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The application of the law of virtual work in the solution of civil engineering structures

Martin Trester Dyke
University of the Pacific

THE APPLICATION OF THE LAW OF VIRTUAL WORK IN THE SOLUTION OF CIVIL ENGINEERING STRUCTURES

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Martin Trester Dyke 3d

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TABLE OF CONTENTS

LIST OF EXAMPLES

THE APPLICATION OF THE LAW OF VIRTUAL WORK IN THE SOLUTION OF CIVIL ENGINEERING STRUCTURES

I. INTRODUCTION

The purpose of this thesis is to explore the civil engineering application of the law of virtual work to the determination of deflections, shears, bending stresses in trussed structures, beams, structures subjected to both direct stress and bending, and indeterminate structures, and further to give examples of their mathematical solutions.

The study of deflections is highly important in structural analysis. The limitations on deflections, as are imposed by materials used in construction or by the nature of the location of the structure or by the loading of the structure, make it imperative that an accurate mathematical analysis of the expected deflection be provided. These deflections of a structure may be due to either elastic or non-elastic distortion of the elements of the structure. Elastic distortion may be caused by stresses developed by the applied loads, or by strains caused by temperature changes; in either case the distortion disappears on removal of its cause, provided of course the elastic limit was not exceeded in the original loading.

Non-elastic distortion can be caused by play in pin joints, settlement of foundations, shrinkage of concrete, *et al*.

In the erection of cantilever and continuous bridges and in the design of lifting devices for swing bridges, a knowledge of deflection is imperative.

An example wherein deflection is limited occurs in the case of ceilings of buildings to prevent excessive cracking of plaster. Here it is necessary to know and limit by construction the magnitude of the deflections.

What is probably most important, the methods of stress analysis of indeterminate structures are based upon an evaluation of their distortion under load.

The Method of Virtual Work is now considered to be one of the more basic methods in analysis of this latter type of structure.

II. HISTORY

As early as 1717, John Bernoulli formulated the principle of virtual work, or virtual displacements which he stated as follows:

> "if a system of fources acting on a rigid body is in equilibrium, the virtual work done by them in any virtual displacement is equal to zero."

Although this became a well recognized mechanical principle, it was not applied as a structural tool for computing deflections until much later. Lamé is credited with first having done so in 1852. Professor Clerk Maxwell in England and Mohr in Germany were the next writers to discuss further applications in papers published in 1864 and 1874. Subsequent German writers, including A. Föppl, Frankel and Müller-Breslau, extended this method. Today it is included in advanced texts such as Fife and Wilbur and Müller-Breslau.

The law of Virtual Work as it now usually is stated follows:

> "If a deformable body is in equilibrium under a system of Q-forces and remains in equilibrium as the body is subjected to a small virtual distortion, the external virtual work done by the external Q-forces is equal to the internal virtual work of distortion done by the internal Q-stresses."

III. GENERAL THEORY

The derivation which follows will be restricted to a consideration of structures whose members are in a state of plane stress, the case in which all stresses in the structure are parallel to one plane. This corresponds to the state of stress ordinarily found in the usual structural problem concerning a truss or beam. This method is not limited to such a case; it is perfectly general and as such is applicable to the most general case of three dimensional stress. The derivation is limited only for simplicity.

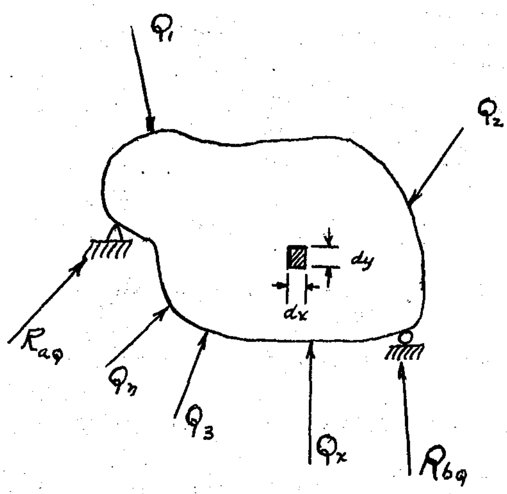


FIG 1a

Any structure in a state of plane stress is considered, such as the body shown in Fig. 1a, which is in equilibrium under a system of external Q-forces and the resulting internal Q-stresses. Now a small change in shape of the body is allowed to occur, the change being to a condition of distortion independent of these Q-forces.

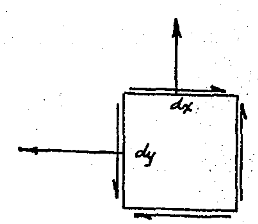


FIG 1b

Such a change in shape will be called a virtual distortion . . . the term virtual meaning that the distortion is independent of the Q-forces.

Next, any small particle of the body under the conditions described above is considered as shown in Fig. 1b. When such a differential particle isisolated, on its internal boundaries with adjacent particles it would be acted upon by internal Q-stresses. If the particle has any external boundary, then it is acted upon by the external Q-forces which may exist on this external boundary. Since the body as a whole is in equilibrium under the external Q-forces, then each and every particle of the body is in equilibrium under the forces to which it is subjected. When the virtual distortion is imposed upon the body, this particle may be translated, rotated and distorted, and the forces and stresses acting on its boundaries will do certain amounts of work, which are designated as virtual work, since the movements of the points of application of the forces are independent of the forces. If dW_s designates the virtual work done by the forces and stresses of the Q-system acting on the boundaries of the particle, this total virtual work dW_s is composed of two parts: first, that done due to only the distortion of the particle, which will be called dW_d ; and second that due to the translation and rotation of the particle, which will be equal to $dW_s - dW_d$. But according to the principle of virtual work, if a system of forces acting on a rigid particle is in equilibrium, the virtual work done by them during any virtual displacement is equal to zero. Hence the virtual work done by the forces and stresses of the Q-system acting on the particle, during only the virtual translation and rotation of the particle as a rigid particle, is equal to zero, that is, $dW_s - dW_d$ or,

$$dW_s = dW_d \qquad \text{Eq. (1)}$$

If the virtual work be integrated over the whole body, this equation becomes,

$$W_s = W_d \qquad \text{Eq. (2)}$$

On the internal boundaries of every particle of the body the forces acting are due to the internal Q-stresses, and represent the action of the particle adjacent to the boundary on the particle being considered. Thus, on every internal boundary between adjacent particles there are two sets of forces, equal, but opposite in direction; consequently, the amounts of virtual work done by the two sets in any translation or rotation of the boundary surface must add up algebraically to zero. Hence, the virtual work done by the surface forces acting on the internal boundaries of all particles must be zero and W_s must be interpreted as being only the virtual work done by the forces acting on the external boundaries of the body. Recognizing this, from equation 2, the law of virtual work may be stated:

"If a body is in equilibrium under a Q-force system, and remains in equilibrium while it is subjected to a small virtual distortion, the virtual work done by the external Q-forces acting on the body is equal to the virtual work of distortion done by the internal Q-stresses."

III. APPLICATION TO TRUSSED STRUCTURES

Before equation 2 may be used as a tool to evaluate the deflection of the joints of a trussed structure, expressions for the virtual work of distortion for trussed structures must be developed. In the most common case, where the usual assumptions of truss stress analysis are permissible, truss members are subjected to only axial stresses when the truss is loaded by joint loads, Hence, if the Q-force system consists of joint loads on the truss, these Q-forces will develop only axial Q-stresses in the members of the truss. Further, if the deflection of the truss is caused by joint loads or by a uniform change in temperature in any one member, the virtual distortion of each member will be only an axial change in length. The virtual work of distortion done by the internal Q-stresses during the virtual distortion of the truss may easily be evaluated. Any prismatic member of the truss is considered, as shown in Fig. 2, having a length, L, a

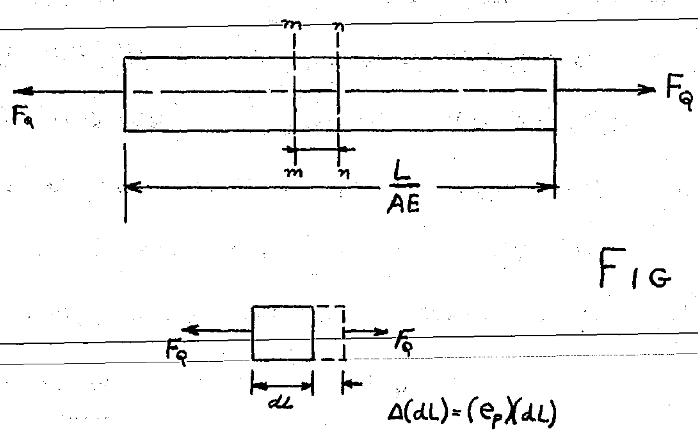


FIG 2

cross sectional area, A, and a modulus of elasticity E. The total axial stress in this member due to the Q-loads acting on the truss is F_Q, which is plus when s tension. An element of the member between two cross sections m-m and n-n which are a differential distance. dL, apart is considered. This element will be equal to $[F_Q]$ $[\Delta(dL)]$. If the axial strain of the member is $[e_p]$, then $\Delta(dL) = (e_p)(dL)$. The virtual work of distortion for the whole member will be the sum of the virtual work for all of the elements, or will equal

$$\int_0^L [F_Q][e_p]\, dL = [F_Q][e_p L] = [F_Q][\Delta L_p]$$

since, if the member is prismatic, e_p is constant over the entire length of the member. Note that ΔL_p designates the change of length of the member due to the cause of distortion of the truss, and is positive when an elongation. The virtual work of distortion for the entire truss will then be the sum of the products $[F_Q][\Delta L_p]$ for each member, or,

$$W_D = \Sigma F_Q \Delta L_p \qquad \text{Eq. (3)}$$

If δ_p denotes the displacement of the point of application of one of the external forces, Q, of the Q-system, along its line of action during the virtual distortion of the truss, then W_s may be represented by

$$W_s = \Sigma Q \delta_p \qquad \text{Eq. (4)}$$

since it has been shown that W_s is equal to the virtual work done by the Q-forces acting on the external boundaries of the body during its virtual distortion.

Hence, by substituting from equations 3 and 4 into equation 2, the law of virtual work in the case of trusses structures may be expressed as,

$$\sum Q \delta_p = \sum F_Q \Delta L_p \qquad \text{Eq. (5)}$$

It is important that at this point that the assumptions and limitations of this derivation be emphasized so that the flexibility and generality of the method be recognized.

(1) The only requirement of the external Q-forces and the internal Q-stresses is that they form a system of force which are in equilibrium throughout the virtual distortion. For this requirement to be satisfied it is necessary to assume that the virtual distortion has not been enough to vary the geometry of the structure appreciably.

(2) The relations derived are independent of the cause of distortion and require only that the changes in length of the members of the truss be compatible with the displacements of the structure, i.e., these relations are true whether the distortion is due to loads, temperature, lack of fit of members, or other causes.

(3) Since, in the general expressions, all virtual work terms are assumed as positive, the sign

convention for the displacements δ_p is that they are positive when in the direction of the corresponding external Q-forces.

