1992

Projective and non-projective systems of first order nonlinear differential equations

Riad A. Rejoub
University of the Pacific

PROJECTIVE AND NON-PROJECTIVE SYSTEMS OF FIRST ORDER NONLINEAR DIFFERENTIAL EQUATIONS

A THESIS

PRESENTED TO

THE GRADUATE FACULTY OF THE

UNIVERSITY OF THE PACIFIC

IN PARTIAL FULFILLMENT

OF THE REQUIRMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

BY

RIAD A. REJOUB

APRIL 1992

AKNOWLEDGEMENTS

I would like to express my extreme gratitude to all professors in this department for their support and help.

I am particularly thankful to Professor Carl E. Wulfman for his thoughtful advice and profound international experience and endless effort during this work.

Special thanks must be extended to Professor Richard L. Perry for giving me a chance to study at U.O.P

Finally my wife and I wish to express our thanks to Mrs. Wulfman and Mrs Perry for their generous hospitality during our stay at Stockton.

TABLE OF CONTENTS

INTRODUCTION

CHAPTER 1: TRANSFORMATIONS CARRIED OUT BY EXP($\alpha\mathfrak{L}$) 1

CHAPTER 2: PROJECTIVE GROUP 5

2.1 DERIVATION OF PROJECTIVE GROUP OF THE PLANE IN TERMS OF A PAIR OF DIFFERENTIAL EQUATIONS 6

2.2 THE NONLINEAR SUBGROUP OF THE PROJECTIVE GROUP 7

CHAPTER 3: NON-PROJECTIVE TRANSFORMATIONS 9

3.1 SIX-PARAMETER FAMILY OF DIFFERENTIAL EQUATIONS WITH QUADRATIC TERMS 9

3.2 THE PHASE SPACE OF SYSTEMS WITH QUADRATIC NONLINEARITY 10

3.3 TOPICS IN ANALYTICAL GEOMETRY 14

3.4 INDEX OF A CRITICAL POINT AND INDEX OF A CURVE 17

3.5 NON-PROJECTIVE ORDINARY DIFFERENTIAL EQUATIONS IN A POLAR COORDINATE SYSTEM 22

CHAPTER 4: CLASSIFICATION OF NON-PROJECTIVE SYSTEMS

IN TERMS OF THE NATURE OF THEIR CRITICAL
POINT AND PHASE PORTRAIT 28

4.1 INTRODUCTION 28

4.2 DEFINITIONS 29

4.3 TYPES OF NON-PROJECTIVE SYSTEMS 32

1) HYPERBOLIC-HYPERBOLIC TYPE 33

2) CIRCULAR-HYPERBOLIC TYPE 46

3) CIRCULAR-ELLIPTICAL TYPE 48

CONCLUSIONS 52

APPENDIX 55

REFERENCES 58

INTRODUCTION

It is well established that many physical and chemical phenomena such as those in chemical reaction kinetics, laser cavities, rotating fluids, and in plasmas and in solid state physics are governed by nonlinear differential equations whose solutions are of variable character and even may lack regularities. Such systems are usually first studied qualitatively by examining their temporal behavior near singular points of their phase portrait.

In this work we will be concerned with systems governed by the time evolution equations

$$\frac{dx_i}{dt} = \sum_j k_{ij}\, x_j + \sum_{j,j'} k_{ijj'} x_j x_{j'} \qquad i,j = 1,2\ ,\ j' \le j\ .$$

The x_i may generally be considered to be concentrations of species in a chemical reaction, in which case the k's are rate constants. In some cases the x_i may be considered to be position and momentum variables in a mechanical system. We will divide the equations into two classes: those in which the evolution can be carried out by the action of one of Lie's transformation groups of the plane, and those for which this is not possible.

Members of the first class can be integrated by quadrature either directly or by use of an integrating factor; those in the second class cannot. Of those in the first class the most interesting evolve by transformations of the projective group, and these, as well as the equations that cannot be integrated by quadrature, we study in some detail. We seek a qualitative analysis of systems which have no linear terms in their evolution equations when the origin from which the x_i are measured is a critical point. The standard, linear, phase plane analysis is of course not adequate for our purposes.

CHAPTER 1

TRANSFORMATIONS CARRIED OUT BY EXP($\alpha\mathfrak{L}$)

Consider a continuous group consisting of transformations which transform the coordinate x_i into $\bar{x}_i$, with

$$\bar{x} = \phi(x, \alpha) \tag{1.1}$$

Here $x = (x_1, x_2, \ldots, x_n)$, and $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ are vectors in n dimensional space and α represents r effective group parameters.

The transformation could be written in detail as[1]

$$\bar{x}_i = \phi_i(x_1, x_2, \ldots, x_n; \alpha_1, \alpha_2, \ldots, \alpha_r) \qquad (i=1,2,\ldots,n) \tag{1.2}$$

The parameters are chosen such that

$$x = \phi(x, 0). \tag{1.3}$$

If we perform an infinitesimal transformation $d\alpha$ from the identity, x is tranformed into $\bar{x} = x + dx$, i.e

$$\bar{x} = x + dx = \phi(x, \alpha + d\alpha) \tag{1.4}$$

This yields

$$dX = \emptyset(x, \alpha - d\alpha) - \emptyset(X, 0) \tag{1.5}$$

Expanding about some neighborhood of $\alpha=0$, we can write to the lowest order of $d\alpha$:

$$\begin{aligned} dX &= \emptyset(X,0) + \left[\frac{\partial}{\partial\alpha}\emptyset(X, \alpha)\right]\Big|_{\alpha=0} \cdot d\alpha - \emptyset(X,0) \\ &= \left[\frac{\partial}{\partial\alpha}\, \emptyset(X, \alpha)\right]\Big|_{\alpha=0} \cdot d\alpha \end{aligned} \tag{1.6}$$

Introducing the abbreviation

$$\mathfrak{U}(X) = \left[\frac{\partial}{\partial\alpha}\, \emptyset(X, \alpha)\right]\Big|_{\alpha=0} \tag{1.7}$$

we have

$$dX = \mathfrak{U}(X) \cdot d\alpha, \tag{1.8}$$

or in standard form,

$$\frac{dx_i}{d\alpha_\mu} = U_{i\mu}(X). \tag{1.8a}$$

Here

$$U_{i\mu} = \left[\frac{\partial}{\partial\alpha_\mu}\, \emptyset_i(X, \alpha)\right]\Big|_{\alpha=0}.$$

We could also write the infinitesimal change of an analytic $F(X)$ under the transformation as

$$dF = \frac{\partial F}{\partial X} \cdot dX = \sum \frac{\partial F(X)}{\partial x_i} dx_i \ .$$

Since

$$dX = \mathfrak{U}(X) \; d\alpha$$

then

$$dF = \frac{\partial F(X)}{\partial X} \cdot \mathfrak{U}(X) \; d\alpha$$

$$= d\alpha \left\{ \mathfrak{U}(X) \cdot \frac{\partial F(X)}{\partial X} \right\}$$

$$= \sum_{\mu, i} d\alpha_\mu \left\{ U_{i\mu}(X) \frac{\partial}{\partial x_i} \right\} F(X)$$

$$= \sum_{\mu, i} d\alpha_\mu \; \mathfrak{L}_\mu(X) \; F(X) \ . \tag{1.9}$$

Here

$$\mathfrak{L}_\mu = \sum U_{i\mu}(X) \frac{\partial}{\partial x_i}$$

$$= \sum \left\{ \frac{\partial}{\partial \alpha\mu} \varnothing_i(X) \right\} \Big|_{\alpha=0} \frac{\partial}{\partial x_i} \tag{1.10}$$

are the generators of the group.

Now because of the group property of infinitesimal transformations, the finite transformation is obtained by a non-denumerable infinity of such transformations. Let

$$d\alpha = \frac{\alpha_\mu}{N} \ , \text{ (with } N \text{ a large number)} \tag{1.11}$$

then

$$\begin{aligned} F(x,\alpha) &= \lim_{N\to\infty} (1 + \mathfrak{L}_\mu\, d\alpha_\mu)^N F(x,0) \\ &= \lim_{N\to\infty} (1 + \mathfrak{L}_\mu\, \alpha_\mu/N)^N F(x,0) \\ &= \exp(\mathfrak{L}_\mu\, \alpha_\mu)\, F(x,0) \\ &= \mathcal{V}(x,\alpha) F(x,0) \ . \end{aligned} \tag{1.12}$$

It is clear that

$$\mathcal{V}(x,\alpha) = \exp(\alpha_\mu\, \mathfrak{L}_\mu),$$

is the group operator and $\mathfrak{L}_\mu$ are the group generators.

The previous discussion may be generalized. If one has a set of n ordinary differential equations of the form

$$\frac{dx_i}{d\alpha_\mu} = U_{i\mu}(x)$$

one may view this as defining an infinitesimal transformation of the variables x_i.

If $d\alpha_\mu$ is replaced by dt (time), then the evolution operator of the system is obtained.

CHAPTER 2

PROJECTIVE GROUP

In this chapter we will focus our study on investigating the projective group of transformations and its properties.

2.1 DERIVATION OF PROJECTIVE GROUP OF THE PLANE IN TERMS OF A PAIR OF DIFFERENTIAL EQUATIONS[2]

Consider the projective group of the plane

$$\bar{x}_1 = \frac{(1+\alpha_1)x_1 + \alpha_2 x_2 + \alpha_3}{-\alpha_7 x_1 - \alpha_8 x_2 + 1}$$

$$\bar{x}_2 = \frac{\alpha_4 x_1 + (1+\alpha_5)x_2 + \alpha_6}{-\alpha_7 x_1 - \alpha_8 x_2 + 1} \tag{2.1}$$

The identity transformation is obtained by making all parameters equal zero.

If we consider the infinitesimal transformation where α_i becomes $d\alpha_i$ and use expression (1.6), we have

$$dx_1 = \sum_{\mu} \left[\frac{\partial}{\partial\alpha_\mu}\left(\frac{(1+\alpha_1)x_1 + \alpha_2 x_2 + \alpha_3}{-\alpha_7 x_1 - \alpha_8 x_2 + 1}\right)\right]\Big|_{\alpha=0} \cdot d\alpha$$

$$= x_1 \cdot d\alpha_1 + x_2 \cdot d\alpha_2 + d\alpha_3 + x_1 \cdot x_1 \cdot d\alpha_7 + x_1 x_2 d\alpha_8 \tag{2.2}$$

$$dx_2 = \sum_{\mu} \left[\frac{\partial}{\partial \alpha_\mu} \left(\frac{\alpha_4 x_1 + (1-\alpha_5)x_2 + \alpha_6}{-\alpha_7 x_1 - \alpha_8 x_2 + 1}\right)\right]\Bigg|_{\alpha=0} \cdot d\alpha$$

$$= x_1 \cdot d\alpha_4 + x_2 \cdot d\alpha_5 + d\alpha_6 + x_2 \cdot x_2 \cdot d\alpha_8 + x_1 \cdot x_2 \cdot d\alpha_7$$

If we set $d\alpha_i = k_i da$ (da is an infinitesimal of first order) we have

$$dx_1 = (\ k_1 x_1 + k_2 x_2 + k_3 + k_7 x_1^2 + k_8 x_1 x_2\ )\ da$$

$$dx_2 = (\ k_4 x_1 + k_5 x_2 + k_6 + k_7 x_1 x_2 + k_8 x_2^2\ )\ da \qquad (2.3)$$

These two equations are a pair of ordinary differential equations whose infinitesimal generators are [3]

$$U_{10} = \frac{\partial}{\partial x_1},\quad U_{20} = \frac{\partial}{\partial x_2},\quad U_{11} = x_1 \frac{\partial}{\partial x_1},$$

$$U_{12} = x_2 \frac{\partial}{\partial x_1},\quad U_{21} = x_1 \frac{\partial}{\partial x_2},\quad U_{22} = x_2 \frac{\partial}{\partial x_2}$$

$$U_{111} + U_{212} = x_1^2 \frac{\partial}{\partial x_1} + x_1 x_2 \frac{\partial}{\partial x_2} \qquad (2.4)$$

$$U_{222} + U_{112} = x_2^2 \frac{\partial}{\partial x_2} + x_1 x_2 \frac{\partial}{\partial x_1}$$

The operator of a general transformation of the projective group is a linear combination of the above generators:

$$\mathfrak{L} = (k_{10} + k_{11} x_1 + k_{12} x_2 + k_{112} x_1 x_2 + k_{111} x_1^2)\ \frac{\partial}{\partial x_1} \qquad (2.5)$$

$$+ (k_{20} + k_{21} x_1 + k_{22} x_2 + k_{112} x_2^2 + k_{111} x_1 x_2) \frac{\partial}{\partial x_2}$$

The finite transformation is carried out by $\exp(\alpha\mathfrak{L})$. As mentioned earlier, an evolution operator is obtained by replacing the parameter da by dt, where t is time.

2.2 THE NON-LINEAR SUBGROUP OF THE PROJECTIVE GROUP

Consider the nonlinear part of the above differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= + k_{111}x_1^2 + k_{112}x_1x_2 \\ \frac{dx_2}{dt} &= + k_{111}x_1x_2 + k_{112}x_2^2 . \end{aligned} \tag{2.6}$$

(Notice that k_{111}, and k_{112} are common in both equations).

If we multiply both sides of these two eqns by x_2 and $-x_1$ respectively, and add them together, we obtain the identity

$$x_2 \frac{dx_1}{dt} - x_1 \frac{dx_2}{dt} = 0. \tag{2.7}$$

If we divide both sides by x_2^2, we have

$$\frac{x_2\, dx_1/dt - x_1\, dx_2/dt}{x_2^2} = 0 \tag{2.7a}$$

The left hand side is simply the time derivative

$$\frac{d}{dt}\left(\frac{x_1}{x_2}\right) = 0 \tag{2.7b}$$

which identifies the bracketed quantity as a constant. This determines the phase curves and suggests that the original equations may be solved exactly.

In fact the general solution of (2.3) is (2.1), if one interprets the x_i as initial values, the $\bar{x}_i$ as final values, and sets $\alpha_i = k_i \alpha$.

CHAPTER 3

NON-PROJECTIVE TRANSFORMATIONS

In this chapter we will consider differential equations whose evolution operator contains no contribution from generators of the projective group.

3.1 THE SIX-PARAMETER FAMILY OF NON-PROJECTIVE DIFFERENTIAL EQUATIONS

Consider the 6-parameter family of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= c_{111}x_1^2 + c_{112}x_1x_2 + c_{122}x_2^2 \\ \frac{dx_2}{dt} &= c_{211}x_1^2 + c_{212}x_1x_2 + c_{222}x_2^2 \end{aligned} \tag{3.1}$$

where the c_{ijk} are parameters.

The evolution operator

$$\begin{aligned} \mathfrak{L} = {} & (c_{111}x_1^2 + c_{112}x_1x_2 + c_{122}x_2^2)\frac{\partial}{\partial x_1} \\ & + (c_{211}x_1^2 + c_{212}x_1x_2 + c_{222}x_2^2)\frac{\partial}{\partial x_2} \end{aligned} \tag{3.2}$$

is not a linear combination including any projective group generators in the sense that $c_{112} \neq c_{222}$, and $c_{122} \neq c_{212}$.

(Notice in the differential equations of the projective group that k_{111}, and k_{112} are common to both differential equations.)

By a suitable choice of t and x scales, we may put $c_{111}=c_{211}=1$ in (3.1). The above equations are then reduced to the 4-parameter family of differential equations.

$$\begin{aligned}\frac{dx_1}{dt} &= x_1^2 + c_{112}x_1x_2 + c_{122}x_2^2 \\ \frac{dx_2}{dt} &= x_1^2 + c_{212}x_1x_2 + c_{222}x_2^2\end{aligned} \tag{3.3}$$

For simplicity we let $c_{112} = \alpha$, $c_{122} = \beta$, $c_{212} = \gamma$, and $c_{222} = \delta$, then the above equations become

$$\begin{aligned}\frac{dx_1}{dt} &= x_1^2 + \alpha x_1 x_2 + \beta x_2{}^2 \\ \frac{dx_2}{dt} &= x_1^2 + \gamma x_1 x_2 + \delta x_2^2\end{aligned} \tag{3.4}$$

The general solution of the above system is of the form

$$\begin{aligned}x_1 &= \mathfrak{X}_1(t,\alpha,\beta,\gamma,\delta,x_1(0),x_2(0)) \\ x_2 &= \mathfrak{X}_2(t,\alpha,\beta,\gamma,\delta,x_1(0),x_2(0))\end{aligned} \tag{3.5}$$

These may be considered as parameter dependent parametric equations of curves in the phase space $\{x_1,x_2\}$, the evolution parameter being t.

3.2 THE PHASE SPACE OF SYSTEMS WITH QUADRATIC NONLINEARITIES[8]

In general the nonlinear differential equations (3-1) can not be solved by quadrature. Non-linear ODE's are studied qualitatively by:

1) Locating the critical points (equilibrium states) of the system and determining whether they are stable, unstable or neutral points.

2) Investigating trajectories $x(t)$ near the critical points to see if points near the critical points remain nearby or move away.

We are considering a system of 2 first-order differential equations of the form

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \alpha, \beta)$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2, \gamma, \delta) \tag{3.6}$$

where f_1 and f_2 are continous and have single-valued and continuous derivatives throughout the phase plane. Thus the values of the phase space coordinates (x_1,x_2), at any instant of time, completely define the state of the system at that time and completely determine its future (and past) evolution.

A given solution to the equations with fixed $\alpha,\beta,\gamma,\delta$ will map out, as a function of time, a smooth curve in the phase space. The motion along the collection of such curves is termed a flow because, due to the single valuedness of f_1 and f_2, no two such curves can cross.

The geometry of the phase space is described by the vector field $(d\vec{x}_1, d\vec{x}_2,\ x_1,x_2)$ which indicates the direction and speed of the evolution of the system at (x_1,x_2). Trajectories in phase space are formed by piecing together infinitesimal steps in the direction indicated by the local vector field. The result is a curve whose tangent is always aligned with the vector field, i.e., with $(d\vec{x}_1, d\vec{x}_2)$ at (x_1, x_2).[7]

Fig. 3.1 illustrates the velocity vector field and the curve describing the motion of a point in the phase plane.

It is worth emphasizing that the vector field as well as the trajectories have the property that only one vector exists at each point and only one trajectory passes through each point in the phase space. The set of all these trajectories define a phase portrait of the system (if the system parameters are considered fixed).

The solutions (trajectories) plotted in this Thesis were obtained by numerical time integration using a fourth order Runge-Kutta routine.

Though two phase curves don't cross one another they may meet at equilibrium points of the motion. At such points, whose coordinates x_1 and x_2 will be denoted by x_{10} and x_{20}, the phase flow is stationary, i.e, these are points for which $f_1 = f_2 = 0$. These points are also called critical points, and there can, in general, be any number of points satisfying these conditions. In our case we have

$$f_1 = \frac{dx_1}{dt} = x_{10}^2 + \alpha x_{10} x_{20} + \beta x_{20}^2 = 0 \tag{3.7}$$

$$f_2 = \frac{dx_2}{dt} = x_{10}^2 + \gamma x_{10}x_{20} + \delta x_{20}^2 = 0$$

Solving these equations one obtains

$$(\alpha-\gamma)x_{10}+ (\beta -\delta)x_{20} = 0 \tag{3.8}$$

It is obvious that the central critical point for the given system is $x_1 = x_{10} = 0$, $x_2 = x_{20} = 0$, when $\alpha \neq \gamma$, and $\beta \neq \delta$. There are also solutions of (3.7) which define critical lines. These will be discussed in Chapter 4.

From a larger standpoint there are critical surfaces in the space $(x_1,x_2,\alpha,\beta,\gamma,\delta)$ defined by the relation

$$\dot{x}_1 = x_1^2 + \alpha x_1x_2 + \beta x_2^2 = 0$$

or

$$x_1 = \frac{x_2}{2} (-\alpha \mp \sqrt{(\alpha^2 - 4\beta)}), \tag{3.9a}$$

and the similar relation

$$\dot{x}_2 = x_1^2 + \gamma x_1x_2 + \delta x_2^2 = 0$$

or

$$x_1 = \frac{x_2}{2} (-\gamma \mp \sqrt{\gamma^2 - 4\delta}) \ . \tag{3.9b}$$

The investigation of this thesis may be thought of as a study of the way in which these surfaces determine properties of the solution of the family of differential equations (3.4) when the RHS are considered functions of x_1,x_2 with coefficients α,β,γ, and δ.

Having identified the critical points, and any critical lines, motion near them can be determined by examining the evolution of small displacments (dx_1, dx_2), about the point (x_{10}, x_{20}).

In order to more easily understand the motion, we investigate equations (3.1) geometrically by considering the conic sections which they define when $\dot{x}_1$, and $\dot{x}_2$ are set equal to constants other than zero.

3.3) TOPICS IN ANALYTICAL GEOMETRY

The conics are circles, ellipses, parabolas, hyperbolas, and straight lines. These curves can be obtained by cutting a cone with a plane either perpendicular to its axis or tilted with respect to the axis.[8]

Consider the following equations

$$
\begin{aligned}
\dot{x}_1 &= c_{111}x_1^2 + c_{112}x_1x_2 + c_{122}x_2 = \lambda_1 \\
\dot{x}_2 &= c_{211}x_1^2 + c_{212}x_1x_2 + c_{222}x_2^2 = \lambda_2
\end{aligned}
\tag{3.10}
$$

Here λ_1, and λ_2 are constants.

Depending on the nature of their roots, the right hand equations of this pair define ellipses, parabolas, hyperbolas, and straight lines. The roots of the equations are

$$
\begin{aligned}
x_1 &= \frac{x_2}{2c_{111}} \mp \sqrt{((c_{112}^2 - 4c_{111}c_{122})x_2^2 + \lambda_1)}/2c_{111} \\
x_1 &= \frac{x_2}{c_{211}} \mp \sqrt{((c_{212}^2 - 4c_{211}c_{222})x_2^2 + \lambda_2)}/2c_{211}.
\end{aligned}
\tag{3.11}
$$

The discriminants are

$$D_1 = ({c_{112}}^2 - 4c_{111}c_{122})x_2^2 + \lambda_1 ,$$

$$D_2 = ({c_{212}}^2 - 4c_{211}c_{222})x_2^2 + \lambda_2 . \qquad (3.12)$$

D_1 and D_2 are the parameters that determine the shapes of the curves defined by (3.11). If D is positive we have hyperbolas, if zero, straight lines. If D is negative we have ellipses. Because the equations (3.10) don't have a linear term, they don't exihibit the standard form of the parabola ($x_1 = 4px_2^2$). Therefore we obtain no parabolic curves in this work.

