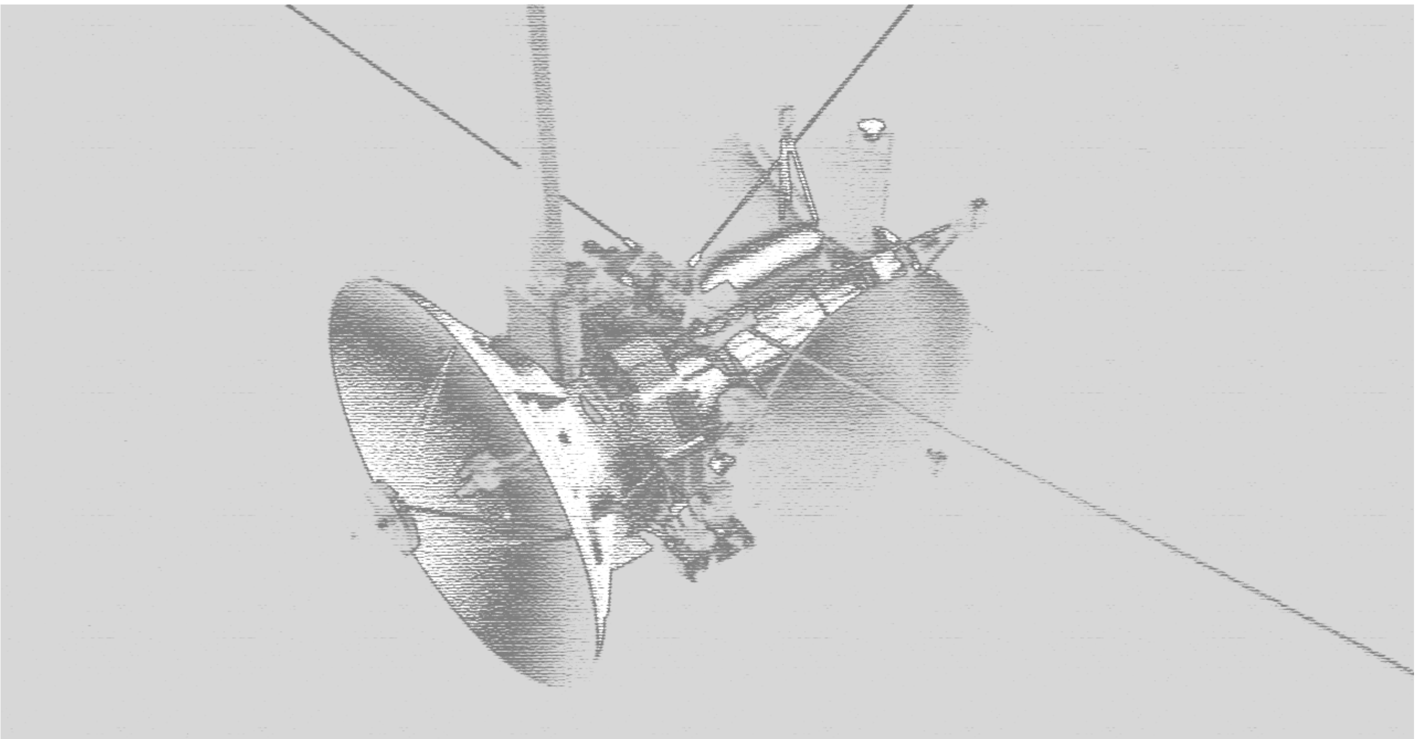
GYROSTAT DYNAMICS ON A CIRCULAR ORBIT

Master Degree Thesis in Aeronautics Engineering



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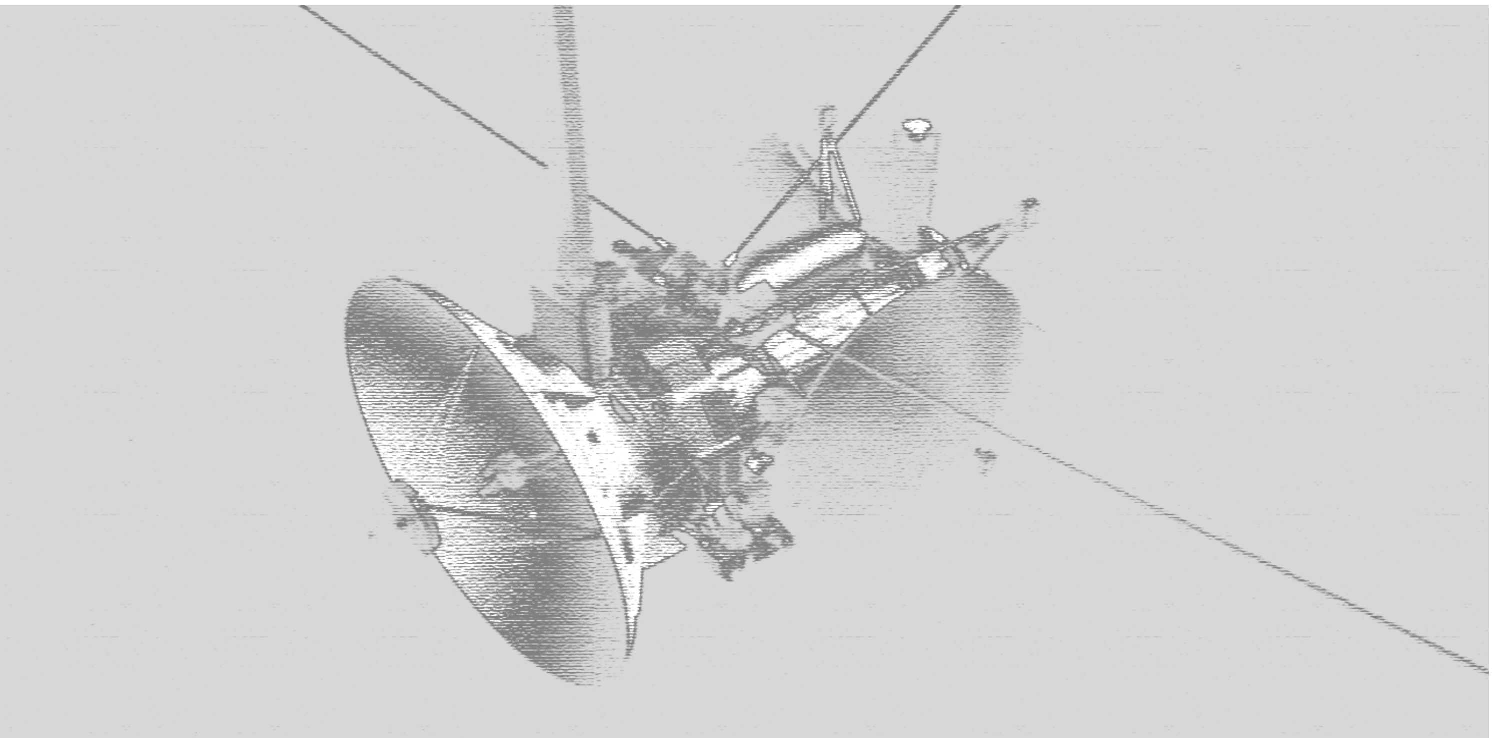
MASTER THESIS



GYROSTAT DYNAMICS ON A CIRCULAR ORBIT

Abstract:

This work deals with a general gyrostat moving along a circular orbit in a central Newtonian force field.

