Jogos Não Cooperativos de Localização de Instalações: uma Resenha

Non-Cooperative Facility Location Games: a Survey

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Resumo: O problema de Localização da Instalações é um problema de otimização combinatória NP-Difícil bem conhecido na literatura. Ele modela uma gama de situações onde se pretende fornecer um conjunto de bens ou serviços através de um conjunto de instalações F para um conjunto de clientes T, também denominados terminais. Há custos de abertura para cada instalação em F e custos de conexão para cada par de instalações e clientes (f,t), onde a instalação f atende o cliente t. Uma autoridade central busca determinar uma solução com um custo mínimo, considerando os custos de abertura e conexão, de tal maneira que todos os clientes sejam supridos por uma instalação. Nesta resenha literária, estamos interessados na versão de jogo não cooperativa desse problema, em que, em vez de existir uma autoridade central, cada cliente é um jogador e decide em qual instalação se conectar. Ao escolher a instalação, cada cliente pretende minimizar seus próprios custos, dado pelos custos de conexão e custos de abertura da instalação, que podem ser compartilhados entre os clientes que usam a mesma instalação. Este problema é relevante a várias aplicações práticas, especialmente em cenários distribuídos, onde uma autoridade central é muito dispendiosa ou impossível de existir. Neste artigo, apresentamos uma resenha da literatura existente descrevendo diferentes variantes desse problema e revisando vários resultados sobre eles, além de adaptar os resultados da literatura existente sobre a existência de equilíbrio, Preço da Anarquia e Preço de Estabilidade. Também destacamos alguns problemas em aberto que ainda não foram abordados na literatura.

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Abstract: The Facility Location problem is a well-know NP-Hard combinatorial optimization problem. It models a diverse set of situations where one aims to provide a set of goods or services via a set of facilities F to a set of clients T, also called terminals. There are opening costs for each facility in F and connection costs for each pair of facility and client, if such facility attends this client. A central authority wants to determine the solution with minimum cost, considering both opening and connection costs, in such a way that all clients are attended by one facility. In this survey we are interested in the non-cooperative game version of this problem, where instead of having a central authority, each client is a player and decides where to connect himself. In doing so, he aims to minimize his own costs, given by the connection costs and opening costs of the facility, which may be shared among clients using the same facility. This problem has several applications as well, specially in distributed scenarios where a central authority is too expensive or even infeasible to exist. In this paper we present a survey describing different variants of this problem and reviewing several results about it, as well as adapting results from existing literature concerning the existence of equilibria, Price of Anarchy and Price of Stability. We also point out open problems that remain to be addressed.

Palavras-chave: Teoria de jogos algorítmica, localização de instalações, preço da anarquia, preço da estabilidade.

Keywords: Algorithmic game theory, facility location, price of anarchy, price of stability.

1 Introduction

The facility location class of problems models a large number of important decision problems that may occur in practice, ranging from traditional areas such as economics and urban planning, to more recent ones such as computer networking. This class of problems is concerned with the placement of facilities that will supply some demand of products or services by clients in order to minimize some cost function. This cost function may be defined in different ways, depending on each specific problem. Generally the costs consider several factors such as competitors, distance from clients, and others.

A common version of the facility location problem can be stated as the problem of choosing from a set of facilities F, a subset of facilities to open and to establish a connection with each client from a set of clients T, also called terminals. The opening and connection costs must be minimized. A formal definition is given bellow.

Definition 1 (Uncapacitated Facility Location Problem)**.** Let F be a set of facilities, T a set of terminals, c_f opening costs for each facility $f \in F$ and d_{tf} connection costs for connecting terminal $t \in T$ to facility $f \in F$. The problem is to find a subset of facilities to open and establish connections from terminals to this subset such that the sum of all costs are minimized.

An integer program formulation for this problem is presented below:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{f\in F} c_f y_f + \sum_{f\in F}\sum_{t\in T} d_{tf} x_{tf} & \\
\text{subject to} \quad & \sum_{f\in F} x_{tf} = 1 & , \forall t \in T \\
& y_f \geq x_{tf} & , \forall f \in F, \forall t \in T \\
& y_f, x_{tf} \in \{0,1\} & , \forall f \in F, \forall t \in T \quad ,
\end{aligned}
$$

where y_f is a binary variable that indicates if a facility f is opened, while the binary variable x_{tf} represents whether terminal t is connected to facility f or not.

From this problem, several possible variants may arise. There might be capacities associated with each facility, as well as quotas for each facility. Furthermore, the opening costs of a facility f may not be constant, but depend on the number of terminals connected to f. In another possible variant the facilities can be any point in a metric space, i.e. F is infinite. All these variants mentioned and the original problem are well studied, with most of them being NP-Hard with known approximation algorithms [1, 2].

In all these problems, it is assumed that both terminals (clients) and facilities are controlled by a single central entity seeking to minimize the total cost of the system. However, in several applications, the terminals or the clients may behave differently, for example being controlled by different agents. It is therefore important to analyse these problems from a game theoretic perspective.

In game theory, a *non–cooperative game* is a scenario where players or agents choose strategies independently trying to either minimize their costs or maximize their utility. For each player i there is a set A_i of actions that it can choose to play. A pure strategy S_i consists of one action from A_i, while a mixed strategy corresponds to a probability distribution over A_i. In a pure game each player choses one action to play, while in a mixed game each player randomizes his action according to the probability distribution. In this survey we assume pure strategies games unless mentioned otherwise. A set of strategies $S = (S_1, S_2, \ldots, S_n)$ consisting of one strategy for each player, is denominated a *strategy profile*. Let $\mathcal{S} = A_1 \times A_2 \times \ldots \times A_n$ be the set of all possible strategy profiles and let $c : \mathcal{S} \to \mathbb{R}^n$ be a cost function that attributes a cost $c_i(S)$ for each player i given a strategy profile S. Define $S_{-i} = (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$ a strategy profile S without i's strategy, so that we can write $S = (S_i, S_{-i})$. If all players other than i decide to play S_{-i}, then player i is faced with the problem of determining a best response to S_{-i}. A strategy S_i^* from a player i is a *best response* to S_{-i}, if there is no other strategy which could yield a better outcome for the player, i.e.

$$c_i(S_i^*, S_{-i}) \leq c_i(S_i, S_{-i}) \ \ , \forall S_i \in A_i.$$

A strategy profile is in a *pure Nash equilibrium* (PNE) if no player can increase his utility or reduce his cost by choosing a different strategy, i.e. for each player, its strategy in the strategy profile is a best response.

Game theory can be used to analyze many aspects of decision problems. It may be used to help design games with desired properties, or be used to measure the inefficiency arising from players selfish behaviour. The social welfare or social cost is a function mapping a strategy profile to a real number, indicating a measure of the total cost or payoff of a game. In a facility location game it can be defined as the cost of the solution induced by the agents choice, i.e. the cost of opened facilities plus connections costs. Two of the most important concepts for efficiency analysis are the *Price of Anarchy* (PoA) and the *Price of Stability* (PoS). The PoA is the ratio between a Nash equilibrium with worst possible social cost and the strategy profile with optimal social cost, while the PoS is the ratio between the best possible Nash equilibrium to the social optimum. In the facility location games analysed in this survey, the optimum social cost is the cost of an optimum solution for the correspondent optimization version of the problem.

One variation of a facility location game occurs when clients behave selfishly connecting to facilities opened by a central authority. If the central authority is aware of the exact location or connection costs of each client, then the problem is equal to the one presented in Definition 1. However, when clients may lie to the central authority about their location, there is a need for such authority to design mechanisms encouraging clients to be truthful. There have been several advancements in this area of mechanism design, in particular on *strategy-proof* mechanisms for these games, with seminal papers by Pal and Tardos [3], Devanur et al. [4, 5] and Leonardi and Schäfer [6] as well as complementary works [7, 8, 9].