Figures 3-2a,b show typical conic sections we do obtain. Each of these figures represents a family of curves plotted for fixed parameters (c_{ijk}) with different values of λ_i. [In addition we obtain straight lines when f_1 or f_2 contains repeated factors; c.f. appendix.]

These figures are not interconverted by rotation or scalings. Consider a rotation through an angle θ that transforms the axes of an x_1, x_2 coordinate system into an x_1', x_2' coordinate system, with

$$x_1 = x_1' cos\theta - x_2' sin\theta$$

$$x_2 = x_2' cos\theta + x_1' sin\theta \qquad (3.13)$$

Under the rotation, the right hand side of the second equation in (3.10) becomes

$$c_{111}(cos^2\theta + \alpha cos\theta sin\theta + \beta sin\theta){x_1'}^2$$

$$+ c_{112}(-2\alpha\cos\theta\sin\theta - \alpha(\sin^2\theta - \cos^2\theta) + 2\beta\sin\theta\cos\theta)x_1'x_2'$$

$$+ c_{122}(\sin^2\theta - 2\alpha\sin\theta\cos\theta + \beta\cos^2\theta)x_2'^2 = \lambda_1 \qquad (3.14)$$

If the parameter θ is chosen such that the cross product coefficients are eliminated, i.e

$$2\sin\theta\cos\theta(\beta - 1) + \alpha(\cos^2\theta - \sin^2\theta) = 0,$$

then
$$\cot 2\theta = \frac{1-\beta}{\alpha} .$$

If equation (3.11) defines an ellipse or a hyperbola in (x_1, x_2), then this rotation puts them in the standard form

$$\left(\frac{x_1'}{a}\right)^2 + \left(\frac{x_2'}{b}\right)^2 = \lambda_1$$

or $\qquad (3.15)$

$$\left(\frac{x_1'}{c}\right)^2 - \left(\frac{x_2'}{d}\right)^2 = \lambda_1$$

respectively. Here a, b, c, d are new parameters.

A rotation which puts $\dot{x}_1$ in standard form will of course seldom simplify the $\dot{x}_2$ equation. We will discuss the $\dot{x}_2$ equation below.

If the x_i' variables are rescaled ($x_1' = \alpha y_1$, and $x_2' = \beta y_2$), the above equations become

$$y_1^2 + y_2^2 = \lambda_1 \qquad (3.16)$$

$$y_1^2 - y_2^2 = \lambda_1$$

respectively.

3.4 INDEX OF A CRITICAL POINT AND INDEX OF A CURVE

In order to give a complete picture of the behavior of equations (3.3) in the phase plane it is usual to classify the system topologically close to its critical points (and far from its critical points). To aid in this task, we introduce the basic concept of the index of a critical point.

i) INDEX OF A CRITICAL POINT

Consider the following system

$$\dot{y}_1 = f_1(y_1,y_2)$$

$$\dot{y}_2 = f_2(y_1,y_2) \qquad (3.17)$$

When the time evolves ($dt>0$), the coordinates of a point P will increase by dy_1, and dy_2, where

$$dy_1 = f_1(y_1,y_2)dt, \quad dy_2 = f_2(y_1,y_2)dt \qquad (3.18)$$

The slope of the vector $\vec{U} = (f_1,f_2)$ can be measured by the angle φ which it makes with the y_2 axis (measured anti-clockwise from the positive y_2 axis). One has

$$\tan\varphi = \frac{f_1(y_1,y_2)}{f_2(y_1, y_2)} \cdot \qquad (3.19)$$

The number of times that f_1/f_2 changes its sign (negative/positive) to (positive/negative), respectively, as P moves completely around a closed curve surrounding the equilibrium point can be used to classify the equilibrium point.

The number of changes in $\tan\omega$ obtained in this way is called the index of the critical point .[9]

The concept of the index of a point discussed above has proved to be very useful in investigating the nature of our equilibrium points. It is of interest, in passing, to generalize it along the following lines.

ii) INDEX OF A CURVE

Consider again the pair of equations given by (3.10). As mentioned in the preceeding section, the $\dot{y}_1$, and $\dot{y}_2$ equations are either of hyperbolic or elliptical type. This suggests the following few paragraphs.

a) CIRCULAR INDEX

Equation 3.9 is supposed converted by rotation and scaling transformations into

$$\begin{aligned} \dot{y}_1 &= (y_1^2 + y_2^2) \\ \dot{y}_2 &= d_{211}y_1^2 + d_{212}y_1y_2 + d_{222}y_2^2 \end{aligned} \tag{3.20}$$

where d_{2ij} are new parameters and the right hand side of dy_1/dt defines a family of circles for $dy_1/dt = \lambda_1$(constant) centered at the origin with axis along the coordinate axes. The equation $dy_2/dt = \lambda_2$ may still define circles, hyperbolas, ellipses, etc.

On the above circles the y_1 component of the vector field is constant. The behavior of the trajectories in the phase space near or close to a circle are determined by considering the other component of dy_2/dt. The number of times dy_1/dy_2 changes sign on the circle is called the index of the circle and it will be used to classify the phase portrait. The index of the circle is necessarily the same as the index of the point at its center. Let r be the radius of a circle, and θ a polar angle, then

$$y_1 = r\sin\theta, \quad y_2 = r\cos\theta$$

This gives $y_1^2 + y_2^2 = r^2$, and $\dot{y}_1 = r^2$. This implies of course that $\dot{y}_1$ is independent of the rotation angle, but

$$\dot{y}_2 = r^2(d_{211}\sin^2\theta + d_{212}\sin\theta\cos\theta + d_{222}\cos^2\theta)$$

Then the co-slope of any point in the phase space is

$$dy_2/dy_1 = d_{211}\sin^2\theta + d_{212}\sin\theta\cos\theta + d_{222}\cos^2\theta \qquad (3.21)$$

<u>That is, the slope of the curve is a function of θ only, and the magnitude of dy_1, and hence, by the above, the magnitude of dy_2 for fixed θ is proportional to r^2</u>.

It is apparent that the index characterizes not just the critical point, but all circles $y_1^2 + y_2^2$ = const. that surround it. The circles themselves are characterized by the number of times dy_1/dy_2 changes sign on them. Now if

$$\frac{dy_1}{dy_2} = 1/(d_{222}.1 + d_{212}sin\theta cos\theta + (d_{211} - d_{222})sin^2\theta),$$

then, taking $cos\theta sin\theta = 1/2 sin2\theta$, and $sin^2\theta = 1/2\ (1-cos2\theta)$, the above equation becomes

$$\frac{dy_1}{dy_2} = 1/((d_{211}+ d_{222})/2 + d_{212}sin2\theta/2 + (d_{211}- d_{222})/2\ cos\theta) \tag{3.21a}$$

Changing to a new polar coordinate φ such that,

$$d_{212}sin2\theta + (d_{211} - d_{222})cos2\theta = Rsin2\varphi,$$

where $R = \dfrac{d_{211} - d_{222}}{d_{212}}$, converts (3.21a) to

$$\frac{dy_1}{dy_2} = 1\ /((d_{211} + d_{222})/2 + Rsin2\varphi/2) \tag{3.21b}$$

This expression has three fundamentally different cases:

i) $\mp R = d_{211} + d_{222}$, or $|R| = d_{211} + d_{222}$, then dy_2/dy_1 will go to zero at just two values of φ separated by 90^0.

ii) $|R| > d_{211} + d_{222}$, dy_2/dy_1 will not vanish any where.

iii) $|R| < d_{211} + d_{222}$, dy_2/dy_1 will vanish for 4 values of φ, separated from each other by two differnt angles.

b) HYPERBOLIC INDEX

We assume the RHS of the first equation in (3.9) defines a hyperbola, so that the standard form becomes

$$\dot{y}_1 = (y_1^2 - y_2^2)$$
$$\dot{y}_2 = d_{211}y_1^2 + d_{212}y_1y_2 + d_{222}y_2^2 \tag{3.22}$$

If the $\dot{y}_2$ equation defines an ellipse then we could have interchanged the roles of y_1, and y_2 and reduced the equations to the previous case. Suppose that, instead, the $\dot{y}_2$ equation defines a hyperbola. Then let

$$y_1 = s\ cosh\alpha, \quad y_2 = s\ sinh\alpha.$$

Then

$$\dot{y}_1 = s^2$$

and

$$\dot{y}_2 = s^2(d_{211}sinh^2\alpha + d_{212}sinh\alpha cosh\alpha + d_{222}cosh^2\alpha)$$

$$= s^2\left(\frac{d_{211} - d_{212}}{2} + \frac{d_{212}}{2}\ sinh2\alpha + \frac{d_{222} + d_{211}}{2}cosh2\alpha\right) \tag{3.23}$$

The inclination of the vector field at any point on the hyperbola $s^2 = \lambda_1$ is then

$$\frac{dy_1}{dy_2} = 1/\left(\frac{d_{211} - d_{222}}{2} + \frac{d_{212}}{2}sinh2\alpha + \frac{d_{222} + d_{211}}{2}cosh2\alpha\right) \tag{3.24}$$

It is clear that the slope of the vector field at any point is independent of s. The magnitude of both dy_1, and dy_2 for fixed α, are proportional to s^2. The number of times dy_1/dy_2 changes sign (negative/positive) to (positive/negative) on the hyperbola $y^2 - y_2^2$ = constant we term the index of the hyperbola.

We investigated its use in classifing phase portraits with open trajectories. However, in the end we found that it was not necessary for this purpose, so we will not pursue the matter further in this Thesis.

3.5 NON-PROJECTIVE ORDINARY DIFFERENTIAL EQUATIONS IN A POLAR COORDINATE SYSTEM

We have found it is helpfull to use polar coordinate systems to provide a new way of analyzing the behavior of trajectories in phase space.

i) CIRCULAR- HYPERBOLIC TYPE

Consider the pair of equations given in (3.20) to be transformed to polar coordinates by the substitutions

$$y_1 = r \sin\theta, \quad y_2 = r \cos\theta. \tag{3.25}$$

Then the left hand sides of equations (3.20) become

$$\dot{y}_1 = \dot{r} \sin\theta + r \cos\theta \, \dot{\theta} \tag{3.26}$$

$$\dot{y}_2 = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

Solving for $\dot{r}$, and $\dot{\theta}$, and replacing the $\dot{y}_i$ in the result by the right hand sides of equations (3.20) we obtain

$$\begin{aligned} \dot{r} &= r^2\,\{(\cos^2\theta + \sin^2\theta)\sin\theta + (d_{211}\sin^2\theta + \\ &\qquad d_{212}\sin\theta\cos\theta + d_{222}\cos^2\theta)\cos\theta\} \\ \dot{\theta} &= r\,\{-(d_{211}\sin^2\theta + d_{212}\sin\theta\cos\theta + \\ &\qquad d_{222}cso^2\theta)\sin\theta + (\cos^2\theta + \sin^2\theta)\cos\theta\}. \end{aligned} \tag{3.27}$$

Since

$$dy_1/dy_2 = \frac{d\theta}{dt}/\frac{dr}{dt} = \frac{d\theta}{dr}$$

we have

$$\frac{d\theta}{dr} = \frac{1}{r}\left\{\frac{-(d_{211}\sin^2\theta + d_{212}\sin\theta\cos\theta + d_{222}\cos^2\theta)\sin\theta + (\cos^2\theta + \sin^2\theta)\cos\theta}{(\cos^2\theta + \sin^2\theta)\sin\theta + (d_{211}\sin^2\theta + d_{212}\sin\theta\cos\theta + d_{222}\cos^2\theta)\cos\theta}\right\}$$

$$\equiv \frac{1}{r}\,Z(\theta) \tag{3.28}$$

If $Z(\theta)$ is zero then $\frac{d\theta}{dr}$ vanishes. This means that θ becomes constant and the trajectories approach radial straight lines for specific values of θ determined by the function $\Xi(\theta)$ when its numerator vanishes:

$$(sin^2\theta + cos^2\theta)cos\theta - (d_{211}sin^2\theta + d_{212}sin\theta cos\theta + d_{222}cos^2\theta)sin\theta = 0$$

or

$$1 = d_{211}tan^3\theta + (d_{212} - 1)\ tan^2\theta + d_{222}tan\theta\ . \qquad (3.29)$$

Solving for $tan\theta$ (which is y_1/y_2), one may find the roots of the above equation and determine the slope of the radial straight lines which are solutions of (3.20), the system of circular-hyperbolic type.

ii) HYPERBOLIC-HYPERBOLIC TYPE

Similar transformatons can be used to write equations (3.22), the system of hyperbolic - hyperbolic type, in polar coordinates. The result is

$$\frac{d\theta}{dr} = \frac{1}{r}\ \{\frac{-(d_{211}sin^2\theta + d_{212}cos\theta sin\theta + d_{222}cos^2\theta)sin\theta + (sin^2\theta - cos^2\theta)cos\theta}{(sin^2\theta - cos^2\theta)sin\theta + (d_{211}sin^2\theta + d_{212}cos\theta sin\theta + d_{222}cos^2\theta)cos\theta}\}$$

$$\equiv \frac{\Xi(\theta)}{r} \qquad (3.30)$$

As in the previous case, if one considers $Z(\theta)$ to be zero $d\theta/dr$ becomes zero. Again the trajectory approaches a radial straight line determined by a real root obtained when the numerator of $Z(\theta)$ is set equal to zero:

$$(d_{211}sin^2\theta + d_{212}cos\theta sin\theta + d_{222}cos^2\theta)sin\theta$$
$$-cos\theta(sin^2\theta - cos^2\theta) = 0$$

In this case

$$d_{211}tan^3\theta + (d_{212} - 1)tan^2\theta + d_{222}tan\theta = -1, \tag{3.31}$$

an equation which differs from (3.29) only by a change in the sign before 1.

iii) CIRCULAR-ELLIPTICAL TYPE

In this case the differential equations of circles and ellipses are written in the following form using the transformation equations (3.25):

$$\dot{r} = r^2 \left\{ sin\theta + \left(\frac{sin^2\theta}{a^2} + \frac{cos^2\theta}{b^2} \right) cos\theta \right\}$$

$$\dot{\theta} = r \left\{ cos\theta - \left(\frac{sin^2\theta}{a^2} + \frac{cos^2\theta}{b^2} \right) sin\theta \right\}. \tag{3.32}$$

Then

$$\frac{d\theta}{dr} = \frac{1}{r}\left\{\frac{\cos\theta - \left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)\sin\theta}{\sin\theta + \left(\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}\right)\cos\theta}\right\}$$

$$\equiv \frac{Z(\theta)}{r} \tag{3.33}$$

It is apparent that the trajectories approach radial straight lines at large r. These lines are defined by the equation obtained by setting the numerator of $Z(\theta) = 0$, viz.

$$\cos\theta - (\sin^2\theta/a^2 + \cos^2\theta/b^2)\sin\theta = 0$$

i.e.

$$1 + \tan^2\theta - \tan^3\theta/a^2 - \tan\theta/b^2 = 0. \tag{3.34}$$

iv) <u>CIRCULAR-CIRCULAR TYPE</u>

Consider the following pair of equations:

$$\begin{aligned} \dot{y}_1 &= y_1^2 + y_2^2 \\ \dot{y}_2 &= k\,(y_1^2 + y_2^2)\,. \end{aligned} \tag{3.35}$$

It immediately follows that

$$dy_1/dy_2 = 1/k \tag{3.36}$$

so

$$y_1 = (1/k)y_2 + \text{const.}$$

The phase plane is filled with a family of parallel straight lines.

In concluding this section we wish to draw attention to several consequences of the invariance of equations (3.1) under the scaling transformation

$$x_i \to y_i = \bar{x}_i = \alpha\, x_i, \quad t \to \bar{t} = \alpha^{-1} t.$$

1. This makes the tangent dy_1/dy_2 independent of the distance of the point $P(y_1, y_2)$ from the origin (Section 3.4), so that the direction of the tangent vector field $(d\vec{y}_1, d\vec{y}_2, y_1, y_2)$ is known everywhere once it is known on any circle centered on the origin.

2. The rate of evolution at a point with angular coordinates r, θ scales as r^2.

CHAPTER 4

CLASSIFICATION OF NON-PROJECTIVE SYSTEMS IN TERMS OF THEIR CRITICAL POINTS AND PHASE PORTRAITS

4.1 INTRODUCTION

Poincare showed that much can be learned about dynamical behavior from an analysis of trajectories in a multidimensional phase space in which a single point characterizes the entire system at an instant of time.[7] A key feature of Poincare's analysis was its emphasis upon topological properties. In this section we will focus our attention on the classification of non-projective systems in terms of their topology.

We will make use of the fact that rotations of coordinates and dilatations of coordinates only change the orientation and scaling of curves. They do not change the basic topological properties of curves. We first concentrate our attention on phase curves that are chosen because of a simple relationship between dy_1/dt and $f_1(y_1,y_2)$ and a simple relationship between dy_2/dt and $f_2(y_1,y_2)$. We consider sets of curves $f_1 = \lambda_1$, and $f_2 = \lambda_2$ in which all members of each set have a common topology and investigate the way in which the topology and the orientation of these curves affects the topology of the phase curves of the system.

Before we proceed to do this, we define the following types of invariant subspaces found in our phase portraits. [A subspace is termed an invariant subspace if the phase curves in the subspace never leave the subspace.]

4-2 DEFINITIONS [6]

i) Inset of an equilibrium point: This is the portion $\{I_s\}$ of the phase portrait where a set of trajectories asymptotically approach the equilibrium point. We distinguish two types of inset regions:

In the first type each trajectory "begins" at an arbitrarily large distance from the critical point. By this we mean that any point on the trajectory (critical point excluded) recedes from the critical point to an arbitrarily large distance when the time is reversed. We denote the region containing such trajectories, and only such trajectories, an open inset. $\{I_{os}\}$.

In the second type all points on each trajectory are a finite distance from the critical point. We denote a region containing only such trajectories, a closed inset, $\{I_{cs}\}$.

ii) Outset of an equilibrium point: This is the portion {Os} of the phase portrait where trajectories permanently depart the region of the equilibrium point.

iii) Some trajectories neither approach nor depart regions arbitrarily close to the critical point; they may get close to it, but not arbitrarily close to it. We term the regions where these trajectories exist intermediate regions, $\{I_t\}$.

Trajectories in all these regions may be characterized by their behavior under space inversion and time reversal. Letting S represent the operation of space inversion, T represent the operation of time reversal, and I represent the identity operation, these operations have the following effect on the pair of vectors $(\vec{r}, d\vec{r})$ which represent the vector field at a point on a trajectory and its direction of motion during time dt:

$$\begin{aligned} S(\vec{r},d\vec{r}) &= (-\vec{r},-d\vec{r}) = -(\vec{r},d\vec{r}) \\ T(\vec{r},d\vec{r}) &= (\vec{r},-d\vec{r}), \end{aligned} \tag{4.1}$$

Note that $T^2 = I$, $S^2 = I$ and the product operation $ST = TS$ leaves our differential equations invariant:

$$\begin{aligned} &TS\{I_{cs}\} = \{I_{cs}'\}, \ TS\{I_{cs}'\} = \{I_{cs}\} \\ &TS(I_{os}) = (O_s), \ \ TS(O_s) = (I_{os}) \\ &TS(I_t) = (I_t'), \ \ TS(I_t') = (I_t). \end{aligned} \tag{4.2}$$

Because $(ST)^2 = I$, intermediate regions can only occur in pairs; one could not have, e.g., $ST(I_t') = (I_t'')$.

"Inset", "outset" and "intermediate" are <u>global</u> terms. Thus in an intermediate region there are local subregions in which a trajectory at first may appear headed toward a critical point but then turn around and head away from the critical point.

We find that, for our non-projective systems, in inset regions there may be a linear attractor - a line which trajectories approach tangentially as they approach the critical point. In some case the linear attractors become critical lines which trajectories approach directly.

We find corresponding linear repellors in some outset regions. We define linear repellors as lines which are converted to linear attractors by time reversal. We also find that there may be linear attractors in outset regions.

Furthermore we find that there may be linear attractors and linear repellors in intermediate regions.

iv) Separating inset from outset, intermediate from inset, and intermediate from outset, are special trajectories. Two regions of the same type can at most meet at a single point. The special trajectories separating regions of different type are called separatrices. Separatrices may themselves be linear attractors and/or repellors.

It will be shown below that all our attractors and repellors, as well as separatrices, are straight lines defined by equations

$$y_1 = m\, y_2 \tag{4.3}$$

where m is the $\tan\theta$ of equations (3.29, 3.31, 3.34).

4.3) TYPES OF NON-PROJECTIVE SYSTEMS

We start with a study of equations of the form

$$\dot{y}_1 = k(y_1^2 \mp y_2^2) \tag{4.4}$$

$$\dot{y}_2 = d_{211}y_1^2 + d_{212}y_1y_2 + d_{222}y_2^2 .$$

We wish to investigate the manner in which changing the d_{2ij} alter the phase portrait. We classify the $\dot{y}_2$ equation into two fundamental cases by setting its RHS equal to a constant. This yields either

1) the equation of an ellipse

or

2) the equation of hyperbola.

We ask how, changing the constants in these cases, without changing the topology of the figures, affects the phase portrait. Then changing the constants in each case simply changes the scale of coordinates and the orientation of the figures defined by $f_2 = \lambda_2$. We may think of our study as a study of the effects of scalings and the orientation of these figures on the phase portrait. We first note that by scaling t we put $k = 1$ in the first equation of (4.4). (This changes the values of the d_{2ij} without changing their relative values.) In the following discussion, it should be kept in mind that it is changing the d_{2ij} that changes the scale and orientation of the figure defined by $f_2 = \lambda_2$. No real rotation or scaling of y_1 and y_2 is intended. [A real rotation, for example, would simply change the orientation of phase curves, a matter which is of little interest here.]

We now turn to a consideration of the various types of phase portraits we have observed.

1) HYPERBOLIC-HYPERBOLIC TYPE

In this type both of the equations (4.4) define hyperbolae and we analyze vector fields, one of whose components in the phase plane is constant in magnitude on a hyperbola $y_1{}^2 - y_2{}^2 =$ const.

Case 1-1

Consider the following pair of equations obtained by setting $\dot{y}_1 = \lambda_1$, $\dot{y}_2 = \lambda_2$

$$\lambda_1 = y_1^2 - y_2^2$$

$$\lambda_2 = y_1^2 + 6y_1y_2 + 4y_2^2 \tag{4.5}$$

Both equations define families of hyperbolae. The first one, represents the y_1 component of the vector field, which has the same magnitude and direction on the hyperbola $y_1^2 - y_2^2 = \lambda_1$.

