2007

Orthogonal Designs of Order 32 and 64 via Computational Algebra

Ilias S. Kotsireas
Wilfrid Laurier University, ikotsire@wlu.ca

Christos Koukouvinos
National Technical University of Athens

AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **39** (2007), Pages 39–48

Orthogonal designs of order 32 and 64 via computational algebra

Ilias S. Kotsireas

Department of Phys. and Comp. Science
Wilfrid Laurier University
Waterloo ON, N2L 3C5
Canada

Christos Koukouvinos

Department of Mathematics
National Technical University of Athens
Zografou 15773, Athens
Greece

Abstract

Baumert and Hall describe a Williamson array construction based on quaternions. We extend by analogy this construction to larger arrays, using the multiplication table of the Cayley-Dickson algebras of dimensions 32 and 64. Then we use Gröbner bases to obtain full orthogonal designs of order 32 with 10 variables and of order 64 in 10 and 11 variables. Finally we use $\mathrm{OD}(32; 1, 1, 2, 4, 4, 4, 4, 4, 4, 4, 4)$ to search for inequivalent Hadamard matrices of orders 96, 160, 224, 288. Such structured matrices are needed in Statistics and Coding Theory applications. This algebraic approach can be extended to larger orders, i.e. $2^n, n \geq 7$, provided that the structural properties of the corresponding polynomial ideals and their Gröbner bases are further investigated and understood.

1 Introduction

Definition Let $x_1, \ldots, x_t$ be commuting indeterminates. An orthogonal design X of order n and type $(s_1, \ldots, s_t)$ denoted $\mathrm{OD}(n; s_1, \ldots, s_t)$, where $s_1, \ldots, s_t$ are positive integers, is a matrix X of order n with entries from $\{0, \pm x_1, \ldots, \pm x_t\}$, such that

$$XX^t = \left(\sum_{i=1}^{t} s_i x_i^2\right) I_n,$$

where X^t denotes the transpose of X and I_n denotes the identity matrix of order n. For applications of ODs to wireless communications see [18] and the references

therein. For more details on orthogonal designs see the book [10] and the survey paper [17].
This paper is organized as follows. First we give an account of the classical Williamson construction for Hadamard matrices, from the point of view of quaternions, following Baumert and Hall. Then we show how the application of the Cayley-Dickson process on the algebra of sedenions, can be used to describe a full orthogonal design of order 32 with 10 variables. We use Gröbner bases [4] [5], [19] to locate a canonical solution of a system of equations that gives the orthogonal design of order 32. The multiplication table of the algebra of dimension 32 is used to construct right multiplication matrices which are used to form orthogonal designs. Using the right multiplication operator is a way to overcome the obstacle of non-associativity of the algebra. Non-associativity is an obstacle, because it is incompatible with the existence of matrix representations, that we could use directly to construct orthogonal designs. To circumvent this obstacle we use the right multiplication operator, as it seems that left multiplication is not suitable for our purposes. We apply this method to construct some full orthogonal designs of order 64 as well. In the last sections of the paper, we use the full orthogonal design of order 32 with 10 variables, to search for inequivalent Hadamard matrices of orders 96, 160, 224, 288. Inequivalent Hadamard matrices with a certain structure, in this case coming from orthogonal designs, are needed in Statistics and Coding Theory applications.
The Cayley-Dickson process in conjunction with Gröbner bases, provides a unifying algebraic framework for the construction of orthogonal designs of orders 32 and 64 with many variables. This approach can be extended to larger orders, i.e. $2^n, n \geq 7$, provided that the structural properties of the corresponding polynomial ideals and their Gröbner bases are better understood.

2 Orthogonal designs of order 32 and 64 via Cayley-Dickson algebras

Baumert and Hall specify in [2, 3] how to construct the Williamson array via quaternions, see also [11].

The Cayley-Dickson process allows us to obtain an algebra of dimension $2n$ from an algebra of dimension n, see [8, 13]. By applying the Cayley-Dickson process successively to the real numbers, one obtains the algebras of quaternions, octonions and sedenions.

The construction by Baumert and Hall can be repeated using octonions and sedenions instead of quaternions. This has been done in [14]. The corresponding orthogonal designs have been found to yield Hadamard matrices.