Another variant of a facility location game considers a cooperative game, where a solution is going to be constructed attending all the players which are the terminals. The problem is how to split the solution cost among all players in such a way that no coalition of players has incentive to leave the grand coalition and form a new solution. This problem was studied by Goemans and Skutella [10] and later in the book Algorithmic Game Theory [11] (Chapter 15), with several related results presented.

These previous versions of facility location games are based on the fact that a central authority is partially present in the problem. Nonetheless, when no authority is dictating where each facility is located, several traditional games may be formed. One possibility is when facilities and clients are controlled by players. The facility players set operating prices for clients, and these last ones behave selfishly, always choosing to connect to the cheapest option available. Games with these premises have been studied by Vetta [12] as valid utility games. Vetta demonstrated that such games always have a pure Nash equilibrium and also showed bounds on the price of anarchy. Later this subject was also covered in the book Algorithmic Game Theory [11] (Chapter 19), and several results for variants of the game are explored in other works [13, 14, 15].

Despite these other versions of facility games the most natural game that arises from facility location problems occurs when players control terminals with the need to connect to a facility. In this case terminals connected to a facility share its opening cost and each player wants to minimize his own total cost. How players share the facility opening costs may vary depending on the specific version of the game being analyzed. When there is no rules on how to share opening costs, some important results have been presented by [16]. However, few direct results have been presented for other variants of this game, with most results being adaptations from other problems such as the network design problem [17]. Therefore, our focus in this work is to study Facility Location games when terminals are controlled by players. We are interested in how much this behaviour may hamper the system cost when compared to the system optimum, both optimistically by considering the Price of Stability of games and pessimistically, with the Price of Anarchy. Furthermore, we adapt and present results from the literature for games where players are not completely selfish in their behaviour.

2 Contributions and Text Organization

In this paper we present a survey on several variants of the facility location game where players control terminals and each player wants to minimize his opening and connection costs. Some of the results were presented in other papers, some of them were presented for related problems and we show how to adapt these results for the facility location. In Section 3, we summarize results for facility location games with no cost sharing rules, mainly from Cardinal and Hoefer [16, 18]. In Section 4, results from network design [17] are adapted to fair cost facility location games, and a short compilation of results for the weighted version of the game is presented. In Section 5, capacity restrictions are added to the previously analyzed games and pure Nash equilibria existence and bounds for PoA and PoS are summarized from previous work from Rodrigues and Xavier [19]. Altruism in facility location games is explored in Section 6, where we adapted results for the cost sharing game [20] to facility location games. In each of these sections we discuss the case where instances of the game satisfies the triangle inequality as well. Conclusions and open problems are presented in Section 7.

3 Facility Location without cost sharing rules

This facility location game can model several practical scenarios. Imagine a situation where some groups are interested in constructing public goods, such as libraries or museums. There is no defined rule on how these groups share the construction costs, and opened facilities do not have ties to the groups which helped build them, being available to anyone willing to use them. These scenarios may be modeled using a game where players controls terminals and need to connect to an opened facility, there being no opening cost sharing rule.

Definition 2 (Facility Location Game without Cost Sharing Rules (FLG))**.** Let $(G = (T \cup F, T \times F), k, c, d)$ be an instance of the FLG, where G is a bipartite graph with vertex sets F of n facilities and T of m terminals, k is the number of players and c and d are opening and connection costs, respectively.

Each facility $f \in F$ has an opening cost c_f, and connection costs d_{tf} for each terminal $t \in T$. In games with general distance costs, some connections (t, f) should be avoided in any solution, because they don't exist for example. In this case we assume they have a prohibitively large constant cost $\mathcal{U}_d$. When a connection is not shown, it is assumed that it has a cost equal to $\mathcal{U}_d$, unless mentioned otherwise.

Each player $i \in [1, \ldots, k]$ controls a subset of terminals $T_i \subseteq T$. These subsets form a partition of T, i.e, each terminal from T is controlled by some player and $T_i \cap T_j = \emptyset$ for $i, j \in [1, \ldots, k]$ with $i \neq j$. Each terminal must be connected to exactly one opened facility. Each possible strategy S_i of player i, is composed of a payment function $p_i^c : F \rightarrow \mathbb{R}_0^+$ indicating how much he offers for opening a facility, as well as a function $p_i^d : T \times F \rightarrow \mathbb{R}_0^+$ which indicates how much he pays for the connection costs.

Let $p^c(f) = \sum_{i=1}^{k} p_i^c(f)$ be the total paid by players for a facility f. If $p^c(f)$ is greater than or equal to the cost c_f, then the facility f is considered opened. Likewise, if the total offered for connection cost of a terminal-facility pair t, f is greater than or equal to d_{tf} then the connection is bought. Each player tries to minimize their payments while ensuring that the terminals they control are connected to an opened facility.

Solutions where terminals do not pay enough to open the facility they are connected to should be avoided. To avoid such solutions we add a prohibitively large constant cost $\mathcal{U}_c$ to the payment of terminals in such situations. For a player i, if there is a connection $(t, f) \in S_i$ where $p^c(f) < c_f$, a prohibitively large constant cost $\mathcal{U}_c > \mathcal{U}_d$ is added to the total amount paid by i, i.e, he pays $p_i(S) + \mathcal{U}_c$.

Note that there is no rule on how players share the costs to open a facility. Therefore, how players share these costs may depend on which player need the facility the most and in how the equilibrium is reached. Consider the game in Figure 1, with two players, each controlling one terminal. One possible strategy is t_1 offers 1 to f_1, while t_2 offers 1 to f_3.

There is no incentive to any player to change their payment scheme, and thus they are in an equilibrium. Suppose now that t_1 chooses instead, as his payment function, to offer 0.75 to facility f_2 and zero to the others, while t_2 chooses to only pay 1 to f_3. The player controlling t_2 then has an incentive to change his strategy to pay 0.75 to open f_2. This strategy profile is an equilibrium in which both players share equally the opening costs of f_2. However, if t_1 had offered only $0.5+\varepsilon$ to f_2, t_2 would still pay less by offering to pay the remaining opening cost of f_2. In fact, there is an infinite number of possible equilibria in this example, since a player may offer to pay for the opening costs of f_2 any amount in the interval $(0.5, 1.0]$ and the other player will, in an equilibrium, complete the offer to open f_2.

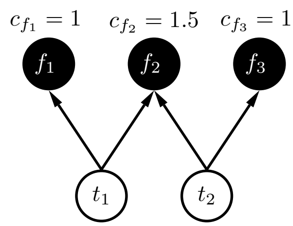

Figura 1. A example of a Facility Location Game. Connection costs are constant.

In [16], Cardinal and Hoefer analyzed a class of covering games which includes FLG and answer a few fundamental questions about it. They proved that there may be instances of FLG with no PNE, also showing that it is NP-Complete to determine whether an instance of FLG has an equilibrium or not. Furthermore, they provided bounds on the Price of Anarchy and Stability for the FLG for the instances that admit equilibria. They also presented approximation algorithms to find an approximated equilibrium based on a well known primal-dual algorithm for the facility location problem. We shortly summarize some of these results below.

Theorem 1 (Pure Nash Equilibrium existence for FLG [16])**.** *There are instances of the FLG where there is no PNE. Moreover it is NP-hard to decide if an instance of the FLG admits a PNE or not*

Demonstração. Consider the instance of FLG showed in Figure 2. Player A controls terminal t_1, while player B controls terminals t_2 and t_3. For all edges shown the connection costs are zero, and infinite otherwise. Suppose facility f_1 is opened. Either A or B paid completely for it, or they shared the costs in some manner. If B paid fully, player A do not need to pay anything to fulfil his constraints, and B would need to pay for either f_2 or f_3 to attend terminal t_2. In this case, B would pay less by not opening f_1 and instead only paying for f_3 for a total payment of $1 + \varepsilon$. However, player A would then need to pay fully for f_1 or f_2, which would make B chooses to use both f_1 and f_2, foregoing f_3 since it would only need to open one facility with total cost equal to 1. Player A would be free in this scenario to not pay for any facility, choosing to connect to the one B opened, and completing a best response cycle. The same occurs when they initially share the costs of f_1, since B would either choose to open fully f_3 or f_2, entering in the same best response cycle.