We define a line on which f_1 or f_2 (and hence dy_1/dy_2) changes sign, to be a "flip line". The total number of radial flip lines is hence equal to the index if two flip lines do not coincide. The equations describing the flip lines are in this case

$$f_1 = y_1^2 - y_2^2 = 0$$

i.e

$$y_1 = \mp y_2$$

and,

$$f_2 = y_1^2 + 6y_1y_2 + 4y_2^2 = 0 \quad ,$$

which gives (4.6)

$$y_1 = y_2/2(-6 \mp\sqrt{(36-16)})$$

i.e.

$$y_1 = -.76x_2, \text{ and } y_2 = -5.24y_2 \ .$$

Fig. 4.1.1.a shows these four flip lines and the signs of f_1, f_2. Note how these functions change their signs from positive to negative and vice versa when a point crosses the corresponding lines(the arrows show the positive signs). It is apparent that dy_1/dy_2 changes its sign 8 times on a circle surrounding the origin if no flip lines coincide; the index of the equilibrium point is 8 in this case.

As the figure shows, the entire phase plane is divided into two primary regions, inset and outset. The inset is separated into three compartments, two of which contain closed trajectories. The trajectories in the inset approach the critical point along two different attractors which are observed to be given by $y_1 = -4.7y_2$ in the lower right hand quadrant, and $y_1 = -.5y_2$ in the upper left hand quadrant. These attractors become repellors as they pass through the origin.

The outset, most of which lies in the upper right quadrant, is contained between the separatrix $y_1 = -4.7y_2$ in the upper left quadrant and the separatrix $y_1 = -.5\ y_2$ in the lower right quadrant. The trajectories in this outset approach the latter line asymptotically as $t \to \infty$, so that this line is an attractor of trajectories in the outset.

The separatrices, attractors and repellors are observed to be defined by straight lines through the origin, and hence have the equation $y_1 = my_2$. Therefore $dy_1 = mdy_2$, and

$$\frac{f_1}{f_2} = \frac{y_1^2 - y_2^2}{y_1^2 + 6y_1y_2 + 4y_2^2} = m \qquad (4.7)$$

Substituting my_2 in place of y_1, one has

$$\frac{m^2 - 1}{m^2 + 6m + 4} = m \qquad (4\text{-}7a)$$

Solving for m one finds two real roots $m_1 = -4.7$, and $m_2 = -.5$, as observed. The existence of these straight lines is consonant with equation (3.30) which shows that the trajectories approach radial straight lines for some values of θ. This follows from the fact that the separatrices (attractors, repellors) are themselves moving solutions of the differential equations determined by eqn.(3.31), so a point on them moves along them.

As two solutions cannot touch for $0 < y_1 < \infty$ and $0 < y_2 < 0$ a solution which is not a separatrix can only tangentially approach a separatrix which is a moving solution of the differential equations.

Notice that y_1 and y_2 don't change very much close to the critical point. As the evolutions are quadratic,

$$\lim_{y_1, y_2 \to 0} \dot{y}_i = 0 \tag{4.8}$$

and the critical point is approached asymptotically.

Now let us see whether the behavior of the trajectories through any given point $P(\vec{r})$ can be qualitatively predicted once one knows:

1. The "flip lines" of the system along which $\dot{y}_1 = 0$, and/or $\dot{y}_2 = 0$, and the sign change at each.

2. The attracting/repelling character of the nearest linear solutions on each side of P.

Let an initial state of the system, defined by the equations (4.5), be contained in the region where f_1 is negative and f_2 is positive, say P(-.2,-.8) in the lower left hand quadrant. After a short time dt, this point moves to $P(-.2+dy_1,-.8+dy_2)$ with

$$dy_1 = f_1 dt$$
$$dy_2 = f_2 dt. \tag{4.9}$$

Since f_1 is negative, dy_1 is negative while dy_2 is positive, so $d\vec{r} = (dy_1, dy_2)$ inclines toward the line $f_1 = 0$ (i.e $y_1 = +y_2$). Once the trajectory governing the motion of the above point hits the line $y_1 = y_2$ (in the lower left hand quadrant), it changes its direction as f_1 (y_1 component) changes its sign. This means that the slope of the vector field dy_1/dy_2 becomes positive in this new region as both f_1, and f_2 are positive. The situation continues until the trajectory reaches a point where dy_2 changes its sign (f_2 becomes negative). This takes place when the trajectory crosses the flip line given by $\dot{y}_2 = 0$ just before the linear solution (the attractor) $y_1 = -4.7y_2$. As two trajectories never coincide, the trajectory in question asymptotically approaches the linear attractor and the point eventually moves almost parallel to the attracting line. The resultant motion will thus be toward the critical point.

With spatial inversion the same point becomes $P'(.2,.8)$, and with time reversal, this point will move to the left with negative slope ($dy_1/dy_2 = -f_1/f_2$) heading toward the flip line ($\dot{y}_1 = 0$, in the upper right quadrant) where the trajectory changes its slope from negative to positive. (Remember time is now running backward.) This motion goes on until the trajectory crosses the flip line $\dot{y}_2 = 0$ where the y_2 component of the vector field is changed from positive to negative. Then the trajectory approaches the repellor (now an attractor by time reversal) and then asymptotically approaches the critical point (again because of time reversal).

Considering the motion of additional points in the same manner, we have been led to the hypothesis that once the attractors, repellors and separatrices are known, the qualitative features of the evolution of any initial point in the phase plane is completely determined by the following two rules:

1) Attractors, repellors dominate the global behavior.
2) Flip lines restrain the response of the trajectory to the attractor/repellor by constraining the slope of the trajectory at every point.

Case 1-2

We consider a new function $g_2(y_1,y_2) = \lambda_2$ to be a mapping of the second function $f_2(y_1,y_2)$ = const. in eqns.(4.5) obtained by an alteration of the parameters d_{2ij} equivalent to a rotation of the figure defined by $f_2 = \lambda_2$ anti-clockwise through an angle of 45^0. Then the equations (4.5) become

$$\begin{aligned} \dot{y}_1 &= y_1^2 - y_2^2 \\ \dot{y}_2 &= 5.5y_1^2 + 3y_1y_2 - .5y_2^2 \end{aligned} \tag{4.10}$$

Fig.4.1.2 shows that equations (4.10) have a different phase portrait which is entirely intermediate. This region is separated into two portions by a straight line trajectory which is a linear attractor on the lower right hand quadrant and a linear repellor on the upper left quadrant. All trajectories, except the straight line one determined by the equation (3.31) avoid the critical point and approach the linear attractor as $t \to \infty$. If the time is reversed these trajectories asymptotically approach the linear repellor. Notice that the repelling and attracting portions of the phase portrait are interchanged: time reversal interconverts pairs of intermediates.

The above mentioned straight line (attractor/repellor), which is the only moving linear solution of the system, is given by eqn. (3.31) and found to be $\tan\theta = y_1/y_2 = -.76$ (considering $d_{211} = 5.5$, $d_{212} = 3$, and $d_{222} = -.5$, the only real root) or

$$\frac{dy_1}{dy_2} = \frac{y_1^2 - y_2^2}{5.5y_1^2 + 3\ y_1y_2 - .5y_2^2} = \beta .$$

As $y_1 = \beta y_2$, we have $dy_1 = \beta dy_2$, the above equation becomes

$$\beta^2 - 1 = 5.5\beta^3 + 3\beta^2 - .5\beta \quad .$$

Solving for β, and taking the real parts, we obtain $\beta = -.76$.

Notice that although the index of the critical point is 8, (as in the previous case), we observed no trajectories that asymptotically approach the critical point. The rotation process has markedly changed the phase portrait.

Now let us examine the qualitative behavior of a point's motion if it is contained in the upper right quadrant and confined between the flip lines $\dot{y}_1 = 0$, and $\dot{y}_2 = 0$. We take into account that we have a repellor in the upper left quadrant and an attractor in the lower right quadrant, and the sign changes across the flip lines shown in Figure 4.1.2 . Note that the sign of f_1 in this region is negative, and the sign of f_2 is positive, so the inclination of the trajectory in this region is negative (toward the line $\dot{y}_2 = 0$). This inclination is changed when the trajectory crosses this line, after which the y_2 component of the vector field becomes negative and the slope becomes positive - pointing toward the flip line $\dot{y}_2 = 0$ (in the lower right quadrant). Again, once the trajectory passes the flip line $\dot{y}_2 = 0$, the y_2 component of the vector field becomes positive while the y_1 component is still negative, so the slope of the trajectory is negative. As we have a linear attractor in this region given by $y_1 = -.76y_2$, the above mentioned trajectory asymptotically approaches it as $t \to \infty$.

Similarly, when the time is reversed, the initial point asymptotically approaches the former linear repellor $y_1 = -.76y_2$.

Note the straight line (the solution of the system, $y_1 = -.76y_2$) separates two regions that are interconverted by space inversion, so under combined space inversion and time reversal the trajectories in the lower side of the straight line behave the same way as the upper trajectories behave.

<u>Case 1-2</u>'

Consider the case

$$\dot{y}_1 = y_1^2 - y_2^2$$
$$\dot{y}_2 = 5.28y_1^2 + 4.99y_1y_2 - .25y_2^2 \ , \tag{4.11}$$

which is a special case of case 1.2 in which one of the asymptotic lines of the $\dot{y}_1$ conic coincides with an asymptote of the $\dot{y}_2$ conic.

As illustrated in Fig. 4.1.2' the straight line trajectory (attractor/ repellor) shown in Figure 4.1.2 is replaced by the critical line at which motions cease. This critical line divides the phase portrait into two parts interconverted by a rotation of 180^0. All trajectories asymptotically approach the critical line as both $t \to +\infty$, and $t \to -\infty$.

Note that the flip lines are now 6 in number and there is no attractor or repellor in the phase plane except the critical line. This line attracts the nearby trajectories in the lower right quadrant, and repells the nearby trajectories in the upper left quadrant. Note also that a point on this line never moves, so trajectories end or begin upon it. This feature allows the trajectories to approach the line, or leave the line, at a non-zero angle.

Case 1-3

Consider now the following pair of equations

$$\dot{y}_1 = f_1 = y_1^2 - y_2^2$$

$$\dot{y}_2 = g_2 = 4y_1^2 - 6y_1y_2 + y_2^2 \qquad (4.12)$$

Where g_2 is a new function obtained from f_2 by anti-clock wise rotation by 90^0. Again when g_2 is set equal to a constant a hyperbola is obtained.

As shown in Fig.4.1.3, the behavior of the system is similar to that of case 1.2, except that there are a number of linear attractors and linear repellors. Every single initial point evolves to one of the attractors as the time $t \to +\infty$, and to one of the repellors as $t \to -\infty$. The three attractors, which become repellors when they pass through the origin, are observed to be given by:

$$y_1 = -.31y_2, \quad y_1 = 1.43y_2, \quad y_1 = +.60y_2 .$$

None of them is a stationary solution. These three relations are in fact the three roots of equation (3.31).

Now let us examine the system qualititively by choosing any point in the phase plane. Let a point $P(\tau)$ be chosen at the upper right quadrant between the repellor $y_1 = .60y_2$ and the flip line $y_1 = y_2$. Here both f_1 and f_2 are negative, so the trajectory's slope is negative (note that although the slope of the repellor is also negative, the trajectory moves away from the repellor because the y_2 component of the vector field is getting bigger and the slope of the trajectory is increasingly negative as the time evolves; there is no way the trajectory can move along the linear solution given by $y_1 = .60y_2$) in the direction pointing out toward the flip line $y_1 = 0$. Once the trajectory hits this flip line it changes its direction as the y_1 component of the vector field flips from negative to positive. Once again the trajectory, as time goes on, changes its direction when it hits the flip line $\dot{y}_1 = 0$, and it ends up asymptotically approaching the linear attractor. When the time is reversed, the point $P(\vec{r})$ approaches the (former) repellor $y_1 = .60y_2$.

With spatial inversion, the point $P(\vec{r})$ becomes $P'(-\vec{r})$ in the lower left hand quadrant contained between the attractor $y_1 = .60y_2$ and the flip line $\dot{y}_1 = 0$. This point, when the time is reversed moves to the right, and when it hits the flip line $\dot{y}_1 = 0$ it changes its direction toward the next flip line $\dot{y}_2 = 0$. Once again the trajectory faces the repellor $y_1 = 1.43y_2$; keeping in mind that the time is reversed and this repellor is now an attractor, the trajectory asymptotically approaches the line $y_1 = 1.43y_2$.

<u>Case 1.3'</u>

If the figure defined by $f_2 = \lambda_2$ is rotated by 82^0, one of the $\dot{y}_2$ asymptotes coincides with an asymptote of the $\dot{y}_1$ RHS, and Figure 4.1.3' is obtained.

As a result, two intermediate regions are eliminated. One now has a critical line, which replaces the attractor in the lower left quadrant of Figure 4.1.3, where nearby trajectories cease.

The trajectories in the remaining part of the phase portrait, depending on their location, asymptotically approach either the attractor given by $y_1 = -.47y_2$ in the lower right quadrant, or $y_1 = 1.6y_2$ in the upper right quadrant. These trajectories asymptotically approach the corresponding repellors when the time is reversed.

Case 1-4

Now consider a rotation of the figure defined by $f_2 = \lambda_2$ in (4.3) anti-clockwise through an angle of 135^0. One obtains

$$\begin{aligned} \dot{y}_1 &= y_1^2 - y_2^2 \\ \dot{y}_2 &= -.5y_1^2 - 3y_1y_2 + 5.5y_2^2 \; . \end{aligned} \tag{4.13}$$

In this case, as shown in Fig.4.1.4, the phase portrait is divided by two separatrices into four regions, an inset, an outset, and two intermediate regions. The inset region contains a linear attractor $y_1 = 1.35y_2$ whose points asymptotically approach the critical point as $t \to \infty$. This attractor is tangent to the nearby trajectories which also head toward the critical point. If the time is reversed these trajectories, depending on their location with respect to this linear attractor, asymptotically approach either the separatrix given by $y_1 = -7.1y_2$, or the separatrix given by $y_1 = -.17y_2$.

In this system the outset contains a linear repellor $y_1 = 1.35y_2$ which separates nearby trajectories. These repelled trajectories, depending on their location with respect to this line, asymptotically approach one of the separatrices $y_1 = -.17y_2$, or $y_1 = -7.1y_2$.

In the intermediate regions, trajectories in the lower right quadrant asymptotically approach the separatrix $y_1 = -.17y_2$ as $t \to \infty$, and the separatrix $y_1 = -7.1y_1$ as $t \to -\infty$. Similarly, trajectories in the upper left quadrant asymptotically approach the separatrix $y_1 = -7.1y_2$ as $t \to \infty$, and asymptotically approach the separatrix $y_1 = -.17y_2$.

Note that the inset and outset are interconverted by spatial inversion and time reversal. As before, combined time reversal and spatial inversion interconverts intermediate regions.

<u>Case 1.4'</u>

If $f_2 = \lambda_2$ in the equations (4.5) is rotated by 120^0, the following equations are obtained

$$\dot{y}_1 = \dot{y}_1 - \dot{y}_2$$
$$y_2 = -.2y_1^2 - 3.2y_1y_2 + 3.2y_2^2 \tag{4.14}$$

As indicated in Fig. 4.1.4' one of the $\dot{y}_1$ asymptotes coincides with one of the $\dot{y}_2$ asymptotes and the rotation of the figure has eliminated the inset and outset regions from the phase portrait. The attractor and the repellor originally in the inset and outset regions are replaced by a critical line which consists of stationary solutions. The behavior of the trajectories in the phase plane is similar to that of case 2.3'

2) CIRCULAR-HYPERBOLIC TYPE:

In this case we consider the RHS of the first equation in (3.4) to define a circle. As dy_1/dt is fixed in magnitude and direction around the circle, the index of the phase portrait is determined by the number of times dy_2/dt changes its sign on the circle.

Case 2-1

Consider The following pair of equations

$$\dot{y}_1 = y_1^2 + y_2^2 \tag{4.15}$$

$$\dot{y}_2 = y_1^2 + 6y_1y_2 + 4y_2^2 ,$$

where the RHS of the first one defines a family of circles when set equal to a constant, and the second RHS defines hyperbolae. As illustrated in Fig.4-2-1, the phase portrait is separated by two separarices into four portions, inset, outset, (each of which has an attractor and repellor respectively given by $y_1 = -3.7y_2$), and two intermediate regions. This figure looks similar to that seen in Fig. 4.1.4. The trajectories in the inset asymptotically approach the linear attractor and eventually approach the critical point as $t \to \infty$. Correspondingly the outset repellor repells the nearby trajectories which, depending on their locations with respect to this repellor, asymptotically approach the separatrix (attractor, in the upper right hand quadrant) $y_1 = .19y_2$, .or the separatrix (attractor in the upper left quadrant) $y_1 = -1.3y_2$.

The trajectories in the upper left and lower right quadrant belong to the intermediate region and asymptotically approach the separatrices $y_1 = -1.3y_2$ and $y_1 = .19y_2$ respectively as $t \to \infty$. They asymptotically approach the separatrices $y_1 = .19y_2$, and $y_1 = -1.3y_2$ as $t \to -\infty$.

These attractors, repellors and separatrices are themselves moving solutions of the system defined by the equation (3.29). and are observed experimentally to be so.

Case 2-2

Consider the following coupled equations

$$\dot{y}_1 = y_1^2 + y_2^2 \qquad (4.16)$$
$$\dot{y}_2 = 4y_1^2 - 6y_1y_2 + y_2^2 \, ,$$

where the second equation is a mapping of the RHS of the second equation in case 4.2.1 by a rotation of 90^0 anti-clock wise.

Fig. 4-2-2 shows that the phase portrait represents an intermediate region separated into two parts (interconverted by a rotation of 180^0) by the straight line $y_1 = 1.6y_2$, a linear solution of the system.

All trajectories asymptotically approach this straight line which is a linear attractor as the time $t \to +\infty$, and a linear repellor as the time $t \to -\infty$ respectively. Notice that this case is similar to the case 4.1.2.

Since dy_1/dy_2 changes its sign only 4 times, and dy_1/dt is always constant in the positive direction of y_1, we observe no curves similar to that in case 4.1.1, and 4.1.3.

3) CIRCULAR-ELLIPSOIDAL TYPE

In this type, one equation defines circles and the other defines circles or ellipses. The vector field and the trajectories can at most change direction by less than 90^{o}.

Case 3.1

Consider the following equations

$$\begin{aligned} \dot{y}_1 &= y_1^2 + y_2^2 \\ \dot{y}_2 &= y_1^2 + y_1 y_2 + .5y_2^2 , \end{aligned} \tag{4.17}$$

equations whose RHS define a family of circles and a family of ellipses respectively.

Figure 4.3.1 shows the behavior of the system in the phase plane (y_1,y_2). All trajectories asymptotically approach a straight line (a linear attractor as $t \to \infty$ and a linear repellor as $t \to -\infty$) which is part of a single solution of the differential equation determined by eqn.(3.34).

Case 3.2

$$\begin{aligned} \dot{y}_1 &= y_1^2 + y_2^2 \\ \dot{y}_2 &= 2(y_1^2 + y_2^2). \end{aligned} \tag{4.18}$$

Here the RHS of both equations define family of circles with different radii. Figure 4.4.2 shows that all trajectories are straight lines. There is no attractor/repellor in this circullar- circullar case; the trajectories don't approach a radial straight line.

In the appendix we consider several degenerate cases not considered above. These develop when f_1 and/or f_2 have repeated factors. However they add nothing new to our essentially topological classification scheme.

We conclude this section with a brief discussion of our observations when they are expressed in canonical coordinates defined by the invariance of the equations under scaling transformations.

Define the new variables

$$\rho = \ln(r), \qquad d\tau = r\, dt \ .$$

Recalling equations (3.28, 3.30), we find that in these new coordinates they take on the form

$$d\rho/d\tau = \mathrm{p}(\theta), \quad d\theta/d\tau = \mathrm{q}(\theta),$$

and

$$d\rho/d\theta = \mathrm{p}/\mathrm{q} = 1 \ / \ Z(\theta).$$

Here, depending on whether one is dealing with equation 3.28 or 3.30 one has

$$\begin{aligned} \mathrm{p} = \ & (sin^2\theta \pm cos^2\theta)\ sin\theta + (d_{211}sin^2\theta + d_{212}sin\theta\ cos\theta \\ & \qquad + d_{222}cos^2\theta)\ cos\theta \end{aligned}$$

$$\mathrm{q} = \ (sin^2\theta \pm cos^2\theta)\ cos\theta - (d_{211}sin^2\theta + d_{212}sin\theta\ cos\theta$$

$$+ d_{222}\cos^2\theta)\ \sin\theta\ .$$

As τ takes on the values $0, 1, \infty$, ρ takes on the values $-\infty$, 0, ∞.

In all the cases where there are attractors or repellors one has three cases depending upon the value of $Z(\theta)$:

1. If $Z(\theta) = 0$, one obtains the separatrices, i.e., attractors and repellors.

2. If $Z(\theta)$ remains positive as ρ grows with τ, then the nearest attractor defined by $Z(\theta) = 0$ is approached as ρ (and τ) approach $+\infty$.

3. If $Z(\theta)$ remains negative as ρ decreases while τ grows, then the nearest attractor defined by $Z(\theta) = 0$ is approached as ρ approaches $-\infty$, that is, as τ approaches zero.

This analysis provides a simple explanation of the observation that trajectories may approach linear attractors *both* as $\tau \to \infty$ and as $\tau \to 0$.

CONCLUSIONS

We have investigated non-projective dynamics globally with a full exploration of the behavior of phase trajectories over a range of parameter values. A wide variety of dynamical behavior has been observed.

The numerical results displayed and analyzed in Chapter 4, showed that every phase portrait contains at least one straight line solution given by $dy_1/dy_2 = f_1/f_2 = \lambda$ (where λ is some constant). Each such line consists of two halves: one half lies on one side of the origin. The half on the other side is obtained from the first half by space inversion. Points on one of these half-lines approach the critical point arbitrarily closely as $t \to \infty$, (but does not reach ∞). This we may term an inset trajectory. The second half-line is obtainable from the first one by space inversion and becomes a correctly oriented trajectory when time is then reversed. This trajectory we may term an outset trajectory. We also observed that the linear inset trajectory is either an attractor which nearby trajectories asymptotically approach as they approach the critical point or it is a repellor from which nearby trajectories depart.

Correspondingly, we found that the linear outset trajectory is either a repellor which nearby trajectories tangentially depart, or an attractor which nearby trajectories tangentially and asymptotically approach at large distances.

The number of these straight lines is determined by the number of real roots of the equation $dy_1/dy_2 = \lambda$, a cubic in λ.

We observed that when a flip line of one equation coincides with a flip line of the other equation, the number of the these straight lines decreases by one and an attractor and a repellor is replaced by a critical line. All trajectories on one side of such a line asymptotically approach it and cease, while on the other side all trajectories are repelled.