The main purpose is to determine all equilibria positions for our spacecraft in the case when internal angular momentum of the gyrostat satellite is collinear to its principal axis of inertia, when the gyrostatic moment vector lies in one of the satellite's principal central plane of inertia and for a general position where the gyrostatic moment doesn't coincide with any of the principal axis of inertia.

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1-BIBLIOGRAPHIC REVISION

Over the past century, an understanding of the torque-free motion of gyrostats has been developed in cases with freely spinning rotors or with rotors constrained to spin at constant speed relative to the platform. Equilibrium motions of orbiting gyrostats, where the gravity gradient torque has been studied for circular orbits. The gravitational moment used in most studies is obtained by truncating the gravitational potential in an inconsistent manner. Furthermore, most results are for spacecraft with free or constant-speed rotors. During rotational maneuvers, the rotors satisfy neither of these conditions. Although many have studied problems of maneuvering gyrostats, virtually no one has used information about equilibria to develop reorientation control laws.

The first and very important problem to be solved in developing Active Control Systems (ACS) consists in a complete analysis of all possible equilibrium orientations and investigation of their stability.

Several authors have studied in the past years this subject, between them Sarychev[1], Mirer[2], Degtyarev[3] and Gutnik[4] has studied the several cases here presented.

This work involves a gyrostat satellite orbiting a circular orbit with internal momentum wheels ("fly wheels"), and has main goal to find all the equilibrium positions for an internal moment along the principal axis [a], also with the vector of gyrostatic moment in the principal plane of inertia [b] and finally with a single non-zero vector of gyrostatic moment [c] using the equations of special motion.

For cases (a) and (b), authors (1), (2) and (3) found analytically solutions that describe the equilibrium positions for our general gyrostat. For case (c) authors (1) and (4) gave a general approach, due to the complexity of this specific case.

Up to know, as referred above, there are analytical solutions for the first two cases. Otherwise, for the last case there is no analytical solution, only numerical approaches.

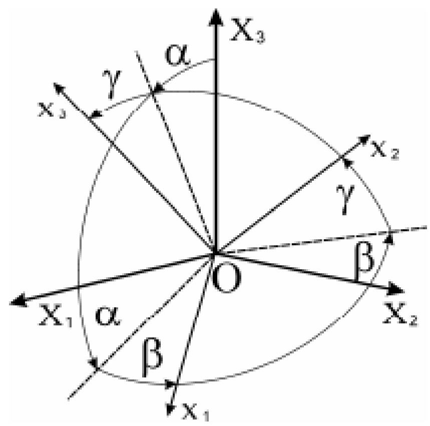
The foremost important interest in this investigation is to confirm and evaluate the results with the ones of publications [3] and [5], and try to perform some developments in the results of [5] and [6]. This work can be tagged as innovative project due to our investigations and precision of results in the non-zero vector of gyrostatic subject.

2-EQUATIONS OF MOTION

Consider the attitude motion of a gyrostat satellite with statically and dynamically balanced rotors inside the satellite body. The rotors angular velocities relative to the satellite body are constant. The gyrostat's center of mass *O* is in a circular orbit.

We now introduce two right Cartesian coordinate systems with common origin at the point *O*: orbital reference frame $OX_1X_2X_3$ (the axis OX_3 is directed from the orbiting body center of mass to the gyrostat's center of mass, the axis OX_1 is along the velocity vector of the point *O*), and the gyrostat body reference frame $Ox_1x_2x_3$ with axes along the gyrostat's principal central axes of inertia. Using angles α, β, γ (Figure 1) we get the direction cosines $\cos(x_i, X_j) = a_{i,j}\,(i, j = 1,2,3)$ in the form:

$$\begin{cases} a_{11} = \cos\alpha\cos\beta \\ a_{12} = sen\,\alpha sen\,\gamma - \cos\alpha sen\,\beta\cos\gamma \\ a_{13} = sen\,\alpha\cos\gamma - \cos\alpha sen\,\beta sen\,\gamma \\ a_{21} = \sin\beta \\ a_{22} = \cos\beta\cos\gamma \\ a_{23} = -\cos\beta sen\,\gamma \\ a_{31} = -sen\,\alpha\cos\beta \\ a_{32} = \cos\alpha sen\,\gamma + sen\,\alpha sen\,\beta\cos\gamma \\ a_{33} = \cos\alpha\cos\gamma - sen\,\alpha sen\,\beta sen\,\gamma \end{cases} \quad (1)$$



$OX_1X_2X_3$ – Orbital Reference Frame

$Ox_1x_2x_3$ – Satellite Body Reference Frame

Figure 1 – Gyrostat Orbital Frame

According to the picture above, our system is described by the following definitions:

$$T=\frac{1}{2}(Ap^2+Bq^2+Cr^2)+\frac{1}{2}\sum_{k=1}^{n}J_k\dot{\varphi}_k^2+\sum_{k=1}^{n}J_k(\hat{\alpha}_k p+\hat{\beta}_k q+\hat{\gamma}_k r)\dot{\varphi}_k. \quad (2)$$

$$U = -\frac{3}{2}\omega_0^2\left[(A-C)a_{31}^2 + (B-C)a_{32}^2\right] \quad (3)$$

Where equations (2) and (3) are respectively the Kinetic Energy and Force Function. For our problem $\dot{\varphi}_k = const.$, so it is more convenient to introduce:

$$\bar{h}_1 = \sum_{k=1}^{n} J_k \hat{\alpha}_k \dot{\varphi}_k, \; \bar{h}_2 = \sum_{k=1}^{n} J_k \hat{\beta}_k \dot{\varphi}_k, \; \bar{h}_3 = \sum_{k=1}^{n} J_k \hat{\gamma}_k \dot{\varphi}_k. \quad (4)$$

The projections of inertia in our different directions are described by:

$$\begin{aligned} p &= (\dot{\alpha} + \omega_o) a_{21} + \dot{\gamma} = \bar{p} + \omega_o a_{21} \\ q &= (\dot{\alpha} + \omega_o) a_{22} + \dot{\beta} \, sen \, \gamma = \bar{q} + \omega_o a \\ r &= (\dot{\alpha} + \omega_o) a_{23} + \dot{\beta} \cos \gamma = \bar{r} + \omega_o a_{23} \end{aligned} \quad (5)$$

Where $\bar{h}_i (i = 1,2,3)$ are the projections of the absolute angular velocity of the gyrostat and the projections of the vector of gyrostatic moment onto the Ox_i axis. A, B and C are the gyrostat principal moments of inertia, and ω_0 is the angular velocity.