By applying the Cayley-Dickson process to the algebra of sedenions, one obtains a Cayley-Dickson algebra of dimension 32 with basis $e_0 = 1, e_1, \ldots e_{31}$ whose multiplication table is omitted.

To associate a 32×32 matrix to each basis element, we use the right multiplication operator, on the column vector

$$v = \left[\begin{array}{cccc} 1 & e_1 & \ldots & e_{31} \end{array} \right]^t .$$

Then the thirty-two right multiplications

$$v \cdot e_0, \;\; v \cdot e_1, \;\; \ldots \;\; v \cdot e_{31},$$

give rise to thirty-two 32×32 matrices $q_0, \ldots, q_{31}$. Let $A_1, \ldots, A_{32}$ be commuting indeterminates. Then the sum $A = \sum_{i=0}^{31} A_{i+1} q_i$ is equal to a 32×32 array with the property that the diagonal elements of AA^t are all equal to $\sum_{i=1}^{32} A_i^2$, but whose other elements are not necessarily all zero. By requiring that all elements of AA^t (except the diagonal ones) are equal to zero, we obtain 252 equations in 30 variables (all variables except A_1, A_{17}).

The reduced Gröbner basis (for a total degree reverse lexicographical ordering) of these 252 equations has 290 elements. The computation was done in Magma V2.11-2. Among the elements in the Gröbner basis, we are interested in the 105 binomials, which can be written concisely as:

$$A_{16-j} A_i - A_{i-16} A_{32-j}, i = 32 - j + 1, \ldots, 32, j = 1, \ldots, 14. \tag{1}$$

For each value of j, there are j binomials, which makes a total of $\sum_{j=1}^{14} j = 105$ binomials. For $j = 1$ the one binomial is equal to $A_{15}A_{32} - A_{16}A_{31}$. For $j = 2$ the two binomials are equal to $A_{14}A_{31} - A_{15}A_{30}$, $A_{14}A_{32} - A_{16}A_{30}$.

Each solution of the 105 equations corresponding to the 105 binomials (1) can be completed to a solution of all the 290 equations coming from the Gröbner basis and thus yield a 32×32 orthogonal design. The structure of these 105 binomials equations points out to the following canonical solution:

$$\begin{aligned} &A_{18} = A_2, A_{19} = A_3, A_{20} = A_4, A_{21} = A_5, A_{22} = A_6, A_{23} = A_7, A_{24} = A_8, A_{25} = A_9, \\ &A_{26} = A_{10}, A_{27} = A_{11}, A_{28} = A_{12}, A_{29} = A_{13}, A_{30} = A_{14}, A_{31} = A_{15}, A_{32} = A_{16} \end{aligned} \tag{2}$$

that can be completed to a solution of the 290 equations by requiring in addition that:

$$A_{10} = A_2, A_{11} = A_3, A_{12} = A_4, A_{13} = A_5, A_{14} = A_6, A_{15} = A_7, A_{16} = A_8. \tag{3}$$

The solution of the 290 equations, comprised of (2) and (3) gives rise to the 10-variable full orthogonal design OD$(32; 1, 1, 2, 4, 4, 4, 4, 4, 4, 4)$ (where we have used the change of variables $A1 = A$, $A2 = B$, $A3 = C$, $A4 = D$, $A5 = E$, $A6 = F$, $A7 = G$, $A8 = H$, $A9 = I$, $A17 = J$ to save space) given in the appendix.
Note that when we take the product of OD$(32; 1, 1, 2, 4, 4, 4, 4, 4, 4, 4)$ with its transpose, the result is:

$$(A^2 + 4B^2 + 4C^2 + 4D^2 + 4E^2 + 4F^2 + 4G^2 + 4H^2 + 2I^2 + J^2) I_{32} \tag{4}$$

A full orthogonal design OD$(32; 1, 1, 2, 4, 4, 4, 4, 4, 4, 4)$ has been found in [10] by a doubling construction on the full orthogonal design of order 16 OD$(16; 1, 1, 2, 2, 2, 2, 2, 2, 2)$ in 9 variables.