We noticed that every trajectory in the phase portrait must asymptotically approach a straight line trajectory (attractor) as $t \to \infty$, and another straight line trajectory (repellor) as $t \to -\infty$. If the attractor is in the inset then a curved trajectory will asymptotically approach the linear trajectory and the critical point. If the attractor is in the outset or in the intermediate then a curved trajectory asymptotically will approach a solution of the system determined by $y_1 = \lambda y_2$.

The behavior of a trajectory as it proceeds on its way toward an linear attractor, or away from a linear repellor, is governed by its position relative to the nearest flip lines, on which either $\dot{y}_1$ or $\dot{y}_2$ vanish. The number of such flip lines is equal to the index of the critical point which itself is equal to the number of different real roots of the equations $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$, equations which are not required to hold simultaneously. We found that the behavior of trajectories is summarized by the rules:

1) Attractors, repellors dominate the global behavior.

2) Flip lines restrain the response of the trajectory to the attractor/repellor by constraining the slope of the trajectory at every point.

We have found that these two rules are sufficient to enable one to qualitatively sketch the trajectory through any point in the phase plane once the linear attractors, repellors, and flip lines are characterized. These lines are all obtained by the solution of simple algebraic equations.

APPENDIX

In this appendix we consider degenerate cases where one or both of the f_i define single straight lines, that is

$$\frac{dy_i}{dt} = f_i = \mp (ay_1 - by_2)^2 \tag{A.1}$$

It is sufficient to let f_1 define a hyperbola, an ellipse, or a single straight line and let f_2 define a single straight line. We wish to emphasize that (A.1) does not define flip lines because $\dot{y}_2$ does not change sign as a point passes through the line $\dot{y}_2 = 0$. The behavior of trajectories is as a consequence dominated by any flip lines of the other equation. Therefore we expect either phase portraits similar to that of cases 4.2.1 and 4.2.2 (circular - hyperbolic cases where the index is four), or phase portraits similar to that of case 4.3.1 (circular - elliptical cases where the index is zero). We expect systems in which both f_1 and f_2 are perfect squares to yeild phase portraits which are limiting cases of the hyperbolic - hyperbolic case obtained in 4.1.4.

Case A.1

Consider the following case

$$\frac{dy_1}{dt} = y_1^2 + 3y_1y_2 + .5y_2^{\,2} \tag{A.2}$$

$$\frac{dy_2}{dt} = -y_1^2 - 2y_1y_2 - y_2^2 = -(y_1 + y_2)^2$$

where the RHS of first equation represents a family of hyperbolae while the RHS of the second one represents a straight line.

Figure A.1 shows that the trajectory does not change the sign of its slope when it crosses the line given by $y_1 = -y_2$.

Again we notice that there is a straight line solution, the single linear solution of the system, given by $y_1 = -.15y_2$. The figure is obviously similar to Fig. 4.2.2, as expected.

Case A.2

$$\frac{dy_1}{dt} = y_1^2 + y_2^2$$

$$\frac{dy_2}{dt} = (.5y_1 - .4y_2)^2 \qquad \text{(A.3)}$$

where the RHS of the first equation defines a circle while the RHS of the second equation defines a single straight line.

Fig. A.2 Shows that the trajectories don't change the sign of their slope any where. The system has a single linear solution (attractor/repellor), which nearby trajectories asymptotically approach as $t\to\infty$, and $t\to-\infty$. Notice that this figure is similar to that of case 4.3.1

Case A.3

Consider the case

$$\frac{dy_1}{dt} = (y_1 - .5y_2)^2$$
$$\frac{dy_2}{dt} = (y_1 - 3y_2)^2 \tag{A.4}$$

where now both equations define straight lines. Figure A.3 verifies that there are three linear solutions of the system given by $y_1 = 1.86\ y_2$, $y_1 = 7.2y_2$, and $y_1 = .15y_2$, as expected from (3.34). The first of these is an attractor/repellor. The latter two are separatrices which divide the phase portrait into four portions: an inset where all nearby trajectories asymptotically approach the critical point (the origin), an outset where the trajectories depart the critical point, and two intermediates where trajectories come in from large distances, turn around, and recede into the distance. Note that this case is the limit of a hyperbolic - hyperbolic case.

REFERENCES

1. W.Greiner, B. Muller, Theoretical Physics (2)-Quantum Mechanics, Symmetries, Springer-Verlag, Berlin 1989

2. John Edward Compbell, Introductory Treatise on Lie's Theory of Finite Continuous Transformation Groups, Chelsea Pub. Co., NY., 1966

3. C.E.Wulfman and H. Rabitz, Global Sensitivity Analysis of Nonlinear Chemical Kinetics Equations using Lie groups: II. Journal Mathematical Chemistry 3(1989)261-297.

4. Abraham Cohen, An Introduction to the Lie Theory, D. C. Health & Co., Publishers, Boston, New York, Chicago, 1911

5. Michael Tabor, Chaos and Integrability in Nonlinear Dynamics, Wiley-Interscience, New York 1989

6. Ralph H. Abraham and Christopher D. Shaw, The Visual Mathematics Library: Vismath Volume 1, Dynamics- The Geometry of Behavior, Aerial Press, Inc. P.O.Box 1360,

Santa Cruz, Ca 95061.

7. Luther Pfahler Eisenhart, Coordinate Geometry, Dover Publications, Inc. New York 1966.

8. D.W.Jordan and P.Smith. Nonlinear Ordinary Differential Equations (Second edition), Oxford Applied Mathematics and Computing Science Series, New York, 1987.

9. Morris W.Hirsch, Stephen Smale, Differeintial Eqautions, Dynamical Systems, and Linear Algebra, University of California, Berkley, Acadmic Press New York 1974.

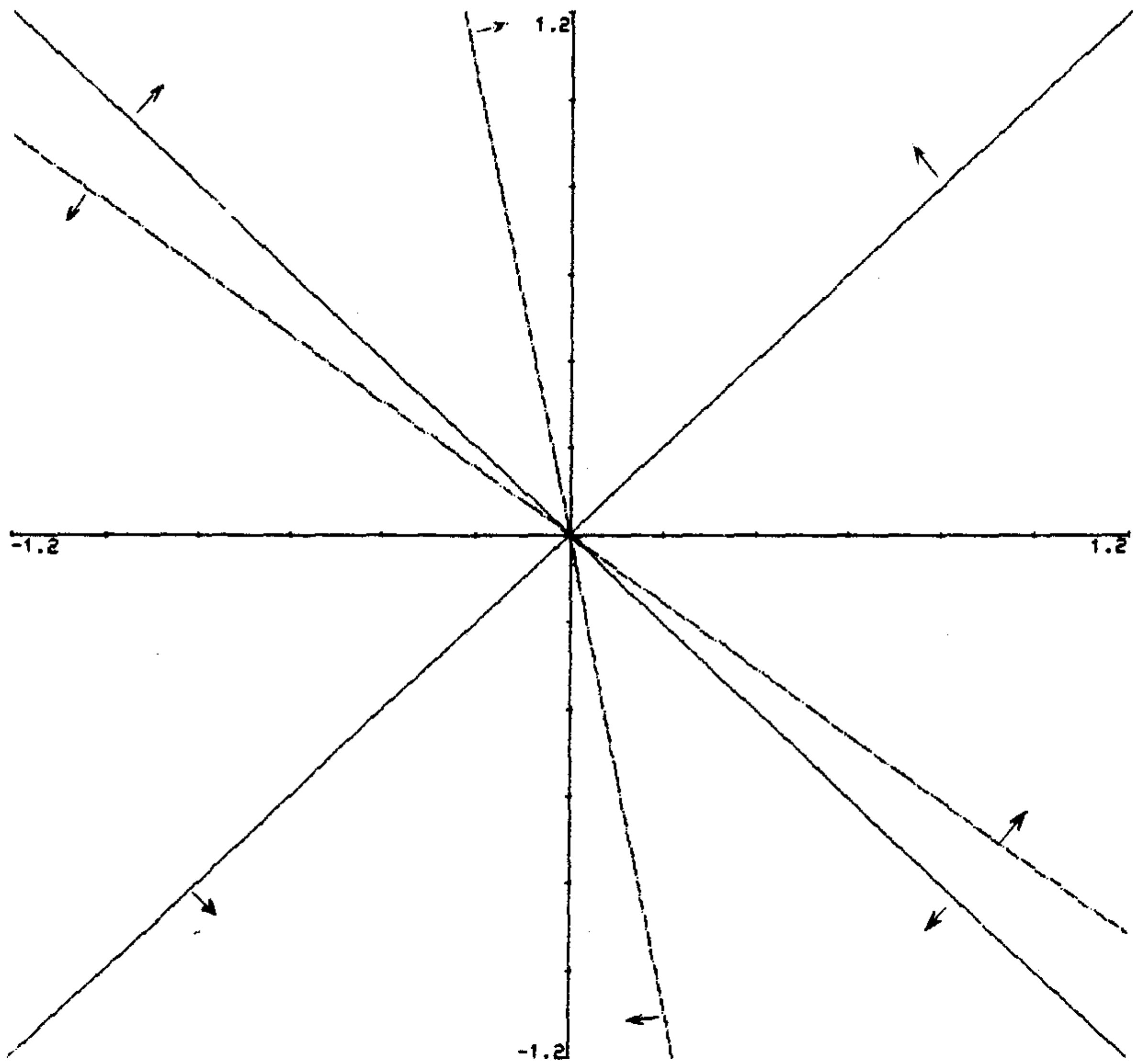


Fig.4.1.1a (flip lines dy1/dt=0 solid lines, dy2/dt=0 dashed lines)

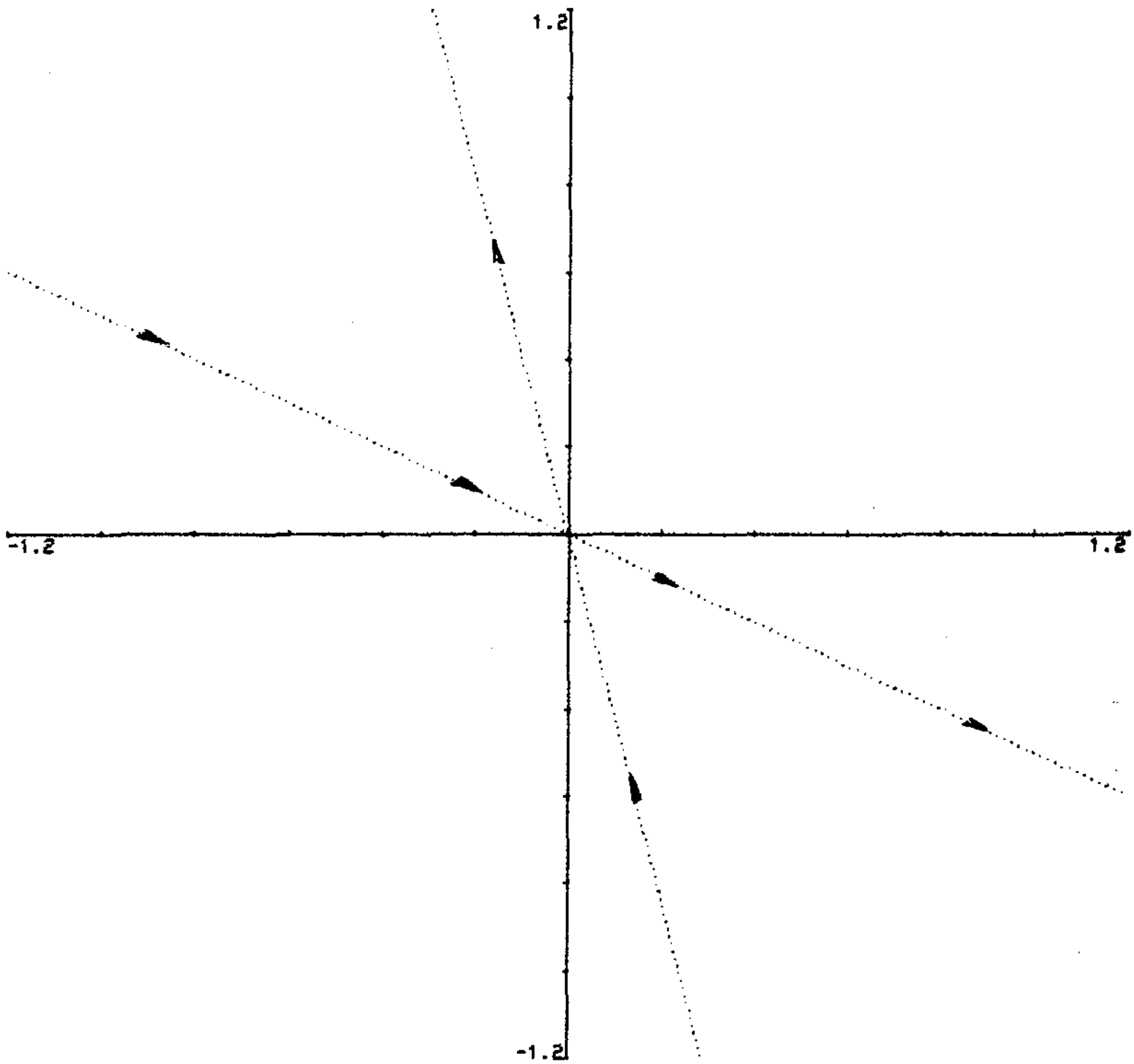


Fig.4.1 1 Attractors/Repellors

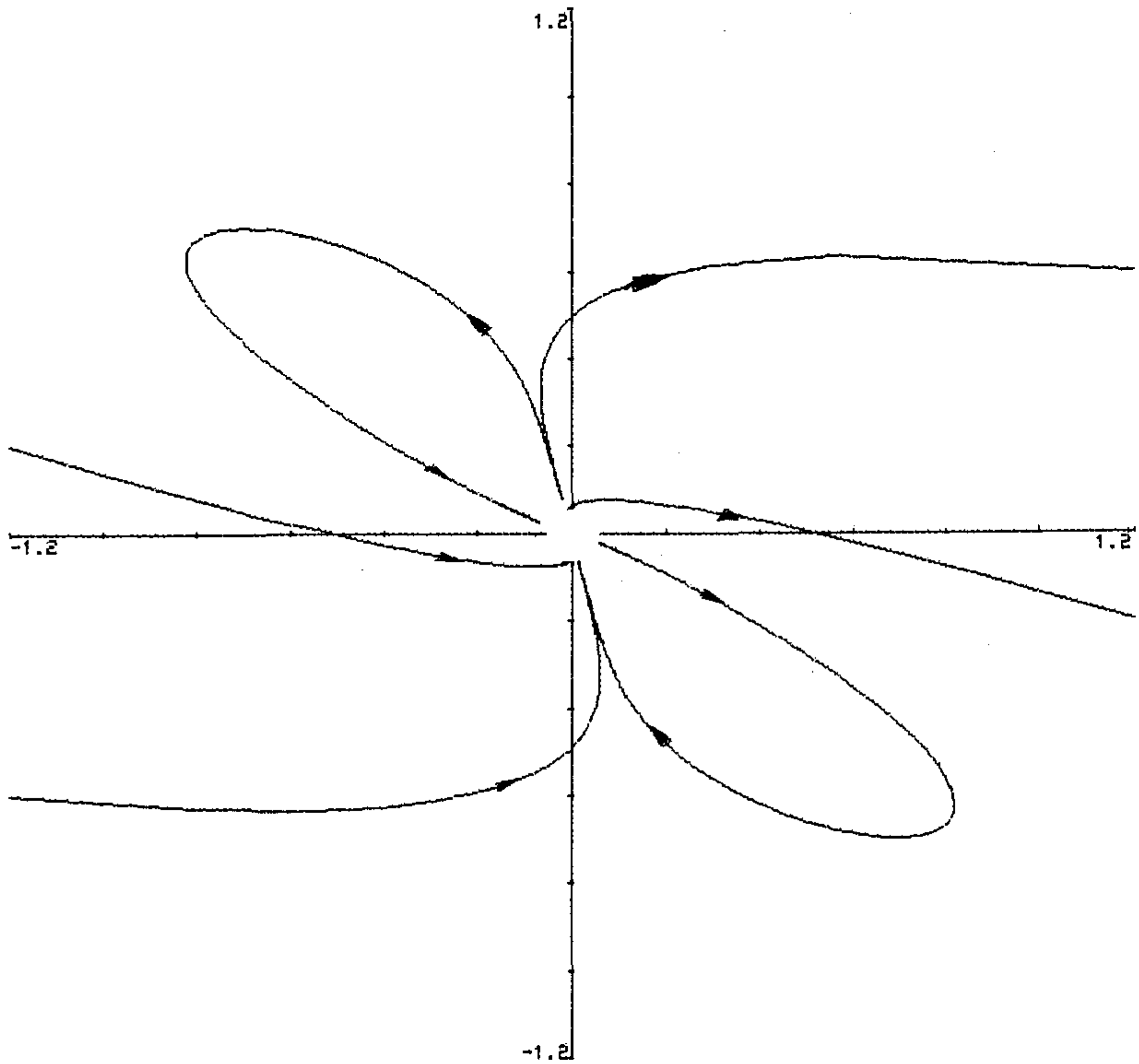


fig.4.1.1 (Hyperbolic- Hyperbolic Type)

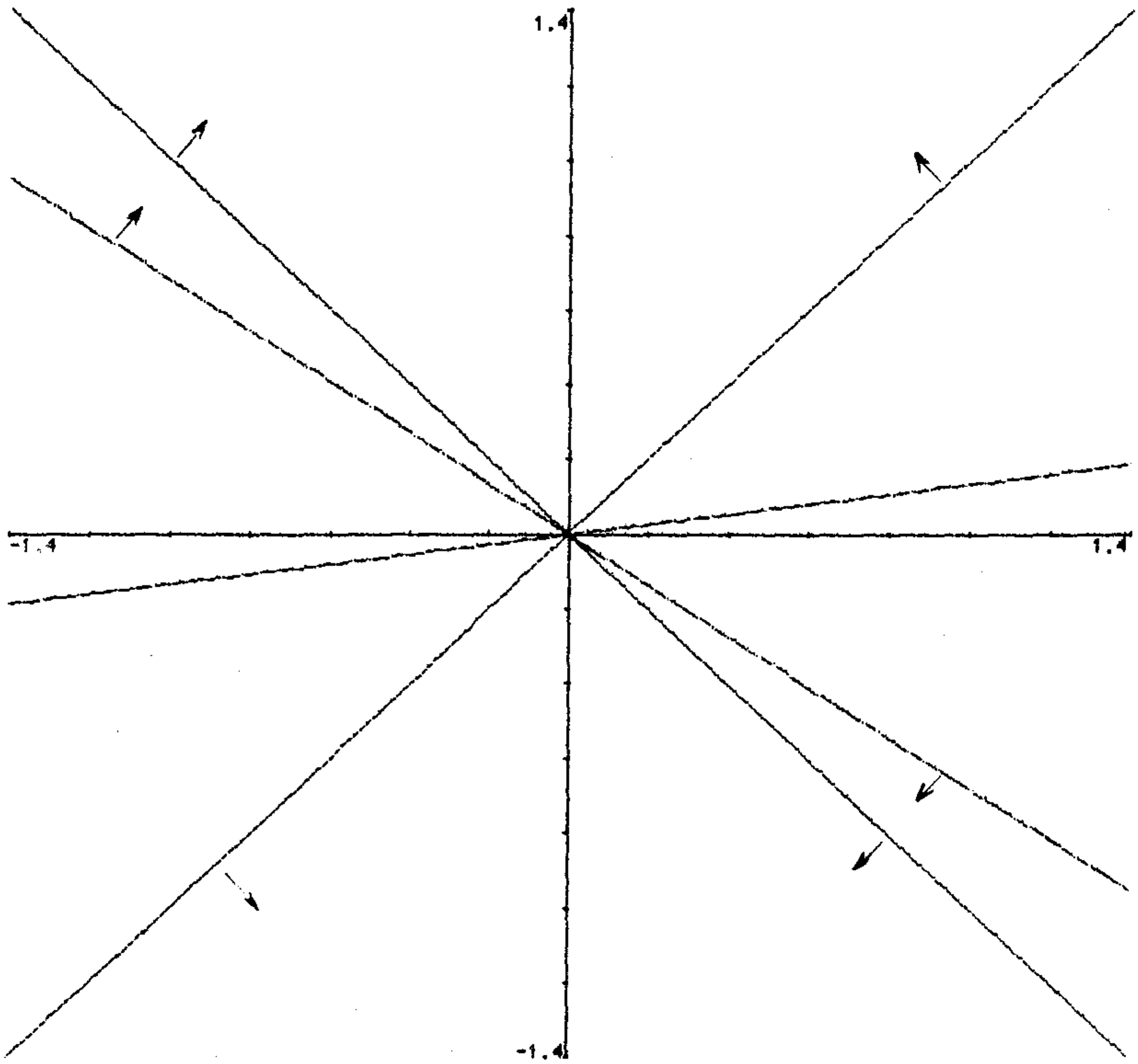


Fig.4.1.2 (Flip linees) dy1/dt=0(solid lines), dy2/dt=0(dashed lines)