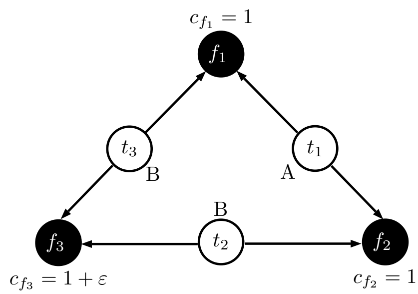

Figura 2. A game instance for the FLG with no PNE, first described in [16]. Letters next to terminals indicate which player controls the terminal.

To determine whether the FLG game does have or not a PNE is NP-Hard. As detailed in [16], the 3-SAT problem [21] can be reduced to this problem. □

When restricted to instances of the game that admits PNE, Cardinal and Hoefer [16] show that the price of anarchy of FLG is k, the number of players. The social cost $C(S)$ for a strategy profile S of an instance of FLG is defined as the sum of all payments made by the k players, i.e.

$$C(S) = \sum_{i=1}^{k} \sum_{f \in F} p_i^c(f) + \sum_{i=1}^{k} \sum_{(t,f) \in T \times F} p_i^d(t, f) \ .$$

Theorem 2 (Price of Anarchy of FLG [16])**.** *The price of anarchy for any FLG instance that admits PNE is at most k, and there is an instance of FLG with price of anarchy of at least k.*

Demonstração. Suppose that there is an equilibrium S which is more than k times the cost of a strategy profile S^* with optimal social cost. Then, at least one player in S is paying more than $C(S^*)$ to cover his terminals. This player could then simply offer the optimal solution S^* as his own payment scheme, and therefore reach a better solution, which means that S is not an equilibrium. Thus, PoA $\leq k$.

Now consider the game of Figure 3, where each terminal is controlled by a different player, and two facilities, f_1 with opening cost 1, and f_2 with opening cost k, with no connection costs. The optimal solution is clearly to connect each terminal to f_1 with total cost 1.

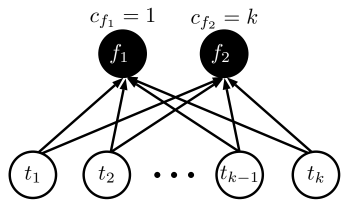

Figura 3. A game instance of the FLG with PoA equal to k.

However, the strategy profile where each player is connected to f_2 paying 1 to open it, is an equilibrium with total cost of k. Therefore, the Price of Anarchy of FLG is k. ☐

By exploiting the fact that there are instances with no equilibrium, Cardinal and Hoefer showed an instance of the game where the Price of Stability is close to the Price of Anarchy.

Theorem 3 (Price of Stability of FLG [16])**.** *There is an instance of the FLG with price of stability of at least $k-2$.*

Demonstração. Consider the game in Figure 4, where player 1 controls terminal t_1, player 2 controls terminals t_2 and t_3, and player $i \in [3, k]$ controls terminal t_{i+1}. Each player i from 3 to k can connect to the center facility f_{k+1} with connection cost ε and opening cost 1, as well as their "leaf" facility f_i with connection cost 1 and opening cost ε. Note that the instance induced by players 1 and 2 is very similar to the one in Figure 2, and also does not have by itself an equilibrium. Clearly, the optimal solution is the one where players 1 and 2 connect to f_1 and f_2, while the remaining players all connect to f_{k+1}, with total cost $1 + (k-2)\varepsilon + 5\varepsilon = 1 + (k+3)\varepsilon$.

If any player chooses to open the center facility f_{k+1}, all other players will eventually connect to the center as well, with the exception of players 1 and 2 which would never reach an equilibrium (see Theorem 1). So in order to exist an equilibrium, player 3 must connect to f_3 paying some value in $[\varepsilon/2, \varepsilon]$ of the facility cost (say $\varepsilon/2$). Then player 2 pays the remaining cost, say ε, connecting both of its terminals to f_3. It is easy to check that this is a best equilibrium. Therefore each player $i \in [4, \dots, k]$ fully pays for facility f_i, each paying a connection cost of 1 and opening cost of ε. Player 3 pays a connection cost of 1 to f_3 and $\varepsilon/2$ of its opening cost. Finally, player 1 will pay 2ε for opening and connecting its terminal to either f_1 or f_2, and player 2 will pay 3ε including connection costs to f_3 and the remaining opening cost. The total cost for this equilibrium is $(k-3)(1+\varepsilon) + 1 + 5.5\varepsilon = (k-2)(1+\varepsilon) + 4.5\varepsilon$. Therefore, when ε tends to 0, the price of stability of this instance tends to $k-2$.

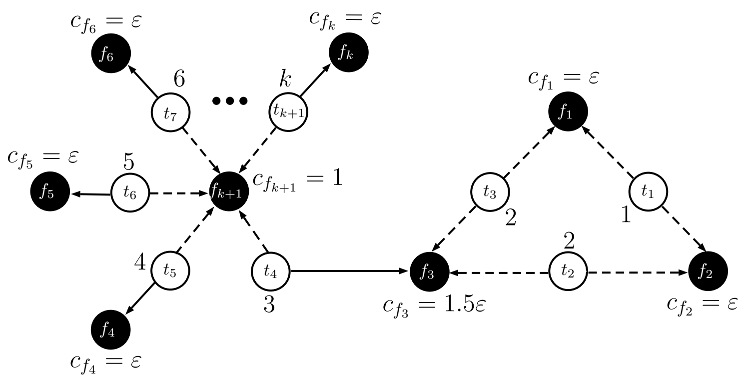

Figura 4. A game instance of the FLG with PoS of $k-2$, first described in [16]. Dashed lines have connection cost ε, while full lines have cost 1. Numbers next to terminals indicate which player controls the terminal.

□

3.1 The Metric FLG

For the metric version of the FLG, where connection costs satisfies the triangle inequality, all results of Theorems 1, 2 and 3 still hold. All instances used in these Theorems can be safely modified to have the additional required connections respecting the triangle inequality without changing the optimal solutions and the possible pure equilibria.

Theorem 4 (Pure Nash Equilibrium existence for the Metric FLG [16])**.** *There are instances of the Metric FLG where there is no PNE.*

Demonstração. Consider the instance of Figure 2 adding the following edge costs: each drawn edge has cost equal to 1 and each not drawn edge between terminals and facilities has cost equal to 3. The same arguments of Theorem 1 works for this metric instance. □

Theorem 5 (Price of Anarchy of the Metric FLG)**.** *The price of anarchy for any Metric FLG instance that admits PNE is at most k, and there is an instance of the Metric FLG with price of anarchy of at least k.*

Demonstração. We can use the same arguments of Theorem 2 using the instance of Figure 3 with a constant cost c to each edge. □

Theorem 6 (Price of Stability of the Metric FLG [16])**.** *There is an instance of the Metric FLG with price of stability of at least* $k - 2$*.*

Demonstração. Consider the instance of Figure 4 by completing the graph with edges between each pair of vertices with cost equal to the cost of the shortest path between them. We can use the same arguments of Theorem 3 because the cost of the new connections between facilities and terminals are higher than the ones presented in the Figure. □

Note that both the Price of Anarchy and the Price of Stability have similar values for this class of games, indicating a large gap between equilibria and social optima. However, in all theorems seen so far, the fact that players do not have a clear way to share facility opening costs plays a major role in making such big differences between the optimal welfare and pure equilibria. If global sharing rules for costs are considered, this gap, and the undesirable fact of the nonexistence of pure equilibria in some games may change as we will see in Section 4.

4 Facility Location with Fair Cost Sharing

In this section, we consider facility location games where, instead of players freely coordinating on how to share facilities' opening costs, they are forced to equally share the costs for each facility they want to open. The game is defined in a similar way as in Section 3, since the only change is in how the players share the facilities opening costs.