The Radon number $\rho(n)$ gives the maximum number of variables allowed in an orthogonal design of order n ([10]). Since we have $\rho(32) = 10$, we conclude that the particular orthogonal design or order 32 that we found, has the maximum permissible number of variables.

The algebraic framework that we present here, based on the Cayley-Dickson process, is a systematic way of constructing Orthogonal Designs of order 2^n. A usual, starting from an Orthogonal Design with many variables one can obtain many other Orthogonal Designs by the Equating and Killing variables technique. The full orthogonal design $\mathrm{OD}(32; 1, 1, 2, 4, 4, 4, 4, 4, 4, 4)$ is just one of the Orthogonal Designs that can be constructed via particular solutions of the equations. To illustrate this point further, we give two full orthogonal designs of order 64 (note that $\rho(64) = 12$)

- $\mathrm{OD}(64; 1, 3, 4, 8, 8, 8, 8, 8, 8, 8)$, with 10 variables

- $\mathrm{OD}(64; 1, 1, 2, 4, 8, 8, 8, 8, 8, 8, 8)$ with 11 variables

in the web page `http://www.cargo.wlu.ca/OD32ineq`. These two full orthogonal designs of order 64 been found by applying the Cayley-Dickson process and finding particular solutions of the systems of polynomial equations that arise after the Gröbner bases calculations.

We note here that the orthogonal design $\mathrm{OD}(64; 1, 3, 4, 8, 8, 8, 8, 8, 8, 8)$ can be constructed from the orthogonal design $\mathrm{OD}(64; 1, 1, 2, 4, 8, 8, 8, 8, 8, 8, 8)$ by Equating variables.

3 Hadamard matrices from the 10-variable full orthogonal design $\mathrm{OD}(32; 1, 1, 2, 4, 4, 4, 4, 4, 4, 4)$

In this section we use the $\mathrm{OD}(32; 1, 1, 2, 4, 4, 4, 4, 4, 4, 4)$, that we denote by H_{32}, to produce structured Hadamard matrices of orders $32n$, where n is the order of the block matrices A, B, C, D, E, F, G, H, I, J. Imitating the classical Williamson construction, we take the ten matrices A, B, C, D, E, F, G, H, I, J to be symmetric circulant matrices of order n each, defined via the matrix U:

$$U = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

which has the property $U^n = I_n$. Take the ten matrices A, B, C, D, E, F, G, H, I, J to be polynomials in U, so that they commute with each other:

$$\begin{array}{ccccccccc}
A & = & a_0 I_n & + & a_1 U & + & \cdots & + & a_{n-1} U^{n-1} \\
B & = & b_0 I_n & + & b_1 U & + & \cdots & + & b_{n-1} U^{n-1} \\
C & = & c_0 I_n & + & c_1 U & + & \cdots & + & c_{n-1} U^{n-1} \\
D & = & d_0 I_n & + & d_1 U & + & \cdots & + & d_{n-1} U^{n-1} \\
E & = & e_0 I_n & + & e_1 U & + & \cdots & + & e_{n-1} U^{n-1} \\
F & = & f_0 I_n & + & f_1 U & + & \cdots & + & f_{n-1} U^{n-1} \\
G & = & g_0 I_n & + & g_1 U & + & \cdots & + & g_{n-1} U^{n-1} \\
H & = & h_0 I_n & + & h_1 U & + & \cdots & + & h_{n-1} U^{n-1} \\
I & = & i_0 I_n & + & i_1 U & + & \cdots & + & i_{n-1} U^{n-1} \\
J & = & j_0 I_n & + & j_1 U & + & \cdots & + & j_{n-1} U^{n-1}
\end{array}$$

Since $U^T = U^{-1}$, the ten matrices A, B, C, D, E, F, G, H, I, J will be symmetric if

$$a_{n-k} = a_k, b_{n-k} = b_k, c_{n-k} = c_k, d_{n-k} = d_k, e_{n-k} = e_k,$$

$$f_{n-k} = f_k, g_{n-k} = g_k, h_{n-k} = h_k, i_{n-k} = i_k, j_{n-k} = j_k.$$

for $k = 1, \dots, n-1$.
When n takes specific values, the requirement

$$H_{32} H_{32}^t = (A^2 + 4B^2 + 4C^2 + 4D^2 + 4E^2 + 4F^2 + 4G^2 + 4H^2 + 2I^2 + J^2) \otimes I_{32}$$

(where $\otimes$ denotes the Kronecker product) can be translated into polynomial equations. Using these polynomial equations we conducted exhaustive and partial searches for the first four odd values of the parameter n.