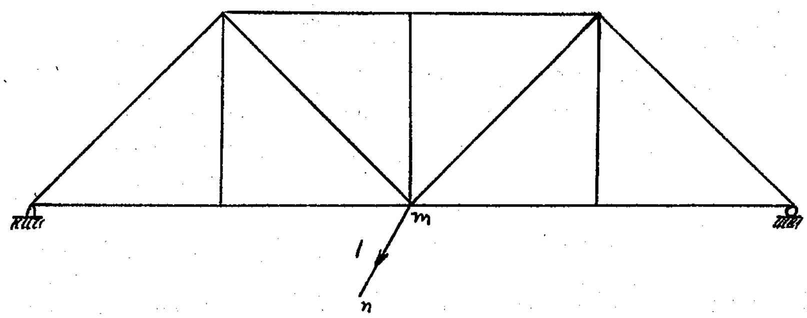


Fig 3

In order to use equation (5) for computation of deflections of joints of a truss, it is necessary to select a suitable Q-force system. For example in Fig. 3, if a unit load, applied at joint m in the direction mn, together with the reactions it caused were selected as the Q-force system, then the external virtual work done by this sytem, as expressed by the left hand side of equation (5), would be equal to $(1)(\delta_m) + W_R$, where δ_m is the deflection of joint m in the direction mn due to the distortion of the truss, and W_R is the virtual work done by the reactions of this unit load. If, due to the cause of distortion, the supports of this truss were unyielding, W_R , the virtual work done by the Q-reactions would be zero. Knowing the condition of distortion of the truss, and hence the change in length of the members of the truss, the right hand side of equation (5) could be evaluated. Hence from equation (5),

$$(1)(\delta_m) = \Sigma\ F_{\varphi}\ \Delta L_p \qquad \text{Eq. (6)}$$

and the value of δ_m may be readily determined. This procedure will be demonstrated by the numerical examples included.

In the evaluation of the right hand side of equations (5) or (6) it is necessary to compute the change in length of each member of the truss due to the cause of distortion being investigated.

If the distortion is due to joint loads P on the truss,

$$\Delta L_p = (e_p)(L) = \left(\frac{f_p}{E}\right)(L) = \left(\frac{F_p}{A}\right)\left(\frac{L}{E}\right) = \frac{F_p L}{AE} \qquad \text{Eq. (7)}$$

If the distortion is due to a change in temperature, t,

$$\Delta L_p = (e_t)(L) = (\varepsilon t)(L) = \varepsilon t L \qquad \text{Eq. (8)}$$

If the distortion is due to both loads P and a change in temperature,

$$\Delta L_p = \frac{F_p L}{AE} + \varepsilon t L$$

In the above expressions and in the computations the following notation has been used,

- total axial stress in any member due to the P-loads
- length of any member

A = gross area of cross ~~section~~ of any member
E = modulus of elasticity
t = change in temperature of any member
ε = coefficient of thermal expansion for the material

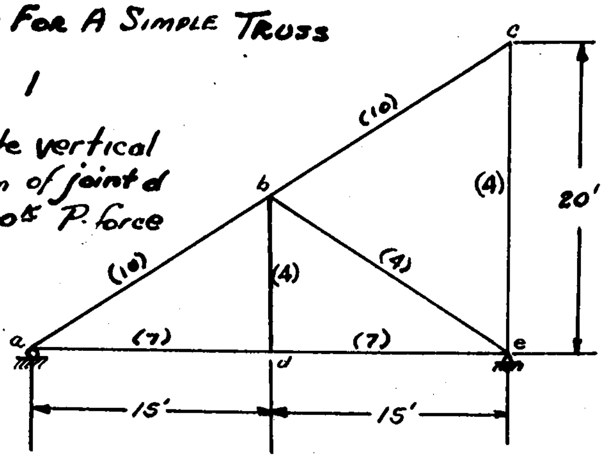
Examples 1, 2, 3, and 4 illustrate the applications of these principles of virtual work to trusses.

Example 1 shows the application in the computation of the vertical deflection of a joint; example 2, the computation of the horizontal component of a joint; example 3, the computation of the horizontal movement of a joint due to a misfit of the members; example 4, the computation of the relative movements of two joints of a truss on account of leading or temperature changes.

SOLUTION FOR A SIMPLE TRUSS

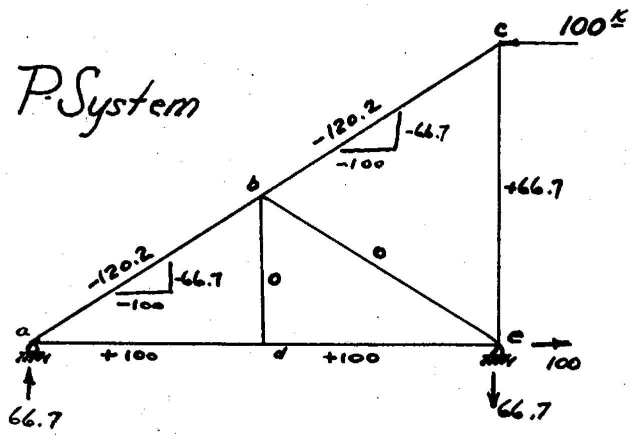
EXAMPLE 1

To Compute vertical deflection of joint d due to 100ᵏ P force shown.

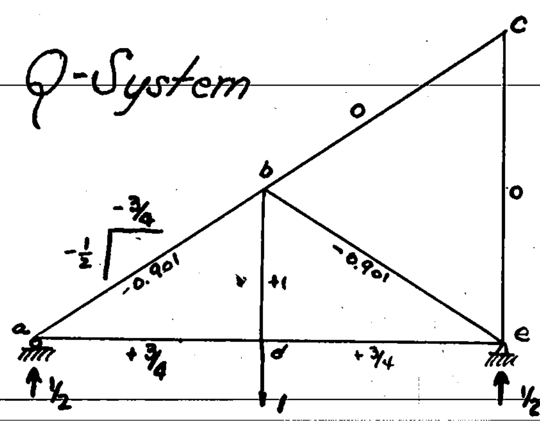


NUMBERS IN PARENTHESES ARE BAR AREAS

P System



Q-System



EXAMPLE 1
cont'd

Bar	L'	A in^2	L/A	F_Q k	F_P k	$F_Q F_P L/A$
ab	18	10	1.8	-0.901	-120.2	+195
bc	18	10	1.8	0	-120.2	
ce	20	4	5	0	+66.7	
bd	10	4	2.5	+1.0	0	
ad	15	7	2.14	+0.75	+100	+160.5
de	15	7	2.14	+0.75	+100	+160.5
be	18	4	4.5	-0.901	0	

$E = 30 \times 10^3$ k/in^2 $\Sigma = +516.0$

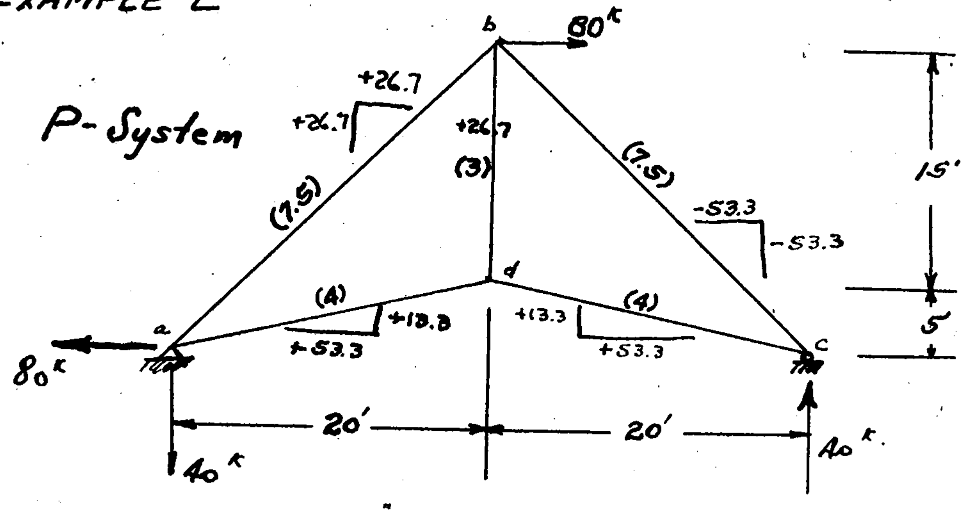
$$(1)(\delta_d) = \frac{1}{E} \Sigma F_Q F_P \frac{L}{A}$$

$$= \frac{1}{30 \times 10^3} \times 516.0$$

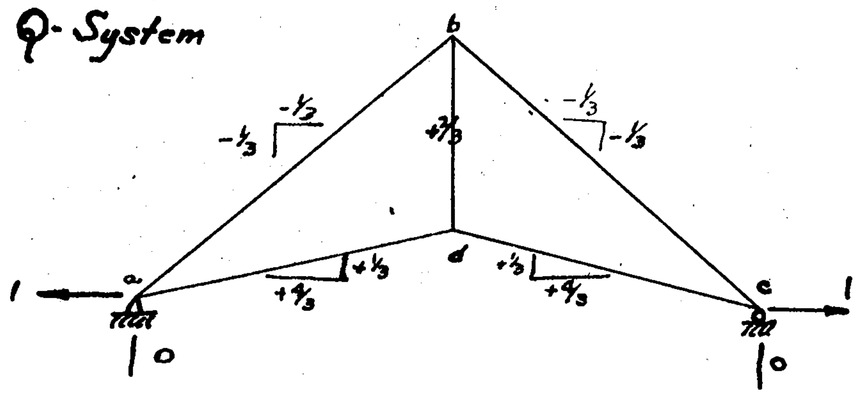
$$= 0.01720' \text{ down}$$

EXAMPLE 2

P-System



Q-System



$Y_{AD} + X_{AB} = 40$

$4\,Y_{AD} + X_{AB} = 80$

$-3\,Y_{AD} = -40$

$Y_{AD} = +13.3$

$X_{AB} = +26.7$

$4\,Y_{AD} = +53.3$

$Y_{AD} + X_{AB} = 0$

$4\,Y_{AD} + X_{AB} = 1$

$Y_{AD} = +1/3$

: Compute the horizontal component of the deflection of joint c due to 80^k load.

EXAMPLE 2
cont'd

Bar	L	A in²	L/A	F_Q	F_P	$F_Q F_P$ L/A
ab	28.28	7.5	3.78	−0.471	+37.8	−67.2
bc	28.28	7.5	3.78	−0.471	−75.4	+134.0
ad	20.6	4	5.15	+1.375	+55.0	+390.0
dc	20.6	4	5.15	+1.375	+55.0	+390.0
bd	15	3	5	+0.667	+26.7	+89.0

$E = 10^3 \times 30$

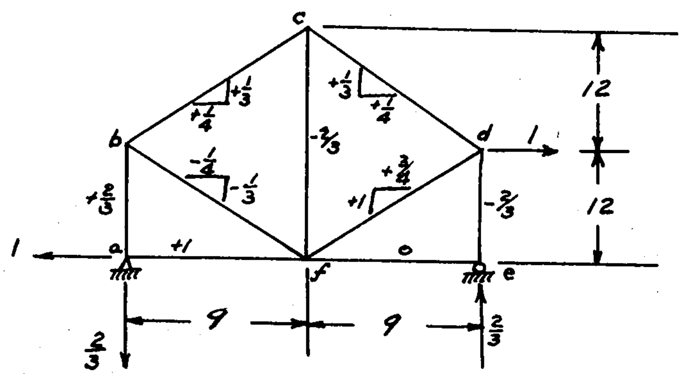
$$
\begin{array}{r}
+1003.0 \\
-\ 67.2 \\
\hline
\Sigma F_P F_Q \, L/A = +935.8
\end{array}
$$

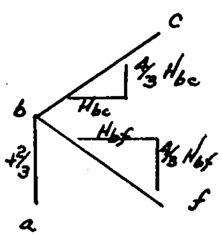
$$(1)(\delta_c) = \frac{1}{E} \Sigma F_P F_Q \frac{L}{A}$$

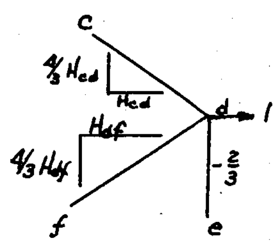
$$\delta_c = \frac{+935.8}{30 \times 10^3} = 0.03119$$

to right.