Definition 3 (Facility Location Game with Fair Cost sharing (FLG-FC))**.** Let $G = (T \cup F, T \times F)$ be a bipartite graph, with vertex sets F of n facilities and T of m terminals. Each facility $f \in F$ has an opening cost c_f, and connection costs d_{tf} for each terminal $t \in T$. Let $K = [1, \ldots, k]$ be the set of players. Each player i controls a subset of terminals $T_i \subseteq T$ (also forming a partition of T), and each terminal must be connected to exactly one opened facility. A player i chooses a strategy $S_i \subset T_i \times F$.

Let $S = (S_1, ..., S_k)$ be a strategy profile and $U(S) = \bigcup_{i \in K} S_i$ be the set of all strategies chosen by players in S. We use the expression $f \in U(S)$ to represent all facilities connected to a terminal in a strategy profile S, while $f \in S_i$ represents all facilities player i uses to connect its terminals in strategy S_i. Each player tries to minimize his own payment

$$p_i(S) = \sum_{f \in S_i} \frac{c_f}{x_f(S)} + \sum_{(t,f) \in S_i} d_{tf} \ ,$$

where $x_f(S) = |\{1 \leq i \leq k : f \in S_i\}|$ is the number of players using facility f in strategy profile S.

The *social welfare cost* for a strategy S is defined as the sum of all player payments, i.e.

$$C(S) = \sum_{i \in K} p_i(S) = \sum_{f \in U(S)} c_f + \sum_{(t,f) \in U(S)} d_{tf} \; .$$

This game can be seen as a specialization of Network Design, first defined and explored in [17]. In the Network Design Game, it is given a graph $G = (V, E)$, where each player i has a set of terminal nodes T_i which he needs to connect, and his strategy is a set of edges $S_i \subset E$ which must form a tree connecting all nodes in T_i. Each edge e has an opening cost c_e associated with it, and players who use this edge share its cost equally.

Theorem 7 (PNE existence for FLG-FC)**.** *Every instance of the FLG-FC game admits a PNE.*

Demonstração. For the network design game, it was proved in [17] that the game always have a PNE, since it is a potential game with potential function

$$\Phi(S) = \sum_{e \in E} \sum_{x=1}^{x_e(S)} \frac{c_e}{x} \tag{1}$$

where x_e is the number of players which have the edge e in the strategy profile S.

A similar proof of the existence of PNE is possible for the Facility Location with Fair Cost sharing. In this case the game admits the potential function

$$\Phi(S) = \sum_{f \in U(S)} \sum_{x=1}^{x_f(S)} \frac{c_f}{x} + \sum_{(t,f) \in U(S)} d_{tf} \tag{2}$$

where $(t, f) \in U(S)$ is a pair terminal–facility used in the strategy profile S, and $x_f(S)$ is the number of players sharing facility f in S. Since FLG admits this potential function and is therefore a potential game, it must always posses a PNE. □

We can use similar arguments to the ones used in Theorem 2 to show that the Price of Anarchy of the FLG-FC is equal to k, the number of players.

Theorem 8 (Price of Anarchy for FLG-FC)**.** *The Price of Anarchy of the FLG-FC is k.*

Demonstração. Let S be a PNE of an instance of the FLG-FC, and S^* an optimal social cost solution. Then,

$$C(S) = \sum_{i \in K} p_i(S_i, S_{-i}) \leq \sum_{i \in K} p_i(S_i^*, S_{-i}) \tag{3}$$

is true since S is a PNE. Now for player i, we have that

$$\begin{aligned}
p_i(S_i^*, S_{-i}) &= \sum_{f \in S_i^*} \frac{c_f}{x_f(S_i^*, S_{-i})} + \sum_{(t,f) \in S_i^*} d_{tf} \\
&\leq \sum_{f \in S_i^*} k \frac{c_f}{x_f(S^*)} + \sum_{(t,f) \in S_i^*} d_{tf} \\
&\leq k \left(\sum_{f \in S_i^*} \frac{c_f}{x_f(S^*)} + \sum_{(t,f) \in S_i^*} d_{tf} \right) \\
&= k \cdot p_i(S^*) \; .
\end{aligned}$$

This must hold since $x_f(S_i^*, S_{-i})$ is at least one for any connected facility f in S_i^*, while $x_f(S^*)$ can be at most k for a facility f in the social optimum S^*. Applying this to the inequality in (3), we obtain a bound of k for the PoA for the FLG-FC:

$$\begin{aligned}
C(S) &\leq \sum_{i \in K} p_i(S_i^*, S_{-i}) = \sum_{i \in K} \sum_{f \in S_i^*} \frac{c_f}{x_f(S_i^*, S_{-i})} + \sum_{(t,f) \in S_i^*} d_{tf} \\
&\leq \sum_{i \in K} k \cdot p_i(S^*) \\
&= k \cdot C(S^*) \; .
\end{aligned}$$

To see that this bound is tight, refer to the instance used in Theorem 2 for FLG, shown in Figure 3. Since all players share evenly the cost for facility f_2 in the worst PNE for FLG, this equilibrium still occurs in FLG-FC, and therefore the PoA for this instance is also at least k for FLG-FC.

□

As for the Price of Stability, Anshelevich et. al [17] proved that for the network design game there is an upper bound of $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$. We can do a similar proof for FLG-FC obtaining the same bound and we can also show that this bound is tight.

Theorem 9 (Price of Stability for FLG-FC (Adapted from [17]))**.** *Consider a facility location game with fair cost sharing* (FLG-FC) *with nondecreasing concave opening cost c_f for each facility f and connection costs d_{tf} for pairs of terminal-facility (t, f). Then the Price of Stability is H_k, where k is the number of players of the game .*

Demonstração. Let $\Phi(S)$ be the potential function defined in Equation 2. Let S^* be a strategy with optimal social cost

$$C(S^*) = \sum_{f \in U(S^*)} c_f + \sum_{(t,f) \in U(S^*)} d_{tf} \ .$$

Then, $\Phi(S^*) \leq H_k C(S^*)$, since

$$\begin{aligned}
\Phi(S^*) &= \sum_{f \in U(S^*)} \sum_{x=1}^{x_f^*(S)} \frac{c_f}{x} + \sum_{(t,f) \in U(S^*)} d_{tf} \\
&\leq \sum_{f \in U(S^*)} c_f H_k + \sum_{(t,f) \in U(S^*)} d_{tf} \\
&\leq H_k \left(\sum_{f \in U(S^*)} c_f + \sum_{(t,f) \in U(S^*)} d_{tf} \right) \\
&\leq H_k C(S^*).
\end{aligned}$$

Since FLG-FC is a potential game, we can start the game using the strategy profile S^* and let each player chooses a best response strategy in a series of rounds. After a finite number of rounds the game will reach a PNE S with $\Phi(S) \leq \Phi(S^*)$. For any strategy profile S', $\Phi(S') \geq C(S')$ and therefore

$$C(S) \leq \Phi(S) \leq \Phi(S^*) \leq H_k C(S^*) \ .$$

Reordering the inequality, we obtain that

$$\text{PoS} \leq \frac{C(S)}{C(S^*)} \leq H_k \ .$$

To see that this bound is tight, consider the example in Figure 5. In this game with $n = k + 1$ facilities and $m = k$ terminals each player $i \leq k$ controls terminal t_i. Clearly the solution with optimal social welfare is the one where all players open facility f_{k+1} with a total cost of $1 + \varepsilon$. However, this strategy profile is not an equilibrium, since player k would be able to pay less by opening facility f_k. This change in strategy from player k consequently would cause player $k - 1$ to also change his strategy to open facility f_{k-1}, which ultimately would cause all players to choose to not open f_{k+1}, resulting in an equilibrium of total cost $1 + \frac{1}{2} + ... + \frac{1}{k} = H_k$. This strategy profile is the only possible equilibrium, since every terminal must connect to a facility and there is no equilibrium in which facility f_{k+1} is open.