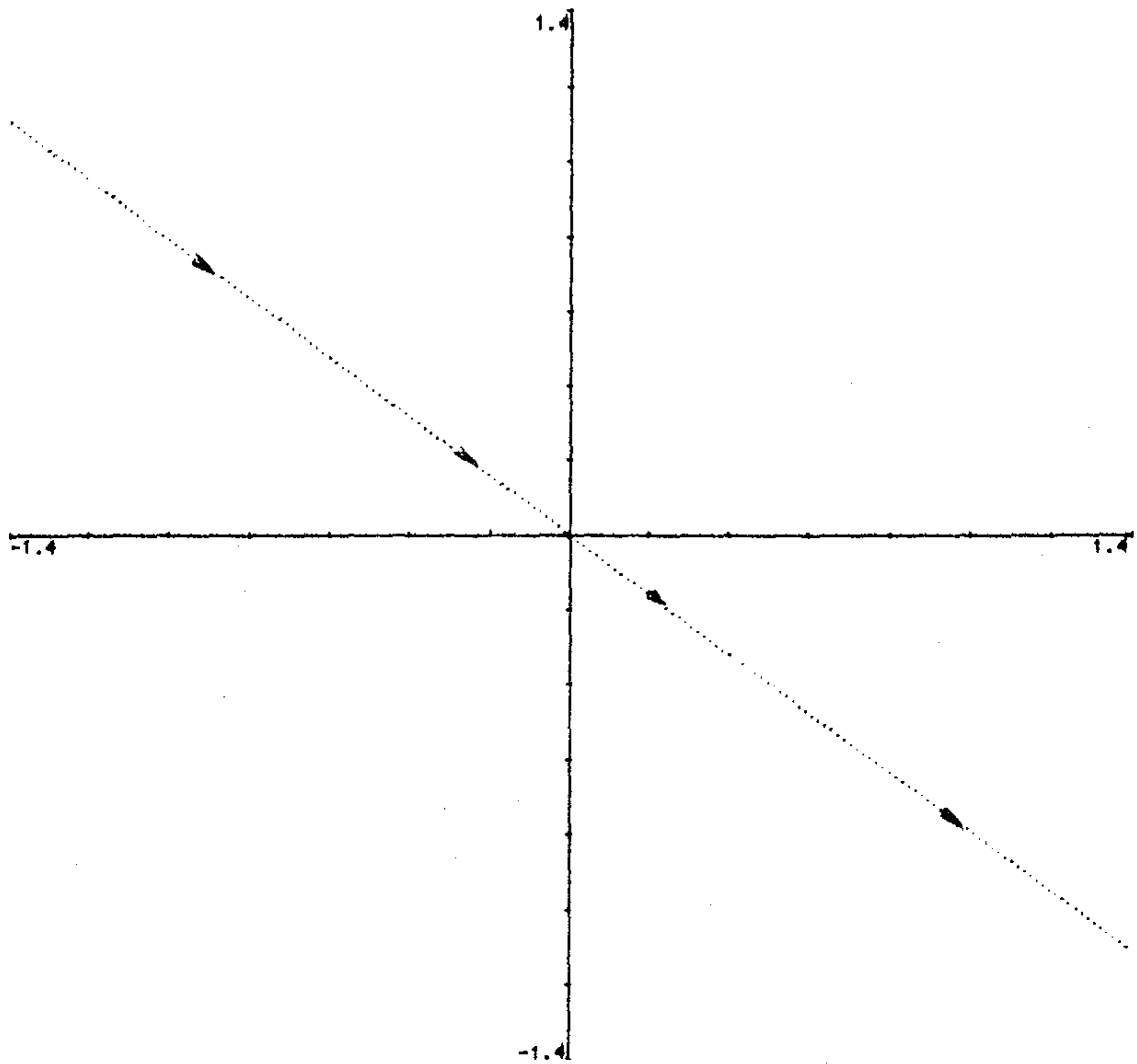


Fig.4.1.2 Attractor/Repellor

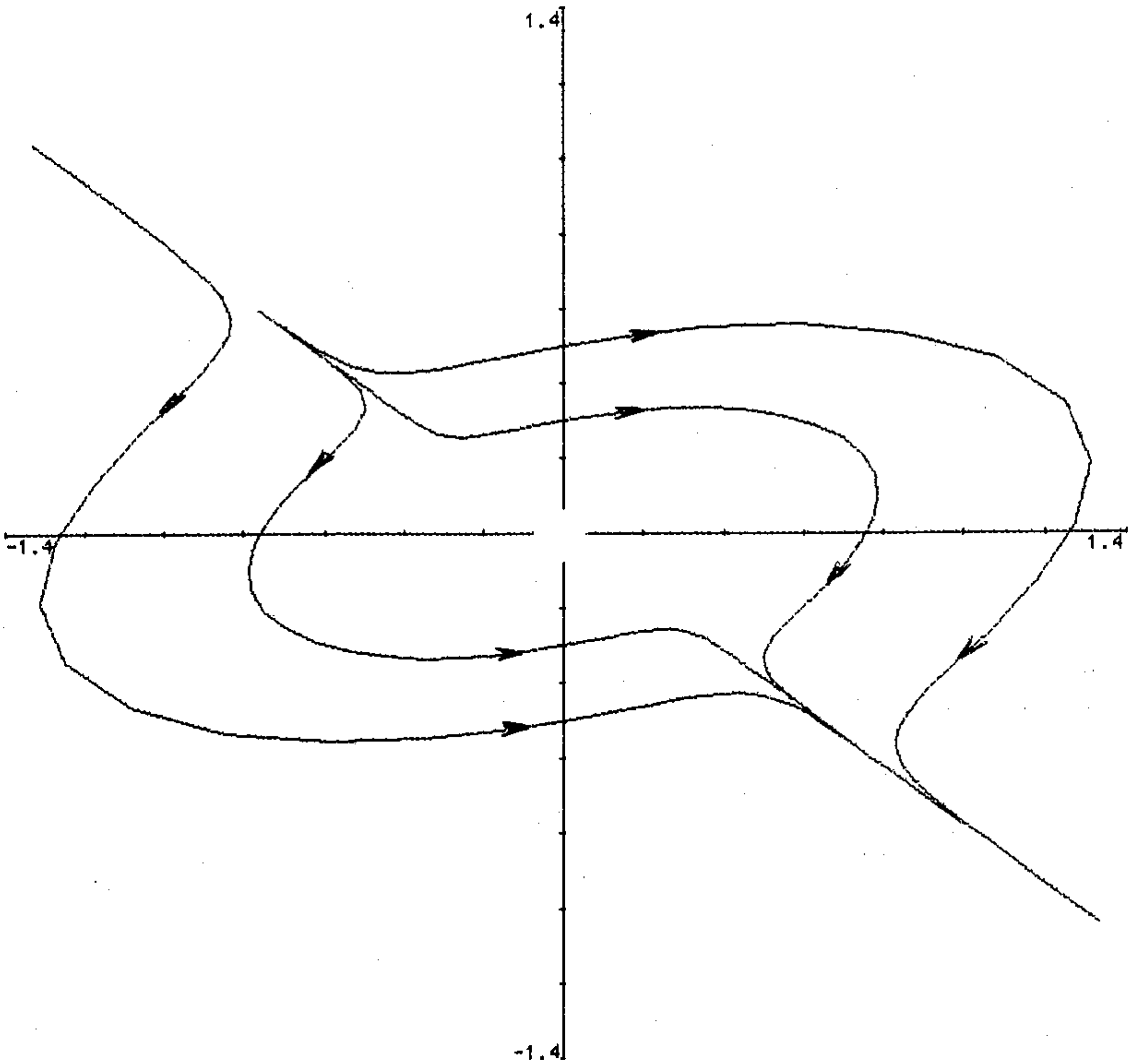
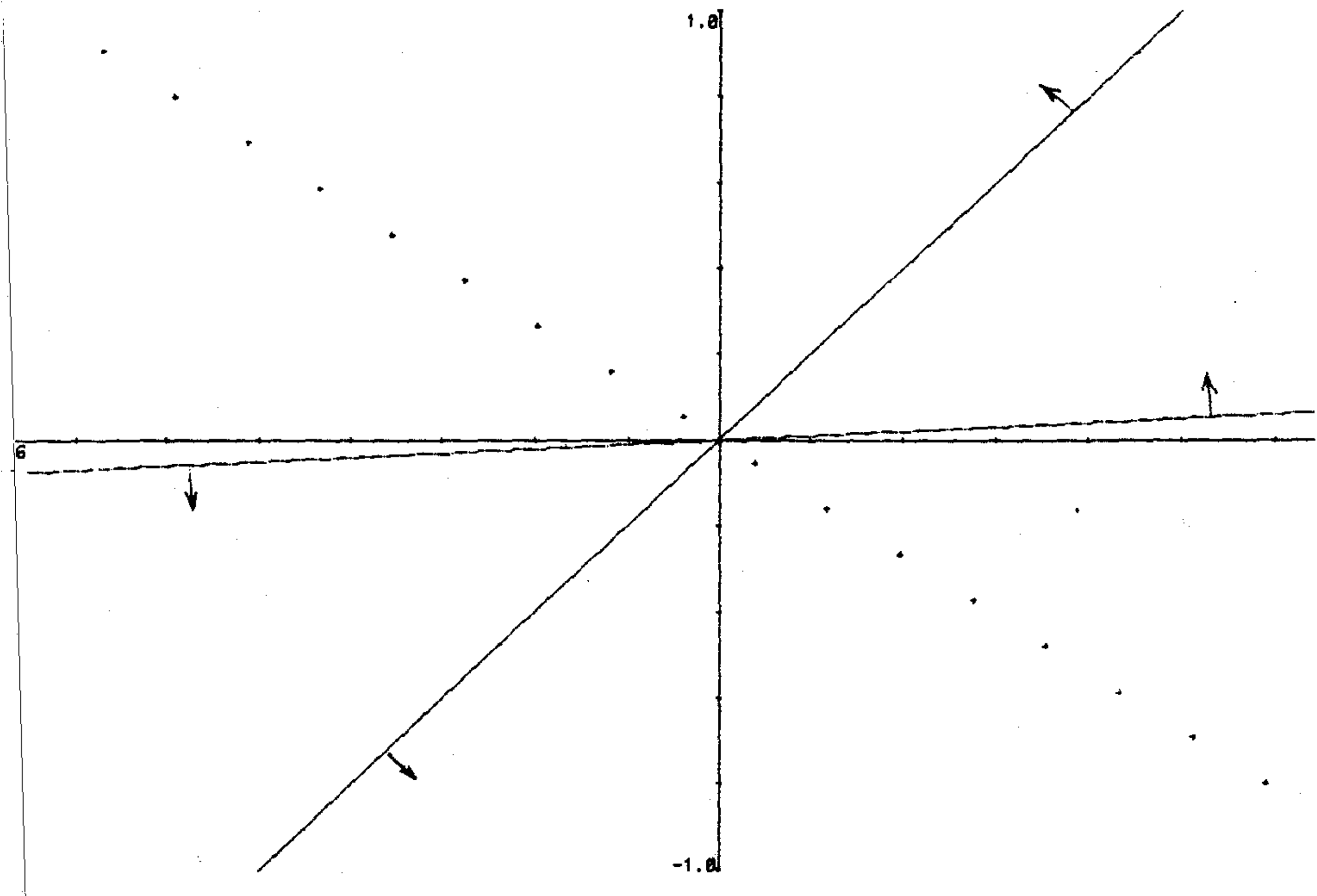


Fig.4.1.2 (Hyperbolic- Hyperbolic Type)



Flip lines(dy1/dt=0 solid line,dy2/dt=0 dashed line),Critical line(dy1/dt=dy2/dt=0 crosses)

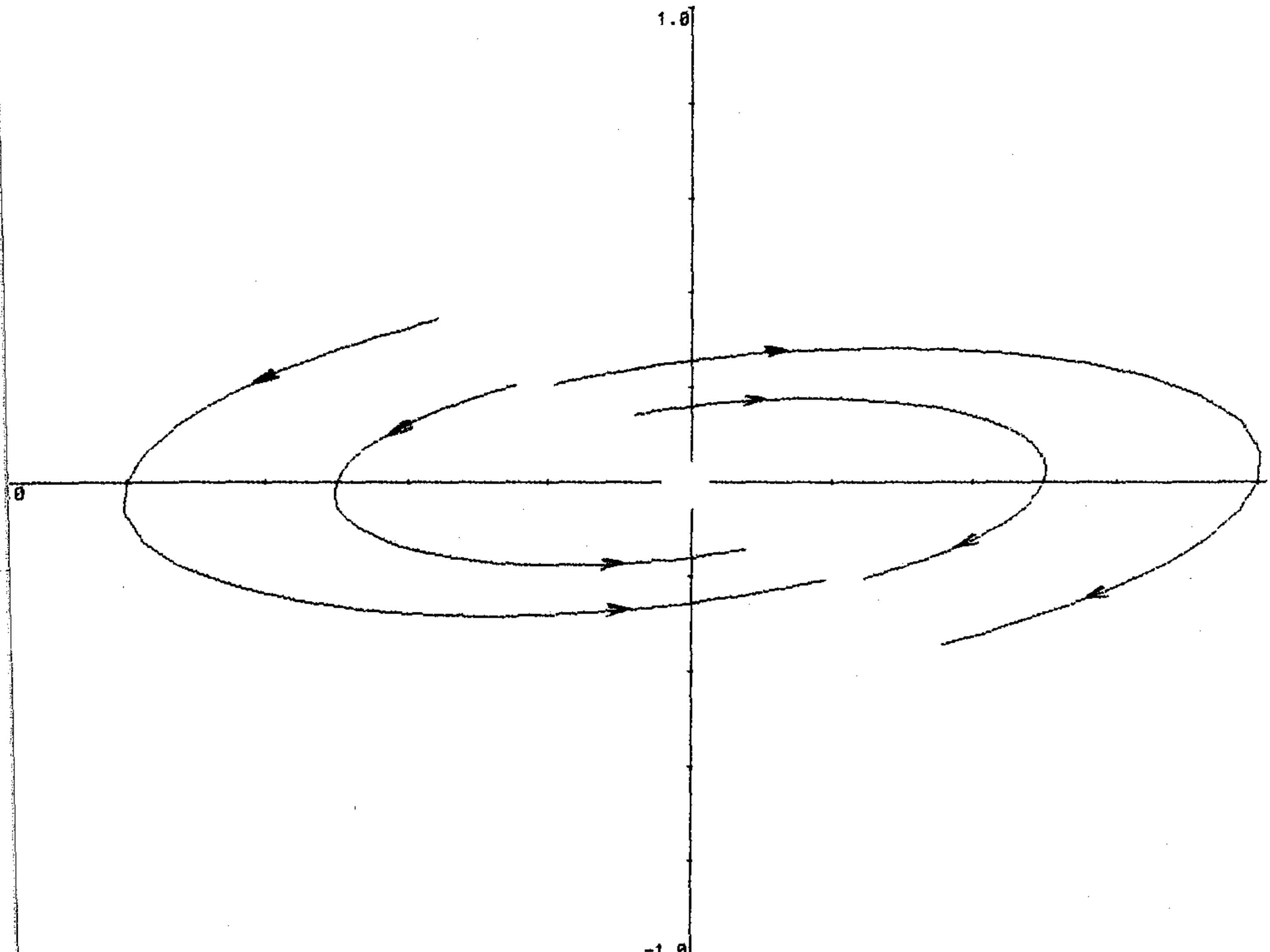


Fig.4.1.2' (Hyperbolic- Hyperbolic Type With Critical Line)

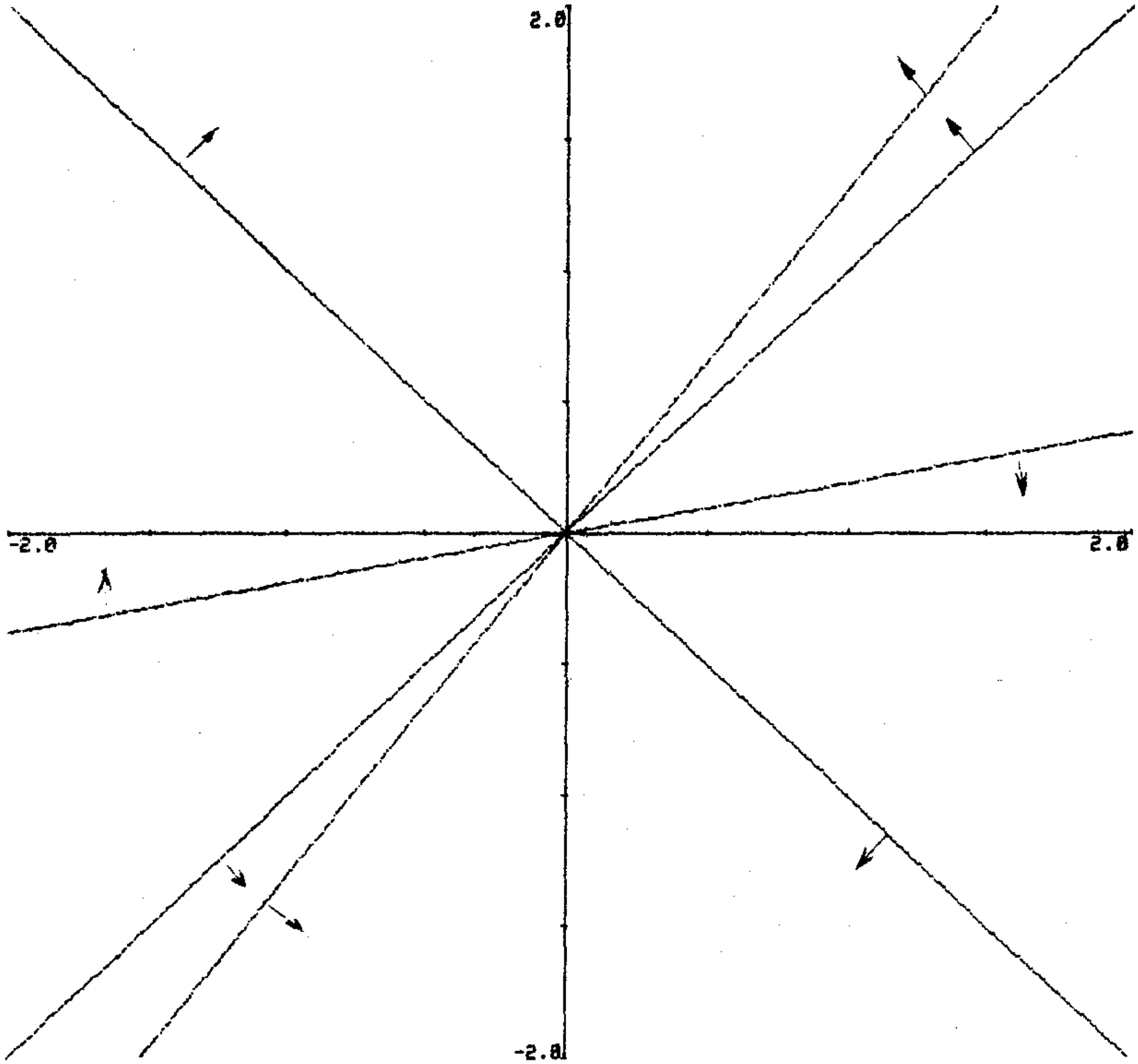


Fig.4.1.3 (flip lines dy1/dt=0(solid lines), dy2/dt=0(dashed lines)

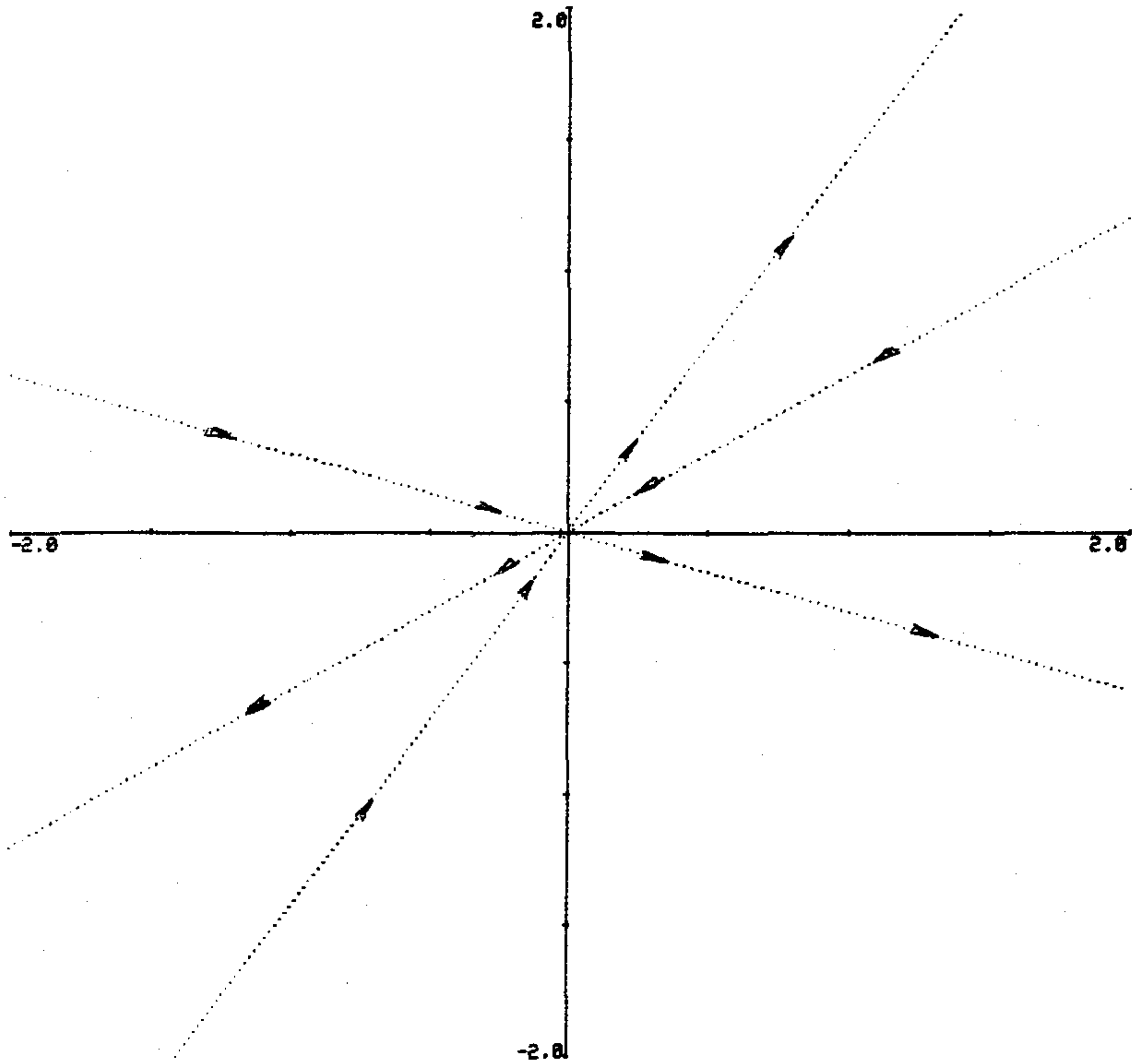


Fig.4.1.3 Attractors/Repellors

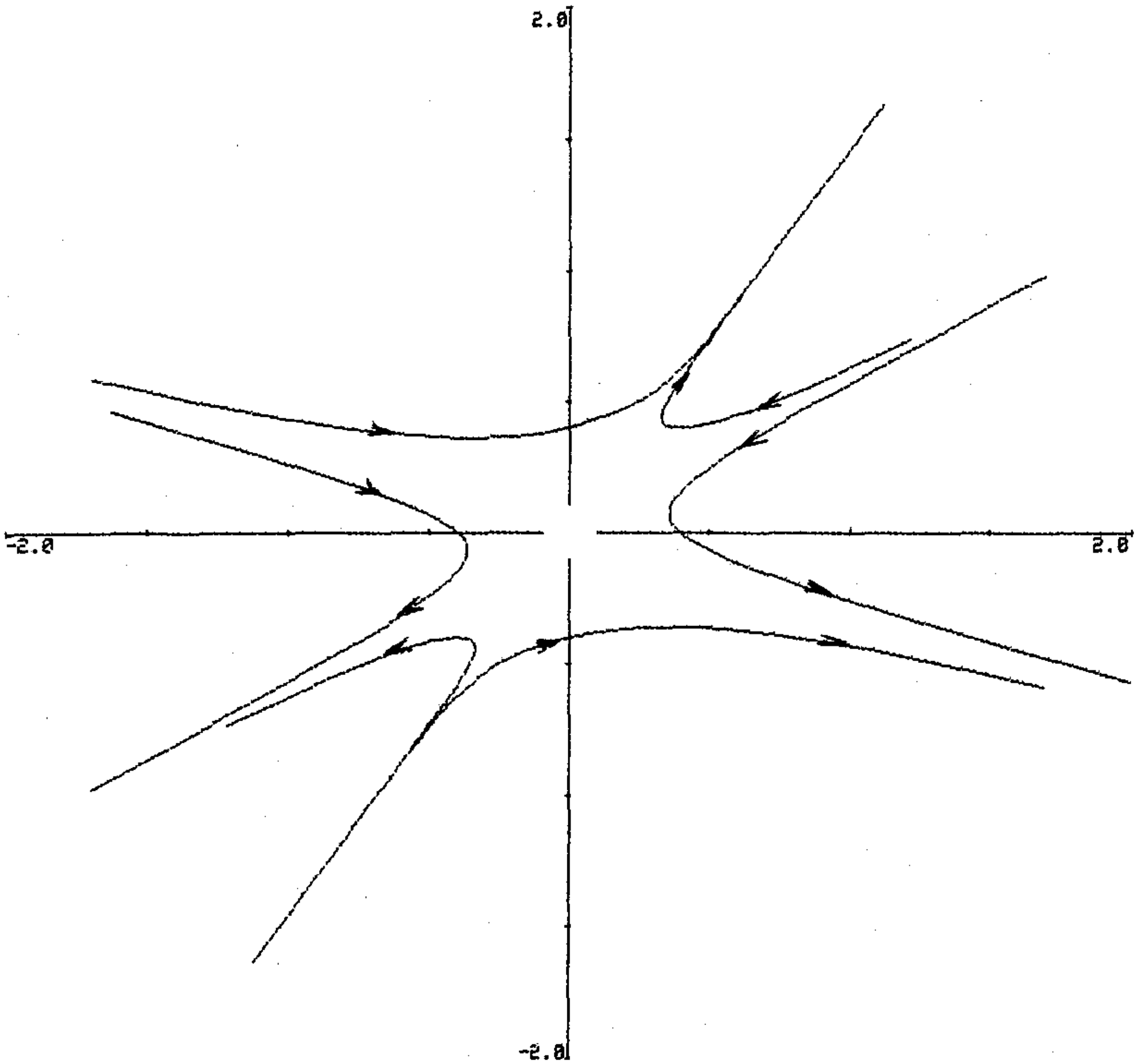


Fig.4.1.3 (Hyperbolic-Hyperbolic type)