EXAMPLE 3 - Compute the horizontal component of the deflection of joint d if bar cf is shortened 1". [$\Delta L = -1"$]





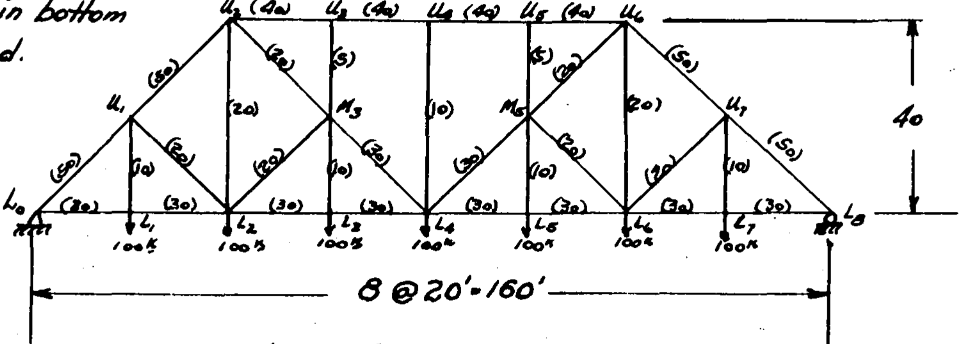


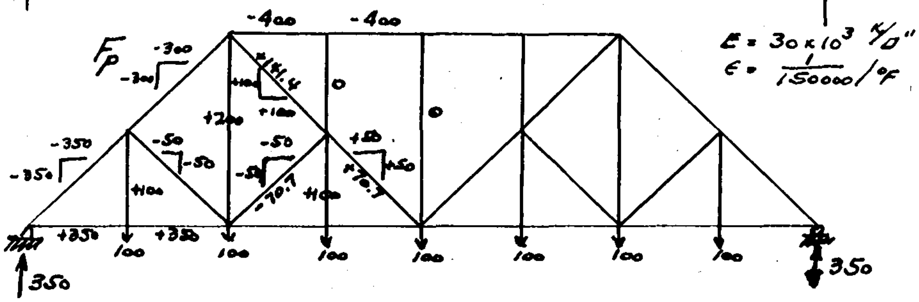
$$\tfrac{4}{3} H_{cd} + \tfrac{4}{3} H_{fd} = \tfrac{4}{3}$$
$$\tfrac{4}{3} H_{cd} - \tfrac{4}{3} H_{fd} = -\tfrac{2}{3}$$
$$\tfrac{8}{3} H_{cd} = +\tfrac{2}{3}$$
$$H_{cd} = +\tfrac{1}{4}$$
$$\tfrac{4}{3} H_{cd} = \tfrac{1}{3}$$
$$H_{df} = +\tfrac{3}{4}$$
$$\tfrac{4}{3} H_{df} = +1$$

$$\tfrac{2}{3} + \tfrac{4}{3} H_{bf} - \tfrac{4}{3} H_{bc} = 0$$
$$H_{bf} - H_{bc} = 0$$
$$-\tfrac{8}{3} H_{bc} = -\tfrac{2}{3}$$
$$H_{bc} = \tfrac{1}{4}$$
$$\tfrac{4}{3} H_{bc} = \tfrac{1}{3}$$
$$H_{bf} = -\tfrac{1}{4}$$
$$\tfrac{4}{3} H_{bf} = -\tfrac{1}{3}$$

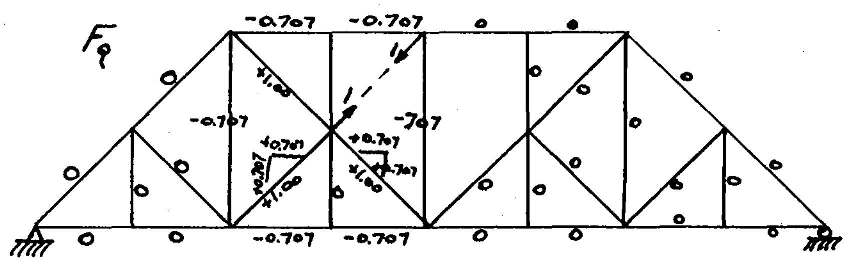
$$\Sigma Q\delta = \Sigma F_Q \Delta L$$
$$(1)(\delta_d) = (-\tfrac{2}{3}) \times (-1) = +\tfrac{2}{3}" \text{ right}$$

EXAMPLE 4: The computation of relative movements of joints M_3 U_4 along the line M_3U_4 a) by loading shown b) by temperature change of 50° in bottom chord.





$E = 30 \times 10^3$ k/□"
$\epsilon = \frac{1}{150000}$/°F



Bar	L	A	L/A	F_Q	F_P	$F_Q F_P \frac{L}{A}$	t	$F_Q t L$
	ft.	in²	ft./in²	K.	K.	K²·ft/in²	°F	K·ft·°F
U_2L_2	40'	20	2	-0.707	+200	-282.5		
U_2L_4	40'	40	1	-0.707	-400	+282.5		
L_2L_4	40	30	1.3	-0.707	+350	-330	+50	-1414.0
U_2M_3	28.28	20	1.414	+1.00	+141.4	+200		
M_3L_4	28.28	30	0.943	+1.00	+70.7	+66.6		
L_2M_3	28.28	20	1.414	+1.00	-70.7	-100 Σ = -163.3		Σ = -1414.0

a) Due to loading $\Sigma (F_Q F_P \frac{L}{A}) \frac{1}{E}$

$$Q^k \delta^{p^k}_{M_3\text{-}U_4} = \frac{-163.3}{30 \times 10^3} = -0.00544' \text{ apart}$$

b) Due to temperature

$$Q \delta^t_{M_3\text{-}U_4} = F_Q \epsilon t L = \frac{1}{150000} \times -1414.0$$

$$\delta = -0.00943 \text{ apart}$$

IV. APPLICATION TO BEAMS

The loads applied to a beam may develop both shearing and normal stresses on the boundaries of an element bounded by two adjacent cross sections and two planes parallel to the axis of the beam, as shown in Fig. 4. Hence the deflection of the beam due to the P-loads is caused by the distortion of such elements due to

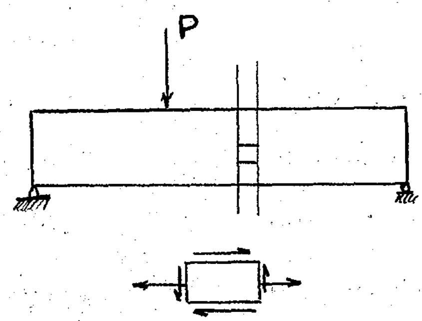


FIG 4

these two types of stress. More detailed analysis shows that, unless a beam is very deep in comparison to its length, the deflection caused by the shearing stresses acting on these elements is a small percentage of the deflection caused by the normal stresses. Therefore in the numerical applications given the effect of the shearing stresses is neglected and only the effect of distortion due to the normal stresses will be considered.

In order to use the law of virtual work to compute the deflection of beams, it is necessary to develop the expression for the virtual work of distortion, W_d, in equation (2): $W_s = W_d$. Since this discussion is limited to the distortion due to the normal stresses only, the beam may be considered as a bundle of fibers, lying parallel to the axis of the beam, which elongatex or contract depending upon whether the fiber is subjected to a normal tensile stress or a normal compressive stress. By injecting this consideration, it follows that the expression for the virtual work of distortion for a

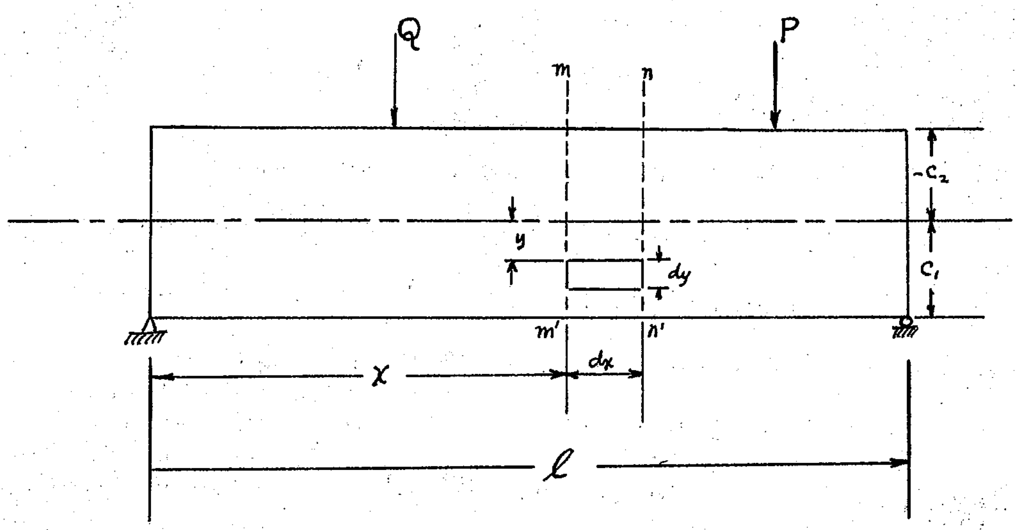


FIG 5

beam may be developed utilizing the expression previously developed for a truss.

Consider any beam such as the simple beam shown in Fig. 5. To develop an expression for the virtual work of distortion done by the stresses due to the Q-load system during the distortion of the beam caused by the P-load system, equation 3 will be used,

$$W_D = \Sigma F_Q \, \Delta L_P$$

M_Q = bending moment on section mm' due to the Q-load system
M_P = bending moment on section mm' due to the P-load system
f_Q = normal fiber stress due to moment at fiber y
f_P = normal fiber stress due to moment at fiber y
I = moment of inertia of cross section nn'
b = width of cross section at fiber y

Considering the element of the beam located at point (s,y), and having a length dx, a width b and a depth dy.

$$F_Q = f_Q\, b\, dy = \frac{M_Q y}{I} b\, dy \; ; \qquad \Delta L_P = \frac{f_P}{E} dx = \frac{M_P y}{EI} dx$$

Designating the virtual work of distortion for this element as dW_d

$$dW_d = (F_Q)(\Delta L_P) = \left(\frac{M_Q y}{I} b\, dy\right)\left(\frac{M_P}{EI} y\, dx\right) = \left(\frac{M_Q M_P}{EI}\right)\left(\frac{y^2 b\, dy}{I}\right)(dx)$$

and W_d for the whole beam will be the sum of the dW_d for all elements or,

$$W_D = \int_0^{\ell}\int_{-c_2}^{c_1}\left(\frac{M_Q M_P}{EI}\right)\left(\frac{y^2 b\, dy}{I}\right) dx \qquad \text{Eq. (9)}$$

Since M_Q and M_P are functions of only x, noting that

$$\int_{-c_2}^{c_1} y^2 b\, dy = I$$

this expression reduces to,

$$W_D = \int_0^{\ell} \frac{M_Q M_P}{EI} dx \qquad \text{Eq. (10)}$$

The expression for W_s will be the same as for trusses and therefore from equation 4,

$$W_s = \Sigma Q \delta_p$$

Substituting in equation 2 from equations 10 and 4,

$$\Sigma Q \delta_p = \int \frac{M_Q M_P}{EI} dx \qquad \text{Eq. (11)}$$

Equation 11 is the expression which can be most easily utilized in the application of the method of virtual work to the solution of a beam deflection problem. As defined previously, M_Q and M_P represent the bending moment acting on any cross section due to the Q- and P-load systems respectively, and δ_P is the displacement, due to the P-load system, of the point of

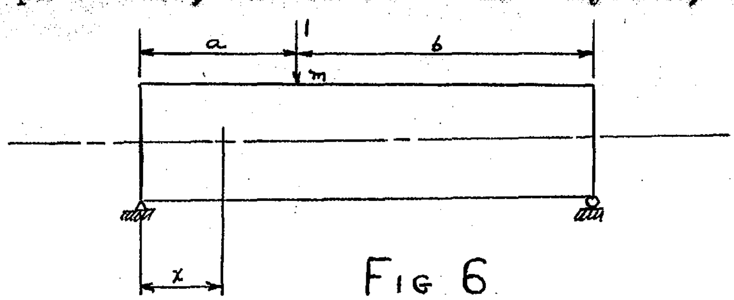


Fig 6

application of any particular force Q along its line of action. A word of caution: any suitable sign convention may be used for M_Q and M_P as long as the same on is used for both, but usually the ordinary beam convention is the most satisfactory; and δ_P is plus when in the direction of the corresponding Q force.