□

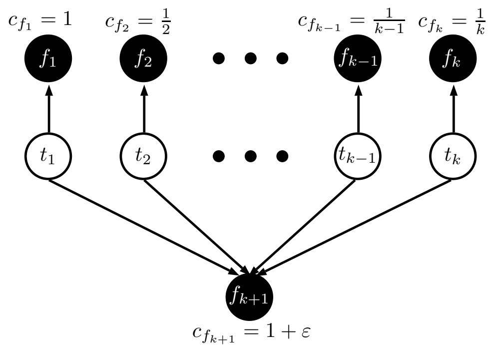

Figura 5. Game instance of the FLG-FC with PoS of H_k. All edges have cost equal to zero.

4.1 The Metric FLG-FC

For the metric version of FLG-FC we can use the same potential function of Theorem 7 since no assumption is made about the connection costs. So every instance of the metric FLG-FC also admits a PNE.

Theorem 10 (PNE existence for the Metric FLG-FC)**.** *Every instance of the Metric FLG-FC game admits a PNE.*

The PoA bound is still the number of players k.

Theorem 11 (Price of Anarchy for the Metric FLG-FC)**.** *The Price of Anarchy of the Metric FLG-FC is* k*.*

Demonstração. The same arguments of Theorem 8 works by considering the instance of Figure 3 with edges cost equal to some constant $c > 0$. □

The upper bound of H_k on the PoS proved in Theorem 9 still works for the metric case since it is derived from the potential function. On the other hand, the lower bound obtained from the instance of Figure 5 does not work anymore. By adding edges respecting the triangle inequality for that instance both the optimal solution and equilibria change since now all terminals can connect to any facility. In fact, Hansen and Telelis in [22] proved a constant upper bound of 2.36 for the PoS of the metric FLG-FC, and also proving a lower bound of 1.45.

Theorem 12 (Price of Stability for the Metric FLG-FC [22])**.** *The Price of Stability of the Metric FLG-FC is at most* 2.36 *and at least* 1.45*.*

4.2 Weighted Players

Suppose a Facility Location game where players have different demands. Suppose player i demands w_i units of some good while player j demands w_j. When sharing the cost of a common facility, this cost should be divided considering these demands.

The facility location game with fair cost sharing can be extended for cases where players may pay a larger or smaller fraction of opening costs for facilities. This is accomplished by changing the cost calculation function by adding weights for each player. Now each player i pays in a strategy profile S,

$$p_i(S) = \sum_{(t,f) \in S_i} d_{tf} + \sum_{f \in S_i} w_i \frac{c_f}{W_{f,S}} \ ,$$

where $W_{f,S}$ is the sum of the weights of all players using f in the strategy profile S.

This extension has been studied by Hansen and Telelis [23, 22]. They prove that e-approximate equilibria exists, i.e. there is a strategy profile S where each player cannot improve by more than a factor e from what he is paying in S. Furthermore, a bound for both the PoA and PoS of $\Theta(\log W)$ is shown, where W is the sum of all player weights.

The Network Design Game with weights was also explored in the literature, particularly by Chen and Roughgarden in [24]. In this paper, the authors proved that this variant do not always have a Pure Nash Equilibrium. However, this proof does not translate directly to the facility location game with weighted players. It is currently an open problem if this version of the Facility Location game always have a Pure Nash Equilibrium. The proof of Theorem 7 using a potential function do not apply to this variant, since the addition of weights for FLG-FC turns that function not potential.

5 Facility Location Games with Capacities

It is not always possible for a facility to provide goods for an unlimited number of terminals. Therefore, extended versions of facility location games where the facilities have limited capacities are also of interest. In the most natural extension, every facility f has a capacity u_f associated with it, which indicates how many terminals can be connected to this facility.

This kind of capacitated game has been recently studied by Rodrigues and Xavier [19], where the authors proved several results concerning both the existence of equilibria and its quality.

For capacitated games with no cost sharing rules, it can be shown that even when all players are singletons, i.e. each player controls only one terminal, there are still games with no equilibria [19], as is exemplified in Figure 6.

Theorem 13 (PNE for the Capacitated FLG [19])**.** *There are instances of the Capacitated FLG that do not admit PNE, even when restricted to singleton players.*

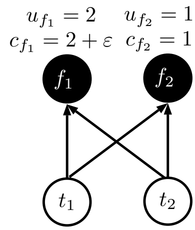

Figura 6. Game instance of the Capacitated FLG without a PNE. Connection costs have a constant value.

Furthermore, Rodrigues and Xavier [19] proved that even when all players are singletons, deciding whether an instance of capacitated FLG has a PNE or not is NP-hard.

Theorem 14 (PNE existence is NP-hard [19])**.** *It is NP-hard to determine if an instance of the metric capacitated FLG has a PNE or not. When restricted to instances with only singleton players, it is NP-complete to determine if an instance of the metric capacitated FLG has a PNE.*

Now consider the game with fair cost sharing. The same potential function of Theorem 7 can be used for the capacitated FLG-FC game, so it always admits PNE.

Theorem 15 (PNE for the Capacitated FLG-FC)**.** *All instances of the Capacitated FLG-FC admit PNE.*

All examples seen in previous Sections can be modified to have capacities, by setting the capacity of each facility equal to the number of terminals. Thus, the price of anarchy is at least the number of players k for this new game. However, with capacities it can be shown that the PoA is unbounded, even in the case of fair cost sharing. As Figure 7 shows, there are instances with unbounded equilibria.

Theorem 16 (Price of Anarchy for Capacitated Facility Location [19])**.** *The PoA for the Capacitated FLG-FC is unbounded. For the Capacitated FLG, there are instances that admits a PNE but whose PoA is unbounded.*

In Figure 7 if t_2 connects to f_2 then t_1 is forced to connect to f_1 paying an unbounded opening cost. Note that the instance in Figure 7 works for capacitated FLG and capacitated FLG-FC as well, since each facility can attend at most one terminal.

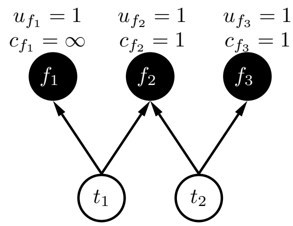

Figura 7. Game instance of the FLG and FLG-FC with unbounded Price of Anarchy.

As for the price of stability, for games with fair cost sharing it remains a potential game with the same potential function. Thus, Theorem 9 still applies, leading to a H_k bound for the PoS.

Theorem 17 (Price of Stability for Capacitated FLG-FC [19])**.** *The PoS for the Capacitated FLG-FC is H_k, where k is the number of players of the game.*

On the other hand for games with no cost sharing rules, Rodrigues and Xavier [19] proved that there are games with unbounded price of stability.

Theorem 18 (Price of Stability for Capacitated FLG [19])**.** *There are instances of the Capacitated FLG that admits PNE but have unbounded PoS.*

Instead of dealing directly with capacities, there is also a possibility of a softer approach. For example we can consider a game where the player cost function has an additional cost that increases with the number terminals using a same facility. In this variant, the cost of a terminal t connecting to a facility f is defined as $\frac{c_f}{x_f} + g(x_f) + d_{tf}$, where x_f is the number of terminals connected to f.

If the added cost function $g(x)$ is monotone increasing, and the opening function cost $c_f(x)$ is monotone increasing and concave then Anshelevich et. al [17] proved for the network design game, that the PoS is bounded by $A \times H_k$. The parameter A depends on the type of the function g. For functions with polynomial degree at most l, this term is equal to $l + 1$. This bound extends for the facility location game with fair cost sharing and with the additional $g(x)$ costs. This occurs because the proof for such bound is based on the potential function of the network design game, and the distance costs d_{tf} that appear in the facility location game do not interfere in the proof presented in [17].

5.1 The Metric Version of Capacitated FL Games

A common restriction for location games is to require all connections to obey the triangle inequality. This restriction do not affect the proof of instances with no PNE in Theorem 13, since the instance used there is metric.

Theorem 19 (PNE for the Metric Capacitated FLG [19])**.** *There are instances of the Metric Capacitated FLG that do not admit PNE, even when restricted to singleton players.*

For the Metric Capacitated FLG-FC, the same potential function of Theorem 7 can be used to show the existence of PNE.