So, the equations of motion of the satellite-gyrostat with respect to its center of mass are written in the form:

$$\begin{cases} A\dot{p} + (C - B)qr - 3\omega_0^2 (C - B) a_{32} a_{33} - \bar{h}_2 r + \bar{h}_3 q = 0 \\ B\dot{q} + (A - C)rp - 3\omega_0^2 (A - C) a_{33} a_{31} - \bar{h}_3 p + \bar{h}_1 r = 0 \\ C\dot{r} + (B - A)pq - 3\omega_0^2 (B - A) a_{31} a_{32} - \bar{h}_1 q + \bar{h}_2 p = 0 \end{cases} \quad (6)$$

Now from systems (5) and (6) we can compute the generalized integral of energy:

$$\frac{1}{2}(A\bar{p}^2 + B\bar{q}^2 + C\bar{r}^2) + \frac{3}{2}\omega_0^2 \left[(A - C) a_{31}^2 + (B - C) a_{32}^2 \right] + + \frac{1}{2}\omega_0^2 \left[(B - A) a_{21}^2 + (B - C) a_{23}^2 \right] - \omega_0 (\bar{h}_1 a_{21} + \bar{h}_2 a_{22} + \bar{h}_3 a_{23}) = H$$

As we can see, the energy remains constant during the entire path.

3- Equilibrium Orientations

Starting from the following system of equations:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial p}\right) - r\dfrac{\partial T}{\partial q} + q\dfrac{\partial T}{\partial r} = a_{33}\dfrac{\partial U}{\partial a_{32}} - a_{32}\dfrac{\partial U}{\partial a_{33}} \\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial q}\right) - p\dfrac{\partial T}{\partial r} + r\dfrac{\partial T}{\partial p} = a_{31}\dfrac{\partial U}{\partial a_{33}} - a_{33}\dfrac{\partial U}{\partial a_{31}} \\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial r}\right) - q\dfrac{\partial T}{\partial p} + p\dfrac{\partial T}{\partial q} = a_{32}\dfrac{\partial U}{\partial a_{31}} - a_{31}\dfrac{\partial U}{\partial a_{32}} \end{cases} \quad (7)$$

Considering now our satellite in a circular orbit and neglecting the atmospheric rotation, and assuming that $\alpha = \alpha_0$, $\beta = \beta_0$ and $\gamma = \gamma_0$, are constants and making $\dfrac{\bar{h}_i}{\omega_0} = h_i$ having into account systems (6) and (7), we get the following set of equations:

$$\begin{aligned} (C - B)(a_{22}a_{23} - 3a_{32}a_{33}) - h_2 a_{23} + h_3 a_{22} &= P = 0 \\ (A - C)(a_{23}a_{21} - 3a_{33}a_{31}) - h_3 a_{21} + h_1 a_{23} &= Q = 0 \\ (B - A)(a_{21}a_{22} - 3a_{31}a_{32}) - h_1 a_{22} + h_2 a_{21} &= P = 0 \end{aligned} \quad (8)$$

Developing set of equation (8) we get:

$$\begin{cases} 4(Aa_{21}a_{31} + Ba_{22}a_{32} + Ca_{23}a_{33}) + (h_1 a_{31} + h_2 a_{32} + h_3 a_{33}) = 0 \\ Aa_{11}a_{31} + Ba_{12}a_{32} + Ca_{13}a_{33} = 0 \\ Aa_{11}a_{21} + Ba_{12}a_{22} + Ca_{13}a_{23} + (h_1 a_{11} + h_2 a_{12} + h_3 a_{13}) = 0 \end{cases} \quad (9)$$

We also know that our linear combination is:

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array} \quad (10)$$

And knowing also the following conditions of orthogonality for the direction cosines:

$$\begin{cases} a_{11}^2 + a_{12}^2 + a_{13}^2 = 1 \\ a_{21}^2 + a_{22}^2 + a_{23}^2 = 1 \\ a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \end{cases} \quad (11) \qquad \begin{cases} a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0 \\ a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0 \\ a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0 \end{cases} \quad (12)$$

Now, from equation 2 on (8), equation 1 on (11) and equation 2 on (12), we have the following system of equations:

$$\begin{cases} a_{11} = 4\dfrac{a_{32}a_{33}(C - B)}{F} \\ a_{12} = 4\dfrac{a_{31}a_{33}(A - C)}{F} \\ a_{13} = 4\dfrac{a_{31}a_{32}(B - A)}{F} \end{cases} \quad (13) \qquad \begin{cases} a_{21} = 4a_{31}\dfrac{(I_3 - A)}{F} \\ a_{22} = 4a_{32}\dfrac{(I_3 - B)}{F} \\ a_{23} = 4a_{33}\dfrac{(I_3 - C)}{F} \end{cases} \quad (14)$$

Where $F = h_1 a_{31} + h_2 a_{32} + h_3 a_{33}$ and $I_3 = Aa_{31}^2 + Ba_{32}^2 + Ca_{33}^2$

The solutions from the system above are solvable only when $A \neq B \neq C$. Let us notice also that the solutions of system (14) exist only when a_{31}, a_{32} and a_{33} none two of them could vanish simultaneously, otherwise we get some special cases. Now, substituting equations (13) and (14) in the first and third equations of (8) and adding the third equation of (11) we finally get these three equations:

$$\begin{cases} 16\left[a_{32}^2a_{33}^2(B-C)^2+a_{31}^2a_{33}^2(C-A)^2+a_{31}^2a_{32}^2(A-B)^2\right]=(h_1a_{31}+h_2a_{32}+h_3a_{33})^2 \\ 4(B-C)(C-A)(A-B)a_{31}a_{32}a_{33}+ \\ +[h_1(B-C)a_{32}a_{33}+h_2(C-A)a_{31}a_{33}+h_3(A-B)a_{31}a_{32}](h_1a_{31}+h_2a_{32}+h_3a_{33})=0 \\ a_{31}^2+a_{32}^2+a_{33}^2=1 \end{cases} \quad (15)$$

To determine the direction cosines a_{31}, a_{32} and a_{33}, if system (15) is solved, then relations (13) and (14) allow us to find the other six director cosines.