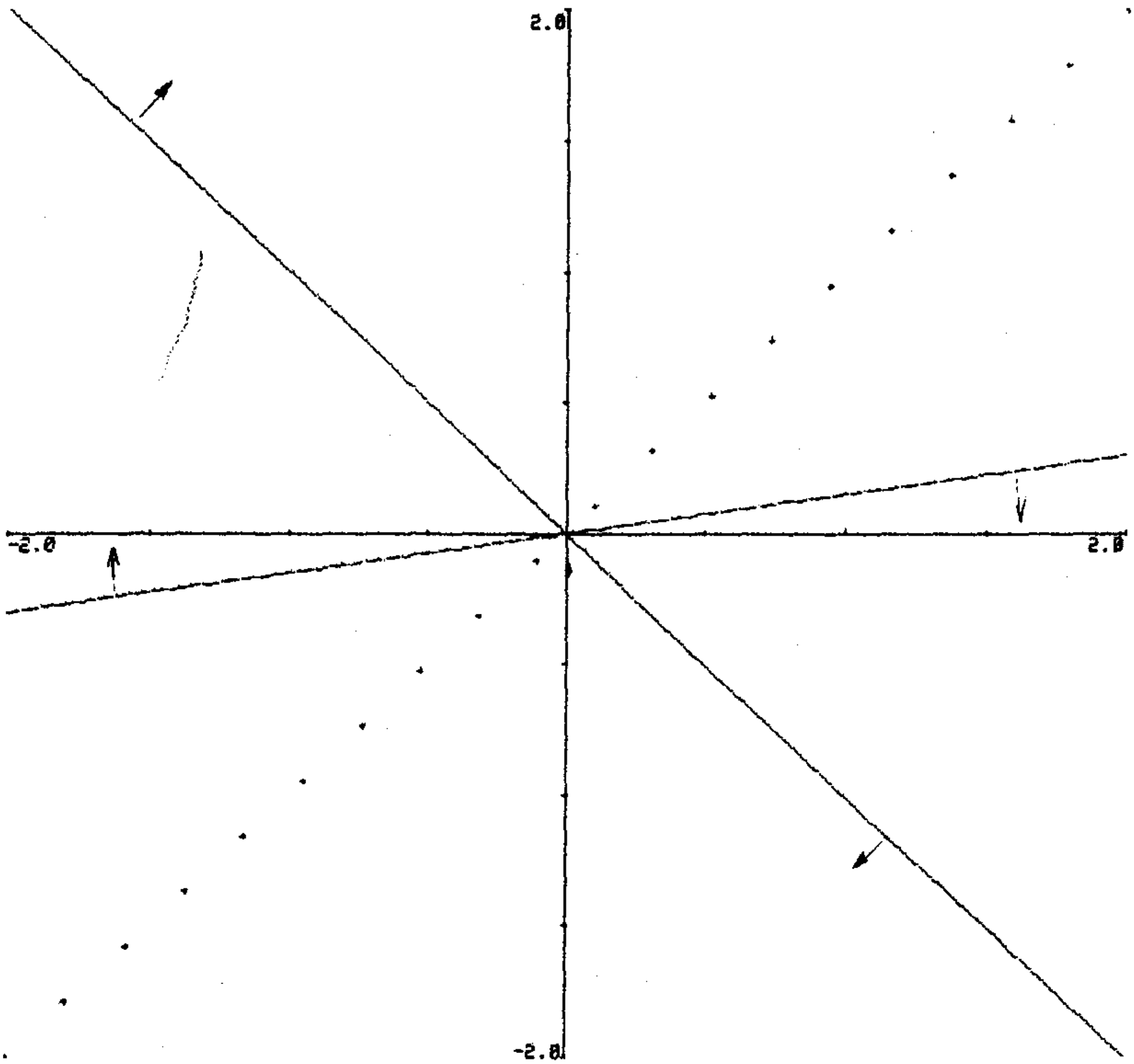


Fig.4.1.3' Flip lines(dy1/dt=0 solid line,dy2/dt=0 dashed line), Critical line(dy1/dt=dy2/dt=0 croesses)

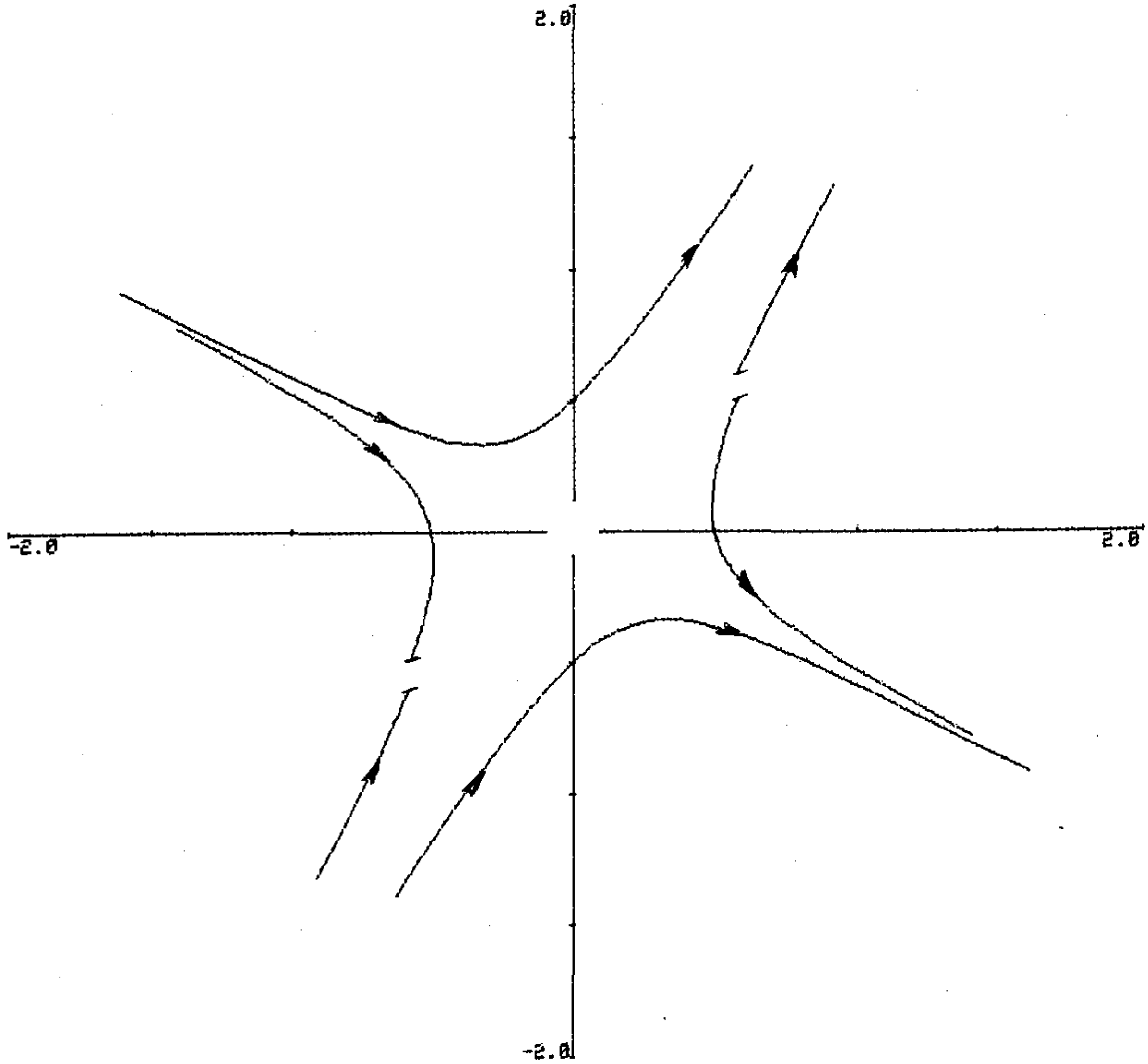


Fig.4.1.3' (Hyperbolic- Hyperbolic Type With Critical line)

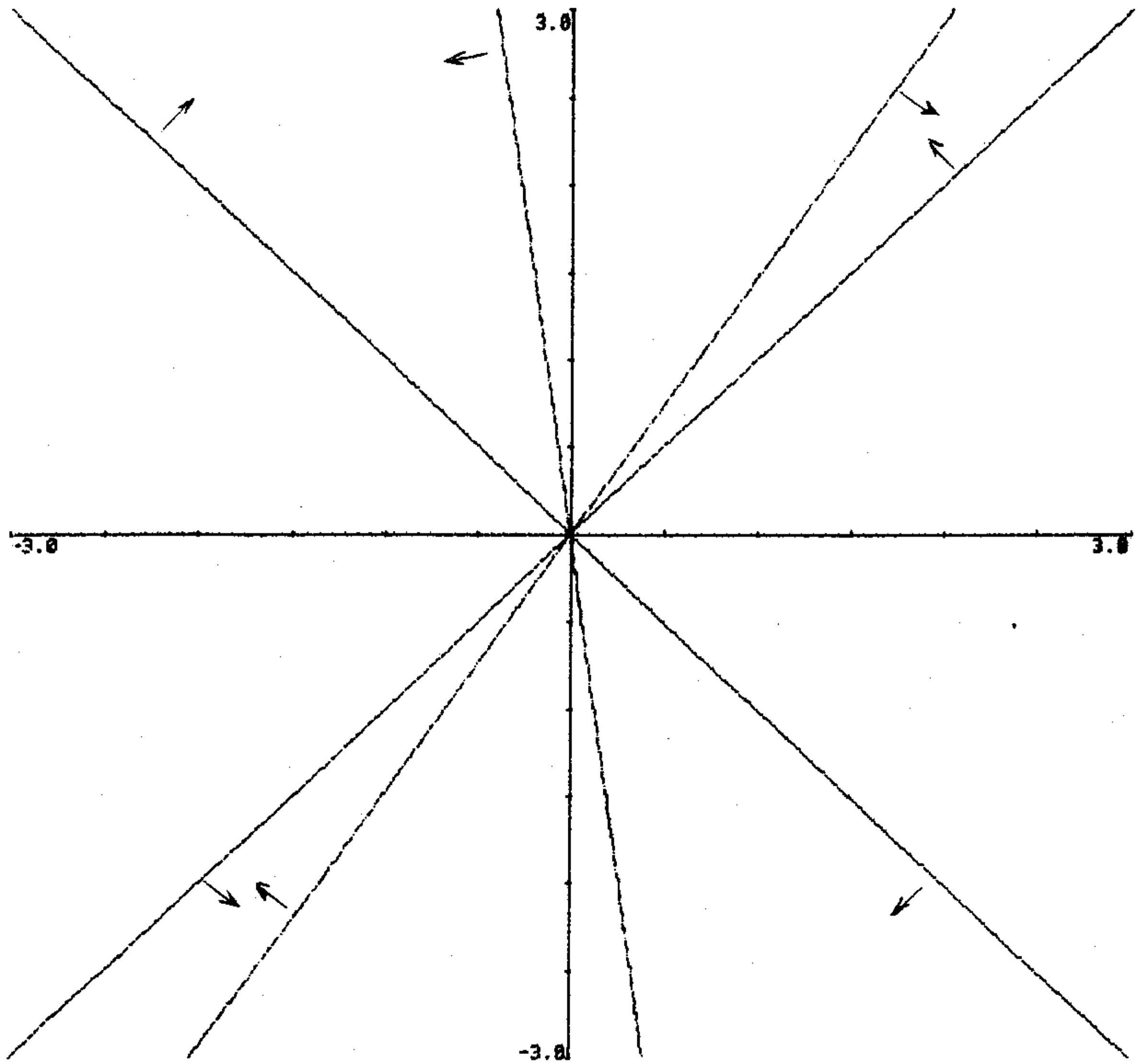


Fig.4.1.4 (flip Lines) dy1/dt=0(solid lines), dy2/dt=0(dashed lines)

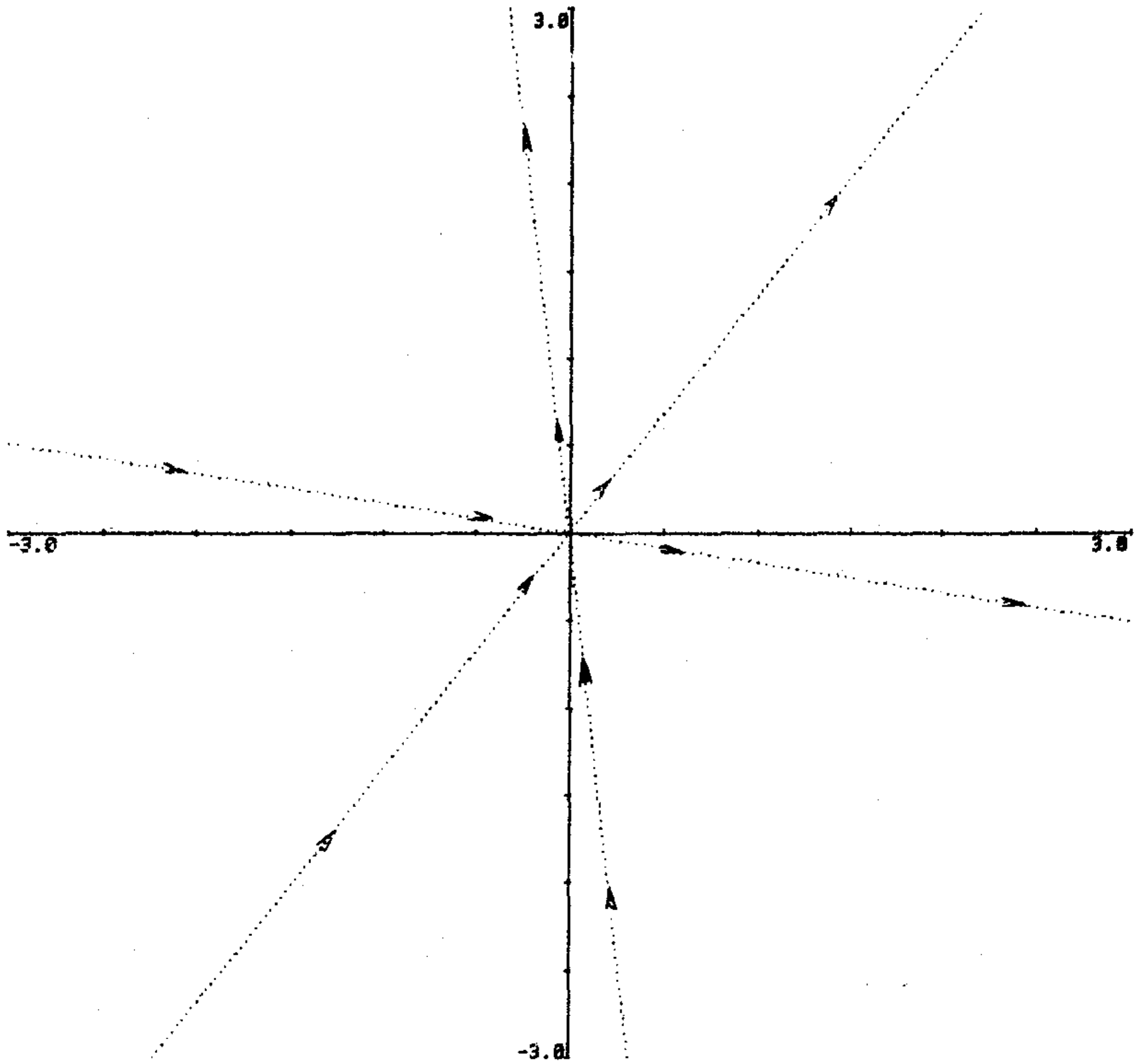


Fig.4.1.4,(attractors-Repellors

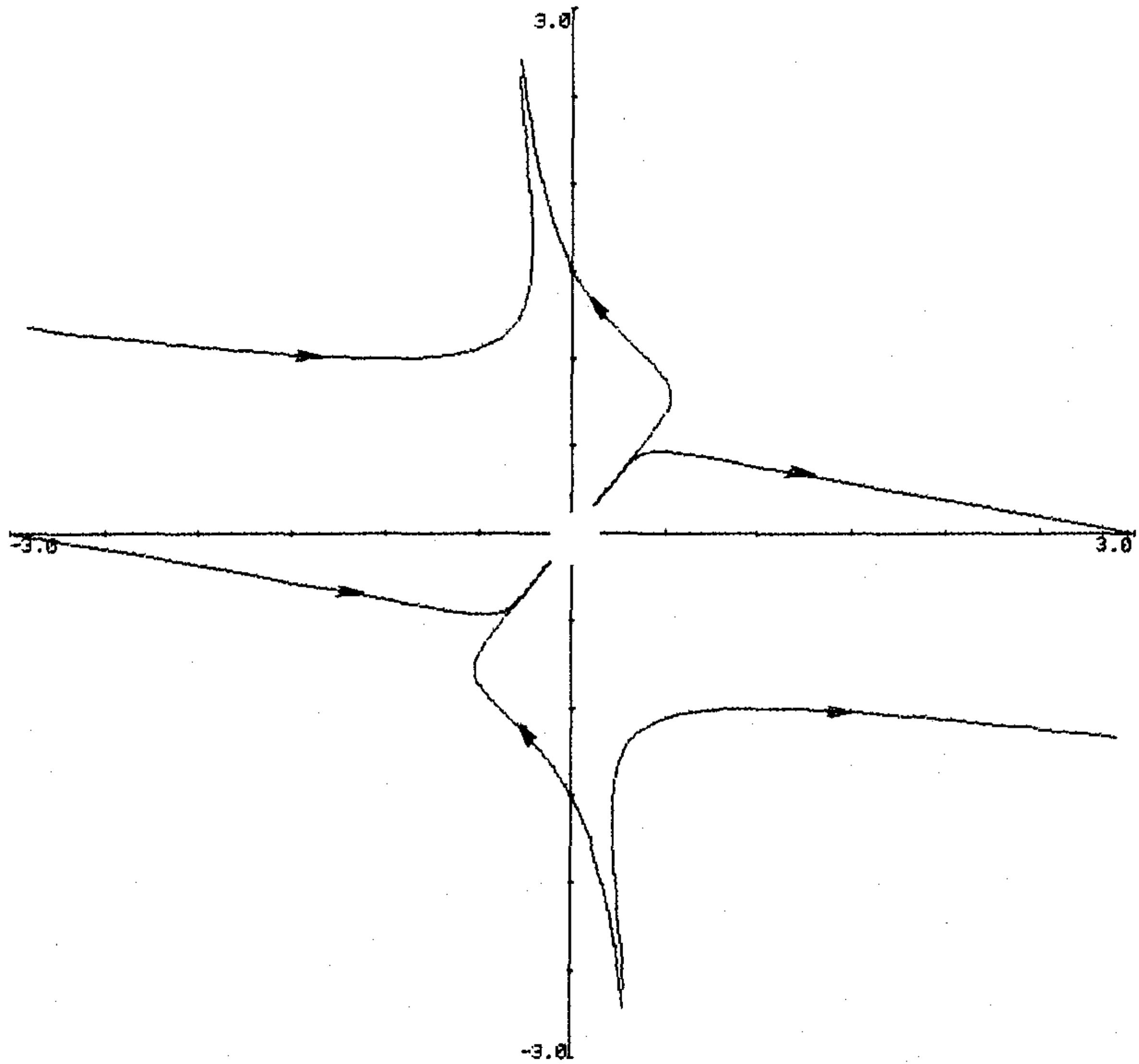


Fig.4.1.4 (Hyperbolic-Hunerbolic Type)