When applying equation 11 to find the deflection of a point on a beam, a suitable Q-load system should first be selected. Suppose it is desired to find the vertical deflection of point m on the beam in Fig. 6 due to the distortion caused by some given P-load system. For this purpose, a Q-load system is selected consisting of a unit vertical load at point m together with the reactions it develops. Then applying equation 11, the external virtual work done by the selected Q-loads, during the deflection of the beam due to the P-loads, would be equated to the virtual work of distortion done by the internal Q-stresses. The external virtual work would be equal to

$$[(1)(\delta_m) + W_R]$$

where W_R represents the virtual work done by the reactions caused by the unit load.

Then, from equation 11

$$(1)(\delta_m) + W_R = \int M_Q M_P \frac{dx}{EI} \qquad \text{Eq. (12a)}$$

or, if the supports are unyielding, then $W_R = 0$, and

$$(1)(\delta_m) = \int M_Q M_P \frac{dx}{EI} \qquad \text{Eq. (12b)}$$

To obtain δ_m, the integral on the right hand side of equations 12a and 12b must be evaluated for the entire beam. Before performing the integration M_Q and M_P must first be expressed as functions of x. Of course, it is usually necessary to separate the integration over the entire beam into the sum of several integrals over portions of the beam; since, whenever there is a change in the functions expressing M_Q, M_P or $\underline{I}$ in terms of $\underline{x}$, it is necessary to break up the integ-

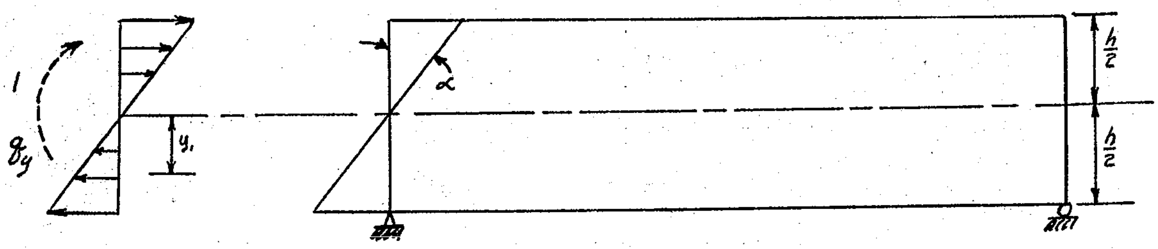


FIG 7

ration at the points where such changes occur. The integration process can often by simplified by selecting different origins for the measurement of $\underline{x}$ for these different portions of the beam. A convenient technique illustrating this point is shown in the numerical examples 5, 6, and 7.

Oftentimes it is necessary to find the change in slope of a cross section of a beam due to the distortion caused by some given P-load system. The essential difference between this problem and the solution for the vertical deflection of a point lies in the selection of the Q-load system. If it can be advantageous to compute the change in slope of the cross section of the left end of the beam, shown in Fig. 7 due to a given P-load system, the Q-load is selected as a uniformly varying distributed pressure over the end cross section of the beam - - having zero intensity at the gravity axis and being a compressive pressure on the top half and a tensile pressure on the bottom half of the cross section. Such a load would have a resultant which is a couple. The magnitude of these pressures should be such that the resultant is a couple of unity, then

$$q_y = \frac{(1)(y)}{I}$$

Assuming that the cross sections remain plane as the beam distorts (which would be the case if shear distortion is neglected), a point on the cross section a distance y from the gravity axis would move through a horizontal deflection equal to αy where α is the change in slope of the end cross section. Then,

$$W_s = \Sigma Q\delta_p = \int_{-\frac{h}{2}}^{+\frac{h}{2}} (q_y\, b\, dy)(\alpha y) + W_R$$

$$= \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(\frac{1y}{I} b\, dy\right)(\alpha y) + W_R) = \frac{1\alpha}{I}\int_{-\frac{h}{2}}^{+\frac{h}{2}} y^2 b\, dy + W_R$$

$$= (1)(\alpha) + W_R \qquad \text{Eq. (13)}$$

where W_R again represents the external virtual work done by the reactions of the Q-load system. From this equation, it can be seen that the virtual work done by this distributed Q-load system is simply equal to its resultant couple of unity times the change in slope of the cross section. Then, substituting in equation 11, the following equation is obtained if the supports are unyielding and therefore W_R is equal to zero,

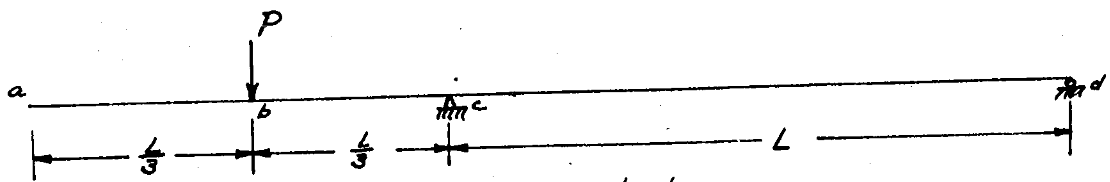
$$(1)(\alpha) = \int \frac{M_q M_p\, dx}{EI} \qquad \text{Eq. (14)}$$

By evaluating the right hand side of this equation in the same manner as discussed above when obtaining the vertical deflection of a point on a beam, the change in slope may be readily determined. To obtain the change in slope of any other cross section of a beam the same method may be employed. The Q-load system selected is a uniformly varying distributed pressure acting on this cross section, which is reduced to its resultant couple of unity so far as the calculations are concerned.

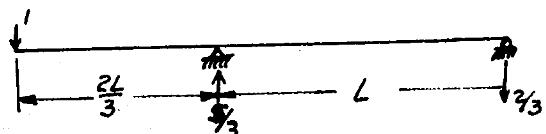
Examples 5, 6, and 7 illustrate the application of virtual work to the solution for the components of the deflection at a point on a beam and for the rotation of any cross section. Example 5 is shown for a beam with a constant moment of inertia; example 6 on a beam with a differing moment of inertia. The effect of distortion due to direct stress and bending is computed in example 7.

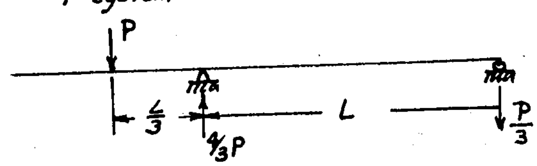
EXAMPLE 5 : The computation of the vertical deflection and change in slope at point a.

a) Vertical deflection computation



E and I are constant





$$\Sigma Q(\delta_a^{\downarrow}) = \int M_Q M_P \frac{dx}{EI} \qquad Q\,\delta_a = \int_a^b + \int_b^c + \int_d^c$$

From	a-b	$M_P = 0$		
	b-c	$M_Q = -\left(\frac{L}{3}+x\right)$	$M_P = -Px$	x from 0 to $\frac{L}{3}$
	d-c	$M_Q = -\frac{2}{3}x$	$M_P = -\frac{Px}{3}$	x from 0 to L

$$Q(\delta_a^{\downarrow}) = \frac{1}{EI}\left[\int_0^{\frac{L}{3}} -\left(\frac{L}{3}+x\right)(-Px)\,dx + \int_0^L -\left(\frac{2}{3}x\right)\left(-\frac{Px}{3}\right)dx\right]$$

$$= \frac{P}{EI}\left[\int_0^{L/3}\left(\frac{Lx}{3}+x^2\right)dx + \int_0^L \frac{2x^2}{9}\,dx\right]$$

$$= \frac{P}{EI}\left[\frac{Lx^2}{6}+\frac{x^3}{3}\Big|_0^{L/3} + \frac{2x^3}{27}\Big|_0^L\right]$$

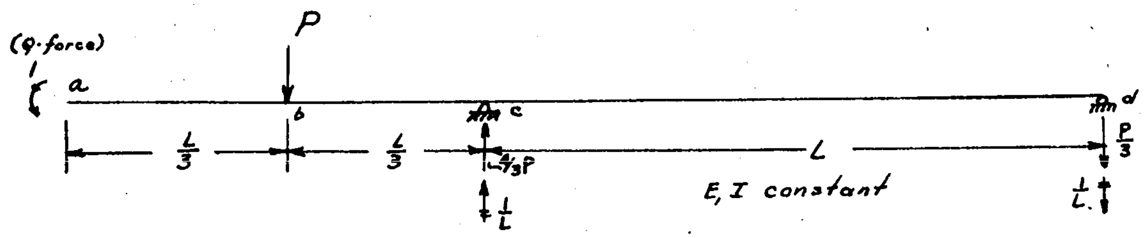
$$= \frac{P}{EI}\left[\frac{L^3}{9\cdot 6}+\frac{L^3}{3\cdot 27}+\frac{2L^3}{27}\right]$$

$$= \frac{PL^3}{EI}\left[\frac{1}{27}\left(\frac{1}{2}+\frac{1}{3}+2\right)\right]$$

$$\delta_a^{\downarrow} = \frac{PL^3}{EI}\left(\frac{1}{27}\,\frac{3+2+12}{6}\right) = +\frac{17PL^3}{162EI} \quad \text{down}$$

EXAMPLE 5
cont'd

b) Rotation of cross section at a. (change in slope)



$$(1)(\alpha) = \Sigma\, M_P M_Q \frac{dx}{EI}$$

From
a-b $\quad M_P = 0$
b-c $\quad M_Q = -1 \quad M_P = -Px \quad x$ from 0 to $\frac{L}{3}$
d-c $\quad M_Q = \frac{x}{L} \quad M_P = -\frac{P}{3}x \quad x$ from 0 to L

$$(1)(\alpha) = \frac{1}{EI}\left[-\int_0^{\frac{L}{3}}(-Px)\,dx - \int_0^L\left(-\frac{Px^2}{3L}\right)dx\right]$$

$$= \frac{P}{EI}\left[\frac{x^2}{2}\Big|_0^{L/3} + \frac{x^3}{9L}\Big|_0^L\right]$$

$$= \frac{P}{EI}\left[\frac{L^2}{18} + \frac{L^2}{9}\right]$$

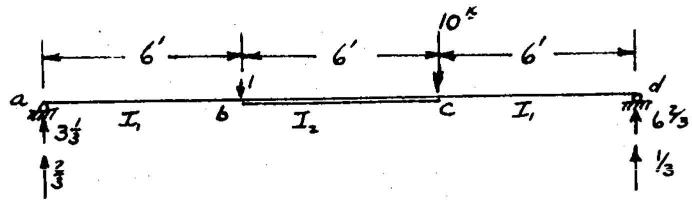
$$= \frac{PL^2}{6EI} \quad \text{counterclockwise}$$

EXAMPLE 6a: The computation of the vertical deflection at b.

