu(s)+1)^{a}}{\mu'(t)^{1/q}(|\mu(t)|+1)^{-\gamma}}\right]\\ &= D(|\mu(s)|+1)^{\varepsilon+\gamma}\sup_{t\geqslant s}\left[\left(\frac{|\mu(t)|+1}{|\mu(s)|+1}\right)^{\gamma}\frac{(\mu(t)-\mu(s)+1)^{a}}{\mu'(t)^{1/q}}\right]\\ &\leqslant D(|\mu(s)|+1)^{\varepsilon+\gamma}\sup_{t\geqslant s}\left[\frac{(\mu(t)-\mu(s)+1)^{a+\gamma}}{\mu'(t)^{1/q}}\right]\end{aligned}$$

and

$$\begin{aligned}\sup_{t\leqslant s}\frac{\alpha_{t,s}}{R(t)} &= D(|\mu(s)|+1)^{\varepsilon}\sup_{t\leqslant s}\left[\frac{(\mu(s)-\mu(t)+1)^{c}}{\mu'(t)^{1/q}(|\mu(t)|+1)^{-\gamma}}\right]\\ &= D(|\mu(s)|+1)^{\varepsilon+\gamma}\sup_{t\leqslant s}\left[\left(\frac{|\mu(t)|+1}{|\mu(s)|+1}\right)^{\gamma}\frac{(\mu(s)-\mu(t)+1)^{c}}{\mu'(t)^{1/q}}\right]\\ &\leqslant D(|\mu(s)|+1)^{\varepsilon+\gamma}\sup_{t\leqslant s}\left[\frac{(\mu(s)-\mu(t)+1)^{c+\gamma}}{\mu'(t)^{1/q}}\right].\end{aligned}$$

Therefore, if

$$\frac{(\mu(t)-\mu(s)+1)^{a+\gamma}}{\mu'(t)^{1/q}}\ \text{is a non-increasing function in the variable } t$$

and

$$\frac{(\mu(s)-\mu(t)+1)^{c+\gamma}}{\mu'(t)^{1/q}}\ \text{is a non-decreasing function in the variable } t.$$

we have

$$\sup_{t\geqslant s}\left[\frac{(\mu(t)-\mu(s)+1)^{a+\gamma}}{\mu'(t)^{1/q}}\right]=\sup_{t\leqslant s}\left[\frac{(\mu(s)-\mu(t)+1)^{c+\gamma}}{\mu'(t)^{1/q}}\right]=\mu'(s)^{-1/q},$$

and (4.6) *is satisfied.*

Moreover, if μ *is twice differentiable we have*

$$\begin{aligned}&\left[\frac{(\mu(t)-\mu(s)+1)^{a+\gamma}}{\mu'(t)^{1/q}}\right]'\\ &=(\mu(t)-\mu(s)+1)^{a+\gamma-1}\mu'(t)^{-1/q-1}\left[(a+\gamma)\mu'(t)^{2}-(\mu(t)-\mu(s)+1)\mu''(t)/q\right]\end{aligned}$$

and

$$\begin{aligned}&\left[\frac{(\mu(s)-\mu(t)+1)^{c+\gamma}}{\mu'(t)^{1/q}}\right]'\\ &=(\mu(s)-\mu(t)+1)^{c+\gamma-1}\mu'(t)^{-1/q-1}\left[-(c+\gamma)\mu'(t)^{2}-(\mu(s)-\mu(t)+1)\mu''(t)/q\right].\end{aligned}$$

and this implies that if

$$q(a+\gamma)\mu'(t)^2 \leqslant \mu''(t) \leqslant -q(c+\gamma)\mu'(t)^2$$

for every $t \in \mathbb{R}$*, then* (4.6) *is true.*

The next result is obvious, making $\mu(t) = t$ in the previous theorem.

Corollary 4.2.6. *Let* X *be a Banach space. Suppose that equation* (1.1) *admits a nonuniform polynomial trichotomy and that* $f\colon \mathbb{R}\times X \to X$ *is a continuous function that verifies* (3.2) *and* (4.5) *with*

$$k(t) = \frac{\delta}{2^q(|t|+1)^{-cq}}$$

with $\delta > 0$*. Let* $R\colon \mathbb{R} \to]0,+\infty[$ *be the function defined by*

$$R(s) = (|s|+1)^{-\gamma}, \quad \gamma > 0.$$

If (3.24) *is fulfilled,* δ *is sufficiently small and*

$$\varepsilon + (c-\gamma)q + 1 < 0, \quad 2\varepsilon + (c-\gamma) \leqslant 0 \quad \text{and} \quad \max\{a,c\} + \gamma < 0,$$

then there is $N \in]0,1[$ *and* $\varphi \in \mathcal{A}_N$ *such that*

$$\Psi_\tau\left(\mathcal{V}^*_{\varphi,\overline{R}}\right) \subset \mathcal{V}^*_{\varphi,R}$$

for every $\tau \in \mathbb{R}$*, where* Ψ_τ *is given by* (3.9)*,* $\overline{R}\colon \mathbb{R} \to]0,\infty[$ *is defined by* (4.3) *and* $\mathcal{V}^*_{\varphi,\overline{R}}$ *and* $\mathcal{V}^*_{\varphi,R}$ *are given by* (4.1)*. Furthermore, we have*

$$\left\|\Psi_{t-s}(p_{s,\xi}) - \Psi_{t-s}(p_{s,\bar{\xi}})\right\| \leqslant \begin{cases} \dfrac{DN}{\widetilde{\omega}}(t-s+1)^a(|s|+1)^\varepsilon \left\|\xi - \bar{\xi}\right\| & \text{for } t \geqslant s, \\ \dfrac{DN}{\widetilde{\omega}}(s-t+1)^c(|s|+1)^\varepsilon \left\|\xi - \bar{\xi}\right\| & \text{for } t \leqslant s, \end{cases}$$

for every $(s,\xi),(s,\bar{\xi}) \in G$ *and* $\xi,\bar{\xi} \in \mathfrak{B}(\overline{R}(s))$ *and where* $p_{s,\xi} = (s,\xi,\varphi(s,\xi))$ *and* $p_{s,\bar{\xi}} = \left(s,\bar{\xi},\varphi(s,\bar{\xi})\right)$*.*

Proof: The result follows immediately by making $\mu(t) = t$ in Theorem 4.2.4 and taking into account last remark and that $a+\gamma \leqslant 0$ and $c+\gamma \leqslant 0$ imply (4.6). □

§4.3 Examples of local invariant manifolds – second type of perturbations

In this section we are going to use perturbations $f\colon \mathbb{R}\times X \to X$ such that (3.2) is satisfied and

$$\|f(t,u)-f(t,v)\| \leqslant k\,\|u-v\|\,(\|u\|+\|v\|)^q \tag{4.9}$$

for every $t\in\mathbb{R}$ and every $u,v\in X$ and with $k,q>0$. In fact, given a function $R\colon \mathbb{R}\to]0,+\infty[$ we have

$$\|f(t,u)-f(t,v)\| \leqslant 2^q k R^q(t)\,\|u-v\|$$

for every $u,v\in\mathfrak{B}(R(t))$ and this shows that $f_t|_{\mathfrak{B}(R(t))}$ is Lipschitz and

$$\operatorname{Lip}\left(f_t|_{\mathfrak{B}(R(t))}\right) \leqslant 2^q k R^q(t). \tag{4.10}$$

In the next examples we are going to consider perturbations of this type.

§4.3.1 Nonuniform $(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d})$−trichotomies

Firstly we will apply Theorem 4.1.2 to nonuniform $(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d})$−trichotomies.