Theorem 20 (PNE for the Metric Capacitated FLG-FC)**.** *All instances of the Metric Capacitated FLG-FC admit PNE.*

On the other hand, the restriction that instances are metric invalidates the example from Figure 7. Nonetheless, there are other more contrived instances that respect the triangle inequality and yet has unbounded PoA [19].

Theorem 21 (Price of Anarchy for Metric Capacitated FLG and FLG-FC [19])**.** *The PoA for the Metric Capacitated FLG-FC is unbounded. For the Metric Capacitated FLG, there are instances that admit PNE but have PoA unbounded.*

These instances explore the fact that players move simultaneously when represented in normal form. When sequentiality [25] is considered, Rodrigues and Xavier [19] prove that the PoA for Metric Capacitated FLG has a bound of $\Theta(2^k)$.

For facility location games, and specially for the capacitated version, it is useful to consider what changes when these games are sequential [25]. To analyze these scenarios, the concept of *Subgame Perfect Equilibrium* (SPE) is used instead of PNE. When represented in the extensive form, games use a tree where each node represents a player and edges represent possible strategies from players. SPE is defined as a strategy profile which is a PNE in every subgame of this tree, so a SPE is also a PNE for the entire game. The *Sequential Price of Anarchy* (SPoA) is defined as the ratio between the cost of the worst subgame perfect equilibrium and the optimal social cost, while the *Sequential Price of Stability* (SPoS) is the ratio between the best SPE and the optimal social cost.

With these concepts in mind, it can be shown that the SPoA for Metric CFLG and CFLG-FC is in $\vartheta(2^k)$.

Theorem 22 (SPoA for Metric CFLG and Metric FLG-FC [19])**.** *Consider an instance of the Metric CFLG-FC and Metric CFLG game with k players where*

- *each player i controls one terminal t_i,*
- *players play in order $1, \ldots, k$, where player i knows every action taken by players $1, \ldots, i-1$,*
- *S is the sub-game perfect equilibrium reached and S^* is a solution with optimum social cost.*

Then the SPoA $\leq 2^k$ and this bound is tight.

To have some intuition on why this is the case, consider instance shown in Figure 8. Allow each player to control a single terminal. Suppose terminals play in order $t_1, \ldots, t_k$. The social optimal clearly happens when any terminal t_i connects to f_i, resulting in a social cost of one. However, there exists another equilibrium where each terminal t_i connects to f_{i+1}, since for t_1 both f_1 and f_2 have the same cost. Such PNE has cost $2^0 + 2^1 + \ldots + 2^{k-1} = 2^k$, and therefore the SPoA is 2^k.

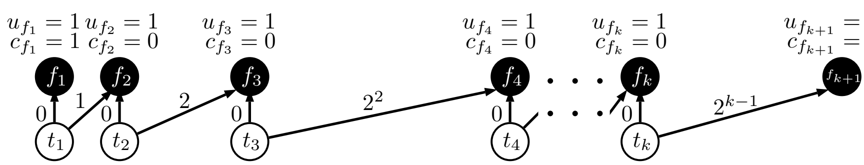

Figura 8. Metric Capacitated FLG instance with SPoA equal to 2^k [19]. Connections not shown have cost equal to the shortest path, and each facility has unitary capacity restrictions.

As for the Price of Stability, for fair cost games Rodrigues and Xavier [19] adapt the instance depicted in Figure 5 to respect the triangle inequality by allowing terminals to connect to any facility at no cost, but restricting facilities f_1 to f_k to connect to only one terminal, as shown in Figure 9.

Theorem 23 (PoS for Metric CFLG-FC [19])**.** *For the Metric CFLG-FC the Price of Stability is H_k, where k is the number of players of the game.*

For the Capacitated FLG, it can be shown by a similar process as used in Theorem 3 that the PoS is $\Omega(2^k)$, as is depicted in Figure 10. Let each player control a single terminal. In order for a PNE to be possible, terminal f_a must have at least some of its cost paid by terminal t_k. Therefore, there needs to be another terminal t_{k+1} blocking f_k, which will only happen when t_1 connects to f_2, terminal t_2 to f_3 and so on.

Theorem 24 (PoS for the Metric CFLG [19])**.** *There are instances of the Metric Capacitated FLG that admit a PNE and have a PoS in $\Omega(2^k)$.*

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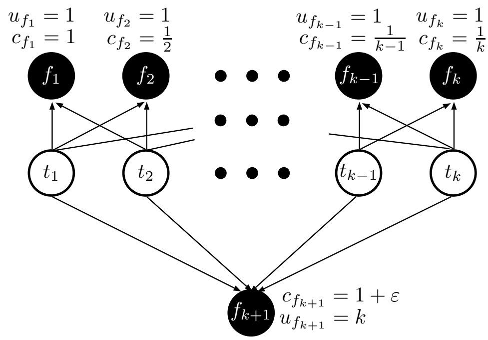

Figura 9. Game instance of the Metric CFLG-FC with PoS of H_k [19]. Each connection has zero cost and the graph is complete. Facilities $f_1, \dots, f_k$ have unitary capacities, while f_{k+1} has capacity k.

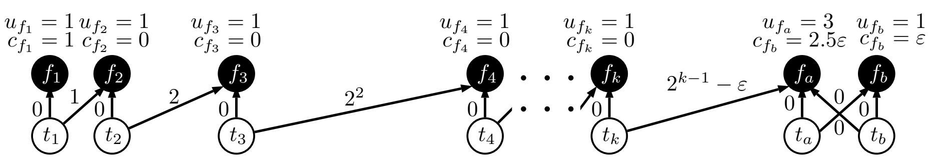

Figura 10. Metric Capacitated FLG instance with PoS equal to $\Theta(2^k)$ [19].

6 Altruism in Facility Location Games

All analysis seen so far for facility location games assume that players are completely selfish. However, this assumption does not always reflect what happens in practice. Players' behavior in practice may be at least partially altruistic [26, 27], indicating a need to incorporate this alternate behavior for games modeling real world scenarios. In light of this, in recent years there has been increasing interest in the study of alternate models on how players behave. A model for altruistic behavior is presented by Chen et al. [20]. It changes how players perceive utility by adding a α_i parameter for each player i indicating how selfless a player behaves.

Definition 4 (Altruism)**.** Let $G = (K, \mathcal{S}, (p_i)_{i \in K})$ be a cost minimization game, where $K = [k]$ is the set of k players, $\mathcal{S} = A_1 \times \ldots \times A_k$ is the set of all possible strategy profiles where A_i is the set of actions (set of pure strategies) of player i. The payment function $p_i : \mathcal{S} \to \mathbb{R}$ defines the cost of a strategy profile for player i.

The *α-altruistic extension* of G, for $\alpha \in [0, 1]^k$, is defined as the game $G^\alpha =$

$(K, \mathcal{S}, (p_i^\alpha)_{i\in K})$, where for every player i and strategy profile $S \in \mathcal{S}$,

$$p_i^\alpha(S) = (1 - \alpha_i)p_i(S) + \alpha_i C(S) \ , \tag{4}$$

where $C : \mathcal{S} \to \mathbb{R}$ is a function defining the social cost of strategy profile S. This function must satisfy the property that for any $S \in \mathcal{S}$, $C(S) \leq \sum_{i\in K} p_i(S)$.

The function $p_i^\alpha(S)$ represents the perceived cost of a strategy profile S for a player i. Note that using this model, when α_i is zero, player i is completely selfish, while a player i is completely altruistic when $\alpha_i = 1$. Therefore, if $\alpha = [0]^k$, the α-altruistic extension G^α is equal to the original game G. We say that a game G^α is *uniformly α-altruistic* when for any player i, $\alpha_i = \alpha$.

Chen et al. in [28] analyzed a few classes of games using this altruistic model. Their analysis extends the definition of smooth games to incorporate altruism. Before using their model, we present some important definitions for smooth games [29].