$E = 30 \times 10^3\ ^{k}/_{in^2}$

$I_1 = 300\ in^4$

$I_2 = 500\ in^4 = \frac{5}{3} I_1$



$$\Sigma Q(\delta_b^{\downarrow}) = \Sigma \int M_p M_Q \frac{dx}{EI}$$

From a·b : $M_Q = +\frac{2}{3}x \quad M_p = +\frac{10}{3}x \quad I_1 \quad x$ from 0-6

b·c : $M_Q = +\frac{2}{3}(6+x) - x \quad M_p = +\frac{10}{3}(x+6) \quad I_2 \quad x$ from 0-6

d·c : $M_Q = +\frac{1}{3}x \quad M_p = +\frac{20}{3}x \quad I_1 \quad x$ from 0-6

$$Q(\delta_b^{\downarrow}) = \frac{1}{E}\left[\frac{20}{9}\int_0^6 x^2 \frac{dx}{I_1} + \frac{20}{9}\int_0^6 (6+x)^2 \frac{dx}{I_1}\left(\frac{3}{5}\right) - \frac{10}{3}\int_0^6 (x^2+6x)\frac{dx}{I_1}\left(\frac{3}{5}\right) + \frac{20}{9}\int_0^6 x^2 \frac{dx}{I_1}\right]$$

$$= \frac{20}{9EI_1}\left[\frac{x^3}{3}\Big|_0^6 + \frac{3}{5}\left(36x + 6x^2 + \frac{x^3}{3}\Big|_0^6 - \frac{3}{2}\cdot\frac{3}{5}\left(\frac{x^3}{3} + 3x^2\Big|_0^6 + \frac{x^3}{3}\Big|_0^6\right]$$

$$= \frac{20}{9EI_1}\left[\frac{216}{3} + \frac{\cancel{3}}{5}\cdot\frac{7}{\cancel{3}}216 - \frac{9}{10}\cdot\frac{5}{3}\cdot 108 + \frac{216}{3}\right] = \frac{20}{9EI_1}\left[\frac{2\cdot 216\cdot 10 + 7\cdot 216\cdot 6 - 45\cdot 108}{30}\right]$$

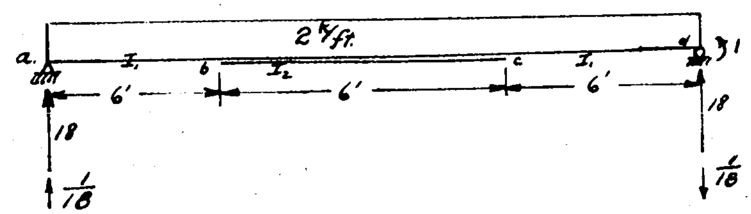
$$= \frac{20}{9EI_1}\left[\frac{4320 + 9072 - 4860}{30}\right]$$

$$= \frac{20}{9EI_1}\cdot\frac{8532}{30} = \frac{17064}{27EI_1}$$

$$= \frac{632}{EI_1}$$

$$\delta_b = \frac{632}{30\times 10^3 \times 144 \cdot \frac{300}{144^2}} = \frac{632 \times 144}{9 \times 10^6} = 0.0102' \text{ down}$$

EXAMPLE 6b: The computation of the change in slope at d due to loading shown.



$$\Sigma Q(\alpha) = \int M_Q M_P \frac{dx}{EI}$$

From a-b $\quad M_Q = \frac{1}{18}x \quad M_p = 18x - x^2 \quad I = I_1 \quad x$ from 0-6

b-c $\quad M_Q = \frac{1}{18}x \quad M_p = 18x - x^2 \quad I = I_2 = \frac{5}{3}I_1 \quad x$ from ~~0~~ 6-12

d-c $\quad M_Q = (1 - \frac{x}{18}) \quad M_p = 18x - x^2 \quad I = I_1 \quad x$ from 0-6

$$\overset{\curvearrowleft}{Q(\alpha)} = \frac{1}{E}\left[\cancel{\int_0^6 \frac{x}{18}(18x - x^2)\frac{dx}{I_1}} + \int_6^{12}\left(\frac{3}{5}\right)\left(\frac{x}{18}\right)(18x - x^2)\frac{dx}{I_1} + \int_0^6\left(1 - \cancel{\frac{x}{18}}\right)(18x - x^2)\frac{dx}{I_1}\right]$$

$$= \frac{1}{EI_1}\left[\frac{3}{5}\left(\frac{x^3}{3} - \frac{x^4}{72}\right)\Big|_6^{12} + \left(9x^2 - \frac{x^3}{3}\right)\Big|_0^6\right]$$

$$= \frac{1}{EI_1}\left[\frac{3}{5}\left(\frac{144 \cdot 12}{3} - \frac{144 \cdot 144}{72} - \frac{36 \cdot 6}{3} + \frac{36 \cdot 36}{72}\right) + 9 \cdot 36 - \frac{216}{3}\right]$$

$$= \frac{1}{EI_1}\left[\frac{3}{5}(576 - 288 - 72 + 18) + 252\right]$$

$$= \frac{702 + 1260}{5EI_1} = \frac{1962}{5EI_1}$$

$$\overset{\curvearrowleft}{Q(\alpha)} = \frac{1962}{5 \times 30 \times 10^3 \times \cancel{144} \times \frac{300}{144}} = 0.00628 \text{ radians}$$

Counterclockwise

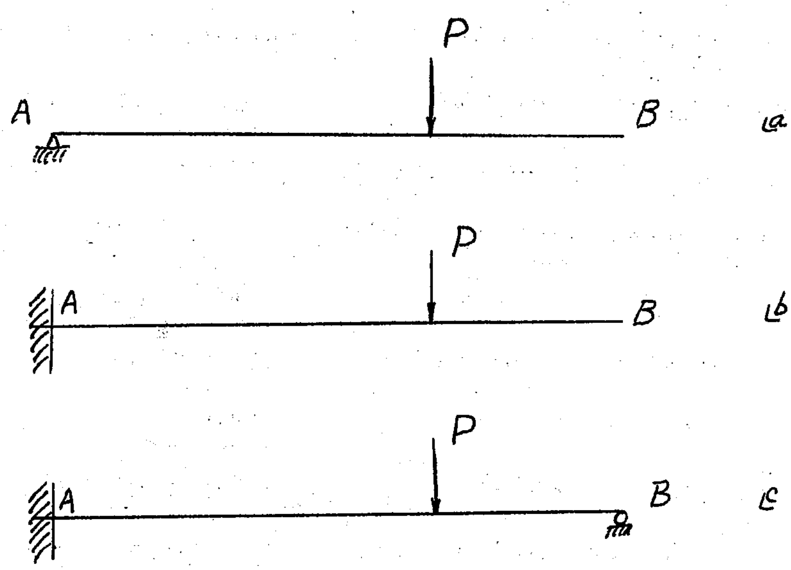
V. APPLICATION TO INDETERMINATE STRUCTURES

In the last twenty years, statically indeterminate structures have steadily become more prevalent in this country. Their more extensive use is no doubt due largely to their economy and increased rigidity under moving loads. Examples of indeterminate structures are: continuous beams and trusses, two-hinged and hingeless arches, rigid frame bridges, suspension bridges, building frames, etc.

Indeterminate structures have certain distinguishing properties which determinate structures do not have; their stress analysis is dependent not only on their geometry but also on their elastic properties, such as modulus of elasticity, and cross-sectional area and moment of inertia of their members. An indeterminate structure must be designed before its stress analysis can be made. The final design of an indeterminate structure must be approached by a series of approximations, starting with a rough preliminary design and approaching a satisfactory final design by a series of red signs.

Further stresses are developed in indeterminate structures not only by loads but also by changes in temperature, settlement of supports and lack of fit of members.

Typical structural action of an indeterminate structure may be studied by considering a structural member AB which is loaded by a force P. If this member is supported only by a hinged support, as shown in Fig. 8a, there are obviously insufficient reactive forces applied to the member to prevent its rotation about A



F I G 8

when the load P is applied. Such a structure is unstable because there are insufficient reactions to prevent its movement as a rigid body. If, however, a moment restraint is inserted at A, being added to the hinged support, converts A into a fixed support, as shown in Fig. 8b. Then, when the load P is applied, there are just enough reactions to prevent movement of member AB as a rigid body, and the equations of static equilibrium may be used to determine these reactions. Such a structure is called a statically determinate structure. In this type of structure, the elastic distortion and resulting deflection may take place without restriction. Now, in addition to the fixed support at A, a vertical reaction is supplied at B also, as shown in Fig. 8c. This addition of a reaction supplies more than enough reactive forces to maintain static equilibrium and restrains the elastic deflection of the member at point B. The magnitude of this reaction could be determined by first computing the deflection at B caused by the application of the load P to the cantilever of Fig. 8b, and by finding the vertical force which would have to be applied at B on the cantilever to bring point B back up so it could fit on the support at that point in the structure shown in Fig. 8c. The remaining reactions at point A may now be easily found by the equations of statics. Since simply the equations of statics are not sufficient to determine the reactions of the structure of 8c, the structure is called a statically indeterminate structure. The degree to which a structure is statically indeterminate is equal to the number of restraints in excess of the minimum necessary for the static equilibrium of the structure. These additional restraints are called redundants. This discussion suggests that a statically indeterminate structure may be considered as a statically determinate structure which is acted upon by the known loads and the unknown redundants; these redundants having such a value that they cause the deflection of their points of application to satisfy the displacement conditions of the same points on the actual indeterminate structure. Mathematically the degree of the statically indeterminate structure can be found by determining the number of unknown stress components and the number of equations of statics; the degree to which the structure is indeterminate is then equal to the number of unknowns in excess of the number of equations.

Considering the indeterminate beam shown in Fig. 9a, whose supports are unyielding, there are four reaction components on this beam: two vertical, one horizontal, and one moment. There are three equations of statics available: $\Sigma H=0, \Sigma V=0$ and $\Sigma M=0$. This beam is indeterminate to the first degree, or there is one more reaction than is necessary for statci equilibrium. Select the vertical reaction at b as the redundant and call it X_b. If the vertical support at b is removed and replaced by the force X_b which it supplies, a statically determinate centilever results, acted upon by the applied load and the unknown redundant force X_b, as shown in Fig. 9b. This statically determinate and stable cantilever which remains after the removal of the redundant support is called the primary structure. If the redundant X_b has the same value as the vertical reaction at b on the actual structure, then the condition of stress in the primary structure and the actual structure will be exactly the same; and the condition of distortion of the two structures

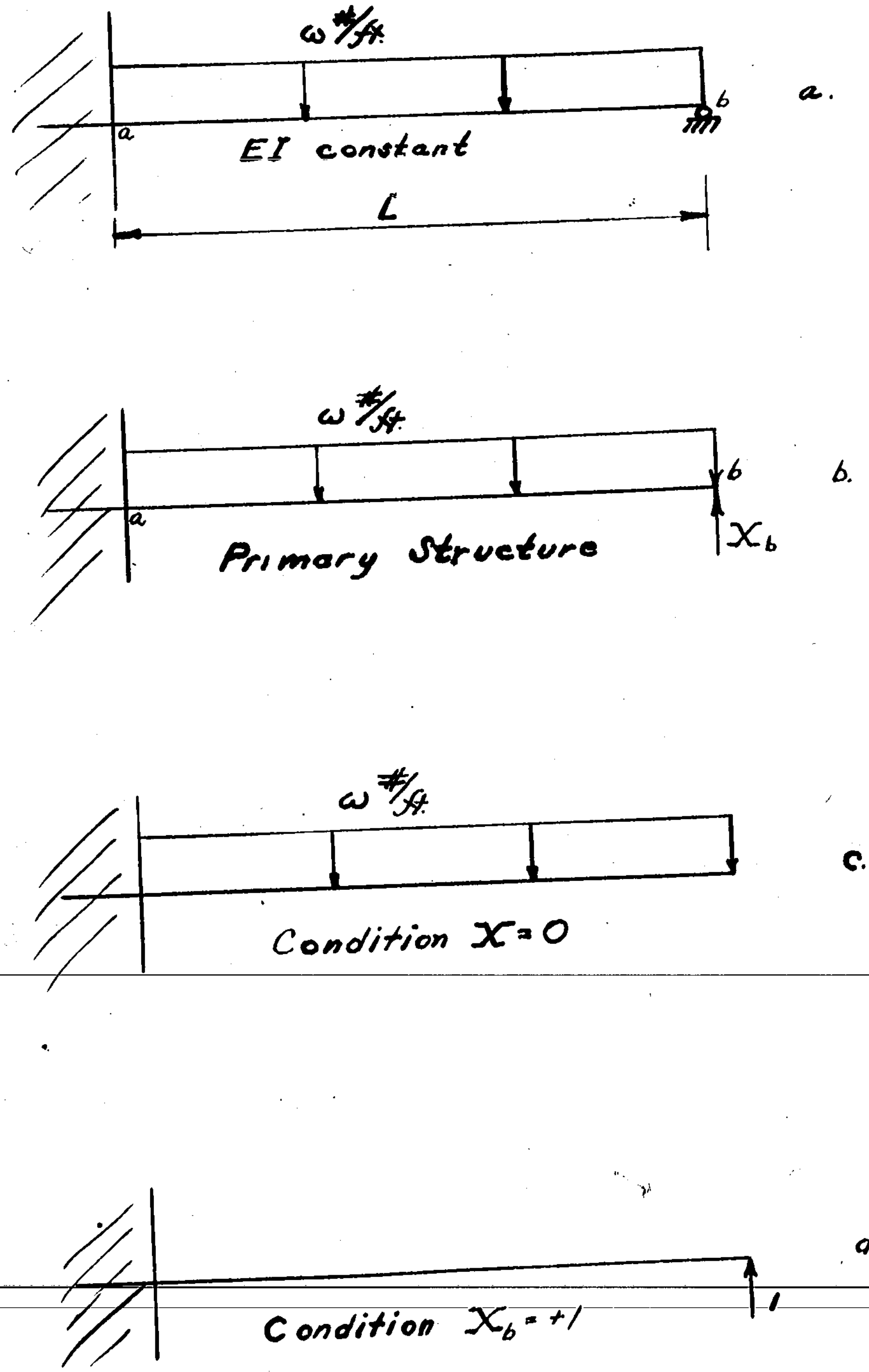

ω #/ft
b
a.
a
EI constant
L
ω #/ft
b
b.
a
X_b
Primary Structure
ω #/ft
c.
Condition X = 0
d.
1
Condition $X_b = +1$