Theorem 4.3.1. *Let X be a Banach space. Suppose that equation* (1.1) *admits a nonuniform $(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d})$−trichotomy. Assume that $f\colon\mathbb{R}\times X\to X$ is a function that verifies* (3.2) *and* (4.9). *Make*

$$R^q(t)=\frac{\delta}{2^q k}\min\left\{\frac{1}{\mathfrak{c}(t)\mathfrak{d}(t)\varepsilon_{\mathfrak{d}}(t)}\left(\frac{\mathfrak{c}(t)\mathfrak{d}(t)}{\varepsilon_{\mathfrak{c}}(t)}\right)',-\frac{\mathfrak{a}(t)\mathfrak{b}(t)}{\varepsilon_{\mathfrak{b}}(t)}\left[\frac{1}{\mathfrak{a}(t)\mathfrak{b}(t)\varepsilon_{\mathfrak{a}}(t)}\right]',\gamma(t)\right\},$$

with $0<\delta<1/12$ and where $\gamma\colon\mathbb{R}\to]0,+\infty[$ is a function such that (3.20) *holds. If* (3.21) *is fulfilled,*

$$\sup_{t\geqslant s}\frac{1}{\mathfrak{a}(t)R(t)}<\infty \quad \textit{and} \quad \sup_{t\leqslant s}\frac{\mathfrak{c}(t)}{R(t)}<\infty,$$

then there is $N\in]0,1[$ and $\varphi\in\mathcal{A}_N$ such that

$$\Psi_\tau\left(\mathcal{V}^*_{\varphi,\overline{R}}\right)\subset\mathcal{V}^*_{\varphi,R}$$

for every $\tau \in \mathbb{R}$*, where* Ψ_τ *is given by* (3.9)*,* $\overline{R}\colon \mathbb{R} \to]0,\infty[$ *is given by* (4.3) *and* $\mathcal{V}^*_{\varphi,\overline{R}}$ *and* $\mathcal{V}^*_{\varphi,R}$ *are given by* (4.1)*. Furthermore, we have*

$$\left\|\Psi_{t-s}(p_{s,\xi}) - \Psi_{t-s}(p_{s,\bar\xi})\right\| \leqslant \begin{cases} \dfrac{N}{\widetilde{\omega}}\dfrac{\mathfrak{a}(s)}{\mathfrak{a}(t)}\varepsilon_{\mathfrak{a}}(s)\left\|\xi-\bar\xi\right\| & \text{for } t \geqslant s,\\ \dfrac{N}{\widetilde{\omega}}\dfrac{\mathfrak{c}(t)}{\mathfrak{c}(s)}\varepsilon_{\mathfrak{c}}(s)\left\|\xi-\bar\xi\right\| & \text{for } t \leqslant s, \end{cases} \tag{4.11}$$

for every $(s,\xi),(s,\bar\xi) \in G$ *and* $\xi,\bar\xi \in \mathfrak{B}(\overline{R}(s))$ *and where* $p_{s,\xi} = (s,\xi,\varphi(s,\xi))$ *and* $p_{s,\bar\xi} = \big(s,\bar\xi,\varphi(s,\bar\xi)\big)$.

Proof: Since

$$\sup_{t\geqslant s}\frac{\alpha_{t,s}}{R(t)} = \sup_{t\geqslant s}\frac{\mathfrak{a}(s)\varepsilon_{\mathfrak{a}}(s)}{\mathfrak{a}(t)R(t)} = \mathfrak{a}(s)\varepsilon_{\mathfrak{a}}(s)\sup_{t\geqslant s}\frac{1}{\mathfrak{a}(t)R(t)} < +\infty$$

and

$$\sup_{t\leqslant s}\frac{\alpha_{t,s}}{R(t)} = \sup_{t\leqslant s}\frac{\mathfrak{c}(t)\varepsilon_{\mathfrak{c}}(s)}{\mathfrak{c}(s)R(t)} = \frac{\varepsilon_{\mathfrak{c}}(s)}{\mathfrak{c}(s)}\sup_{t\leqslant s}\frac{\mathfrak{c}(t)}{R(t)} < +\infty,$$

(4.2) is satisfied. As (3.21) holds and as in the proof of Theorem 3.2.1

$$\widetilde{\sigma} < \delta \quad \text{and} \quad \widetilde{\omega} < 2\delta$$

choosing $\delta < 1/12$ we have

$$4\widetilde{\sigma} + 4\widetilde{\omega} < 1$$

and all the hypotheses of the Theorem 4.1.2 are verified. Furthermore, in this case (4.3) becomes

$$\begin{aligned}\overline{R}(s) &= \frac{\widetilde{\omega}}{N\max\left\{\mathfrak{a}(s)\varepsilon_{\mathfrak{a}}(s)\sup\limits_{t\geqslant s}\dfrac{1}{\mathfrak{a}(t)R(t)}, \dfrac{\varepsilon_{\mathfrak{c}}(s)}{\mathfrak{c}(s)}\sup\limits_{t\leqslant s}\dfrac{\mathfrak{c}(t)}{R(t)}\right\}}\\ &= \frac{\widetilde{\omega}}{N}\max\left\{\frac{1}{\mathfrak{a}(s)\varepsilon_{\mathfrak{a}}(s)}\inf_{t\geqslant s}\left[\mathfrak{a}(t)R(t)\right], \frac{\mathfrak{c}(s)}{\varepsilon_{\mathfrak{c}}(s)}\inf_{t\leqslant s}\left[\frac{R(t)}{\mathfrak{c}(t)}\right]\right\}\end{aligned}$$

and (4.4) obviously becomes (4.11) for every $\xi,\bar\xi \in \mathfrak{B}(\overline{R}(s)) \cap E_s$. □

§4.3.2 $\rho-$nonuniform exponential trichotomies

In this subsection we are going to consider $\rho-$nonuniform exponential trichotomies and perturbation functions f satisfying (4.9).

Theorem 4.3.2. *Let X be a Banach space. Suppose that equation* (1.1) *admits a $\rho-$nonuniform trichotomy. Assume that $f\colon \mathbb{R}\times X \to X$ is a function that verifies* (3.2) *and* (4.9). *Let*

$$R^q(t) = \frac{\delta}{kD^2 2^q}\tau(t)\rho'(t)\,\mathrm{e}^{-2\varepsilon|\rho(t)|} \tag{4.12}$$

with $0<\delta<1/12$ *and*

$$\tau(t) := \min\left\{-c-d-\varepsilon\,\mathrm{sgn}(t), -a-b+\varepsilon\,\mathrm{sgn}(t), \frac{D\gamma}{2}\,\mathrm{e}^{(\varepsilon-\gamma)|\rho(t)|}\right\},$$

where γ is given by (3.23). *If* (3.24) *holds and*

$$\max\left\{\sup_{t\geqslant s}\frac{\mathrm{e}^{a\rho(t)+(\varepsilon+\gamma)|\rho(t)|/q}}{\rho'(t)^{1/q}},\ \sup_{t\leqslant s}\frac{\mathrm{e}^{-c\rho(t)+(\varepsilon+\gamma)|\rho(t)|/q}}{\rho'(t)^{1/q}}\right\}<+\infty, \tag{4.13}$$

then there is $N\in]0,1[$ and $\varphi\in\mathcal{A}_N$ such that

$$\Psi_\tau\left(\mathcal{V}^*_{\varphi,\overline{R}}\right)\subset\mathcal{V}^*_{\varphi,R}$$

for every $\tau\in\mathbb{R}$, where Ψ_τ is given by (3.9), *$\overline{R}\colon\mathbb{R}\to]0,\infty[$ is given by* (4.3) *and $\mathcal{V}^*_{\varphi,\overline{R}}$ and $\mathcal{V}^*_{\varphi,R}$ are given by* (4.1). *Furthermore, we have*

$$\left\|\Psi_{t-s}(p_{s,\xi})-\Psi_{t-s}(p_{s,\bar\xi})\right\|\leqslant\begin{cases}\dfrac{DN}{\widetilde{\omega}}\,\mathrm{e}^{a[\rho(t)-\rho(s)]+\varepsilon|\rho(s)|}\left\|\xi-\bar\xi\right\| & \text{for } t\geqslant s,\\[2ex] \dfrac{DN}{\widetilde{\omega}}\,\mathrm{e}^{c[\rho(s)-\rho(t)]+\varepsilon|\rho(s)|}\left\|\xi-\bar\xi\right\| & \text{for } t\leqslant s,\end{cases} \tag{4.14}$$

for every $(s,\xi),(s,\bar\xi)\in G$ and $\xi,\bar\xi\in\mathfrak{B}(\overline{R}(s))$ and where $p_{s,\xi}=(s,\xi,\varphi(s,\xi))$ and $p_{s,\bar\xi}=\big(s,\bar\xi,\varphi(s,\bar\xi)\big)$.