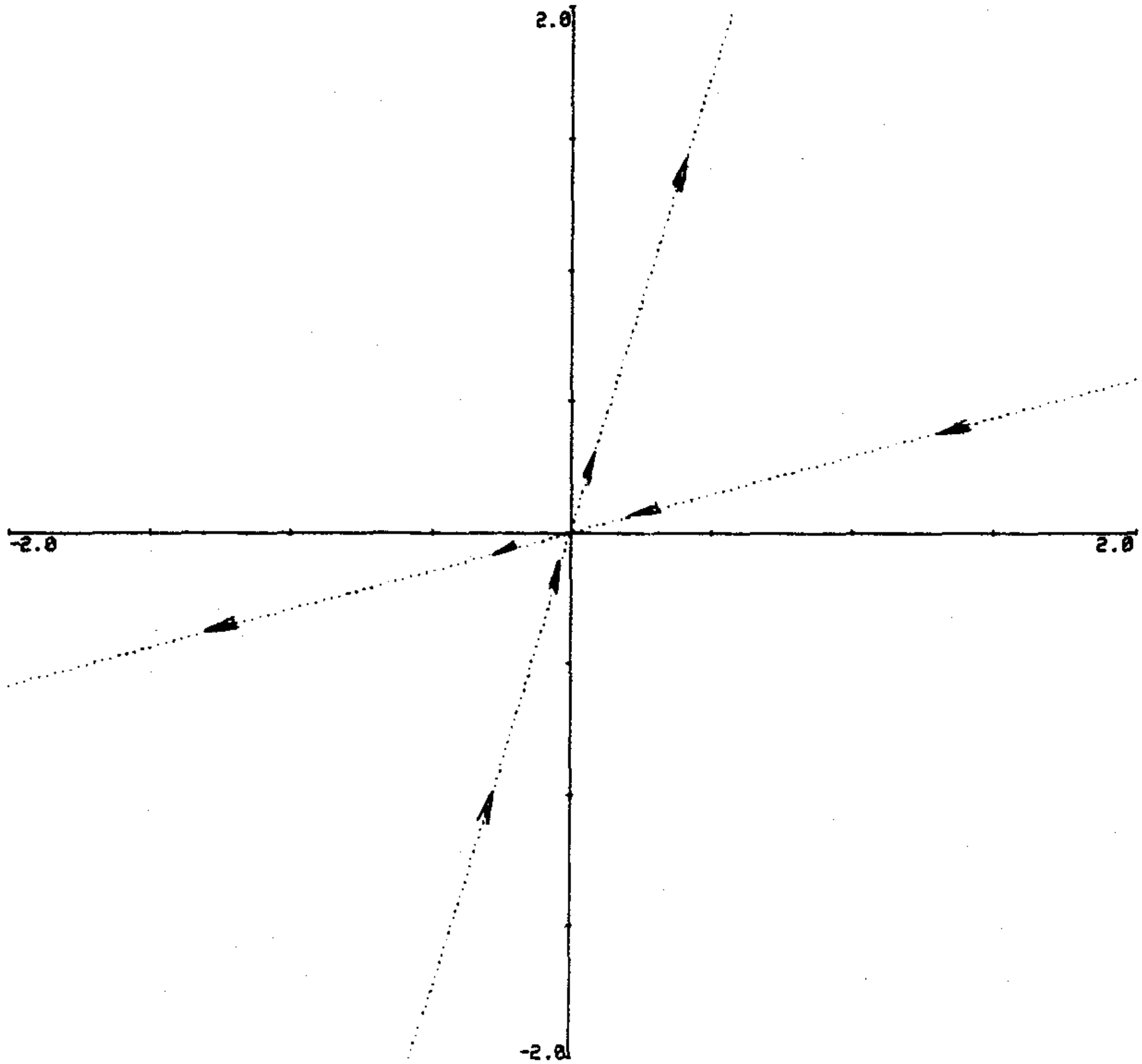


Fig.4.1.4' Attractors/Repellors

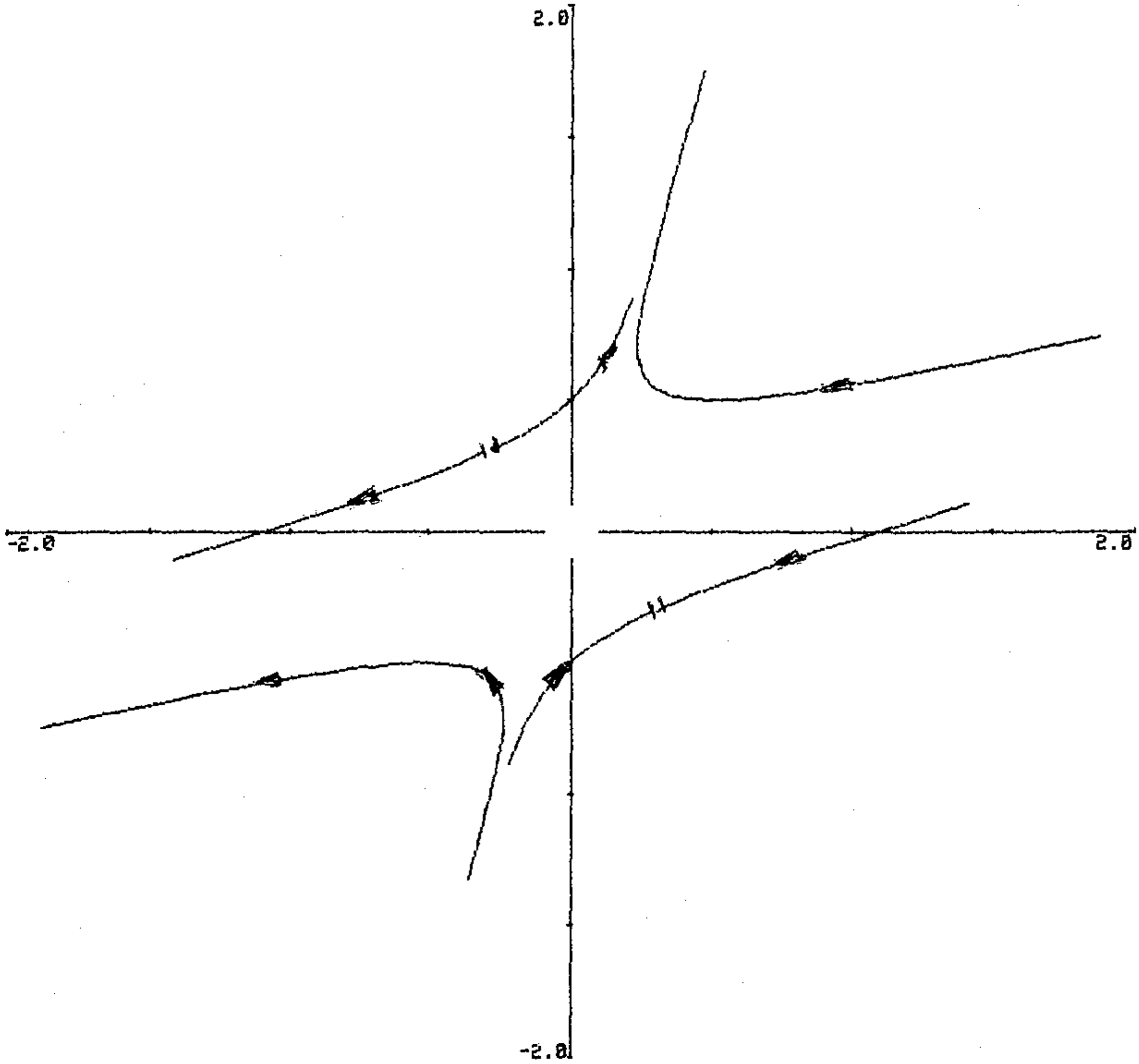


Fig.4.1.4′ (Hyperbolic- Hyperbolic Type With Critical line)

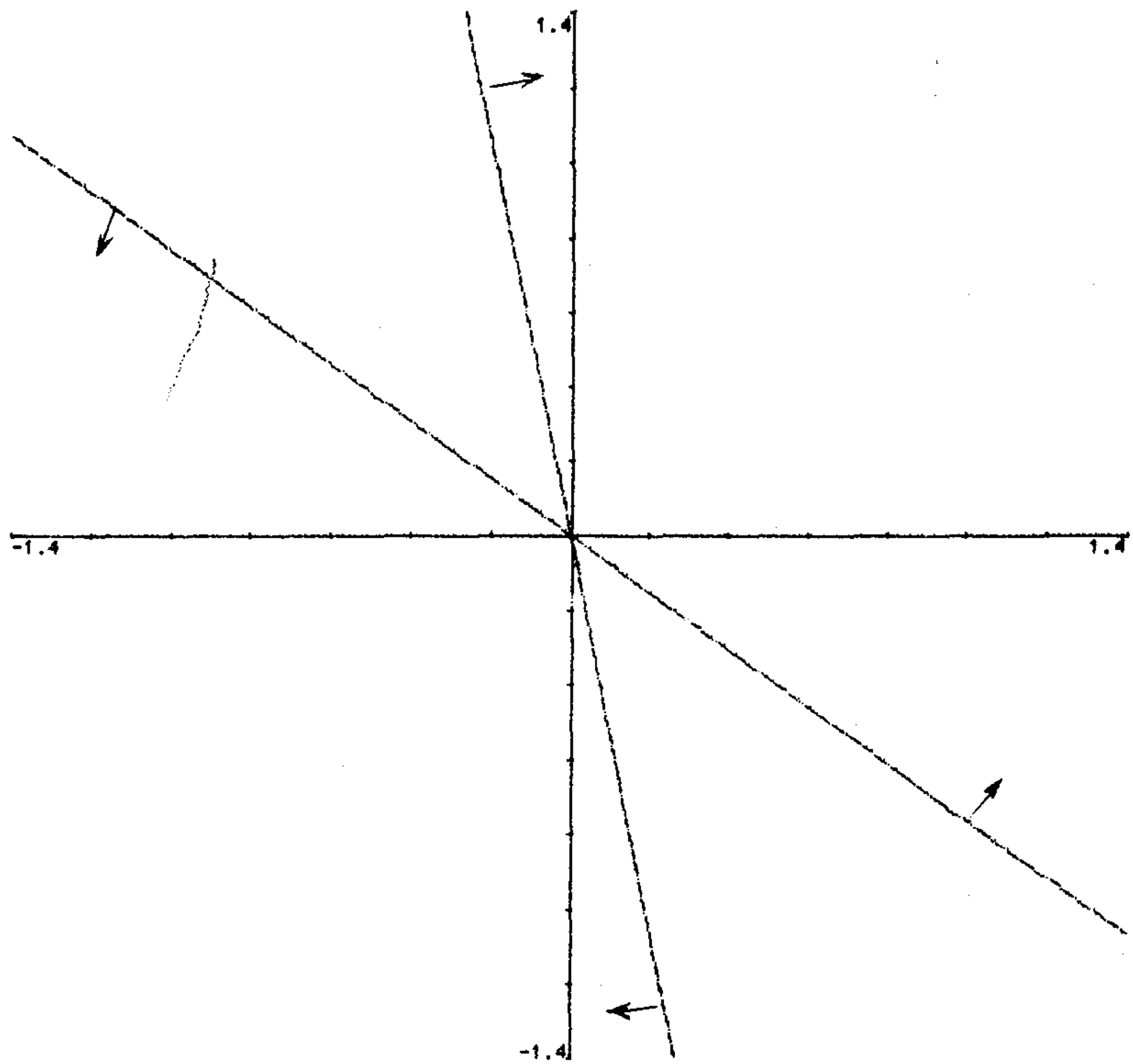


Fig.4.2.1 (Flip Lines) dy2/dt=0(dashed lines)

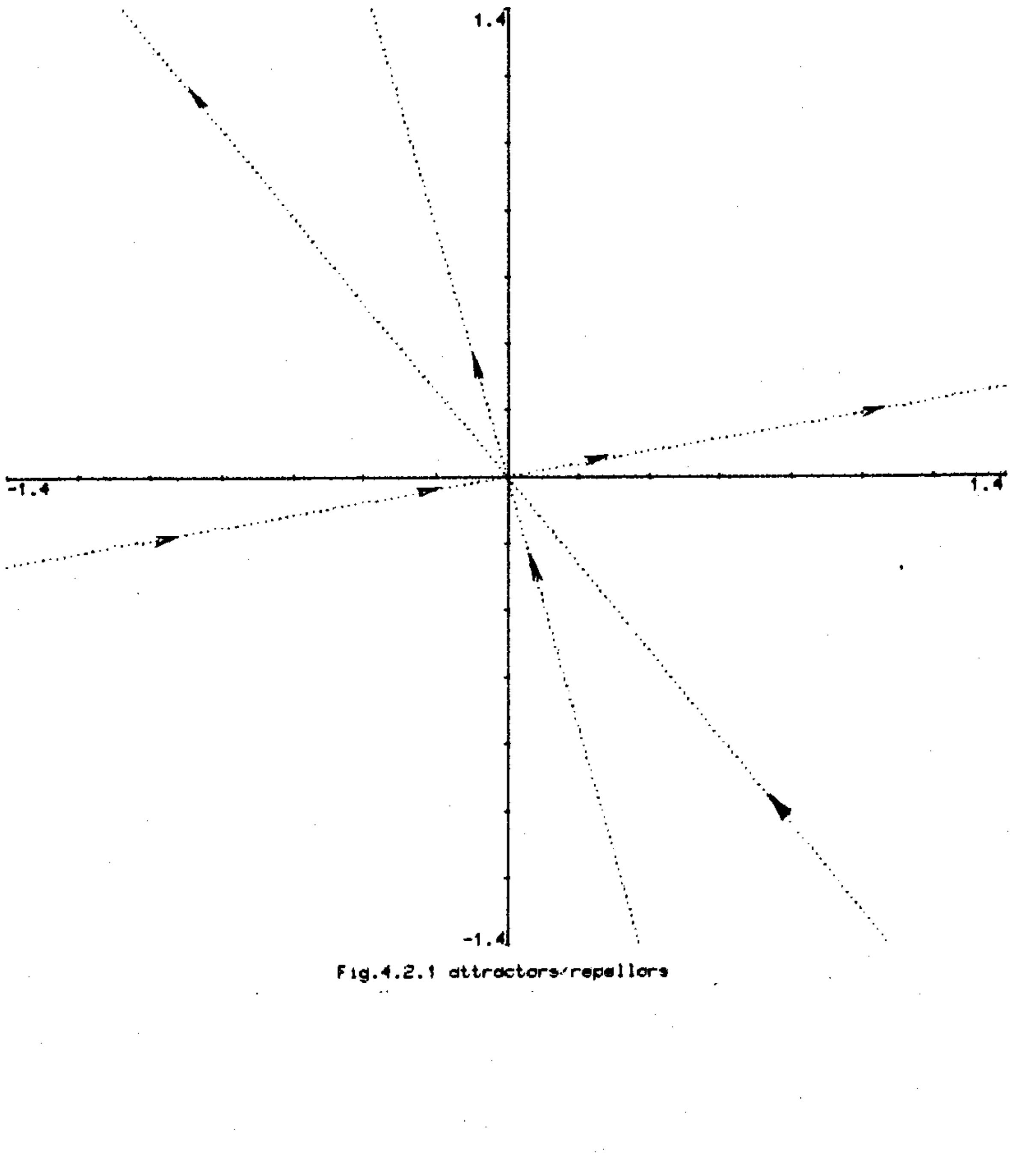


Fig.4.2.1 attractors/repellors

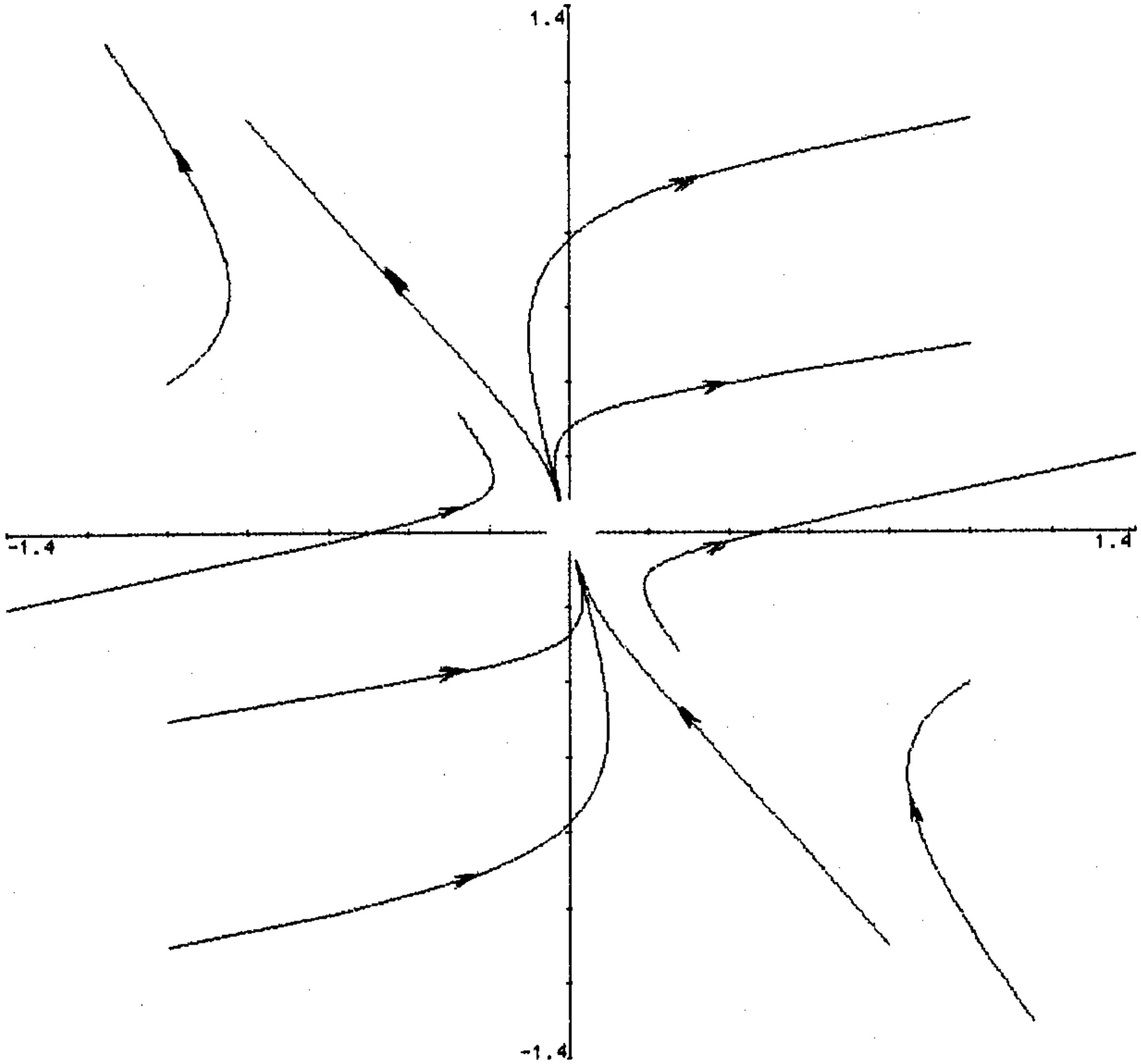


Fig.4.2.1(Circular-Hyperbolic Type

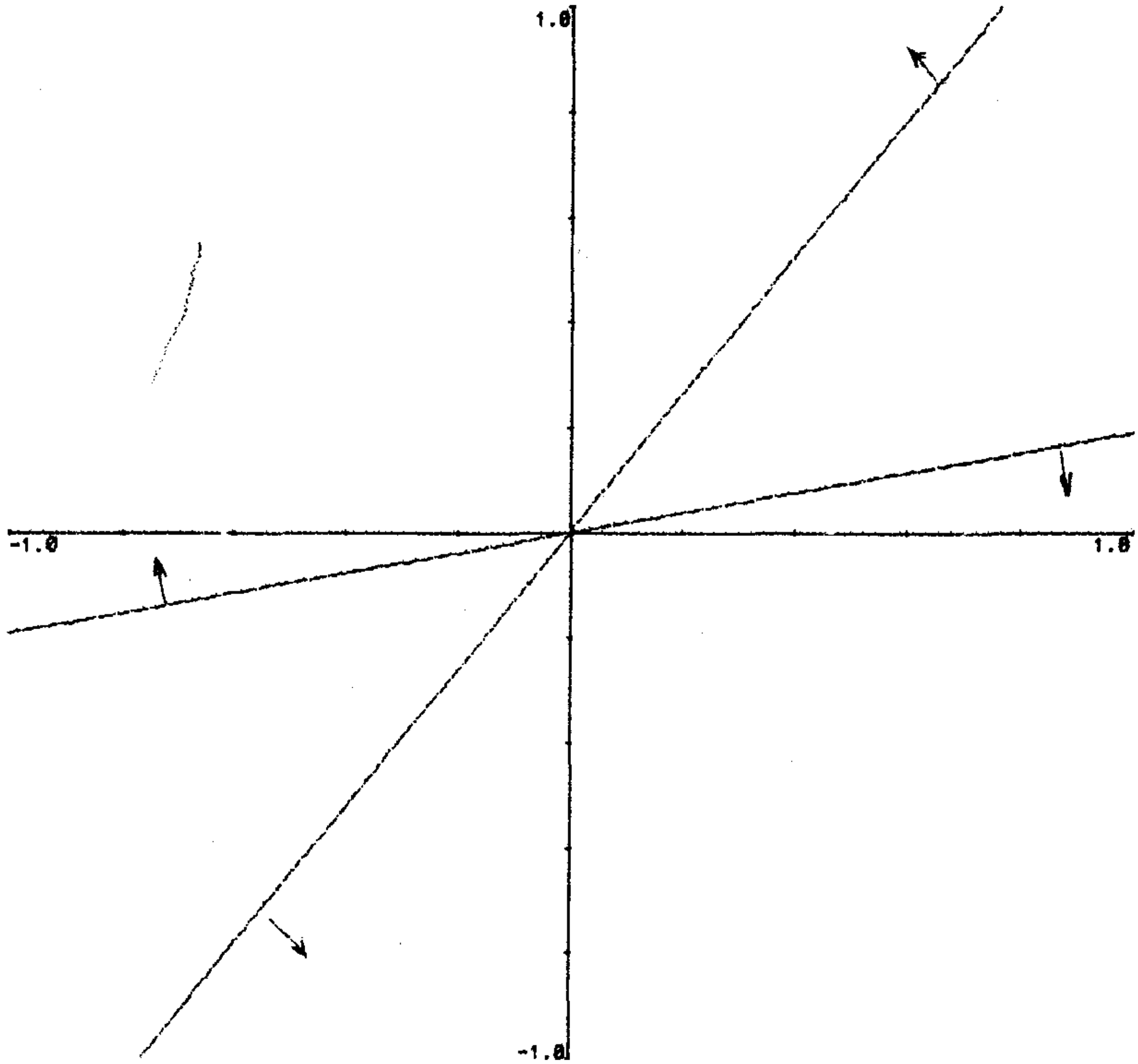


Fig.4.2.2 (Flip Lines) dy2/dt=0 (dashed Lines)