Fig 9

must also be alike. In order for the conditions of distortion to be the same, it is necessary that the vertical deflection at point b on the primary structure, due to both the applied load and X_b, be equal to zero, since all the supports of the actual structure are unyielding. The method of virtual work may now be used to find the vertical deflection of point b on the primary structure.

The vertical deflection of point b, δ_b, in the same direction as that assumed for the force X_b - or in this case upward - will now be evaluated using equation 11 and selecting as a Q-system a unit upward load at point b. The following notation is introduced:

M_P = bending moment at any pint on primary structure due to all the loads causing distortion;

M_b = bending moment at any point on primary structure due to a unit toward load at b, hereinafter called "Condition $X_b = +1$", as shown in Fig. 9d;

M_o = bending moment at any point on primary structure due only to applied loads with redundants removed, hereinafter called "Condition $X = 0$", as shown in Fig. 9c.

Then from equation 11, $\Sigma Q \delta_P = \int \frac{M_Q M_P}{EI} dx$

in which $M_Q = M_b$

and $M_P = M_o + X_b M_b$

since M_P is the sum of the bending moments on the primary structure due to both the applied load and the redundant X_b. Or, since the supports are unyielding $\sum Q\delta_p = (1)(\delta_b)$, and

$$(1)(\delta_b) = \int (M_b)(M_o + X_b M_b)\frac{dx}{EI}$$

$$(1)(\delta_b) = \int M_b M_o \frac{dx}{EI} + X_b \int M_b^2 \frac{dx}{EI} \qquad Eq.\ 15$$

But δ_b must equal zero in order for the deflection of point b to be the same on the primary and the actual structure; hence, substitution for δ_b, equation 15 becomes

$$X_b \int M_b^2 \frac{dx}{EI} + \int M_b M_o \frac{dx}{EI} = 0$$

and

$$X_b = -\frac{\int M_b M_o \frac{dx}{EI}}{\int M_b^2 \frac{dx}{EI}} \qquad \text{Eq. (16)}$$

The integrals may easily be evaluated and the magnitude of the vertical reaction X_b on the actual structure may be computed from equation 16. If X_b is found to be plus, it acts in the direction assumed or upwards; if it is negative, it acts downward. Having the value of the redundant reaction, the remaining reactions, and the shear and bending moment at any point may be computed by statics.

This same procedure may be applied to any indeterminate structure. Fig. 10 illustrates a continuous

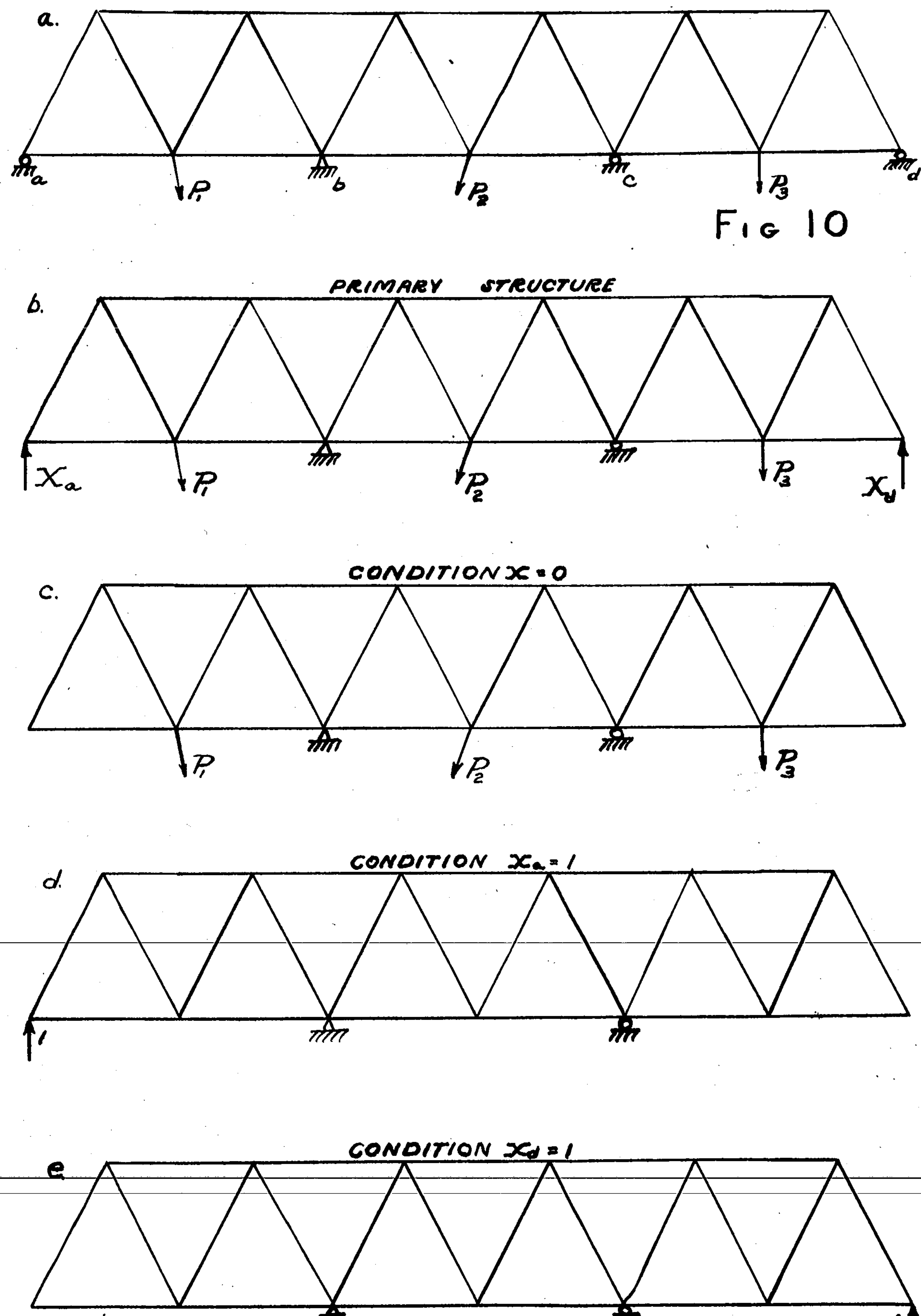

a.
a
P1
b
P2
c
P3
d
PRIMARY STRUCTURE
b.
Xa
P1
P2
P3
Xd
CONDITION X = 0
c.
P1
P2
P3
CONDITION Xa = 1
d.
1
CONDITION Xd = 1
e
1

Fig 10

truss whose supports are unyielding. The unknowns in this structure are 5 rea ction components and 23 bar stresses, a total of 28. There are 13 joints so that 26 equations of statics are available. Therefore this truss is indeterminate to the 2d degree. The redundants properly selected are the vertical reactions at a and d called X_a and X_d, respectively. By removing the vertical supports at a and d and replacing them by the forces X_a and X_d which they supply, the primary structure which is the statically determinate and stable truss left, as shown in Fig. 10b, is then acted upon by the applied loads and the unknown redundants X_a and X_d. As before, if the redundants X_a and X_d have the same value as the vertical reactions at a and d on the actual structure, the condition of stress in the primary structure will be exactly the same as the actual structure; and the condition of distortion of the two structures must also be the same. In order for the condition of distortion to be the same, it is necessary that the vertical deflection of both points a and d on the primary structure, due to the applied loads, X_a and X_d, be equal to zero, since all of the supports of the actual structure are unyielding. That is, X_a and X_d must have such values that δ_a and δ_d are simultaneously equal to zero. Now if the method of virtual work is to express the vertical upward deflections of a and d on the primary structure (that is, in the assumed directions of X_a and X_d) due to the applied loads, X_a and X_d, two equations involving the unknown redundants, X_a and X_d, would be obtained and could be solved simultaneously for their values.

Using equation 5 which expresses the law of virtual work as applied to trusses, δ_a and δ_d may now be evaluated. In doing this, the following notation is adopted:

F_p = total stress in any bar of primary structure due to all the loads causing distortion;

F_a = stress in any bar of primary structure due to a unit upward load at a, hereinafter called "Condition ", as shown in Fig. 10d.

F_d = stress in any bar of primary structure due to a unit upward load at d, hereinafter called "Condition ", as shown in Fig. 10e.

F_o = stress in any bar of primary structure due to applied loads with redundants removed, hereinafter "Condition ", as shown in Fig. 10c.

To evaluate δ_a, a unit upward load at a would be selected as the Q-force system; then applying equation 5 remembering that the supports of the primary structure are unyielding,

$$(1)(\delta_a) = \sum F_Q \frac{F_p L}{AE}$$

where $\quad F_Q = F_a \quad$ and $\quad F_P = F_o + X_a F_a + X_d F_d$

Then,

$$(1)(\delta_a) = \Sigma F_a (F_o + X_a F_a + X_a F_d) \frac{L}{AE} \qquad \text{Eq. (17)}$$

To evaluate δ_d , a unit upward load at d would be selected as the Q-force system; then, applying equation 5 remembering that the supports of the primary structure are unyielding,

$$(1)(\delta_d) = \Sigma F_Q \frac{F_P L}{AE}$$

where now $F_Q = F_d$ and F_P is the same as above.