Proof: For this bounds we have that (3.21) is equivalent to (3.24) and

$$\frac{\alpha_{t,s}}{R(t)} = \begin{cases} D\,\mathrm{e}^{a(\rho(t)-\rho(s))+\varepsilon|\rho(s)|}\,\dfrac{2k^{1/q}D^{2/q}}{\delta^{1/q}\rho'(t)^{1/q}\,\mathrm{e}^{-2\varepsilon|\rho(t)|/q}\,\tau(t)^{1/q}} & \text{if } t \geqslant s,\\ D\,\mathrm{e}^{c(\rho(s)-\rho(t))+\varepsilon|\rho(s)|}\,\dfrac{2k^{1/q}D^{2/q}}{\delta^{1/q}\rho'(t)^{1/q}\,\mathrm{e}^{-2\varepsilon|\rho(t)|/q}\,\tau(t)^{1/q}} & \text{if } t \leqslant s,\end{cases}$$
$$= \begin{cases} 2D\left[\dfrac{D^2k}{\delta}\right]^{1/q}\mathrm{e}^{-a\rho(s)+\varepsilon|\rho(s)|}\,\dfrac{\mathrm{e}^{a\rho(t)+2\varepsilon|\rho(t)|/q}}{[\rho'(t)\tau(t)]^{1/q}} & \text{if } t \geqslant s,\\ 2D\left[\dfrac{D^2k}{\delta}\right]^{1/q}\mathrm{e}^{c\rho(s)+\varepsilon|\rho(s)|}\,\dfrac{\mathrm{e}^{-c\rho(t)+2\varepsilon|\rho(t)|/q}}{[\rho'(t)\tau(t)]^{1/q}} & \text{if } t \leqslant s.\end{cases} \tag{4.15}$$

From (4.13) the functions

$$\chi^+(s) = \sup_{t\geqslant s}\frac{\mathrm{e}^{a\rho(t)+(\varepsilon+\gamma)|\rho(t)|/q}}{\rho'(t)^{1/q}} \quad \text{and} \quad \chi^-(s) = \sup_{t\leqslant s}\frac{\mathrm{e}^{-c\rho(t)+(\varepsilon+\gamma)|\rho(t)|/q}}{\rho'(t)^{1/q}}$$

are well defined.

First we consider the case $\varepsilon > 0$. Then $\gamma = \varepsilon$ and it is trivial to see that, when $\varepsilon > 0$, τ is a bounded function with $\tau^- = \min_{t\in\mathbb{R}}\tau(t) > 0$ and $\tau^+ = \max_{t\in\mathbb{R}}\tau(t) > 0$. Hence, from (4.15) we have

$$\frac{\alpha_{t,s}}{R(t)} \leqslant \begin{cases} 2D\left[\dfrac{D^2k}{\delta\tau^-}\right]^{1/q}\mathrm{e}^{-a\rho(s)+\varepsilon|\rho(s)|}\,\chi^+(s) & \text{if } t \geqslant s,\\ 2D\left[\dfrac{D^2k}{\delta\tau^-}\right]^{1/q}\mathrm{e}^{c\rho(s)+\varepsilon|\rho(s)|}\,\chi^-(s) & \text{if } t \leqslant s,\end{cases}$$

and this implies that

$$\sup_{t\in\mathbb{R}}\frac{\alpha_{t,s}}{R(t)} \leqslant 2D\left[\frac{D^2k}{\delta\tau^-}\right]^{1/q}\mathrm{e}^{\varepsilon|\rho(s)|}\max\left\{\mathrm{e}^{c\rho(s)}\,\chi^-(s), \mathrm{e}^{-a\rho(s)}\,\chi^+(s)\right\} < +\infty. \tag{4.16}$$

Now we consider the case $\varepsilon = 0$. By (3.23) we have

$$\tau(t) = \min\left\{-c-d, -a-b, \frac{D\gamma}{2}\,\mathrm{e}^{-\gamma|\rho(t)|}\right\} = \frac{D\gamma}{2}\,\mathrm{e}^{-\gamma|\rho(t)|}.$$

From (4.15) we have

$$\frac{\alpha_{t,s}}{R(t)} \leqslant \begin{cases} 2D\left[\dfrac{2Dk}{\delta\gamma}\right]^{1/q}\mathrm{e}^{-a\rho(s)}\,\chi^+(s) & \text{if } t \geqslant s,\\ 2D\left[\dfrac{2Dk}{\delta\gamma}\right]^{1/q}\mathrm{e}^{c\rho(s)}\,\chi^-(s) & \text{if } t \leqslant s\end{cases} \tag{4.17}$$

and thus

$$\sup_{t\in\mathbb{R}} \frac{\alpha_{t,s}}{R(t)} \leqslant 2D\left[\frac{2Dk}{\delta\gamma}\right]^{1/q} \max\left\{e^{c\rho(s)}\,\chi^-(s), e^{-a\rho(s)}\,\chi^+(s)\right\} < +\infty. \tag{4.18}$$

Therefore, all conditions of Theorem 4.3.1 are satisfied and it is easy to see that (4.4) becomes (4.14) for every $\xi, \bar{\xi} \in \mathfrak{B}(\overline{R}(s)) \cap E_s$. □

Remarks 4.3.3.

a) When $\varepsilon > 0$, from (4.15) *we also have*

$$\sup_{t\geqslant s} \frac{\alpha_{t,s}}{R(t)} \geqslant 2D\left[\frac{D^2k}{\delta\tau^+}\right]^{1/q} e^{-a\rho(s)+\varepsilon|\rho(s)|}\,\chi^+(s)$$

and

$$\sup_{t\leqslant s} \frac{\alpha_{t,s}}{R(t)} \geqslant 2D\left[\frac{D^2k}{\delta\tau^+}\right]^{1/q} e^{c\rho(s)+\varepsilon|\rho(s)|}\,\chi^-(s)$$

and this implies

$$\sup_{t\in\mathbb{R}} \frac{\alpha_{t,s}}{R(t)} \geqslant 2D\left[\frac{D^2k}{\delta\tau^+}\right]^{1/q} e^{\varepsilon|\rho(s)|} \min\left\{e^{-a\rho(s)}\,\chi^+(s), e^{c\rho(s)}\,\chi^-(s)\right\}.$$

Hence

$$\overline{R}(s) \geqslant \frac{\widetilde{\omega}\,(\delta\tau^-)^{1/q}}{2ND\,(D^2k)^{1/q}\, e^{\varepsilon|\rho(s)|} \max\{e^{-a\rho(s)}\,\chi^+(s), e^{c\rho(s)}\,\chi^-(s)\}} \tag{4.19}$$

and from (4.16) *we also have*

$$\overline{R}(s) \leqslant \frac{\widetilde{\omega}\,(\delta\tau^+)^{1/q}}{2ND\,(D^2k)^{1/q}\, e^{\varepsilon|\rho(s)|} \min\{e^{-a\rho(s)}\,\chi^+(s), e^{c\rho(s)}\,\chi^-(s)\}}. \tag{4.20}$$