3.1- Equilibrium Orientations. Special Case (h1=h3=0, h2≠0)

This case $(h_1=h_3=0, h_2\neq 0)$ means that the attitude of the gyrostat is turned around the local tangent though a roll angle with value γ_0 $(h_1=0)$ and also turned around the local vertical through a yaw angle with value β_0 $(h_3=0)$. This case can be physically interpretated as:

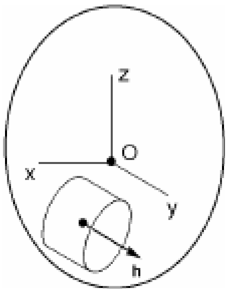


Figure 2 – Physical Interpretation of our case 3.1

Taking into account our system (15), and making

$h_1=h_3=0$ and $h_2\neq 0$, our system is transformed into:

$$\begin{cases} 16\left[a_{32}^2a_{33}^2(B-C)^2+a_{31}^2a_{33}^2(C-A)^2+a_{31}^2a_{32}^2(A-B)^2\right]=a_{32}^2h_2^2 \\ 4(B-C)(C-A)(A-B)a_{31}a_{32}a_{33}+h_2^2(C-A)a_{31}a_{32}a_{33}=0 \\ a_{31}^2+a_{32}^2+a_{33}^2=1 \end{cases} \quad (16)$$

By the analysis of the 2nd equation from system (16) we reach the following conclusions:

Or $h_2^2=4(B-A)(B-C)$ (17) or $a_{31}a_{32}a_{33}=0$ (18)

So, let's suppose that $h_2^2=4(B-A)(B-C)$ and $a_{31}a_{32}a_{33}\neq 0$.

From our system of equations (16) let's equalize the 1st and 2nd equations to get:

$$(4-\nu)\bar{y}+4\bar{x}(1-\nu)^2+\nu(4\nu-1)\bar{x}\bar{y}-\bar{y}^2\nu=0 \quad (19)$$

Where $\bar{x}=\frac{a_{31}^2}{a_{33}^2}$, $\bar{y}=\frac{a_{32}^2}{a_{33}^2}$ and $\nu=\frac{(B-A)}{(B-C)}$ (20)

When $\nu\neq 1/4$ then our equation (19) determines a hyperbola branch lying in the first quadrant. Using the first and second relations of (20) and the third equation from our system (15), we can obtain the expressions for every point $(\bar{x},\bar{y})$ of that curve, as shown below.

$$\begin{cases} a_{31}^2=\frac{\bar{x}}{\bar{x}+\bar{y}+1} \\ a_{32}^2=\frac{\bar{y}}{\bar{x}+\bar{y}+1} \\ a_{33}^2=\frac{1}{\bar{x}+\bar{y}+1} \end{cases} \quad (21)$$

The expressions (21) imply that the considered problem has eight one-parameter families of solutions.

Now let's consider that $h_2^2\neq 4(B-A)(B-C)$.

Then the second equation from system (16) takes the form of equation (18). Specific examination from $a_{31}=0$, $a_{32}=0$ and $a_{33}=0$ results in the following groups of isolated solutions:

To simplify our calculations let's introduce the following terms:

$$x=\frac{h_2}{4(B-C)} \text{ and } y=\frac{h_2}{4(B-A)}$$

CASE 1 - $a_{31}=0, a_{32}\neq 0, a_{33}\neq 0$

From the first equation of (16) we have:

$$a_{33}=\pm\frac{h_2}{4(B-C)}=\pm x$$

This represents physically:

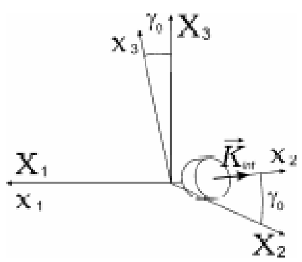


Figure 3 – Physical Interpretation of our case 1

And from (13) and (14) and the third equation of (11) we get:

$$\begin{bmatrix} -\frac{a_{33}}{x} & 0 & 0 \\ 0 & -x & \frac{a_{32}a_{33}}{x} \\ 0 & \pm\sqrt{1-x^2} & \pm x \end{bmatrix}$$

Note that every group involves four solutions, because they correspond to definite choice of signs.

CASE 2 - $a_{31} \neq 0, a_{32} \neq 0, a_{33} = 0$

Again from the first equation of (16) we have:

$$a_{31} = \pm\frac{h_2}{4(B-A)} = \pm y$$

This represents physically:

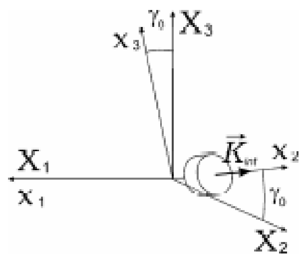


Figure 4 – Physical Interpretation of our case 2

And from (13) and (14) and the third equation of (11) we get:

$$\begin{bmatrix} 0 & 0 & \frac{a_{31}}{y} \\ \frac{a_{31}a_{32}}{y} & -y & 0 \\ \pm y & \pm\sqrt{1-y^2} & 0 \end{bmatrix}$$

Following the same procedure as for cases 1 and 2, we get:

CASE 3 - $a_{31}=0, a_{32}=0, a_{33}\neq 0$: $\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & a_{11}a_{33} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$

CASE 4 - $a_{31}\neq 0, a_{32}=0, a_{33}=0$: $\begin{bmatrix} 0 & 0 & \pm 1 \\ 0 & -a_{13}a_{31} & 0 \\ \pm 1 & 0 & 0 \end{bmatrix}$

CASE 5 - $a_{31}\neq 0, a_{32}=0, a_{33}=0$: $\begin{bmatrix} 0 & a_{23}a_{31} & 4xa_{31} \\ 0 & -4x & \pm\sqrt{1-16x^2} \\ \pm 1 & 0 & 0 \end{bmatrix}$

CASE 6 - $a_{31}=0, a_{32}=0, a_{33}\neq 0$: $\begin{bmatrix} -4ya_{33} & -a_{21}a_{33} & 0 \\ \pm\sqrt{1-16y^2} & -4y & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$

Note that every group of the above results involves four solutions, this is because they correspond to definite choice of signs. For example in the case 5 the solutions correspond to definite choice of signs that determines the following set of solutions:

$$(a_{11}, a_{22}, a_{23}) = (1,1,1), (1,-1,-1), (-1,-1,1), (-1,1,-1)$$

Conclusions

Now let's prove the domains for our mathematical model. Let's take for example our case 5.

When $a_{23} = \pm\sqrt{1-16x^2} \Leftrightarrow x^2 < 1/16$.

Doing the same for our case 6 we have:

$a_{21} = \pm\sqrt{1-16y^2} \Leftrightarrow y^2 < 1/16$.

Getting now into our case 1, we have $a_{32} = \pm\sqrt{1-x^2}$, developing it we get that $x^2 > 1$.

Doing again the same for our case 2 we have: $a_{32} = \pm\sqrt{1-y^2}$, developing it again we get that $y^2 > 1$.