Then,

$$(1)(\delta_d) = \Sigma F_d (F_o + X_a F_a + X_d F_d) \frac{L}{AE} \qquad \text{Eq. (18)}$$

Now in equations 17 and 18, δ_a and δ_d must be equal to zero in order for the deflection of both points a and d to be the same on the primary and the actual structure. Then, the following two equations are obtained which must be solved simultaneously for X_a and X_d .

$$0 = \Sigma F_a F_o \frac{L}{AE} + X_d \Sigma F_a^2 \frac{L}{AE} + X_d F_a F_d \frac{L}{AE} \qquad \text{Eq. (19)}$$

$$0 = \Sigma F_d F_o \frac{L}{AE} + X_a \Sigma F_a F_d \frac{L}{AE} + X_d \Sigma F_d^2 \frac{L}{AE} \qquad \text{Eq. (20)}$$

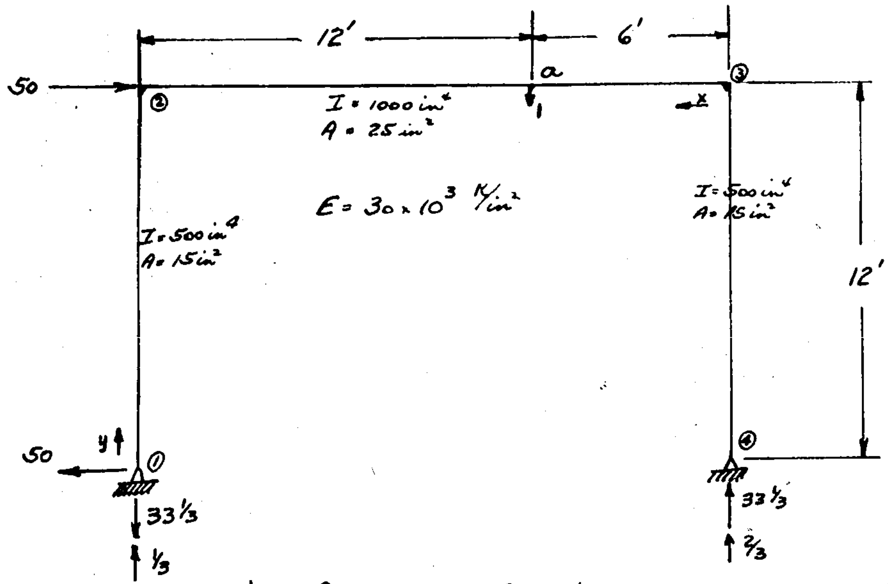
The summations may be easily evaluated and it should be noted that each of these summations includes every bar of the primary structure. Of course, some bars may be found to have zero values for some of the products shown in these equations. If the sign of either X_a or X_d is plus, the redundant acts in the assumed direction or upward; if negative, it acts downward. Having the value of the redundants the remaining reactions and bar stresses may be found by statics.

The method of virtual work may be used as described above for the stress analysis of indeterminate structures regardless whether the structure is composed of members subjected to direct stress, or bending moment or both, and whether the distortion is caused by applied loads, temperature change, settlement of the supports, or any other cause. The applications to these problems are shown in the numerical examples 8 and 9.

The general procedure may be briefly summarized as follows: If any structure is indeterminate to the nth degree, n of the stress components may be selected as the redundants. These n restraints are removed and are replaced by the n stress components which they supply on the structure, all of these acting together with the applied loads on the primary structure which remains after the removal of the n restraints. The only requirement in selecting the n redundants is that the primary structure, which remains after the removal of their restraints, be statically determinate and stable. If these n redundants have the same value on the primary structure as they do on the actual structure, then the condition of stress of the two structures is the same; and the condition of distortion must also be exactly the same. If the method of virtual work is then used to express the n distortion conditions of the points of application of the n redundants on the primary structure, n equations will be obtained in terms of the applied loads and the n redundants, whose value will be determined by solving these n equations simultaneously.

Examples 8 and 9 show the methods of solution and computation for indeterminate structures. Example 8 shows the computations of the component of a reaction on an arch due to loading, temperature changes, errors in fabrication, and support settlements. Example 9 is the computation of an influence line of the structure.

EXAMPLE 7: The computation of the vertical deflection of point a considering the effect of distortion due to direct stress and bending.



$$(Q)(\delta_a^{\downarrow}) = \int M_P M_Q \frac{dx}{EI} + \int F_P F_Q \frac{L}{AE}$$

From: ①②	$M_P = +50y$	$M_Q = 0$		
	$F_P = +33\frac{1}{3}$	$F_Q = -\frac{1}{3}$	$L = 12'$	
②-ⓐ	$M_P = 600 - \frac{100}{3}x$	$M_Q = +\frac{1}{3}x$	$I = 1000\ in^4$	x from 0-12
	$F_P = -50$	$F_Q = 0$		
④③	$M_P = 0$	$M_Q = 0$		
	$F_P = -33\frac{1}{3}$	$F_Q = -\frac{2}{3}$	$L = 12'$	
③-ⓐ	$M_P = 33\frac{1}{3}x$	$M_Q = \frac{2}{3}x$	$I = 1000\ in^4$	x from 0-6
	$F_P = 50$	$F_Q = 0$		

$$(Q)(\delta_a^{\downarrow}) = \int_0^{12}\left(600 - \frac{100}{3}x\right)\left(\frac{1}{3}x\right)\frac{dx}{EI_1} + \int_0^{6}\left(33\tfrac{1}{3}x\right)\left(\frac{2}{3}x\right)\frac{dx}{EI_1} + \left(-\frac{1}{3}\right)\left(33\tfrac{1}{3}\right)\left(\frac{12}{15}\right)\left(\frac{1}{E}\right) + \left(-33\tfrac{1}{3}\right)\left(-\frac{2}{3}\right)\left(\frac{12}{15E}\right)$$

$$= \left[100x^2 - \frac{100x^3}{27}\right|_0^{12}\frac{1}{EI_1} + \frac{200}{9}\left[\frac{x^3}{3}\right|_0^{6}\frac{1}{EI_1} + \frac{1}{3}\,\frac{100}{3}\cdot\frac{12}{15}\,\frac{1}{E}$$

$$= \left(14400 - \frac{100\cdot 1728}{27}\right)\frac{1}{EI_1} + \frac{200 \times 216}{27}\,\frac{1}{EI_1} + \frac{400}{45E}$$

$$= \frac{9600}{EI_1} + \frac{400}{45E}$$

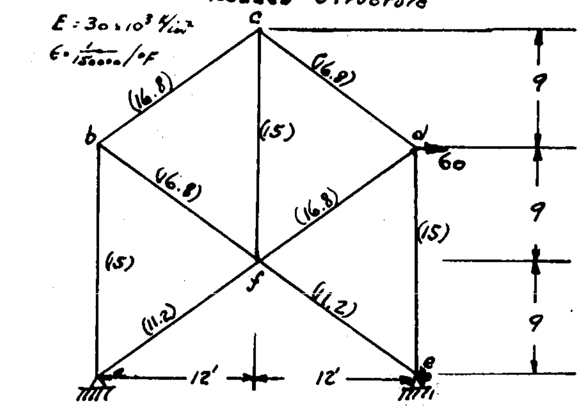
$$= \frac{9600 \times 144}{30 \times 10^3 \times 10^3} + \frac{400}{45 \times 30 \times 10^3}$$

$$= +0.046100' + 0.000296'$$

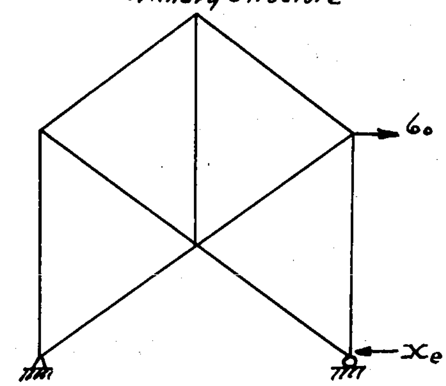
$$= +0.0464' \text{ down}$$

EXAMPLE 8: The computation of the horizontal component of the right reaction of the arch -- indeterminate to the first degree

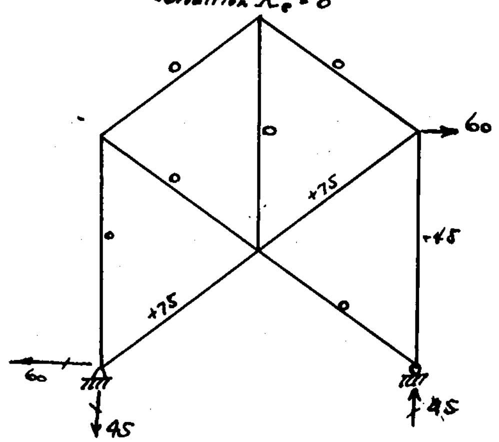
Loaded Structure



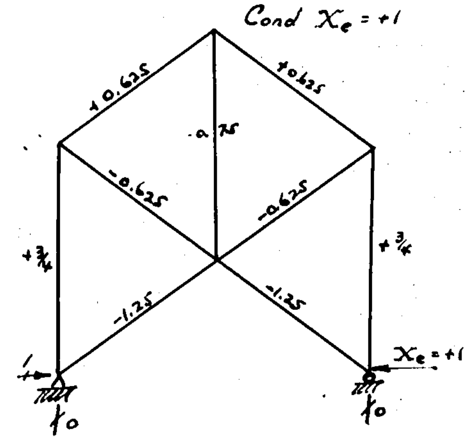
Primary Structure



Condition $X_e = 0$



Cond $X_e = +1$



$\Delta_e = \Delta_{eo} + \Delta_{ec}$

$\Delta_{ee} = X_e \delta_e$

$\Delta_e = \Delta_{eo} + X_e \delta_e$

$$\Delta_{eo} = \frac{1}{E} \Sigma F_e F_o \frac{L}{A}$$

Bar	L	A	L/A	F_e	F_o	$F_e F_o \frac{L}{A}$	$F_e^2 \frac{L}{A}$
af	15	11.2	1.34	-1.25	+75	-125.5	2.09
fd	15	16.8	0.893	-0.625	+75	-41.8	0.349
de	18	15	1.20	+0.75	-45	-40.5	0.674
fe	15	11.2	1.34	-1.25		-207.8	2.09
ab	18	15	1.2	+0.75			0.674
bc	15	16.8	0.893	+0.625			0.349
cd	15	16.8	0.893	+0.625			0.349
bf	15	16.8	0.893	-0.625			0.349
cf	18	15	1.2	-0.75			0.674
							+7.598

$(1)(\Delta_{eo}) = \frac{\cancel{L}}{\cancel{A}}\left(-\frac{207.9}{E}\right)$

$(1)(\delta_{ee}) = +\frac{7.598}{E}$

$$\Delta_e = 0 = -\frac{207.9}{E} + \frac{7.598}{E} X_e = 0$$

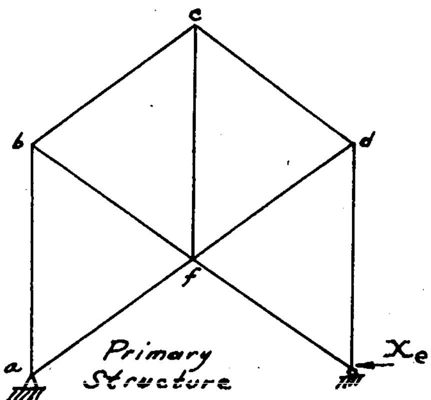
$X_e = +27.3$ to left.