In the case $\varepsilon = 0$, from (4.17) *and* (4.18) *we have*

$$\overline{R}(s) \geqslant \frac{\widetilde{\omega}\,(\delta\gamma)^{1/q}}{2DN\,(2Dk)^{1/q} \max\{e^{-a\rho(s)}\,\chi^+(s), e^{c\rho(s)}\,\chi^-(s)\}} \tag{4.21}$$

and

$$\overline{R}(s) \leqslant \frac{\widetilde{\omega}\,(\delta\gamma)^{1/q}}{2DN\,(2Dk)^{1/q} \min\{e^{-a\rho(s)}\,\chi^+(s), e^{c\rho(s)}\,\chi^-(s)\}}. \tag{4.22}$$

b) Note that if ρ is twice differentiable, we have for $t \neq 0$

$$\begin{aligned}&\left[\rho'(t)^{-1/q}\,\mathrm{e}^{a\rho(t)+(\varepsilon+\gamma)/q)|\rho(t)|}\right]' =\\&= -\frac{1}{q}\rho'(t)^{-1/q-1}\rho''(t)\,\mathrm{e}^{[a+\operatorname{sgn} t(\varepsilon+\gamma)/q]\rho(t)}\\&\quad+\rho'(t)^{-1/q}\left[a\rho'(t)+\operatorname{sgn} t\frac{\varepsilon+\gamma}{q}\rho'(t)\right]\mathrm{e}^{(a+\operatorname{sgn} t(\varepsilon+\gamma)/q)\rho(t)}\\&= \frac{1}{q}\rho'(t)^{-1/q-1}\,\mathrm{e}^{(a+\operatorname{sgn} t(\varepsilon+\gamma)/q)\rho(t)}\left[-\rho''(t)+(aq+\operatorname{sgn} t(\varepsilon+\gamma))(\rho'(t))^2\right]\end{aligned}$$

and

$$\begin{aligned}&\left[\rho'(t)^{-1/q}\,\mathrm{e}^{(-c\rho(t)+(\varepsilon+\gamma)/q)|\rho(t)|}\right]' =\\&= -\frac{1}{q}\rho'(t)^{-1/q-1}\rho''(t)\,\mathrm{e}^{[-c+\operatorname{sgn} t(\varepsilon+\gamma)/q]\rho(t)}\\&\quad+\rho'(t)^{-1/q}\left[-c\rho'(t)+\operatorname{sgn} t\frac{\varepsilon+\gamma}{q}\rho'(t)\right]\mathrm{e}^{(-c+\operatorname{sgn} t(\varepsilon+\gamma)/q)\rho(t)}\\&= \frac{1}{q}\rho'(t)^{-1/q-1}\,\mathrm{e}^{(-c+\operatorname{sgn} t(\varepsilon+\gamma)/q)\rho(t)}\left[-\rho''(t)+(-cq+\operatorname{sgn} t(\varepsilon+\gamma))(\rho'(t))^2\right],\end{aligned}$$

and this two last equalities allow to conclude that if $\rho'(t)^{-1/q}\,\mathrm{e}^{a\rho(t)+(\varepsilon+\gamma)/q)|\rho(t)|}$ *is non-increasing and* $\rho'(t)^{-1/q}\,\mathrm{e}^{(-c\rho(t)+(\varepsilon+\gamma)/q)|\rho(t)|}$ *is non-decreasing, that is, if*

$$(aq+\operatorname{sgn} t(\varepsilon+\gamma))\,[\rho'(t)]^2 \leqslant \rho''(t) \leqslant (-cq+\operatorname{sgn} t(\varepsilon+\gamma))\,[\rho'(t)]^2 \tag{4.23}$$

for every $t \in \mathbb{R}$, then

$$\chi^+(s) = \sup_{t\geqslant s}\frac{\mathrm{e}^{a\rho(t)+(\varepsilon+\gamma)|\rho(t)|/q}}{\rho'(t)^{1/q}} = \frac{\mathrm{e}^{a\rho(s)+(\varepsilon+\gamma)|\rho(s)|/q}}{\rho'(s)^{1/q}}$$

and

$$\chi^-(s) = \sup_{t\leqslant s}\frac{\mathrm{e}^{-c\rho(t)+(\varepsilon+\gamma)|\rho(t)|/q}}{\rho'(t)^{1/q}} = \frac{\mathrm{e}^{-c\rho(s)+(\varepsilon+\gamma)|\rho(s)|/q}}{\rho'(s)^{1/q}}.$$

Since

$$\mathrm{e}^{-a\rho(s)}\,\chi^+(s) = \mathrm{e}^{c\rho(s)}\,\chi^-(s) = \frac{\mathrm{e}^{(\varepsilon+\gamma)|\rho(s)|/q}}{\rho'(s)^{1/q}}$$

then we have

$$\begin{aligned}&\max\left\{\mathrm{e}^{-a\rho(s)}\chi^+(s),\mathrm{e}^{c\rho(s)}\chi^-(s)\right\}\\&= \min\left\{\mathrm{e}^{-a\rho(s)}\chi^+(s),\mathrm{e}^{c\rho(s)}\chi^-(s)\right\}\\&= \frac{\mathrm{e}^{(\varepsilon+\gamma)|\rho(s)|/q}}{\rho'(s)^{1/q}}.\end{aligned}$$

Therefore from (4.20) *and* (4.19) *we have*

$$\overline{R}(s) \leqslant \frac{\widetilde{\omega}}{2DN}\left[\frac{\delta\gamma\tau^+}{2Dk}\right]^{1/q}\rho'(s)^{1/q}\,\mathrm{e}^{-(1+2/q)\varepsilon|\rho(s)|}$$

and

$$\overline{R}(s) \geqslant \frac{\widetilde{\omega}}{2DN}\left[\frac{\delta\gamma\tau^-}{2Dk}\right]^{1/q}\rho'(s)^{1/q}\,\mathrm{e}^{-(1+2/q)\varepsilon|\rho(s)|}$$

when $\varepsilon > 0$ *and from* (4.22) *and* (4.21) *we get*

$$\overline{R}(s) = \frac{\widetilde{\omega}}{2DN}\left[\frac{\delta\gamma}{2Dk}\right]^{1/q}\rho'(s)^{1/q}\,\mathrm{e}^{-\gamma|\rho(s)|/q}$$

for $\varepsilon = 0$.

Making $\rho(t) = t$ in the last theorem we have the following result.

Corollary 4.3.4. *Let X be a Banach space. Suppose that equation* (1.1) *admits a nonuniform exponential trichotomy. Assume that $f\colon \mathbb{R}\times X \to X$ is a function that verifies* (3.2) *and* (4.9)*. Make*

$$R^q(t) = \frac{\delta\,\mathrm{e}^{-2\varepsilon|t|}}{kD^2 2^q}\tau(t)$$

with $0 < \delta < 1/12$ *and*

$$\tau(t) := \min\left\{-c-d-\varepsilon\,\mathrm{sgn}(t), -a-b+\varepsilon\,\mathrm{sgn}(t), \frac{D\gamma}{2}\,\mathrm{e}^{(\varepsilon-\gamma)|t|}\right\},$$

where γ is given by (3.23)*. If* (3.24) *holds and*

$$\max\{a,c\} + (\varepsilon+\gamma)/q \leqslant 0, \tag{4.24}$$

then there is $N \in\,]0,1[$ and $\varphi \in \mathcal{A}_N$ such that

$$\Psi_\tau\left(\mathcal{V}^*_{\varphi,\overline{R}}\right) \subset \mathcal{V}^*_{\varphi,R}$$

for every $\tau \in \mathbb{R}$, where Ψ_τ is given by (3.9)*, $\overline{R}\colon \mathbb{R} \to]0,\infty[$ is given by* (4.3) *and $\mathcal{V}^*_{\varphi,\overline{R}}$ and $\mathcal{V}^*_{\varphi,R}$ are given by* (4.1)*. Furthermore, we have*

$$\left\|\Psi_{t-s}(p_{s,\xi}) - \Psi_{t-s}(p_{s,\bar{\xi}})\right\| \leqslant \begin{cases} \dfrac{DN}{\widetilde{\omega}}\,\mathrm{e}^{a(t-s)+\varepsilon|s|}\left\|\xi-\bar{\xi}\right\| & \text{for } t \geqslant s,\\[2mm] \dfrac{DN}{\widetilde{\omega}}\,\mathrm{e}^{c(s-t)+\varepsilon|s|}\left\|\xi-\bar{\xi}\right\| & \text{for } t \leqslant s,\end{cases}$$

for every $(s,\xi),(s,\bar{\xi}) \in G$ *and* $\xi,\bar{\xi} \in \mathfrak{B}(\overline{R}(s))$ *and where* $p_{s,\xi} = (s,\xi,\varphi(s,\xi))$ *and* $p_{s,\bar{\xi}} = \big(s,\bar{\xi},\varphi(s,\bar{\xi})\big)$.