Moreover, we need to take into account the diophantine equation

$$a^2 + 4b^2 + 4c^2 + 4d^2 + 4e^2 + 4f^2 + 4g^2 + 4h^2 + 2i^2 + j^2 = 32n \qquad (5)$$

where $a, b, c, d, e, f, g, h, i, j$ are the sums of the elements of the first rows of the matrices $A, B, C, D, E, F, G, H, I, J$ respectively.

We computed all solutions of equation (5) in positive integers, for $n = 3, 5, 7, 9$. For $n = 3$, there are 46 solutions. For $n = 5$, there are 344 solutions. For $n = 7$, there are 1360 solutions. For $n = 9$, there are 3731 solutions.

Each solution of equation (5) furnishes additional linear constraints that can be added into the system of polynomial equations we will derive below and thus restrict the search space dramatically.

3.1 Exhaustive searches for $n = 3, 5, 7$ and partial search for $n = 9$

In this subsection we report the results of our exhaustive searches for $n = 3, 5, 7$ and our partial search for $n = 9$.

- For $n = 3$ we obtain the one equation

$$a0 \cdot a1 + 4 \cdot b0 \cdot b1 + 4 \cdot c0 \cdot c1 + 4 \cdot d0 \cdot d1 + 4 \cdot e0 \cdot e1 + 4 \cdot f0 \cdot f1 + 4 \cdot g0 \cdot g1 + 4 \cdot h0 \cdot h1 + 2 \cdot i0 \cdot i1 + j0 \cdot j1 + 16 = 0$$

  This equation has exactly $28,672$ solutions when all 20 variables take ± 1 values. These solutions give rise to Hadamard matrices of order 96;

- For $n = 5$ we obtain the two equations

$$\begin{aligned} & a0 \cdot a1 + a1 \cdot a2 + 4 \cdot b0 \cdot b1 + 4 \cdot b1 \cdot b2 + 4 \cdot c0 \cdot c1 + 4 \cdot c1 \cdot c2 \\ & + 4 \cdot d0 \cdot d1 + 4 \cdot d1 \cdot d2 + 4 \cdot e0 \cdot e1 + 4 \cdot e1 \cdot e2 + +4 \cdot f0 \cdot f1 + 4 \cdot f1 \cdot f2 \\ & + 4 \cdot g0 \cdot g1 + 4 \cdot g1 \cdot g2 + 4 \cdot h0 \cdot h1 + 4 \cdot h1 \cdot h2 + 2 \cdot i0 \cdot i1 + 2 \cdot i1 \cdot i2 \\ & + j0 \cdot j1 + j1 \cdot j2 + 16 = 0 \end{aligned}$$

$$\begin{aligned} & a0 \cdot a2 + a1 \cdot a2 + 4 \cdot b0 \cdot b2 + 4 \cdot b1 \cdot b2 + 4 \cdot c0 \cdot c2 + 4 \cdot c1 \cdot c2 \\ & + 4 \cdot d0 \cdot d2 + 4 \cdot d1 \cdot d2 + 4 \cdot e0 \cdot e2 + 4 \cdot e1 \cdot e2 + +4 \cdot f0 \cdot f2 \\ & + 4 \cdot f1 \cdot f2 + 4 \cdot g0 \cdot g2 + 4 \cdot g1 \cdot g2 + 4 \cdot h0 \cdot h2] + 4 \cdot h1 \cdot h2 \\ & + 2 \cdot i0 \cdot i2 + 2 \cdot i1 \cdot i2 + j0 \cdot j2 + j1 \cdot j2 + 16 = 0 \end{aligned}$$

  These two equations have exactly $1,576,960$ solutions when all 30 variables take ± 1 values. These solutions give rise to Hadamard matrices of order 160;

- For $n = 7$ we obtain the three equations in 40 variables. These three equations have exactly $45,158,400$ solutions when all 40 variables take ± 1 values. These solutions give rise to Hadamard matrices of order 224.