EXAMPLE 8b: The computation of the horizontal component of the right reaction of the arch due to:

a) Increase in temperature of 50°F in bars ab, bc, cd, de.

b) Bars bc and cd being $\frac{1}{4}$" too short and bar cf $\frac{1}{8}$" too long on account of errors in fabrication, and it being necessary to force them into place.

c) Support Settlements as follows:
Left Support: Vertical .048" ↓ Horizontal 0.24" ←
~~Right Support~~: Vertical: 0.24" ↓ Horizontal: 0.36" →



Primary Structure

$\epsilon = \frac{1}{150000}$ per °F.

a)

Bar	F_Q	L	t	$F_Q tL$
ab	+0.75	18	+50	675
bc	+0.625	15	+50	468.5
cd	+0.625	15	+50	468.5
de	+0.75	18	+15	675.0
				+ 2287

$$(1)(\Delta_{et}) = \frac{+2287}{150000} = +0.01527$$

$$+0.01527 + X_e \frac{7.605}{30\times10^3} = 0$$

$$X_e = -\frac{457}{7.605} = -60.0$$

Reaction is 60ᵏ to right

b)

$$\Delta_e = \Sigma F_Q \, \Delta L_e$$

$$\Delta_e = -0.0332$$

$$-0.0332 + X_e \frac{7.605}{30\times10^3} = 0$$

$$X_e = \frac{996}{7.605} = +131^k \text{ to left.}$$

Bar	ΔL_e	F_Q	$F_Q \Delta L_e$
bc	$-\frac{1}{48}'$	+0.625	−0.0130
cd	$-\frac{1}{48}'$	+0.625	−0.0130
cf	$+\frac{1}{96}'$	−0.75	−0.00782
			−0.0332

c) Settlement of Supports

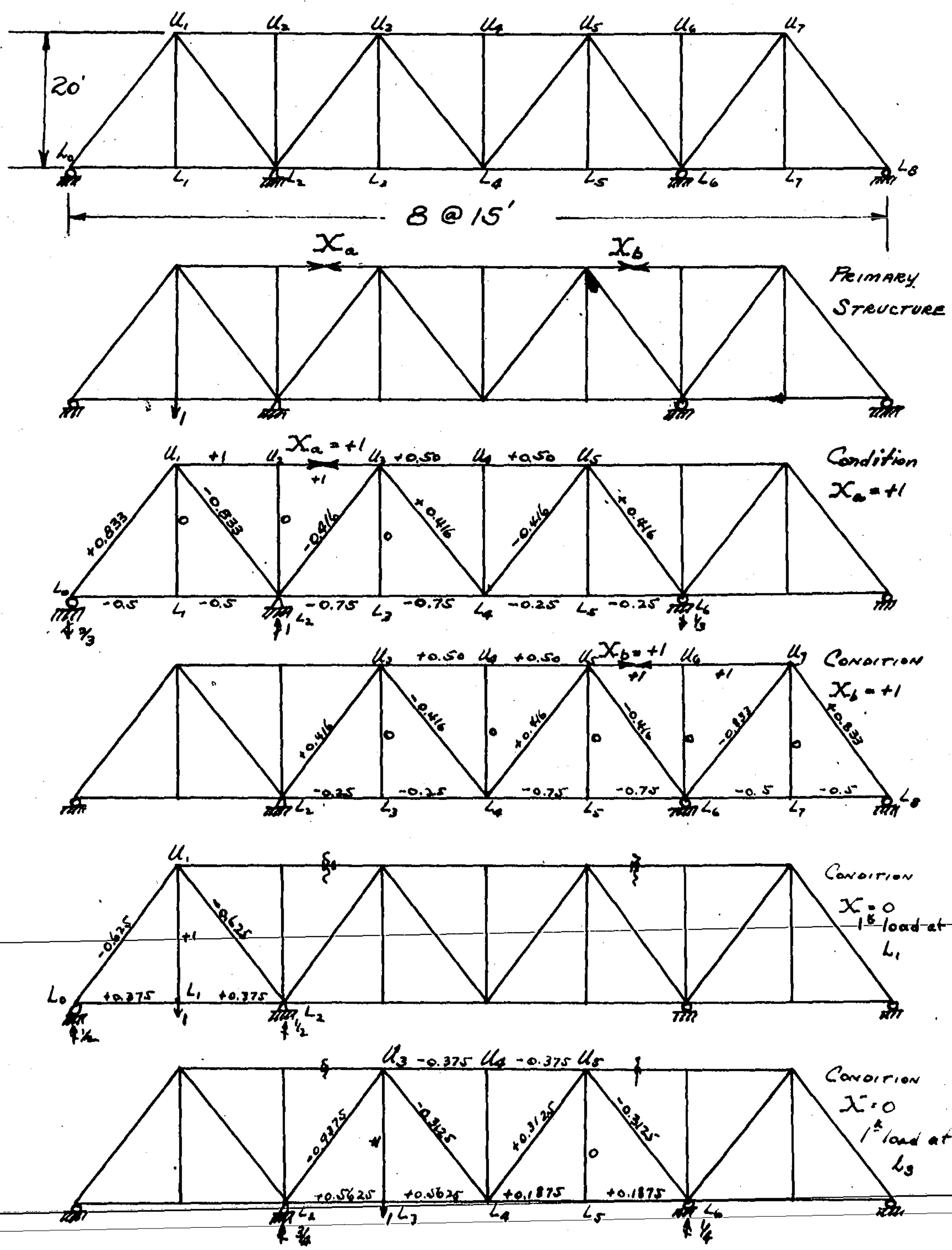
$$-(1 \times 0.02') + (1)(\Delta_{e_s}) = 0$$

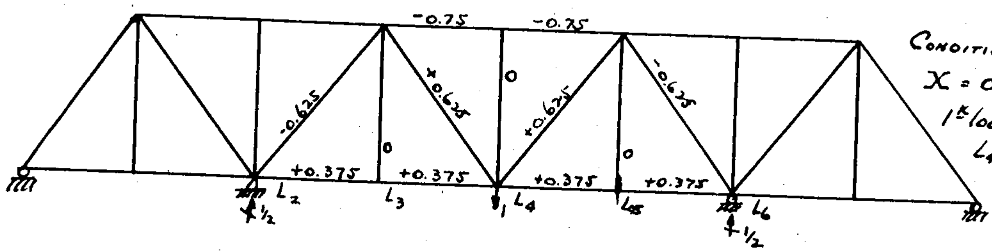
$$\Delta_{es} = +0.02'$$

$$-0.03 = +0.02 + X_e \frac{7.605}{30\times10^3}$$

$$X_e = -\frac{1500}{\cancel{30\times10^3}\; 7.605} = -197.1^k \text{ to right}$$

EXAMPLE 9: The computation of the influence line ordinates for the stress in U_2U_3.
$A=1$ for all members





CONDITION
$X = 0$
1^k load at L_4

$$A = \Sigma F_a^2 \frac{L}{A}$$

$$B = \Sigma F_a F_b \frac{L}{A}$$

EVALUATION OF A, B

Since $A = 1$ $\frac{L}{A}$ numerically equals L

Bar	$\lvert L/A \rvert = \lvert L \rvert$	F_a	F_b	$F_a^2 \frac{L}{A}$	$F_a F_b \frac{L}{A}$
	ft/in²	k	k	k²·ft/in²	k²·ft/in²
$L_0 U_1$ $L_2 U_0$	25	+0.833 -0.833	— —	+34.7	— —
$L_2 U_3$ $U_3 L_4$ $L_4 U_5$ $U_5 U_6$	25	-0.416 +0.416 -0.416 +0.416	+0.416 -0.416 +0.416 -0.416	+17.35	-17.35
$L_0 L_1$ $L_1 L_2$	15	-0.5	—	7.5	—
$L_2 L_3$ $L_3 L_4$	15	-0.75	-0.25	16.85	+11.25
$L_4 L_5$ $L_5 L_6$	15	-0.25	-0.75	1.87	
$U_1 U_2$ $U_2 U_3$	15	+1	—	30.0	
$U_4 U_3$ $U_4 U_5$	15	+0.50	+0.50	7.5	+7.5

$\Sigma =$ +115.77 +1.40

$$\therefore \quad A = \Sigma F_a^2 \frac{L}{A} = +115.77 \ \frac{K^2 ft}{in^2} \qquad A^2 = 13400$$

$$B = \Sigma F_a F_b \frac{L}{A} = +1.40 \ \frac{K^2 ft.}{in^2} \qquad B^2 = 1.96 \qquad A^2 - B^2 = 13400$$

Load term M in Span 1

Bar	$\frac{L}{A}$	F_a	F_o	$F_o F_a \frac{L}{A}$
	ft^2/in^2	K.	K.	$\frac{K^2 ft.}{in^2}$
L_0U_1	25	+0.833	-0.625	-13.0
U_1L_2	25	-0.833	-0.625	+13.0
L_0L_2	30	-0.5	+0.375	-5.625

$$\Sigma = -5.625 \frac{K^2 ft.}{in^2} = M_{span\,1}$$

Load term M and N in Span 2

Bar	$\frac{L}{A}$	F_a	F_b	F_o (1^k load at L_3)	$F_a F_o \frac{L}{A}$	$F_b F_o \frac{L}{A}$	F_o (1^k load at L_4)	$F_a F_o \frac{L}{A}$	$F_b F_o \frac{L}{A}$
L_2U_3	25	-0.416	+0.416	-0.9375	+9.75	-9.75	-0.625	+6.5	-6.5
U_3L_4	25	+0.416	-0.416	-0.3125	-3.25	+3.25	+0.625	+6.5	-6.5
L_4U_5	25	-0.416	+0.416	+0.3125	-3.25	+3.25	+0.625	-6.5	+6.5
U_5L_6	25	+0.416	-0.416	-0.3125	-3.25	+3.25	-0.625	-6.5	+6.5
U_3U_5	30	+0.50	+0.50	-0.375	-5.62	-5.62	-0.75	-11.25	-11.25
L_2L_4	30	-0.75	-0.25	+0.5625	-12.65	-4.22	+0.375	-8.43	-2.81
L_4L_6	30	-0.25	-0.75	+0.1875	-1.4	-4.22	+0.375	-2.81	-8.43
					-19.67	-14.06		-22.49	-22.49

With load at L_3

$$M = \Sigma F_a F_o \frac{L}{A} = -19.67$$

$$N = \Sigma F_b F_o \frac{L}{A} = -14.06$$

With load at L_4

$$M = \Sigma F_a F_o \frac{L}{A} = -22.49$$

$$N = \Sigma F_b F_o \frac{L}{A} = -22.49$$

With load in Span 1 =

$$A X_a + B X_b = -M$$
$$B X_a + A X_b = -N = 0$$
$$X_a = -\frac{AM}{A^2 - B^2} = -\frac{(115.77)(-5.625)}{13400} = +0.04854$$
$$X_b = -\frac{B}{A} X_a = -\frac{+1.40}{+115.77}(+0.04854) = -0.000588$$

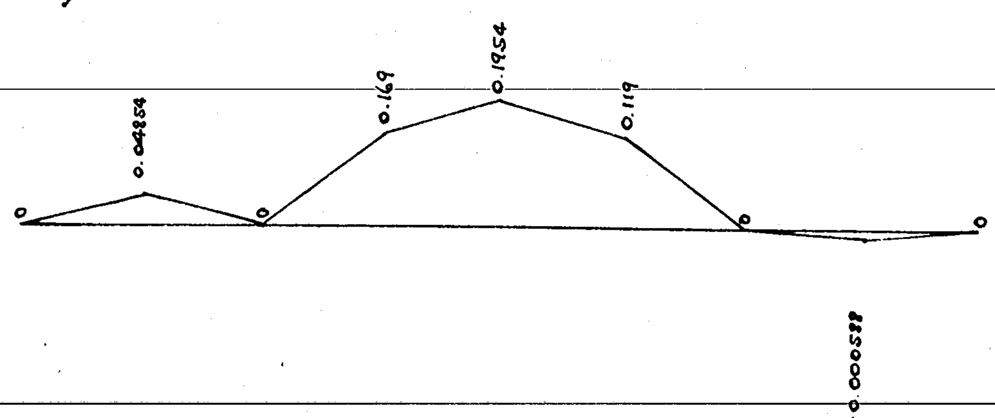
With load at L_3

$$A X_a + B X_b = -M$$
$$B X_a + A X_b = -N$$
$$X_a = -\frac{(AM - BN)}{A^2 - B^2} = -\frac{(115.77)(-19.77) - (+1.40)(-14.0)}{13400} = +0.169$$
$$X_b = -\frac{AN - BM}{A^2 - B^2} = -\frac{(115.77)(-14.0) - (1.40)(-19.77)}{13400} = +0.1190$$

With load at L_4

$$X_a = X_b = -\frac{(AM - BN)}{A^2 - B^2} = -\frac{(115.77)(-22.49) - (1.40)(-22.49)}{13400} = +0.1954$$

Influence Line



VI. SUMMARY

The determination of the reactions of an indeterminate structure by the method of virtual work involves removing as many reactions as are necessary to permit a solution by the equations of statics. Stress analyses are performed first assuming all the replaced forces to be zero then by using each replaced force separately as a unity force. The effect of the unity force on the structure is compared with the deflection obtained with the extraneous forces equalled to zero but with the structure loaded, and the additional reactions are thereby ascertained.

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