Proof: It follows immediately taking into account that $\rho(t) = t$ is twice differentiable and (4.23) follows from (4.24), since

$$\begin{aligned}
&(aq + \operatorname{sgn} t(\varepsilon+\gamma))\,[\rho'(t)]^2 \leqslant \rho''(t) \leqslant (-cq + \operatorname{sgn} t(\varepsilon+\gamma))\,[\rho'(t)]^2\\
&\Leftrightarrow aq + \operatorname{sgn} t(\varepsilon+\gamma) \leqslant 0 \leqslant -cq + \operatorname{sgn} t(\varepsilon+\gamma)
\end{aligned}$$

and having $\max\{a,c\} + (\varepsilon+\gamma)/q \leqslant 0$, implies

$$aq + \varepsilon + \gamma \leqslant 0 \quad \text{and also} \quad -cq - (\varepsilon+\gamma) \geqslant 0$$

which obviously implies

$$aq + \operatorname{sgn} t(\varepsilon+\gamma) \leqslant 0 \leqslant -cq + \operatorname{sgn} t(\varepsilon+\gamma)$$

for every $t \in \mathbb{R}$. In this case we have, when $\varepsilon > 0$

$$\frac{\widetilde{\omega}}{2DN}\left[\frac{\delta\gamma\tau^-}{2Dk}\right]^{1/q} \mathrm{e}^{-(1+2/q)\varepsilon|s|} \leqslant \overline{R}(s) \leqslant \frac{\widetilde{\omega}}{2DN}\left[\frac{\delta\gamma\tau^+}{2Dk}\right]^{1/q} \mathrm{e}^{-(1+2/q)\varepsilon|s|}$$

and

$$\overline{R}(s) = \frac{\widetilde{\omega}}{2DN}\left[\frac{\delta\gamma}{2Dk}\right]^{1/q} \mathrm{e}^{-\gamma|s|/q}$$

for $\varepsilon = 0$. □

Corollary 4.3.5. *Let* X *be a Banach space. Suppose that equation* (1.1) *admits a* $\rho-$*nonuniform trichotomy with* ρ *given by*

$$\rho(t) = \operatorname{sgn}(t)\ln(1+|t|) = \ln\left([1+|t|]^{\operatorname{sgn}(t)}\right).$$

Assume that $f\colon \mathbb{R}\times X \to X$ *is a function that verifies* (3.2) *and* (4.9)*. Make*

$$R^q(t) = \frac{\delta}{kD^2 2^q}\tau(t)(1+|t|)^{-2\varepsilon-1}, \tag{4.25}$$

with $0 < \delta < 1/12$ *and*

$$\tau(t) := \min\left\{-c-d-\varepsilon\operatorname{sgn}(t), -a-b+\varepsilon\operatorname{sgn}(t), \frac{D\gamma}{2}(1+|t|)^{\varepsilon-\gamma}\right\}, \tag{4.26}$$

where γ *is given by* (3.23)*. If* (3.24) *holds and*

$$\max\{a,c\}+(\varepsilon+\gamma+1)/q \leqslant 0,$$

then there is $N \in]0,1[$ *and* $\varphi \in \mathcal{A}_N$ *such that*

$$\Psi_\tau\left(\mathcal{V}^*_{\varphi,\overline{R}}\right) \subseteq \mathcal{V}^*_{\varphi,R}$$

for every $\tau \in \mathbb{R}$*, where* Ψ_τ *is given by* (3.9)*,* $\overline{R}\colon \mathbb{R} \to]0,\infty[$ *is given by* (4.3) *and* $\mathcal{V}^*_{\varphi,\overline{R}}$ *and* $\mathcal{V}^*_{\varphi,R}$ *are given by* (4.1)*. Furthermore, we have*

$$\left\|\Psi_{t-s}(p_{s,\xi})-\Psi_{t-s}(p_{s,\bar{\xi}})\right\| \leqslant \begin{cases} \dfrac{DN}{\widetilde{\omega}}\left[\dfrac{(1+|t|)^{\operatorname{sgn}(t)}}{(1+|s|)^{\operatorname{sgn}(s)}}\right]^a (1+|s|)^\varepsilon \left\|\xi-\bar{\xi}\right\| & \textit{if } t \geqslant s, \\ \dfrac{DN}{\widetilde{\omega}}\left[\dfrac{(1+|s|)^{\operatorname{sgn}(s)}}{(1+|t|)^{\operatorname{sgn}(t)}}\right]^c (1+|s|)^\varepsilon \left\|\xi-\bar{\xi}\right\| & \textit{if } t \leqslant s, \end{cases} \tag{4.27}$$

for every $(s,\xi),(s,\bar{\xi}) \in G$ *and* $\xi,\bar{\xi} \in \mathfrak{B}(\overline{R}(s))$ *and where* $p_{s,\xi}=(s,\xi,\varphi(s,\xi))$ *and* $p_{s,\bar{\xi}}=\big(s,\bar{\xi},\varphi(s,\bar{\xi})\big)$*.*

Proof: It is clear that (1.6) is an odd differentiable function with

$$\rho'(t)=\begin{cases}\dfrac{1}{1+t}, & t \geqslant 0\\ \dfrac{-(-1)}{1-t}, & t<0\end{cases} = \frac{1}{1+|t|} \qquad \text{for all } t \in \mathbb{R},$$

always positive. For this choice of ρ in (1.5) we have

$$\alpha_{t,s}=\begin{cases} D\left[\dfrac{(1+|t|)^{\operatorname{sgn}(t)}}{(1+|s|)^{\operatorname{sgn}(s)}}\right]^a (1+|s|)^\varepsilon & \text{for } (t,s)\in\mathbb{R}^2_\geqslant,\\ D\left[\dfrac{(1+|s|)^{\operatorname{sgn}(s)}}{(1+|t|)^{\operatorname{sgn}(t)}}\right]^c (1+|s|)^\varepsilon & \text{for } (t,s)\in\mathbb{R}^2_\leqslant,\end{cases}$$

$$\beta^+_{t,s}=D\left[\frac{(1+|t|)^{\operatorname{sgn}(t)}}{(1+|s|)^{\operatorname{sgn}(s)}}\right]^d (1+|s|)^\varepsilon \qquad \text{for } (t,s)\in\mathbb{R}^2_\geqslant,$$

$$\beta^-_{t,s}=D\left[\frac{(1+|s|)^{\operatorname{sgn}(s)}}{(1+|t|)^{\operatorname{sgn}(t)}}\right]^b (1+|s|)^\varepsilon \qquad \text{for } (t,s)\in\mathbb{R}^2_\leqslant.$$

Suppose that $\varepsilon>0$ and by (3.23) this implies $\gamma=\varepsilon$. Then taking into account (4.12) and (1.6) we have (4.25), and (4.26) becomes

$$\tau(t)=\min\left\{-c-d-\varepsilon\operatorname{sgn}(t), -a-b+\varepsilon\operatorname{sgn}(t), \frac{D\gamma}{2}\right\}.$$