- For $n = 9$ we obtain four equations in 50 variables. We computed more than 51 million solutions of the system, with a serial C program. These solutions give rise to Hadamard matrices of order 288.

We omit the equations for $n = 7, 9$, as and their structure can be easily deduced from the corresponding equations for $n = 3, 5$.

4 Inequivalent Hadamard matrices from the full orthogonal design OD$(32; 1, 1, 2, 4, 4, 4, 4, 4, 4, 4)$

In this section we summarize the computational results on locating inequivalent Hadamard matrices within the sets of Hadamard matrices computed in the previous section. We analyzed the corresponding solution sets with Magma V2.11 to search for inequivalent Hadamard matrices. We used the profile criterion to distinguish between inequivalent Hadamard matrices. The profile criterion is a sufficient (but not necessary) condition for Hadamard inequivalence. Hadamard matrices with unequal profiles are inequivalent. However, Hadamard matrices with equal profiles may or may not be inequivalent. For $n = 3$, we located inequivalent matrices of order $32 \cdot 3 = 96$. For $n = 5$, we located inequivalent matrices of order $32 \cdot 5 = 160$. For

$n = 7$, we located inequivalent matrices of order $32 \cdot 7 = 224$. For $n = 9$, we located inequivalent matrices of order $32 \cdot 9 = 288$.

The doubling method of [9] can be used to produce vast numbers of inequivalent Hadamard matrices of all orders $8t$. Nevertheless, it is important to have at our disposal inequivalent Hadamard matrices that don't come from doubling, for coding theory investigations. More specifically, inequivalent Hadamard matrices constructed by the doubling method are not too useful in constructing extremal self-dual codes, see [1]. For this reason, although the lower bounds on the numbers of inequivalent matrices for orders greater than 28 are astronomical, inequivalent Hadamard matrices which are not constructed by a doubling method are greatly needed. Similar facts are of importance in some statistical applications, where inequivalent Hadamard matrices and orthogonal arrays are used as screening designs in some experiments and the related classification problems are of considerable interest, see [12].

The theoretical results of Lam, Lam and Tonchev [15, 16] imply that the number of inequivalent Hadamard matrices of order 96 is greater than or equal to 10^{49} and that the number of inequivalent Hadamard matrices of order 160 is greater than or equal to 10^{105}. Even though these lower bounds are astronomical, the actual numbers of available inequivalent Hadamard matrices (needed for applications) for those orders, are quite small. These theoretical results contain the most powerful lower bounds on Hadamard designs and matrices known presently.

Our computations using the full orthogonal design $OD(32; 1, 1, 2, 4, 4, 4, 4, 4, 4, 4)$ yield many inequivalent Hadamard matrices of orders 96, 160, 224, 288. The results are summarized in the web page `http://www.cargo.wlu.ca/OD32ineq`

5 Discussion and Future Research

The method presented in this paper is more general than the doubling method, because it can produce some orthogonal designs that cannot be constructed by doubling. Such an example is the orthogonal design $OD(16; 1, 1, 7, 7)$ presented in [14] (see overleaf).

The orthogonal design $OD(16; 1, 1, 7, 7)$ cannot be obtained from the orthogonal design $OD(16; 1, 1, 2, 2, 2, 2, 2, 2, 2)$, or by doubling of an orthogonal design of order 8. The orthogonal design $OD(16; 1, 1, 7, 7)$ can be constructed from simple Paley matrices.