So we can conclude that nine domains exists in the plane (x, y), and the fixed number of solutions is possible in every of these domains. For example, if $x^2 < \frac{1}{16}$ and $y^2 < \frac{1}{16}$ all the 24 solutions exists, while for $x^2 > 1$ and $y^2 > 1$ only 8 solutions exist.

If $h_2 = 0$ $(x = y = 0)$, from case 1 to 6 coincide with the well-known equilibria positions of a rigid body in the orbital reference frame.

3.2- Equilibrium Orientations. Special Case

(h1≠0,h2=0,h3≠0)

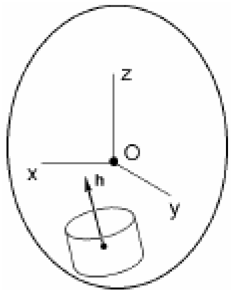


Figure 5 – Physical Interpretation of our case 3.2

This case means that the axis of rotor rotation is not directed along the gyrostat axis of Ox_2.

For this specific situation, and after introducing $\nu = \dfrac{A-B}{C-A}$, $H_1 = \dfrac{h_1}{C-A}$ and $H_3 = \dfrac{h_3}{C-A}$ our system (15) is now transformed into:

$$\begin{cases} 16\left[a_{32}^2 a_{33}^2 (1+\nu)^2 + a_{31}^2 a_{33}^2 + a_{31}^2 a_{32}^2 \nu^2\right] = (H_1 a_{31} + H_3 a_{33})^2 \\ a_{32}^2 \{4\nu(1+\nu) a_{31} a_{33} + [H_1(1+\nu) a_{33} - H_3 \nu a_{31}](H_1 a_{31} + H_3 a_{33})\} = 0 \\ a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \end{cases} \quad (22)$$

With a quick look to our system (22), for investigating these equations we need to consider two cases: $a_{32} \neq 0$ and $a_{32} = 0$.

For $a_{32} \neq 0$, the second equation of (22) takes this shape:

$$4\nu(1+\nu) a_{31} a_{33} + [H_1(1+\nu) a_{33} - H_3 \nu a_{31}](H_1 a_{31} + H_3 a_{33}) = 0 \quad (23)$$

To fulfill (23) we can easily find that only at $a_{31} = 0$, and then at $a_{33} = 0$ satisfy our conditions. The existence of a solution for which $a_{31} = a_{33} = 0$ requires an investigation analyzing the original equations (9), (11) and (12).

Case Where: $a_{31} = 0, a_{32} \neq 0, a_{33} = 0$

After making $x_1 = a_{23}$, and solving (9), (11) and (12) we get the following set of equilibrium conditions which we will refer as Group of Solutions I:

$$\begin{bmatrix} -x_1 a_{32} & 0 & a_{21} a_{32} \\ \dfrac{x_1 H_1}{x_1 + H_3} & 0 & x_1 \\ 0 & \pm 1 & 0 \end{bmatrix}$$

Again manipulating equations (9), (11) and (12) we get a fourth order equation presented below:

$$x_1^4 + 2H_3 x_1^3 + (H_1^2 + H_3^2 - 1)x_1^2 - 2H_3 x_1 - H_3^2 = 0 \quad (24)$$

We can easily find that the equation (24) can have either 2 or 4 real roots. The number of roots changes on the surface determined by the following conditions:

Has four roots if $H_1^{2/3} + H_3^{2/3} < 1$ and two roots if $H_1^{2/3} + H_3^{2/3} > 1$. Hence, the total number of equilibrium orientations for the case where we have the $a_{31} = a_{33} = 0$, i.e. the number of solutions for this case, can be either 8 or 4, depending on the relation between dimensionless parameters H_1 and H_3.

Case Where: $a_{31} \neq 0, a_{32} \neq 0, a_{33} \neq 0$

Making $x_2 = \dfrac{a_{31}}{a_{33}}$ the second equation of (22) becomes:

$$H_1 H_3 \nu x_2^2 + x_2[H_3^2 \nu - H_1^2(1+\nu) - 4\nu(1+\nu)] - H_1 H_3 (1+\nu) = 0 \quad (25)$$

Solving equation (25) we can find that:

$$x_2 = \frac{-[H_3^2 \nu - H_1^2(1+\nu) - 4\nu(1+\nu)] \pm \sqrt{\Delta}}{2H_1 H_3 \nu} \quad (26)$$

where,

$$\Delta = [H_3^2 \nu - H_1^2(1+\nu) - 4\nu(1+\nu)]^2 + 4H_1^2 H_3^2 \nu(1+\nu) \quad (27)$$

Then, going back to system (22), and catching the respective first and third equations, and having into consideration that $a_{31} = x_2 a_{33}$, $a_{31} \neq 0$, $a_{32} \neq 0$ and $a_{33} \neq 0$ we get:

$$\begin{cases} a_{31} = x_2 a_{33} \\ a_{32}^2 = \dfrac{(x_2^2 + 1)(H_1 x_2 + H_3)^2 - 16 x_2^2}{16[(1+\nu) + \nu x_2^2]^2} \\ a_{33}^2 = \dfrac{16[(1+\nu)^2 + x_2^2 \nu^2] - (H_1 x_2 + H_3)^2}{16[(1+\nu) + \nu x_2^2]^2} \end{cases} \quad (28)$$

In order to found the solution that will correspond to the equilibrium position of the gyrostat satellite, the conditions $\Delta \geq 0$, $a_{32}^2 \geq 0$ and $a_{33}^2 \geq 0$ must be met. Let's first analyse the determinant sign. It's pretty clear that $\Delta \geq 0$ if $\nu(1+\nu) \geq 0$, i.e., either at $\nu \leq -1$ or at $\nu \geq 0$.

After fullfilling our calculations we can finnaly get our equilibrium conditions, refered as Solutions II, that does not exceed eight:

$$\begin{cases} a_{11} = \dfrac{4(\nu+1)a_{32}a_{33}}{H_1 a_{31} + H_3 a_{33}} \\ a_{12} = \dfrac{-4a_{31}a_{33}}{H_1 a_{31} + H_3 a_{33}} \\ a_{13} = \dfrac{-4\nu a_{31}a_{32}}{H_1 a_{31} + H_3 a_{33}} \\ a_{21} = 4a_{31}\dfrac{-\nu a_{32}^2 + a_{33}^2}{H_1 a_{31} + H_3 a_{33}} \\ a_{22} = 4a_{32}\dfrac{\nu a_{31}^2 + a_{33}^2(1+\nu)}{H_1 a_{31} + H_3 a_{33}} \\ a_{23} = -4a_{33}\dfrac{a_{31}^2 + a_{32}^2(1+\nu)}{H_1 a_{31} + H_3 a_{33}} \end{cases} \quad (29)$$