6.1 Smooth Games and Altruism

The notion of smooth games, first defined by Roughgarden in [29], is an important tool in the analysis of inefficiency in games. It provides bounds not only for pure and mixed equilibria, but also for both correlated and coarse correlated equilibria [30].

Definition 5 (Smooth Games)**.** A cost minimization game $G = (K, \mathcal{S}, (p_i)_{i\in K})$ is (λ, μ)–*smooth* if for any pair of strategy profiles $S, S^* \in \mathcal{S}$,

$$\sum_{i\in K} p_i(S_i^*, S_{-i}) \leq \lambda \cdot C(S^*) + \mu \cdot C(S). \tag{5}$$

where $C : \mathcal{S} \to \mathbb{R}$ is again mapping strategy profiles to social costs such that for any $S \in \mathcal{S}, C(S) \leq \sum_{i\in K} p_i(S)$.

If a minimization game is (λ, μ)–smooth then it is possible to assert several facts about such game. Among them, a bound for the price of anarchy. If a game is (λ, μ)–smooth, with ($\lambda \geq 0$ and $\mu < 1$), then every equilibria S has cost at most $\frac{\lambda}{1-\mu}$ times that of a social optimal solution S^*.

The *robust price of anarchy* is defined as the best upper bound that is possible to prove using smoothness analysis.

Definition 6 (Robust Price of Anarchy)**.** The *robust price of anarchy* of a cost-minimization game is defined as

$$\inf \left\{ \frac{\lambda}{1-\mu} : (\lambda, \mu) \text{ s.t. the game is } (\lambda, \mu)\text{-smooth} \right\} \ , \tag{6}$$

where $\lambda \geq 0$ and $\mu < 1$.

Definition 5 can be relaxed by allowing the inequality to hold only for an optimal solution S^* and all other strategy profiles S, while still retaining the properties based on the smoothness property [29]. Note that we already used this type of analysis to prove PoA bounds for FC-FLG in Theorem 8, with an implicit use of λ of k and a μ of zero.

In [20], the definition of smooth games is extended to incorporate altruism, while maintaining most of the properties proved for the original concept.

Definition 7 ((λ, μ, α)-smoothness)**.** Let G^α be an α-altruistic game with social cost function C. G^α is (λ, μ, α)-smooth iff for any two strategy profiles $S, S^* \in \mathcal{S}$, the following is satisfied:

$$\sum_{i \in K} \left[p_i(S_i^*, S_{-i}) + \alpha_i(C_{-i}(S_i^*, S_{-i}) - C_{-i}(S)) \right] \leq \lambda C(S^*) + \mu C(S) \ , \tag{7}$$

where $C_{-i}(S) = C(S) - p_i(S)$ and for any S, $C(S) \leq \sum_{i \in K} p_i(S)$.

If a game is (λ, μ, α)-smooth with $\mu < 1$, then the price of anarchy of the game is at most $\frac{\lambda}{1-\mu}$, even for coarse correlated equilibria.

6.2 Fair Cost Sharing Games with Altruism

A similar game to the Facility Location Game with Fair Cost sharing (FLG-FC) seen in Section 4, denominated Fair-Cost Sharing Game, has been considered in [20]. In this game, there are no connection costs between terminals and facilities. Furthermore, while in the FLG-FC each terminal may choose any facility to open, in the Fair-Cost Sharing Game each player has to connect his clients to some subset of facilities given as an input to the game. Below we adapt the results in [20] to the FLG-FC game.

Recall the FLG-FC specified in Definition 3. Here we use $d(S_i) = \sum_{(t,f) \in S_i} d_{tf}$ as the sum of all connection costs for a player i in strategy S_i, and $U(S) = \bigcup_{i \in K} S_i$ as the set with all strategies in the strategy profile S.

Theorem 25 (Altruistic FLG-FC smoothness (Adapted from [20]))**.** *For any FLG-FC game G with k players, the α–altruistic extension G^α is $(k, \hat{\alpha}, \alpha)$–smooth, where $\hat{\alpha} = \max_{i \in K} \alpha_i$.*

Demonstração. Let S and S^* be two strategy profiles for G, with social cost function $C(S) = \sum_{f \in U(S)} c_f + \sum_{j \in K} d(S_j)$. Fix an arbitrary player $i \in K$. Then,

$$C(S_i^*, S_{-i}) - C(S)$$
$$=$$
$$\sum_{f \in U(S_i^*, S_{-i})} c_f + \sum_{j \in K : j \neq i} d(S_j) + d(S_i^*) - \sum_{f \in U(S)} c_f - \sum_{j \in K} d(S_j)$$
$$\leq$$
$$\sum_{f \in S_i^* \setminus U(S)} c_f + \sum_{(t,f) \in S_i^* \setminus U(S)} d_{tf} \ .$$

This inequality can be used to establish the following bound:

$$(1 - \alpha_i) p_i(S_i^*, S_{-i}) + \alpha_i (C(S_i^*, S_{-i}) - C(S))$$
$$\leq$$
$$(1 - \alpha_i) \left(\sum_{f \in S_i^*} \frac{c_f}{x_f(S_i^*, S_{-i})} + d(S_i^*) \right) +$$
$$\alpha_i \left(\sum_{f \in S_i^* \setminus U(S)} \frac{c_f}{x_f(S_i^*, S_{-i})} + \sum_{(t,f) \in S_i^* \setminus U(S)} d_{tf} \right)$$
$$\leq$$
$$\sum_{f \in S_i^*} \frac{c_f}{x_f(S_i^*, S_{-i})} + d(S_i^*) \leq k \left(\sum_{f \in S_i^*} \frac{c_f}{x_f(S^*)} + d(S_i^*) \right).$$

The last inequality follows from the fact that $x_f(S_i^*, S_{-i}) \geq 1$ and $x_f(S_i^*, S_{-i}) \geq x_f(S^*)/k$ for every $f \in S_i^*$.

From this, we can conclude that the game is $(k, \hat{\alpha}, \alpha)$–smooth as defined in Definition 7:

$$\sum_{i\in K}\Big[p_i(S_i^*,S_{-i})+\alpha_i[C_{-i}(S_i^*,S_{-i})-C_{-i}(S)]\Big]=$$
$$\sum_{i\in K}\Big[p_i(S_i^*,S_{-i})+\alpha_i[C(S_i^*,S_{-i})-p_i(S_i^*,S_{-i})-C(S)+p_i(S)]\Big]=$$
$$\sum_{i\in K}\Big[(1-\alpha_i)p_i(S_i^*,S_{-i})+\alpha_i[C(S_i^*,S_{-i})-C(S)]+\alpha_i p_i(S)\Big]\leq$$
$$k\sum_{i\in K}\Big[\sum_{f\in S_i^*}\frac{c_f}{x_f(S^*)}+d(S_i^*)\Big]+\sum_{i\in K}\alpha_i p_i(S)\leq$$
$$kC(S^*)+\hat{\alpha}C(S)\ .$$

□

Corollary 1 (Robust Price of Anarchy for α-altruistic FLG-FC (Adapted from [20]))**.** *The robust price of anarchy (RPoA) of the α-altruistic FLG-FC game is at most $\frac{k}{1-\hat{\alpha}}$, where $\hat{\alpha}=\max_{i\in P}\alpha_i$, and there is an α-altruistic instance of the FLG-FC with a RPoA of $\frac{k}{1-\hat{\alpha}}$.*

Demonstração. As seen in Theorem 25, for any instance G of the FLG-FC game, the α-altruistic extension G^α is $(k,\hat{\alpha},\alpha)$-smooth and therefore has a robust price of anarchy of $\frac{k}{1-\hat{\alpha}}$.

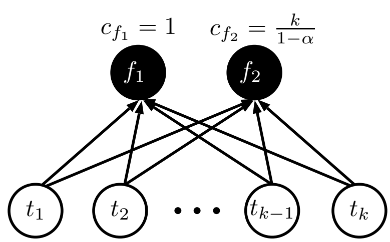

Figura 11. Instance of FLG-FC with PoA equal to $\frac{k}{1-\alpha}$ where every player is α-altruistic, first described in [20]. Connections cost are equal to zero.