This implies that

$$\frac{\delta\tau^-}{kD^2 2^q}(1+|t|)^{-2\varepsilon-1} \leqslant R^q(t) \leqslant \frac{\delta\tau^+}{kD^2 2^q}(1+|t|)^{-2\varepsilon-1},$$

where $\tau^- = \min_{t\in\mathbb{R}} \tau(t) > 0$ and $\tau^+ = \max_{t\in\mathbb{R}} \tau(t) > 0$. In this case we have

$$\chi^+(s) = \sup_{t\geqslant s} \frac{\mathrm{e}^{a\rho(t)+(\varepsilon+\gamma)|\rho(t)|/q}}{\rho'(t)^{1/q}} = \sup_{t\geqslant s}(1+|t|)^{a\,\mathrm{sgn}(t)+(2\varepsilon+1)/q}$$

and, since $a+(2\varepsilon+1)/q \leqslant 0$ and $-a+(2\varepsilon+1)/q > 0$ (because $a \leqslant 0$), the function given by

$$(1+|t|)^{a\,\mathrm{sgn}(t)+(2\varepsilon+1)/q} = \begin{cases} (1+t)^{a+(2\varepsilon+1)/q} & \text{if } t>0 \\ 1 & \text{if } t=0 \\ (1-t)^{-a+(2\varepsilon+1)/q} & \text{if } t<0 \end{cases}$$

is decreasing and this implies

$$\chi^+(s) = (1+|s|)^{a\,\mathrm{sgn}(s)+(2\varepsilon+1)/q}.$$

On the other hand, we have

$$\chi^-(s) = \sup_{t\leqslant s} \frac{\mathrm{e}^{-c\rho(t)+(\varepsilon+\gamma)|\rho(t)|/q}}{\rho'(t)^{1/q}} = \sup_{t\leqslant s}(1+|t|)^{-c\,\mathrm{sgn}(t)+(2\varepsilon+1)/q}$$

and since $c+(2\varepsilon+1)/q \leqslant 0$ and $-c+(2\varepsilon+1)/q > 0$ (also because $c \leqslant 0$) the function given by

$$(1+|t|)^{-c\,\mathrm{sgn}(t)+(2\varepsilon+1)/q} = \begin{cases} (1+t)^{-c+(2\varepsilon+1)/q} & \text{if } t>0 \\ 1 & \text{if } t=0 \\ (1-t)^{c+(2\varepsilon+1)/q} & \text{if } t<0 \end{cases}$$

is increasing and this implies

$$\chi^-(s) = (1+|s|)^{-c\,\mathrm{sgn}(s)+(2\varepsilon+1)/q}.$$

Hence, for $\varepsilon > 0$,

$$\max\left\{\mathrm{e}^{-a\rho(s)}\,\chi^+(s), \mathrm{e}^{c\rho(s)}\,\chi^-(s)\right\} = (1+|s|)^{(2\varepsilon+1)/q}.$$

Therefore we get

$$\sup_{t\in\mathbb{R}} \frac{\alpha_{t,s}}{R(t)} \leqslant 2\left[\frac{D^{q+2}k}{\delta\tau^-}\right]^{1/q} (1+|s|)^{\varepsilon+(2\varepsilon+1)/q} < +\infty.$$

If $\varepsilon = 0$, by (3.23) we have

$$\tau(t) = \min\left\{-c-d-\varepsilon\operatorname{sgn}(t), -a-b+\varepsilon\operatorname{sgn}(t), \frac{D\gamma}{2}(1+|t|)^{\varepsilon-\gamma}\right\} = \frac{D\gamma}{2}(1+|t|)^{-\gamma},$$

and this implies that

$$R^q(t) = \frac{\delta(1+|t|)^{-1}}{kD^2 2^q}\frac{D\gamma}{2}(1+|t|)^{-\gamma} = \frac{\delta\gamma(1+|t|)^{-1-\gamma}}{kD2^{q+1}}.$$

In this case we have

$$\chi^+(s) = \sup_{t\geqslant s} \frac{\mathrm{e}^{a\rho(t)+(\varepsilon+\gamma)|\rho(t)|/q}}{\rho'(t)^{1/q}} = \sup_{t\geqslant s}(1+|t|)^{a\operatorname{sgn}(t)+(\gamma+1)/q}$$

$$\chi^-(s) = \sup_{t\leqslant s} \frac{\mathrm{e}^{-c\rho(t)+(\varepsilon+\gamma)|\rho(t)|/q}}{\rho'(t)^{1/q}} = \sup_{t\leqslant s}(1+|t|)^{-c\operatorname{sgn}(t)+(\gamma+1)/q}.$$

Because $a+(1+\gamma)/q \leqslant 0$ we have

$$\begin{aligned}\sup_{t\geqslant s} \frac{\alpha_{t,s}}{R(t)} &= \left[\frac{(2D)^{q+1}k}{\delta\gamma}\right]^{1/q} (1+|s|)^{(1+\gamma)/q} \frac{\chi^+(s)}{(1+|s|)^{a\operatorname{sgn}(s)+(1+\gamma)/q}} \\ &= \left[\frac{(2D)^{q+1}k}{\delta\gamma}\right]^{1/q} (1+|s|)^{(1+\gamma)/q} < +\infty\end{aligned} \tag{4.28}$$

and because $c+(1+\gamma)/q \leqslant 0$ we get

$$\begin{aligned}\sup_{t\leqslant s} \frac{\alpha_{t,s}}{R(t)} &= \left[\frac{(2D)^{q+1}k}{\delta\gamma}\right]^{1/q} (1+|s|)^{(1+\gamma)/q} \frac{\chi^-(s)}{(1+|s|)^{-c\operatorname{sgn}(s)+(1+\gamma)/q}} \\ &= \left[\frac{(2D)^{q+1}k}{\delta\gamma}\right]^{1/q} (1+|s|)^{(1+\gamma)/q} < +\infty.\end{aligned} \tag{4.29}$$

It is easy to see that (4.14) becomes (4.27) for every $\xi, \bar\xi \in \mathfrak{B}(\overline{R}(s)) \cap E_s$. □

Remark 4.3.6.

Note that, for $\varepsilon > 0$, we have

$$\begin{aligned}\max\left\{\mathrm{e}^{-a\rho(s)}\,\chi^+(s), \mathrm{e}^{c\rho(s)}\,\chi^-(s)\right\} &= \min\left\{\mathrm{e}^{-a\rho(s)}\,\chi^+(s), \mathrm{e}^{c\rho(s)}\,\chi^-(s)\right\} \\ &= (1+|s|)^{(2\varepsilon+1)/q}\end{aligned}$$

and then

$$2\left[\frac{D^{q+2}k}{\delta\tau^{+}}\right]^{1/q}(1+|s|)^{\varepsilon+(2\varepsilon+1)/q} \leqslant \sup_{t\in\mathbb{R}}\frac{\alpha_{t,s}}{R(t)} \leqslant 2\left[\frac{D^{q+2}k}{\delta\tau^{-}}\right]^{1/q}(1+|s|)^{\varepsilon+(2\varepsilon+1)/q}.$$

Therefore for $\varepsilon>0$, *from* (4.3) *we have*

$$\frac{\widetilde{\omega}}{2N}\left[\frac{\delta\tau^{-}}{D^{q+2}k}\right]^{1/q}(1+|s|)^{-\varepsilon-(2\varepsilon+1)/q} \leqslant \overline{R}(s) \leqslant \frac{\widetilde{\omega}}{2N}\left[\frac{\delta\tau^{+}}{D^{q+2}k}\right]^{1/q}(1+|s|)^{-\varepsilon-(2\varepsilon+1)/q}.$$

In the case that $\varepsilon=0$, *from* (4.28) *and* (4.29), (4.3) *becomes*

$$\overline{R}(s)=\frac{\widetilde{\omega}}{2DN}\left[\frac{\delta\gamma}{2Dk}\right]^{1/q}(1+|s|)^{-(1+\gamma)/q}.$$

§4.3.3 $\mu-$nonuniform polynomial trichotomies

At last, we are going to apply the main theorem of this chapter to $\mu-$nonuniform polynomial trichotomies for this type of perturbations f satisfying (4.9).