Case Where: $a_{31} \neq 0, a_{32} = 0, a_{33} \neq 0$

For our last case $a_{31} \neq 0, a_{32} = 0, a_{33} \neq 0$, with the help of (22), and making $x_3 = a_{31}/a_{33}$, we can find the borders for our solutions are described as following:

$$H_1^2 x^4 + 2H_1H_3x^3 + x^2(H_1^2 + H_3^2 - 16) + 2H_1H_3x + H_3^2 = 0 \quad (30)$$

And the equilibria solutions for this last case are (Solutions III):

$$\begin{bmatrix} 0 & \dfrac{H_1x_3 + H_3}{4x_3} & -\dfrac{H_1x_3 + H_3}{4}a_{12} \\ \pm 1 & 0 & -\dfrac{H_1x_3 + H_3}{4} \\ 0 & 0 & -\dfrac{H_1x_3 + H_3}{4x_3}a_{12} \end{bmatrix}$$

The geometrical meaning of the found equilibrium positions of groups I and III means for group I that the axes OX_3 and Ox_2 are collinear, the angle between axes OX_2 and Ox_1 are determined from the relation $\cos\delta = \pm\dfrac{H_1 x_1}{H_3 + x_1}$, while position of the gyrostatic moment vector in the plane Ox_1x_3 is determined by angle ε for which $\tan\varepsilon = \dfrac{H_3}{H_1}$. For group III the axes OX_1 and Ox_2 are collinear, and the angle between axes OX_3 and Ox_1 is determined from the relation $\cos\delta = \mp\dfrac{(H_1x_3 + H_3)}{4}$, and the position of the gyrostatic moment vector is determined in the same way as for group I. Thus, for all solutions of **group I the angular momentum vector is turned around the radius vector (in yaw angle)**, and for solutions of **group III it is turned relative to the transversal (in roll)**. In both cases at the equilibrium position the gyroscopic moment caused by misalignment of the angular momentum vector and the normal to the orbit plane is balanced by the gravitational moment acting upon the satellite deflected from the orbital trihedral. In both cases, gyroscopic torque due to the no-coincidence of angular momentum and being normalto the orbital plane, the gyrostat is balanced by the gravity-gradient torque. For solutions of group III none of the axes of the body reference frame coincide with any of the axes of the orbital reference frame, position of the gyrostatic moment vector in the plane Ox_1x_3, as for two other groups, is determined from the relation $\tan\varepsilon = \dfrac{H_3}{H_1}$. The illustrated meaning for this explanation is shown bellow.

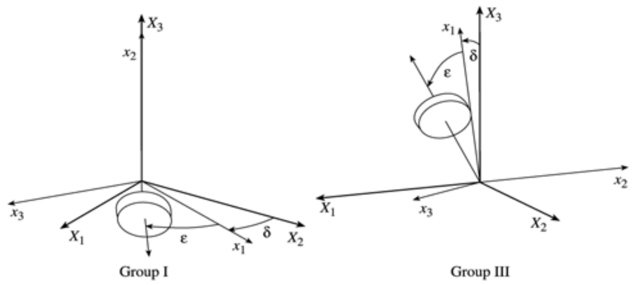


Figure 6 - Geometric interpretation of the equilibrium of groups I and III

The Figure 7 bellow shows the final partitioning of the plane (H_3, H_1) by the border of curves (24), (30) and $a_{32} = 0$, $a_{33} = 0$ into sub-regions, in each of which there are a certain number of equilibrium positions. Notice that curves a_{32} and a_{33} are symmetrical about the coordinate axes, which follows immediately from (26) and (28).

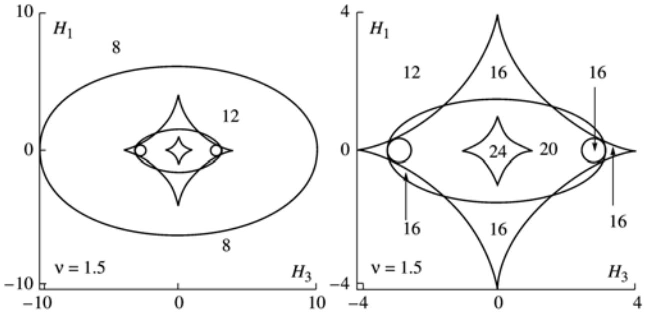


Figure 7 - The regions of existence of the fixed number of equilibrium orientations

3.3- Equilibrium Orientations. Special Case (h1≠0,h2≠0,h3≠0)

Let's start this part of the problem making:

$$x = \frac{a_{31}}{a_{33}},\ \ y = \frac{a_{32}}{a_{33}},\ \ \nu = \frac{B-A}{B-C},\ \ H_i = \frac{h_i}{B-C},\ \ (i = 1, 2, 3)$$

So, after applying the above simplifications, our system (15) becomes:

$$\begin{cases} 16\left[y^2 + x^2(\nu-1)^2 + \nu^2 x^2 y^2\right] = (H_1x + H_2y + H_3)^2(x^2+y^2+1) \\ 4\nu(1-\nu)xy + \left[H_1y + H_2(\nu-1)x - H_3\nu xy\right](H_1x + H_2y + H_3) = 0 \\ a_{33}^2 = \dfrac{1}{x^2+y^2+1} \end{cases} \quad (31)$$

Now let's divide our system (31) into two equations, according to the ones below:

$$\begin{aligned} &a_0y^2 + a_1y + a_2 = 0, \\ &b_0y^4 + b_1y^3 + b_2y^2 + b_3y + b_4 = 0 \end{aligned} \quad (32)$$

And applying the following resultant:

$$R(x) = \begin{vmatrix} a_0 & a_1 & a_2 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & 0 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & 0 \\ 0 & 0 & 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & b_3 & b_4 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 \end{vmatrix} = 0 \quad (33)$$

The result for the determinant (33) is represented as shown below:

$$p_0x^{12} + p_1x^{11} + p_2x^{10} + p_3x^9 + p_4x^8 + p_5x^7 + p_6x^6 + p_7x^5 + p_8x^4 + p_9x^3 +$$
$$p_{10}x^2 + p_{11}x + p_{12} = 0 \quad (34)$$

Being impossible to perform analytically calculations due to the complexity and size of the resultant equations we need to run some numerical simulations.

We need to find the zeros from equation (34). The zeros will correspond to the equilibrium conditions from our gyrostat in this general case.

The number of its real roots in (34) is even and does not exceed 12. Substituting the value of a real root of the algebraic equation (34) into the first two equations of system (31) we can find coinciding root y_1 of these equations. For each solution x_1, y_1 one can determine from the last equation of system (31) two values of a_{33}, and then the values of a_{31} and a_{32} corresponding to them. Thus, each real root of the algebraic equation corresponds to two sets of values of a_{31}, a_{32}, and a_{33} which, by virtue of (13) and (14), uniquely determines the remaining direction cosines a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, and a_{23}. It follows from these considerations that the satellite-gyrostat in a circular orbit may have no more than 24 orientations in the orbital reference frame.