To show that this bound is tight, even for pure Nash equilibria, we can slightly alter the example of Figure 3. Instead of a facility with cost equal to the number of players, we now have a facility with cost $\frac{k}{1-\alpha}$, with every player being uniformly α-altruistic, as shown in Figure 11. In this instance each player $i\in[1,k]$ controls a terminal t_i, and can choose between facilities f_1 with cost 1 and f_2 with cost $\frac{k}{1-\alpha}$. Consider the strategy profile $S^*=((t_1,f_1),...,(t_k,f_1))$ where every player chooses f_1 and $S=((t_1,f_2),...,(t_k,f_2))$

where f_2 is chosen by all players. Clearly S^* is the strategy profile with optimal social cost, with $C(S^*) = 1$, while S has cost $C(S) = \frac{k}{1-\alpha}$. The strategy profile S is a PNE of the uniformly α-altruistic extension G^α, since for any player i,

$$p_i^\alpha(S) = (1-\alpha)p_i(S) + \alpha C(S) = 1 + \alpha \frac{k}{1-\alpha} = p_i^\alpha((t_i, f_1), S_{-i}) \ .$$

Therefore, the price of anarchy of G^α is at least $\frac{k}{1-\alpha}$, and the bound for the robust price of anarchy for the α-altruistic extension of FLG-FC is tight. □

The PoS for uniformly α-altruistic games can be determined in a similar way as was done in Theorem 9, for a bound of $(1-\alpha)H_k + \alpha$.

Theorem 26 (Price of Stability for α-altruistic FLG-FC (Adapted from [20]))**.** *The Price of Stability for uniformly α-altruistic fair cost facility location games is at most* $(1-\alpha)H_k + \alpha$.

Demonstração. Let G^α be an uniformly α-altruistic facility location game. Then it is a potential game with potential function

$$\Phi^\alpha(S) = (1-\alpha)\Phi(S) + \alpha C(S) \ ,$$

where

$$\Phi(S) = \sum_{f \in U(S)} \sum_{x=1}^{x_f(S)} \frac{c_f}{x} + \sum_{(t,f) \in U(S)} d_{tf} \ .$$

We have that

$$\Phi^\alpha(S) = (1-\alpha)\left(\sum_{f \in U(S)} \sum_{x=1}^{x_f(S)} \frac{c_f}{x} + \sum_{(t,f) \in U(S)} d_{tf} \right) + \alpha C(S)$$
$$\leq$$
$$((1-\alpha)H_k + \alpha)\left(\sum_{f \in U(S)} c_f + \sum_{(t,f) \in U(S)} d_{tf} \right)$$
$$=$$
$$((1-\alpha)H_k + \alpha)C(S) \ .$$

Let S^* be the social optimum strategy profile. From this strategy, we can derive an equilibrium S by best response dynamics such that $\Phi^\alpha(S^*) \geq \Phi^\alpha(S)$. Since $C(S) \leq \Phi^\alpha(S)$, we have

$$C(S) \leq \Phi^\alpha(S) \leq \Phi^\alpha(S^*) \leq ((1-\alpha)H_k + \alpha)C(S^*) \ ,$$

which means that the price of stability is at most $((1-\alpha)H_k + \alpha)$.

□

We note that in this section a tight bound of a linear function of k was given for the robust price of anarchy, while for the price of stability a considerably more restricted bound was proven (considering only games with uniform altruism). It is possible that this bound can indeed be much closer to the optimal social welfare if completely altruistic players are mixed with selfish players. For example, in the instance in Figure 5, used to prove the tightness of the bound H_k for the PoS of FLG-FC, if the player controlling terminal t_k is completely altruistic, the best PNE is the optimal solution.

Another variant not taken in consideration for the altruistic extension is the metric case, where any terminal can be connected to any facility and connection costs are bounded by the triangle inequality. Clearly the bound for the PoA for α-altruistic FLG-FC holds for the metric variant, since connection costs are not important in Theorem 25 and the instance in Figure 11. However while the bound for the PoS is still valid, it may be that for the metric variant a better PoS for α-altruistic FLG-FC is possible.

We analyzed in this section altruistic versions of the FLG-FC game. It remains an open problem to explore altruism for the facility location game without cost sharing rules. An interesting question for these games, where there are instances with no pure Nash equilibria, is whether a certain amount of altruism in the game can guarantee the existence of such equilibrium. And if altruism can guarantee the existence of equilibrium, how this altruistic behaviour must be distributed between the players. It may be the case that a certain amount of completely altruistic players are needed to guarantee PNE existence, or that every player must be at least a α amount altruistic.

7 Conclusions and Open Problems

In this survey, we combined results from several works related to facility location games. We focused on proving existence of pure Nash equilibria and bounds for the price of anarchy and stability.

For facility location games without cost sharing rules, we presented results from Cardinal and Hoefer [16] for the uncapacitated version, and provided results from Rodrigues and Xavier [19] for the capacitated game proving unbounded PoA, PoS and instances with no PNE even when players control only a single terminal. For fair cost sharing facility location games, we adapted results from Anshelevich et al. [17, 24] for the network design game. In these proofs we needed to take in consideration the additional connection costs and show that they do not interfere in the bounds for equilibria. For the capacitated version, Rodrigues and Xavier [19] proved that the PoA can be worse than the uncapacitated for the metric version, and unbounded in the general case, while the PoS has the same bound as the uncapacitated version. Furthermore, we analyzed facility location using an altruistic model for player behavior [20], and adapted recent results for the fair cost sharing games to facility location games. A summary of the known bounds are presented in Table 1. Note that for Metric Capacitated FLG and FLG-FC results refer to the Sequential Price of Anarchy.

Tabela 1. Known results for Facility Location Games.

Game	PNE	PoA	PoS	Reference
FLG (Facility Location Game)	×	k	$k-2$	[16]
Capacitated FLG	×	∞	∞	[19]
Metric Cap. FLG	×	2^k	$\Theta(2^k)$	[19]
FLG-FC (Fair Cost sharing)	✓	k	$H(k)$	adapted from [17]
Metric FLG-FC	✓	k	2.36	[22]
FLG-FC with Weights	?	$\Theta(\log W)$	$\Theta(\log W)$	[23, 22]
Capacitated FLG-FC	✓	∞	$H(k)$	[19]
Metric Cap. FLG-FC	✓	2^k	$H(k)$	[19]
Altruistic FLG-FC	✓	$\frac{k}{1-\hat{\alpha}}$	$((1-\alpha)H(k)+\alpha)$	adapted from [20]

While several bounds for these games can be adapted directly from network design and fair cost games, some questions remain undiscovered. One example is the question of the existence of PNE for weighted FLG-FC. The examples of instances with no equilibria for network design do not translate to instances of the FLC-FG, and there is still no proof for PNE existence.

Since the addition of hard capacities to facility location games can lead to unbounded PoA and PoS, other models that impose capacities in facility location games may be of interest. Nonetheless, few have been explored in the context of facility location. Furthermore, while sequentiality has been investigated for capacitated facility location games [19] and for games similar to fair cost facility location [25], to the best of our knowledge there is no results concerning the sequential Price of Anarchy or Stability when players have no rules on how to share opening costs.

Another interesting question is to consider altruism for facility location games outside of the fair cost sharing model, for example with no cost sharing rules. Several questions can be explored with altruism, specially in games with no guaranteed equilibria.

Finally, while there are some research in altruism in the context of location games, no results are known when spiteful behaviour from players are considered. We note however, that there are still no completely accepted model for spiteful player behaviour. If we model players as completely spiteful, selfish or altruistic, perhaps some results can be extended from distributed networks, as disrupting agents may be considered completely spiteful players. If someone models spite in a similar manner to altruism, then solution feasibility is another possible concern.

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9 Authors' Contributions

- Félix Carvalho Rodrigues: bibliographic research, writing, revisions, adaptation of proofs and theorems;

- Eduardo Candido Xavier: bibliographic research, revisions, correction of proofs and theorems, final revision.

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