Theorem 4.3.7. *Let* X *be a Banach space. Suppose that equation* (1.1) *admits a* $\mu-$*nonuniform polynomial trichotomy. Assume that* $f\colon\mathbb{R}\times X\to X$ *verifies* (3.2) *and* (4.9). *Make*

$$R(t)=\delta\mu'(t)^{1/q}(|\mu(t)|+1)^{-\gamma}$$

for δ *sufficiently small and* $\gamma>0$. *If we assume* (3.24), $a,c\leqslant 0$, $2\varepsilon-\gamma q+1<0$ *and for every* $s\in\mathbb{R}$, (4.6) *holds then there is* $N\in]0,1[$ *and* $\varphi\in\mathcal{A}_N$ *such that*

$$\Psi_\tau\left(\mathcal{V}^*_{\varphi,\overline{R}}\right)\subseteq\mathcal{V}^*_{\varphi,R}$$

for every $\tau\in\mathbb{R}$, *where* Ψ_τ *is given by* (3.9), $\overline{R}\colon\mathbb{R}\to]0,\infty[$ *is given by* (4.3) *and* $\mathcal{V}^*_{\varphi,\overline{R}}$ *and* $\mathcal{V}^*_{\varphi,R}$ *are given by* (4.1). *Furthermore, we have*

$$\left\|\Psi_{t-s}(p_{s,\xi})-\Psi_{t-s}(p_{s,\bar\xi})\right\|\leqslant\begin{cases}\dfrac{DN}{\widetilde{\omega}}(\mu(t)-\mu(s)+1)^{a}(|\mu(s)|+1)^{\varepsilon}\left\|\xi-\bar\xi\right\| & \text{if } t\geqslant s,\\[2mm] \dfrac{DN}{\widetilde{\omega}}(\mu(s)-\mu(t)+1)^{c}(|\mu(s)|+1)^{\varepsilon}\left\|\xi-\bar\xi\right\| & \text{if } t\leqslant s,\end{cases}$$

for every $(s,\xi),(s,\bar\xi)\in G$ *and* $\xi,\bar\xi\in\mathfrak{B}(\overline{R}(s))$ *and where* $p_{s,\xi}=(s,\xi,\varphi(s,\xi))$ *and* $p_{s,\bar\xi}=\left(s,\bar\xi,\varphi(s,\bar\xi)\right)$.

Proof: For this R it follows that (4.10) becomes

$$\mathrm{Lip}\left(f_t|_{\mathfrak{B}(R(t))}\right) \leqslant k2^q\delta^q\mu'(t)(|\mu(t)|+1)^{-\gamma q}. \tag{4.30}$$

In this case (3.18) becomes (3.24), and since (3.29) and (3.28), choosing $\varepsilon < \gamma q - 1$, $a, c \leqslant 0$ and using (4.30) we have

$$\begin{aligned}
\widetilde{\sigma} &= \sup_{(t,s)\in\mathbb{R}^2} \left| \int_s^t \frac{\alpha_{t,r}\,\mathrm{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right)\alpha_{r,s}}{\alpha_{t,s}}\,dr \right| \\
&\leqslant 2^q Dk\delta^q \sup_{(t,s)\in\mathbb{R}^2} \left| \int_s^t \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma q}\,dr \right| \\
&= 2^q Dk\delta^q \int_{-\infty}^{+\infty} \mu'(r)(|\mu(r)|+1)^{\varepsilon-\gamma q}\,dr \\
&= 2^q Dk\delta^q \int_{-\infty}^{+\infty} (|\tau|+1)^{\varepsilon-\gamma q}\,d\tau \\
&= \frac{2^{q+1}Dk\delta^q}{|\varepsilon-\gamma q+1|},
\end{aligned}$$

where we made the substitution $\tau = \mu(r)$.

Assuming (4.30) and making the substitution $\tau = \mu(s) - \mu(r)$ we have

$$\begin{aligned}
&\int_{-\infty}^{s} \beta^+_{s,r}\,\mathrm{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right)\alpha_{r,s}\,dr \\
&\leqslant 2^q D^2\delta^q k \int_{-\infty}^{s} \mu'(r)(\mu(s)-\mu(r)+1)^{c+d}(|\mu(r)|+1)^{\varepsilon-\gamma q}(|\mu(s)|+1)^{\varepsilon}\,dr \\
&= 2^q D^2\delta^q k \int_0^{+\infty} (\tau+1)^{c+d}(|\tau-\mu(s)|+1)^{\varepsilon-\gamma q}(|\mu(s)|+1)^{\varepsilon}\,d\tau
\end{aligned}$$

and making the substitution $\tau = \mu(r) - \mu(s)$ we have

$$\begin{aligned}
&\int_s^{+\infty} \beta^-_{s,r}\,\mathrm{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right)\alpha_{r,s}\,dr \\
&\leqslant 2^q D^2\delta^q k \int_s^{+\infty} \mu'(r)(\mu(r)-\mu(s)+1)^{a+b}(|\mu(r)|+1)^{\varepsilon-\gamma q}(|\mu(s)|+1)^{\varepsilon}\,dr \\
&= 2^q D^2\delta^q k \int_0^{+\infty} (\tau+1)^{a+b}(|\tau+\mu(s)|+1)^{\varepsilon-\gamma q}(|\mu(s)|+1)^{\varepsilon}\,d\tau.
\end{aligned}$$

Since $c+d+\varepsilon < 0$, $a+b+\varepsilon < 0$, $\varepsilon-\gamma q+1 < 0$, $2\varepsilon-\gamma q \leqslant 0$ and using

Lemma 3.2.6 it follows that

$$\begin{aligned}\widetilde{\omega} &= \sup_{s\in R}\left[\int_{-\infty}^{s} \beta^{+}_{s,r} \operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right) \alpha_{r,s}\, dr + \int_{s}^{+\infty} \beta^{-}_{s,r} \operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right) \alpha_{r,s}\, dr\right]\\ &\leqslant \frac{2^q D^2 \delta^q k(2^{\varepsilon+1}+1)}{|\max\{a+b,c+d\}+2\varepsilon-\gamma q+1|} + \frac{2^q D^2 \delta^q k}{|\min\{a+b,c+d\}+2\varepsilon-\gamma q+1|}.\end{aligned}$$

Since for every $s \in \mathbb{R}$, we have

$$\sup_{t\geqslant s}\frac{\alpha_{t,s}}{R(t)} = \frac{D}{\delta}(|\mu(s)|+1)^{\varepsilon} \sup_{t\geqslant s}\left[\frac{(\mu(t)-\mu(s)+1)^a}{\mu'(t)^{1/q}(|\mu(t)|+1)^{-\gamma}}\right] \tag{4.31}$$

and

$$\sup_{t\leqslant s}\frac{\alpha_{t,s}}{R(t)} = \frac{D}{\delta}(|\mu(s)|+1)^{\varepsilon} \sup_{t\leqslant s}\left[\frac{(\mu(s)-\mu(t)+1)^c}{\mu'(t)^{1/q}(|\mu(t)|+1)^{-\gamma}}\right], \tag{4.32}$$

therefore assuming that (3.24) holds, the last two supremuns are finite and choosing δ sufficiently small such that $4\widetilde{\sigma}+4\widetilde{\omega} < 1$, it follows that all conditions of Theorem 4.1.2 are satisfied. □

Remark 4.3.8. *As in Remarks* 4.2.5 *in last theorem we have*

$$\overline{R}(s) = \frac{\widetilde{\omega}\delta}{DN\vartheta(s)}\,(|\mu(s)|+1)^{-\varepsilon},$$

where $\vartheta(s)$ *is given by* (4.8)*. Note that in order to have* (4.6)*, it is sufficient that*

$$\frac{(\mu(t)-\mu(s)+1)^{a+\gamma}}{\mu'(t)^{1/q}} \text{ is a non-increasing function in the variable } t$$

and

$$\frac{(\mu(s)-\mu(t)+1)^{c+\gamma}}{\mu'(t)^{1/q}} \text{ is a non-decreasing function in the variable } t.$$