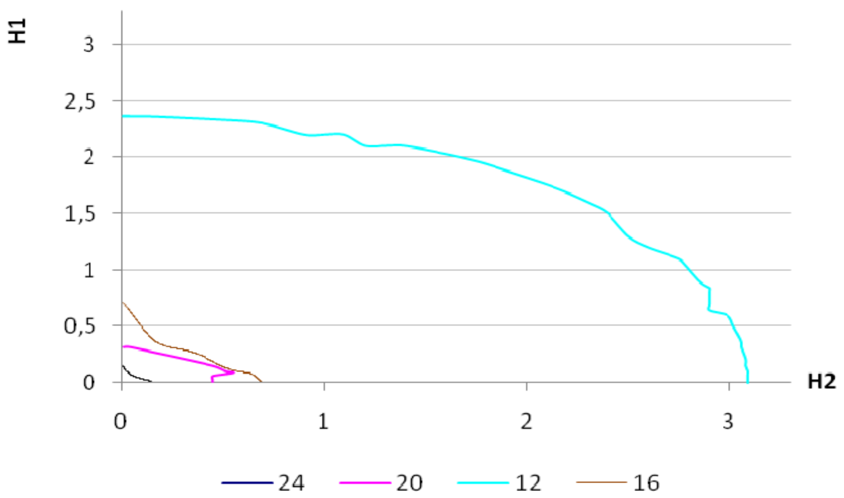


Figure 8 – Equilibrium Regions for H_3=0.25 and ν=0.2

The above picture reflects for this special case when $H_3 = 0.25$ and $\nu = 0.2$ the equilibrium regions for this general case.

And we can easily see that only for relative small values of H_1 and H_2 exists a 24-equilibrium region.

With a more general approach and a more intensive then in our particular case above studied, we can also conclude that besides only for relative small values of H_1 and H_2, its necessary that **exists also a relative small number of ν for the 24 equilibrium regions be present.**

4- Sufficient Conditions of Stability of Equilibria (S-Conditions)

In this subject is convenient to use the Liapunov integral of energy to find our conditions of stability more easily.

$$\begin{vmatrix} A_{\alpha\alpha} & A_{\alpha\beta} & A_{\alpha\gamma} \\ A_{\beta\alpha} & A_{\beta_\beta} & A_{\beta\gamma} \\ A_{\gamma\alpha} & A\gamma_{\beta} & A_{\gamma\gamma} \end{vmatrix} \quad (35)$$

The Liapunov theorem tells us that there is stability in case of Liapunov matrix is positively defined, i.e., the entire square triangular from our matrix must be positive, and so we need to fulfill the following conditions:

$$\begin{cases} A_{\alpha\alpha} > 0 \\ A_{\alpha\alpha}A_{\alpha\beta} - A^2{}_{\alpha\beta} > 0 \\ A_{\alpha\alpha}A_{\beta\beta}A_{\gamma\gamma} + 2A_{\alpha\beta}A_{\beta\gamma}A_{\alpha\gamma} - A_{\alpha\alpha}A^2{}_{\beta\gamma} - A_{\beta\beta}A^2{}_{\alpha\gamma} - A_{\gamma\gamma}A^2{}_{\alpha\beta} > 0 \end{cases} \quad (36)$$

Let's now perform some small variations in the vicinity of $\overline{\alpha}$, $\overline{\beta}$ e $\overline{\gamma}$, where:

$\alpha = \alpha_0 + \overline{\alpha}$, $\beta = \beta_0 + \overline{\beta}$ and $\gamma = \gamma_0 + \overline{\gamma}$.

And expand our direction cosines according the Taylor Series below described:

$$\begin{aligned} a_{ij}(\alpha,\beta,\gamma) &= a_{ij}(\alpha_0+\alpha, \beta_0+\overline{\beta}, \gamma_0+\overline{\gamma}) = a_{ij}(\alpha_0,\beta_0,\gamma_0) + \\ &+ \left(\frac{\partial \overline{a}_{ij}}{\partial \alpha_0}\overline{\alpha} + \frac{\partial \overline{a}_{ij}}{\partial \beta_0}\overline{\beta} + \frac{\partial \overline{a}_{ij}}{\partial \gamma_0}\overline{\gamma}\right) + \frac{1}{2}\left(\frac{\partial^2 \overline{a}_{ij}}{\partial {\alpha_0}^2}\overline{\alpha}^2 + \frac{\partial^2 \overline{a}_{ij}}{\partial {\beta_0}^2}\overline{\beta}^2 + \right. \\ &\left. + \frac{\partial^2 \overline{a}_{ij}}{\partial {\gamma_0}^2}\overline{\gamma}^2 + 2\frac{\partial^2 \overline{a}_{ij}}{\partial \alpha_0 \partial \beta_0}\overline{\alpha}\overline{\beta} + 2\frac{\partial^2 \overline{a}_{ij}}{\partial \alpha_0 \partial \gamma_0}\overline{\alpha}\overline{\gamma} + 2\frac{\partial^2 \overline{a}_{ij}}{\partial \beta_0 \partial \gamma_0}\overline{\beta}\overline{\gamma}\right) \end{aligned} \quad (37)$$

To apply the above Taylor series we need to use the equations (1) in the vicinity of $\overline{\alpha}$, $\overline{\beta}$ e $\overline{\gamma}$.

4.1- S-Conditions for Special Case (h1=h3=0,h2≠0)

Applying (37) on equations (1) and substituting in our Integral of Energy, and applying in each one of our previous studied cases (Case 1 to 6), we reach the following sufficient conditions for each one of our cases:

Case 1

$$\frac{{h_2}^2}{(B-C)} - 16(B-A) > 0$$

$$\frac{{h_2}^2}{(B-C)} - 16(B-C) > 0$$

$$(A-C)\left[4(B-A) - \frac{{h_2}^2}{(B-C)}\right] > 0$$

Case 2

$$\frac{{h_2}^2}{(B-A)} - 16(B-C) > 0$$

$$\frac{{h_2}^2}{(B-A)} - 16(B-A) > 0$$

$$(A-C)\left[\frac{{h_2}^2}{(B-A)} - 4(B-C)\right] > 0$$

Case 3

$$(A-C) > 0$$

$$(B-A) + h_2 a_{22} > 0$$

$$4(B-C) + a_{22} h_2 > 0$$

Case 4

$$-(A-C)>0$$

$$4(B-A)+h_2 a_{22}>0$$

$$(B-C)+h_2 a_{22}>0$$

Case 5

$$(B-A)-\frac{h_2{}^2}{(B-C)}>0$$

$$\frac{h_2{}^2}{(B-C)}-(B-C)>0$$

$$(A-C)\left[\frac{h_2{}^2}{(B-C)}-4(B-A)\right]>0$$

Case 6

$$(B-C)-\frac{h_2{}^2}{(B-A)}>0$$

$$\frac{h_2{}^2}{(B-A)}-(B-A)>0$$

$$(A-C)\left[4(B-C)-\frac{h_2{}^2}{(B-A)}\right]>0$$

Conclusions

Now it is more convenient and simpler to analyze the inequalities from our cases 1 to 6 separately, for the case when we have gyrostats with different correlations between their moments of inertia.