The orthogonal design $OD(128; 8_{16})$ is given as conjecture 2.117 (page 295) in the second edition of the Handbook of Combinatorial Designs [7]. We believe that it is possible to construct $OD(128; 8_{16})$ with the method presented in this paper.

$$\begin{bmatrix}
A & H & H & H & H & H & H & H & I & J & J & J & J & J & J & J \\
-H & A & -H & H & -H & H & H & -H & -J & I & J & -J & J & -J & -J & J \\
-H & H & A & -H & -H & -H & H & H & -J & -J & I & J & J & J & -J & -J \\
-H & -H & H & A & -H & H & -H & H & -J & J & -J & I & J & -J & J & -J \\
-H & H & H & H & A & -H & -H & -H & -J & -J & -J & -J & I & J & J & J \\
-H & -H & H & -H & H & A & H & -H & -J & J & -J & J & -J & I & -J & J \\
-H & -H & -H & H & H & -H & A & H & -J & J & J & -J & -J & J & I & -J \\
-H & H & -H & -H & H & H & -H & A & -J & -J & J & J & -J & -J & J & I \\
-I & J & J & J & J & J & J & J & A & -H & -H & -H & -H & -H & -H & -H \\
-J & -I & J & -J & J & -J & -J & J & H & A & H & -H & H & -H & -H & H \\
-J & -J & -I & J & J & J & -J & -J & H & -H & A & H & H & H & -H & -H \\
-J & J & -J & -I & J & -J & J & -J & H & H & -H & A & H & -H & H & -H \\
-J & -J & -J & -J & -I & J & J & J & H & -H & -H & -H & A & H & H & H \\
-J & J & -J & J & -J & -I & -J & J & H & H & -H & H & -H & A & -H & H \\
-J & J & J & -J & -J & J & -I & -J & H & H & H & -H & -H & H & A & -H \\
-J & -J & J & J & -J & -J & J & -I & H & -H & H & H & -H & -H & H & A
\end{bmatrix}$$

Acknowledgments

The first author thanks Murray Bremner for useful discussions and Maple code on the Cayley-Dickson process. The authors thank the anonymous referee for their detailed comments that helped them to improve the quality of the paper. This work is supported in part by an NSERC grant. The prototype C programs were generated automatically with Maple at the *Computer Algebra Research Group, CARGO, Wilfrid Laurier University*, Waterloo ON, Canada and executed at *SHARCnet* and *WestGrid* high-performance computing clusters. All computations in Magma have been performed remotely at the *Centre de calcul formel MEDICIS, École Polytechnique* Paris, France.

References

[1] M. Araya, M. Harada, and H. Kharaghani, Some Hadamard matrices of order 32 and their binary codes, *J. Combin. Des.* 12 (2004), 142–146.

[2] L. D. Baumert and M. Hall, Jr., Hadamard matrices of the Williamson type, *Math. Comput.* 19 (1965), 442–447.

[3] L. D. Baumert and M. Hall, Jr., A new construction for Hadamard matrices, *Bull. Amer. Math. Soc.* 71 (1965), 169–170.

[4] T. Becker and V. Weispfenning, *Gröbner bases. A computational approach to com mutative algebra*, Graduate Texts in Mathematics 141, Springer Verlag, New York, 1993.