Moreover, if μ *is twice differentiable and*

$$q(a+\gamma)\mu'(t)^2 \leqslant \mu''(t) \leqslant -q(c+\gamma)\mu'(t)^2,$$

for every $t \in \mathbb{R}$*, then* (4.6) *is also satisfied.*

Corollary 4.3.9. *Let X be a Banach space. Suppose that equation* (1.1) *admits a nonuniform polynominal trichotomy. Assume that $f\colon \mathbb{R}\times X \to X$ is a function that verifies* (3.2) *and* (4.9) *and make*

$$R(t) = \delta(|t|+1)^{-\gamma}$$

with δ sufficiently small and $\gamma > 0$. If (3.24) *holds and*

$$2\varepsilon - \gamma q + 1 < 0, \qquad a+\gamma \leqslant 0 \qquad \text{and} \qquad c+\gamma \leqslant 0$$

then there is $N \in]0,1[$ and $\varphi \in \mathcal{A}_N$ such that

$$\Psi_\tau\left(\mathcal{V}^*_{\varphi,\overline{R}}\right) \subseteq \mathcal{V}^*_{\varphi,R}$$

for every $\tau \in \mathbb{R}$, where Ψ_τ is given by (3.9)*, $\overline{R}\colon \mathbb{R} \to]0,\infty[$ is given by* (4.3) *and $\mathcal{V}^*_{\varphi,\overline{R}}$ and $\mathcal{V}^*_{\varphi,R}$ are given by* (4.1)*. Furthermore, we have*

$$\left\|\Psi_{t-s}(p_{s,\xi}) - \Psi_{t-s}(p_{s,\bar{\xi}})\right\| \leqslant \begin{cases} \dfrac{DN}{\widetilde{\omega}}(t-s+1)^a(|s|+1)^\varepsilon \left\|\xi - \bar{\xi}\right\| & \text{for } t \geqslant s,\\ \dfrac{DN}{\widetilde{\omega}}(s-t+1)^c(|s|+1)^\varepsilon \left\|\xi - \bar{\xi}\right\| & \text{for } t \leqslant s, \end{cases}$$

for every $(s,\xi),(s,\bar{\xi}) \in G$ and $\xi,\bar{\xi} \in \mathfrak{B}(\overline{R}(s))$ and where $p_{s,\xi} = (s,\xi,\varphi(s,\xi))$ and $p_{s,\bar{\xi}} = \left(s,\bar{\xi},\varphi(s,\bar{\xi})\right)$.

Proof: This follows immediately making $\mu(t) = t$ in Theorem 4.3.7 and taking into account the last remark and that $a+\gamma \leqslant 0$ and $c+\gamma \leqslant 0$ imply that (4.31) and (4.32) are finite. □

§4.4 Proof of Theorem 4.1.2

In the proof of this theorem we are going to use Theorem 3.1.3.

Let $\bar{f}\colon \mathbb{R}\times X \to X$ be the function defined by

$$\bar{f}(r,x) = \begin{cases} f(r,x) & \text{if } x \in \mathfrak{B}(R(r)),\\ f\left(r, \dfrac{R(r)}{\|x\|}x\right) & \text{if } x \notin \mathfrak{B}(R(r)). \end{cases}$$

Clearly $\bar{f}$ is a continuous function. Making $\bar{f}_r(x) = \bar{f}(r,x)$, for every $r \in \mathbb{R}$, $x \in X$, and since $f_r|_{\mathfrak{B}(R(r))} \colon \mathfrak{B}(R(r)) \to X$ is a Lipscthitz function for each $r \in \mathbb{R}$, then we can say that $\bar{f}_r : X \to X$ is Lipschitz and

$$\operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right) \leqslant \operatorname{Lip}(\bar{f}_r) \leqslant 2\operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right)$$

for every $r \in \mathbb{R}$.

Now we will apply Theorem 3.1.3 with f_r replaced by $\bar{f}_r$. Thus, we have

$$\begin{aligned}\sigma &= \sup_{(t,s)\in\mathbb{R}^2} \left| \int_s^t \frac{\alpha_{t,r}\operatorname{Lip}(\bar{f}_r)\alpha_{r,s}}{\alpha_{t,s}}\,dr \right| \\ &\leqslant 2\sup_{(t,s)\in\mathbb{R}^2} \left| \int_s^t \frac{\alpha_{t,r}\operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right)\alpha_{r,s}}{\alpha_{t,s}}\,dr \right| \\ &= 2\widetilde{\sigma}.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\omega &= \sup_{s\in\mathbb{R}} \left[\int_{-\infty}^s \beta^+_{s,r}\operatorname{Lip}(\bar{f}_r)\alpha_{r,s}\,dr + \int_s^{+\infty} \beta^-_{s,r}\operatorname{Lip}(\bar{f}_r)\alpha_{r,s}\,dr \right] \\ &\leqslant 2\sup_{s\in\mathbb{R}} \left[\int_{-\infty}^s \beta^+_{s,r}\operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right)\alpha_{r,s}\,dr + \int_s^{+\infty} \beta^-_{s,r}\operatorname{Lip}\left(f_r|_{\mathfrak{B}(R(r))}\right)\alpha_{r,s}\,dr \right] \\ &= 2\widetilde{\omega}.\end{aligned}$$

Hence, if

$$2\sigma + 2\omega \leqslant 4\widetilde{\sigma} + 4\widetilde{\omega} < 1$$

and considering $\widetilde{\Psi}_\tau$ the flow given by (3.9) then by Theorem 3.1.3, applied to the problem (3.1) with f replaced by $\bar{f}$ we may say that there is only one function

$$\varphi = (\varphi^+, \varphi^-) \in \mathcal{A}_N \;\text{ such }\; \widetilde{\Psi}_\tau(\mathcal{V}_\varphi) \subseteq \mathcal{V}_\varphi,$$

for every $\tau \in \mathbb{R}$ and

$$\left\| \widetilde{\Psi}_{t-s}(s,\xi,\varphi(s,\xi)) - \widetilde{\Psi}_{t-s}(s,\bar{\xi},\varphi(s,\bar{\xi})) \right\| \leqslant \frac{N}{\omega}\,\alpha_{t,s}\left\|\xi - \bar{\xi}\right\| \tag{4.33}$$

for every $(t,s) \in \mathbb{R}^2$ and every $\xi, \bar{\xi} \in E_s$.

In particular, if $\xi \in \mathfrak{B}\left(\overline{R}(s)\right) \cap E_s$, by (4.2), (4.3) and since we have that $\widetilde{\sigma} \leqslant \sigma$ and $\widetilde{\omega} \leqslant \omega$, it follows that

$$\left\|\widetilde{\Psi}_{t-s}(s,\xi,\varphi(s,\xi))\right\| \leqslant \frac{N}{\omega}\alpha_{t,s}\left\|\xi\right\| \leqslant \frac{N}{\widetilde{\omega}}\alpha_{t,s}\overline{R}(s) \leqslant R(t),$$

for every $(t,s) \in \mathbb{R}^2$ and this implies

$$\widetilde{\Psi}_\tau\left(\mathcal{V}^*_{\varphi,\overline{R}}\right) \subseteq \mathcal{V}^*_{\varphi,R} \tag{4.34}$$

for every $\tau \in \mathbb{R}$.

Since $\bar{f}_r|_{\mathfrak{B}(R(r))} = f_r|_{\mathfrak{B}(R(r))}$ then $\widetilde{\Psi}|_{\mathcal{V}^*_{\varphi,\overline{R}}} = \Psi|_{\mathcal{V}^*_{\varphi,\overline{R}}}$, from (4.34) we get

$$\Psi_\tau\left(\mathcal{V}^*_{\varphi,\overline{R}}\right) \subseteq \mathcal{V}^*_{\varphi,R}$$

for every $\tau \in \mathbb{R}$ and (4.33) implies that

$$\left\|\Psi_{t-s}(s,\xi,\varphi(s,\xi)) - \Psi_{t-s}(s,\bar{\xi},\varphi(s,\bar{\xi}))\right\| \leqslant \frac{N}{\widetilde{\omega}}\,\alpha_{t,s}\left\|\xi-\bar{\xi}\right\|$$

for every $(t,s) \in \mathbb{R}^2$ and every $\xi,\bar{\xi} \in \mathfrak{B}(\overline{R}(s)) \cap E_s$ and this finishes the proof of Theorem 4.1.2.

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Symbols

Index