So there are six specific correlations:

1) $A>B>C$

2) $A>C>B$

3) $B>A>C$

4) $B>C>A$

5) $C>A>B$

6) $C>B>A$

For example, if $B>A>C$, then the sufficient conditions for stability in our Case 1, Case 4 and Case 5 are not fulfilled.

Again, for example our Case 3 is stable if:

$$(B-A)+h_2 a_{22}>0$$

$$4(B-C)+a_{22}h_2>0$$

Our Case 2 is stable if $h_2^2>16(B-A)(B-C)$

And case 6 is stable if:

$$(B-A)^2<h_2^2<(B-A)(B-C)$$

However, it is seen from our Case 2 and Case 6 that these solutions exist only if $y^2\le 1$ and $y^2\le 1/16$ respectively. Therefore, the obtained conditions from our Case 2 and Case 6 also will not be fulfilled. So, only the sufficient conditions for stability from our Case 3 and Case 4 for can be fulfilled in this case.

4.2- S-Conditions for Special Case (h1≠0,h2=0,h3≠0)

Applying (37) on equations (1) and substituting in our Integral of Energy, and applying in each one of our previous studied cases (Case 1 to 6) we reach the following sufficient conditions for each one of our cases:

Case 1

$$(A-C)x_1^2+(C-B)>0$$

$$(C-A)(2x_1^2-1)+\frac{h_1^2x_1}{(C-A)x_1+h_3}+h_3x_1>0$$

$$\left[(A-C)x_1^2+(C-B)\right]\left[(C-B)(3+x_1^2)+h_3x_1\right]-\frac{3(C-B)^2h_1^2x_1^2}{\left[(C-A)x_1+h_3\right]^2}>0$$

Case 2

Investigation of the stability of the steady-state solutions on this case 2 turns out to be a much more difficult problem. In this case, one should use relations (26), (27) and (28) in order to determine a_{31}, a_{32}, a_{33} and after that the rest of the elements from the direction cosines matrix.

After that, we can use mathematic software in order to find the steady-state direction cosines resulting from the Liapunov theorem. In this particular case, those solutions are huge. The analysis for this case should be done numerically due to the size of the direction cosines.

Case 3

$$\frac{(C-A)x_3^2}{1+x_3^2}+(B-C)>0$$

$$\frac{(C-A)(h_3-h_1x_3^3)}{(x_3^2+1)(h_1x_3+h_3)}>0$$

$$(B-C)^2(3-x_3^2)+4\frac{(A-C)(B-C)h_3x_3^2}{h_1x_3+h_3}-\frac{(A-C)(B-C)(3-x_3^2)x_3^2}{x_3^2+1}-$$
$$-4\frac{(A-C)^2h_3x_3^4}{(x_3^2+1)(h_1x_3+h_3)}-48\frac{(A-C)^2(B-C)^2x_3^2}{(x_3^2+1)(h_1x_3+h_3)^2}>0$$

Conclusions

In this particular case, three groups of steady-state solutions are determined each of which describes up to eight equilibrium orientations of the gyrostat.

<table>
<tr><th colspan="2">Group I</th><th colspan="3">Group II</th><th colspan="2">Group III</th></tr>
<tr><th>Root eq. 24</th><th>Sign a32</th><th>Root eq. 25</th><th>Sign a32</th><th>Sign a33</th><th>Root eq. 30</th><th>Sign a32</th></tr>
<tr><td rowspan="2">1</td><td>-</td><td rowspan="4">1</td><td>-</td><td>-</td><td rowspan="2">1</td><td>-</td></tr>
<tr><td>+</td><td>-</td><td>+</td><td>+</td></tr>
<tr><td rowspan="2">2</td><td>-</td><td>+</td><td>-</td><td rowspan="2">2</td><td>-</td></tr>
<tr><td>+</td><td>+</td><td>+</td><td>+</td></tr>
<tr><td rowspan="2">3</td><td>-</td><td rowspan="4">2</td><td>-</td><td>-</td><td rowspan="2">3</td><td>-</td></tr>
<tr><td>+</td><td>-</td><td>+</td><td>+</td></tr>
<tr><td rowspan="2">4</td><td>-</td><td>+</td><td>-</td><td rowspan="2">4</td><td>-</td></tr>
<tr><td>+</td><td>+</td><td>+</td><td>+</td></tr>
</table>

To determine exactly these regions we need to solve the equations from our Case 1 to 3 analytically and numerically to be more precise in the limits of those regions.

5 – Main Conclusions

For the case when internal angular momentum of a gyrostat is collinear to one of its principal axes of inertia, the present work found all equilibria positions of a gyrostat in a circular orbit. The sufficient conditions for stability of these equilibria positions were also derived. The proposed method can be successfully applied to solve another problems (equilibria of a satellite under the action of gravitational and aerodynamic torques or equilibria of a system satellite–pendulum).

All the obtained results coincide with the results found in [3].

In this part of the work, the motion of a gyrostat satellite relative to its center of mass has been investigated, the satellite being in a circular orbit under the action of a gravitational moment. In the particular case when $h_1 \neq 0, h_2 = 0, h_3 \neq 0$, three groups of steady-state solutions are determined each of which describes up to eight equilibrium orientations of the gyrostat.

Expressions for direction cosines are presented in the explicit form as functions of parameters H_1, H_3 and ν for all equilibriums positions. The conditions of existence are obtained for these equilibrium positions as functions of dimensionless parameters of the problem.

It seems for the general case of our problem, a gyrostat where $h_1 \neq 0, h_2 \neq 0, h_3 \neq 0$, we get a 12th order equation that describes all the equilibrium positions. Solving this equation precisely takes a lot of resources because requires a great precision.

For the case performed in this paper ($H_3 = 0.25$ and $\nu = 0.2$), we can easily see that **only** for relative small values of H_1 and H_2 exists a 24-equilibrium region.

With a more general approach we can also conclude that besides only for relative small values of H_1 and H_2, its necessary that exists also a relative small number of ν for the 24 equilibrium regions be present.

6 – Future Work

Future work that might be done in this field can pass to increase the precision on our numerical simulations, with this we can plot more precise regions of stability for our system, as well to plot other regions then the ones from Figure 8, in order to get a more general visualization how the equilibrium regions change with different values of H_3 and ν.

Also for the future, is important to consider a situation in an elliptic orbit and also add damping to our gyrostat system.

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