Appendix

The 10 variable full orthogonal design OD$(32; 1, 1, 2, 4, 4, 4, 4, 4, 4, 4)$

$$
\left[\begin{array}{cccccccccccccccccccccccccccccccc}
A & B & C & D & E & F & G & H & I & B & C & D & E & F & G & H & J & B & C & D & E & F & G & H & I & B & C & D & E & F & G & H \\
-B & A & -D & C & -F & E & H & -G & -B & I & D & -C & F & -E & -H & G & -B & J & D & -C & F & -E & -H & G & B & -I & -D & C & -F & E & H & -G \\
-C & D & A & -B & -G & -H & E & F & -C & -D & I & B & G & H & -E & -F & -C & -D & J & B & G & H & -E & -F & C & D & -I & -B & -G & -H & E & F \\
-D & -C & B & A & -H & G & -F & E & -D & C & -B & I & H & -G & F & -E & -D & C & -B & J & H & -G & F & -E & D & -C & B & -I & -H & G & -F & E \\
-E & F & G & H & A & -B & -C & -D & -E & -F & -G & -H & I & B & C & D & -E & -F & -G & -H & J & B & C & D & E & F & G & H & -I & -B & -C & -D \\
-F & -E & H & -G & B & A & D & -C & -F & E & -H & G & -B & I & -D & C & -F & E & -H & G & -B & J & -D & C & F & -E & H & -G & B & -I & D & -C \\
-G & -H & -E & F & C & -D & A & B & -G & H & E & -F & -C & D & I & -B & -G & H & E & -F & -C & D & J & -B & G & -H & -E & F & C & -D & -I & B \\
-H & G & -F & -E & D & C & -B & A & -H & -G & F & E & -D & -C & B & I & -H & -G & F & E & -D & -C & B & J & H & G & -F & -E & D & C & -B & -I \\
-I & B & C & D & E & F & G & H & A & -B & -C & -D & -E & -F & -G & -H & -I & -B & -C & -D & -E & -F & -G & -H & J & B & C & D & E & F & G & H \\
-B & -I & D & -C & F & -E & -H & G & B & A & D & -C & F & -E & -H & G & -B & I & -D & C & -F & E & H & -G & -B & J & -D & C & -F & E & H & -G \\
-C & -D & -I & B & G & H & -E & -F & C & -D & A & B & G & H & -E & -F & -C & D & I & -B & -G & -H & E & F & -C & D & J & -B & -G & -H & E & F \\
-D & C & -B & -I & H & -G & F & -E & D & C & -B & A & H & -G & F & -E & -D & -C & B & I & -H & G & -F & E & -D & -C & B & J & -H & G & -F & E \\
-E & -F & -G & -H & -I & B & C & D & E & -F & -G & -H & A & B & C & D & -E & F & G & H & I & -B & -C & -D & -E & F & G & H & J & -B & -C & -D \\
-F & E & -H & G & -B & -I & -D & C & F & E & -H & G & -B & A & -D & C & -F & -E & H & -G & B & I & D & -C & -F & -E & H & -G & B & J & D & -C \\
-G & H & E & -F & -C & D & -I & -B & G & H & E & -F & -C & D & A & -B & -G & -H & -E & F & C & -D & I & B & -G & -H & -E & F & C & -D & J & B \\
-H & -G & F & E & -D & -C & B & -I & H & -G & F & E & -D & -C & B & A & -H & G & -F & -E & D & C & -B & I & -H & G & -F & -E & D & C & -B & J \\
-J & B & C & D & E & F & G & H & I & B & C & D & E & F & G & H & A & -B & -C & -D & -E & -F & -G & -H & -I & -B & -C & -D & -E & -F & -G & -H \\
-B & -J & D & -C & F & -E & -H & G & B & -I & -D & C & -F & E & H & -G & B & A & D & -C & F & -E & -H & G & B & -I & -D & C & -F & E & H & -G \\
-C & -D & -J & B & G & H & -E & -F & C & D & -I & -B & -G & -H & E & F & C & -D & A & B & G & H & -E & -F & C & D & -I & -B & -G & -H & E & F \\
-D & C & -B & -J & H & -G & F & -E & D & -C & B & -I & -H & G & -F & E & D & C & -B & A & H & -G & F & -E & D & -C & B & -I & -H & G & -F & E \\
-E & -F & -G & -H & -J & B & C & D & E & F & G & H & -I & -B & -C & -D & E & -F & -G & -H & A & B & C & D & E & F & G & H & -I & -B & -C & -D \\
-F & E & -H & G & -B & -J & -D & C & F & -E & H & -G & B & -I & D & -C & F & E & -H & G & -B & A & -D & C & F & -E & H & -G & B & -I & D & -C \\
-G & H & E & -F & -C & D & -J & -B & G & -H & -E & F & C & -D & -I & B & G & H & E & -F & -C & D & A & -B & G & -H & -E & F & C & -D & -I & B \\
-H & -G & F & E & -D & -C & B & -J & H & G & -F & -E & D & C & -B & -I & H & -G & F & E & -D & -C & B & A & H & G & -F & -E & D & C & -B & -I \\
-I & -B & -C & -D & -E & -F & -G & -H & -J & B & C & D & E & F & G & H & I & -B & -C & -D & -E & -F & -G & -H & A & B & C & D & E & F & G & H \\
-B & I & -D & C & -F & E & H & -G & -B & -J & -D & C & -F & E & H & -G & B & I & -D & C & -F & E & H & -G & -B & A & -D & C & -F & E & H & -G \\
-C & D & I & -B & -G & -H & E & F & -C & D & -J & -B & -G & -H & E & F & C & D & I & -B & -G & -H & E & F & -C & D & A & -B & -G & -H & E & F \\
-D & -C & B & I & -H & G & -F & E & -D & -C & B & -J & -H & G & -F & E & D & -C & B & I & -H & G & -F & E & -D & -C & B & A & -H & G & -F & E \\
-E & F & G & H & I & -B & -C & -D & -E & F & G & H & -J & -B & -C & -D & E & F & G & H & I & -B & -C & -D & -E & F & G & H & A & -B & -C & -D \\
-F & -E & H & -G & B & I & D & -C & -F & -E & H & -G & B & -J & D & -C & F & -E & H & -G & B & I & D & -C & -F & -E & H & -G & B & A & D & -C \\
-G & -H & -E & F & C & -D & I & B & -G & -H & -E & F & C & -D & -J & B & G & -H & -E & F & C & -D & I & B & -G & -H & -E & F & C & -D & A & B \\
-H & G & -F & -E & D & C & -B & I & -H & G & -F & -E & D & C & -B & -J & H & G & -F & -E & D & C & -B & I & -H & G & -F & -E & D & C & -B & A
\end{array}\right]
$$

[5] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties, and Algorithms : An Introduction to Computational Algebraic Geometry and Commutative Algebra* UTM, Springer Verlag, New York, 1992.

[6] R. Craigen, Hadamard Matrices and Designs, in *The CRC Handbook of Combinatorial Designs*, (eds. C.J. Colbourn and J.H. Dinitz), CRC Press, Boca Raton, Fla., 1996, 370–377.

[7] R. Craigen and H. Kharaghani, Orthogonal Designs, in *The CRC Handbook of Com binatorial Designs*, (eds. C.J. Colbourn and J.H. Dinitz), 2nd ed. CRC Press, Boca Raton, Fla., 2006, 280–295.

[8] G. M. Dixon, *Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics*, Mathematics and its Applications, 290, Kluwer Academic Publishers Group, Dordrecht, 1994.

[9] S. Georgiou, I. S. Kotsireas and C. Koukouvinos, Inequivalent Hadamard matrices of order $2n$ constructed from Hadamard matrices of order n, *J. Combin. Math. Combin. Comput.*, (to appear).

[10] A. V. Geramita and J. Seberry, *Orthogonal designs. Quadratic forms and Hadamard matrices*, Lecture Notes in Pure and Applied Mathematics, 45. Marcel Dekker, Inc., New York, 1979.

[11] M. Hall, Jr. *Combinatorial Theory*, Reprint of the 1986 second edition, Wiley Classics Library, Wiley, New York, 1998.

[12] A. S. Hedayat, N. J. A. Sloane, and J. Stufken, *Orthogonal Arrays. Theory and Appli cations*, Springer Series in Statistics, Springer Verlag, New York, 1999.

[13] N. Jacobson, *Basic Algebra I*, W.H. Freeman, San Francisco, 1974.

[14] I. S. Kotsireas and C. Koukouvinos, Orthogonal Designs via Computational Algebra, *J. Combin. Designs* 14 (2006), 351–362.

[15] C. Lam, S. Lam and V. D. Tonchev, Bounds on the number of affine, symmetric, and Hadamard designs and matrices, *J. Combin. Theory Ser. A*, 92 (2000), 186–196.

[16] C. Lam, S. Lam and V. D. Tonchev, Bounds on the number of Hadamard designs of even order, *J. Comb. Designs* 9 (2001), 363–378.

[17] J. Seberry and R. Craigen, Orthogonal Designs, in *The CRC Handbook of Combina torial Designs*, (eds. C. J. Colbourn and J. H. Dinitz), CRC Press, Boca Raton, Fla., 1996, 400–406.

[18] J. Seberry, S. A. Seberry and T. A. Wysocki, A construction technique for general ized complex orthogonal designs and applications to wireless communications, *Linear Algebra Applic.* 405 (2005), 163–176.

[19] J. von zur Gathen and J. Gerhard, *Modern Computer Algebra*, Second Edition. Cam bridge University Press, Cambridge, 2003.

(Received 19 June 2006; revised 1 Mar 2007)