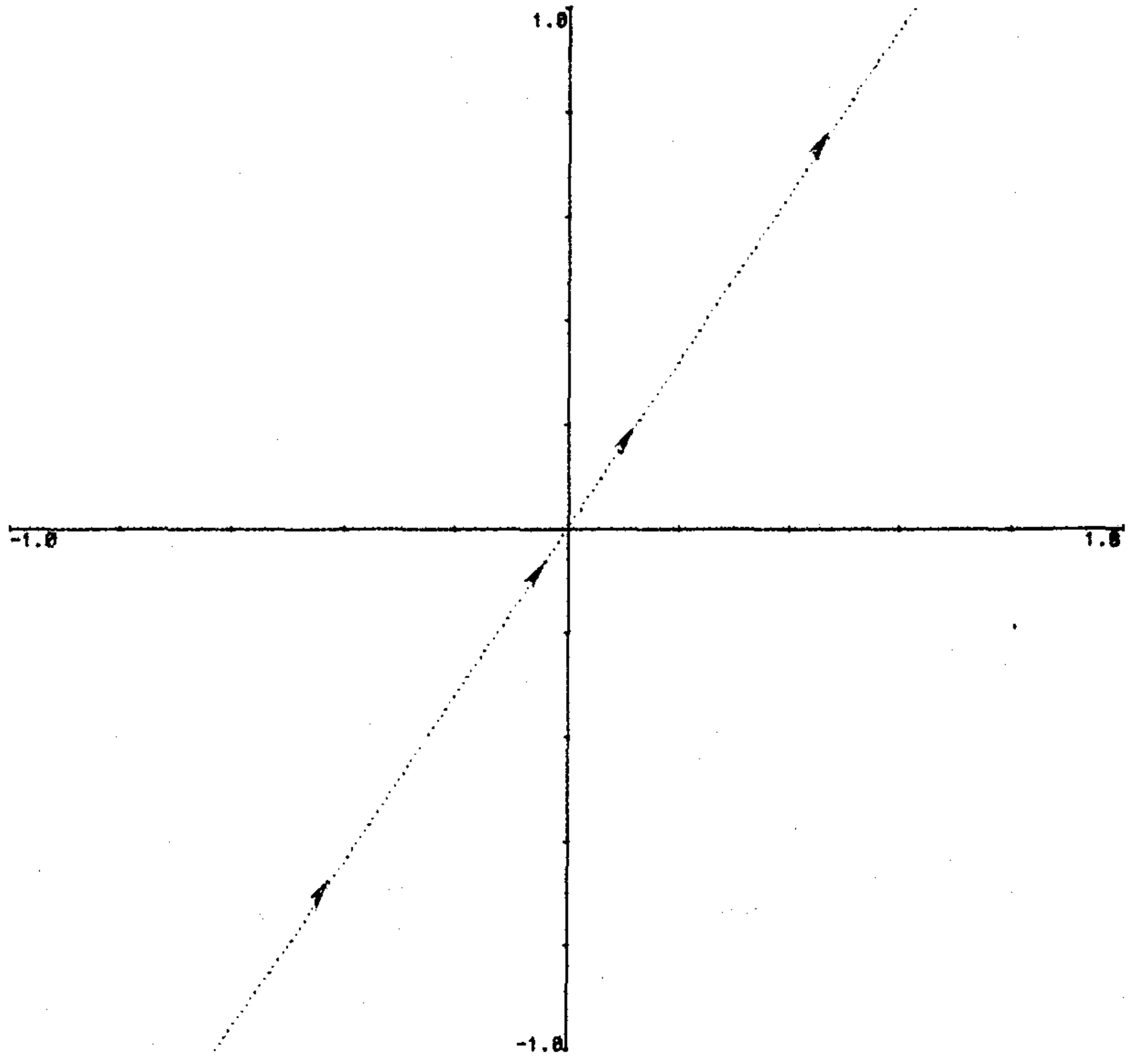


Fig.4.2.2 Attractor/Repellor

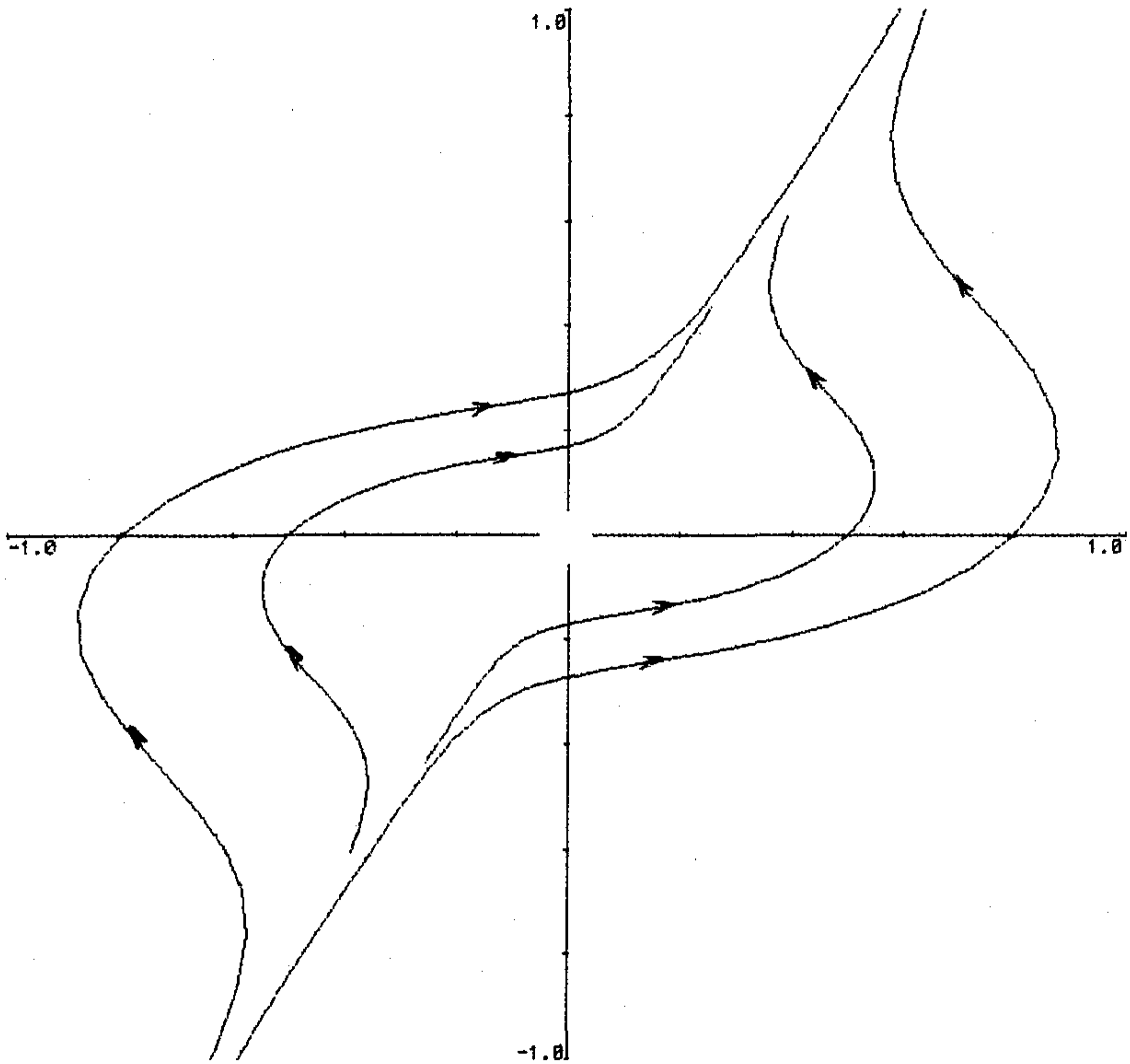


Fig.4.2.2 (Circular- Hyperbolic Type)

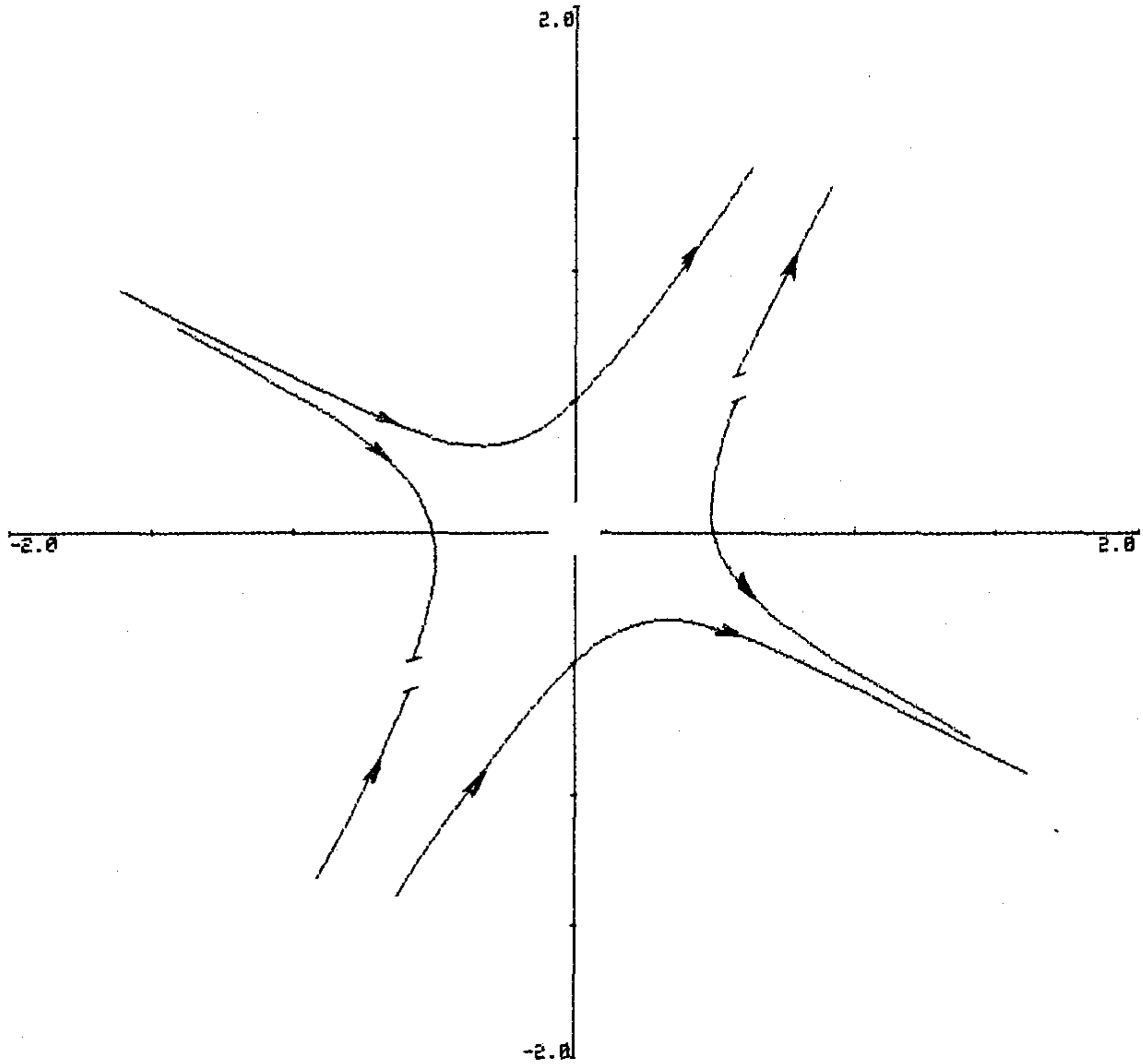


Fig.4.1.3' (Hyperbolic- Hyperbolic Type With Critical line)

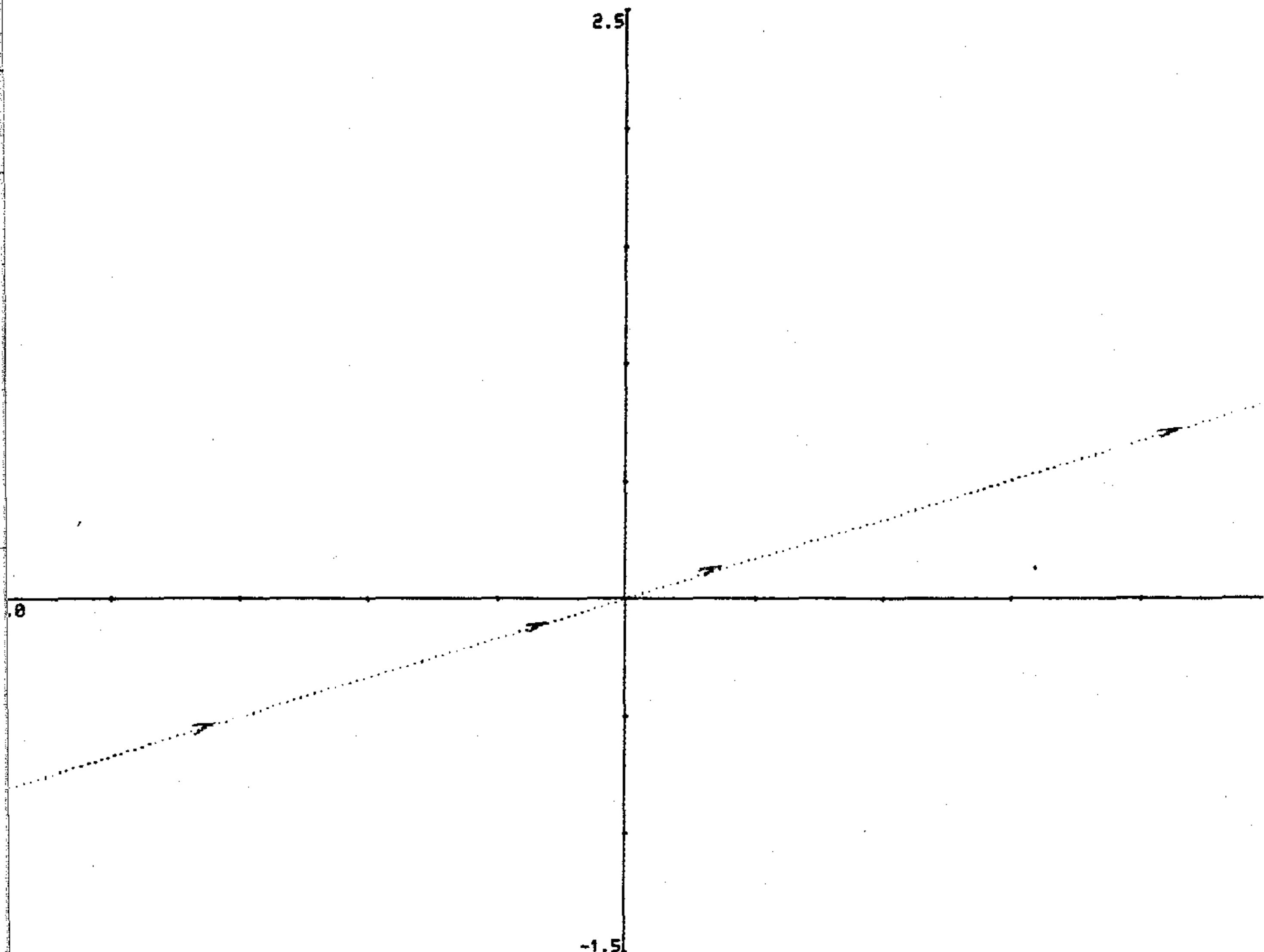


Fig.4.3.1 Attractor/Repellor

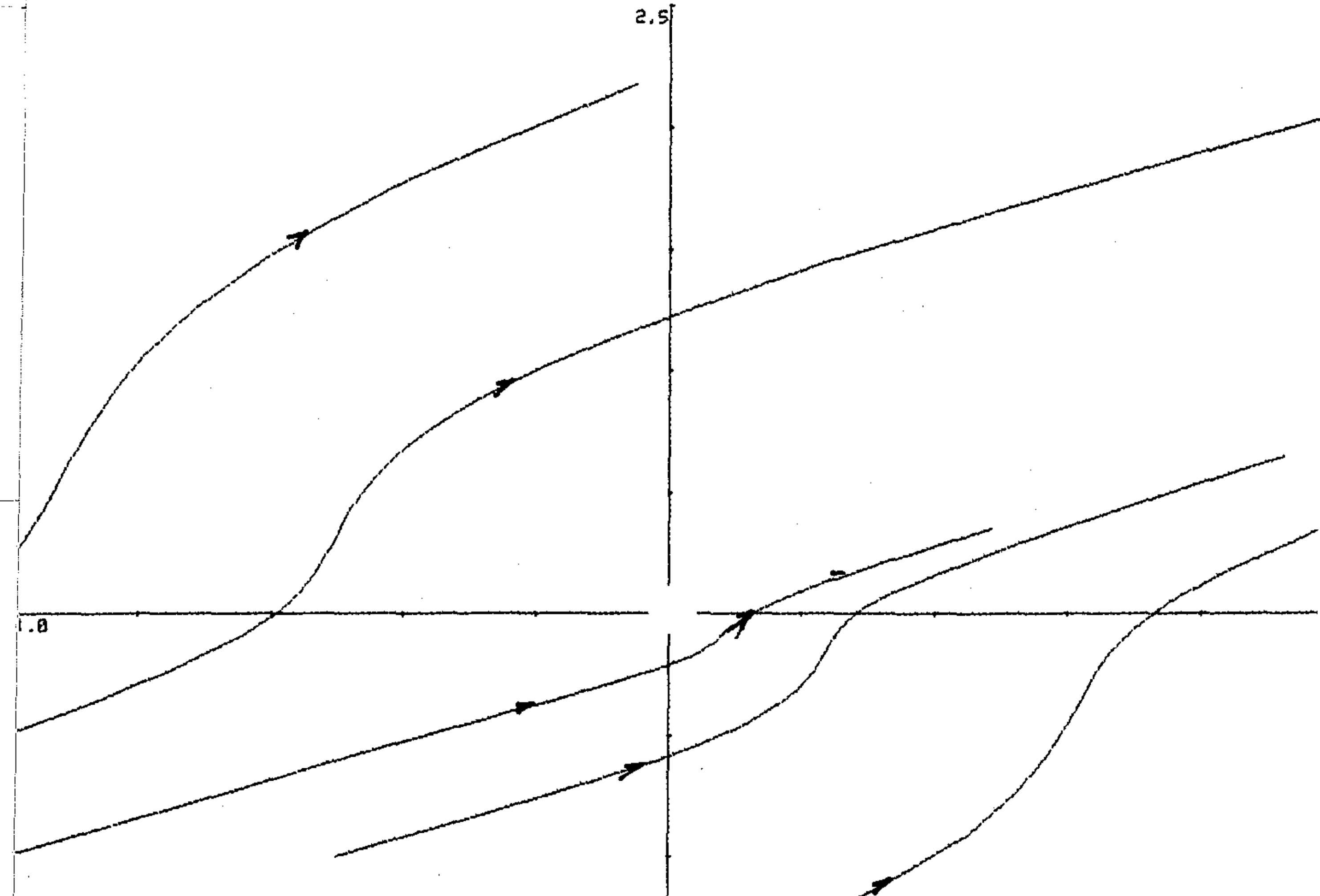


Fig.4.3.1 (Circular- Elliptical Type)

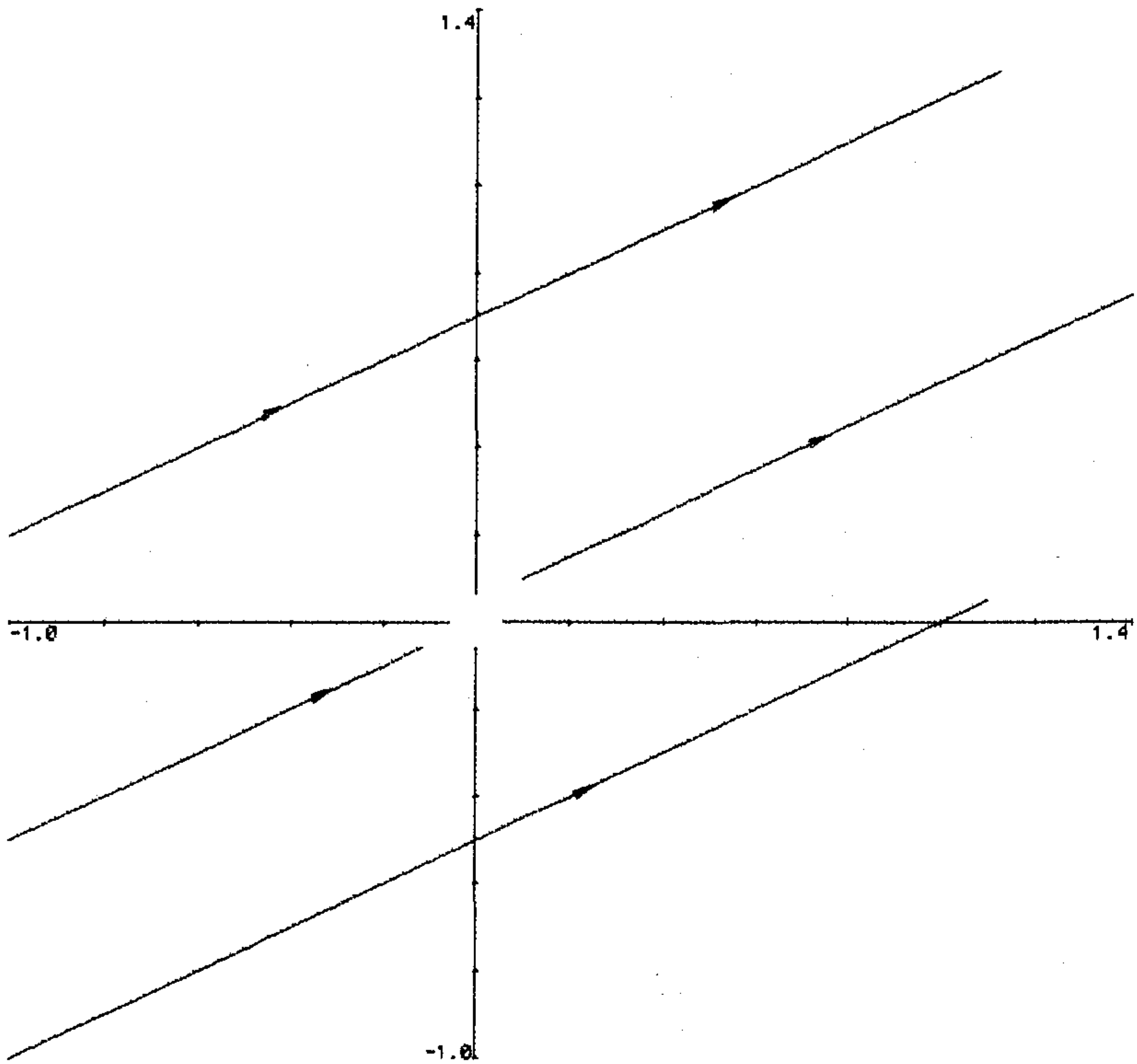
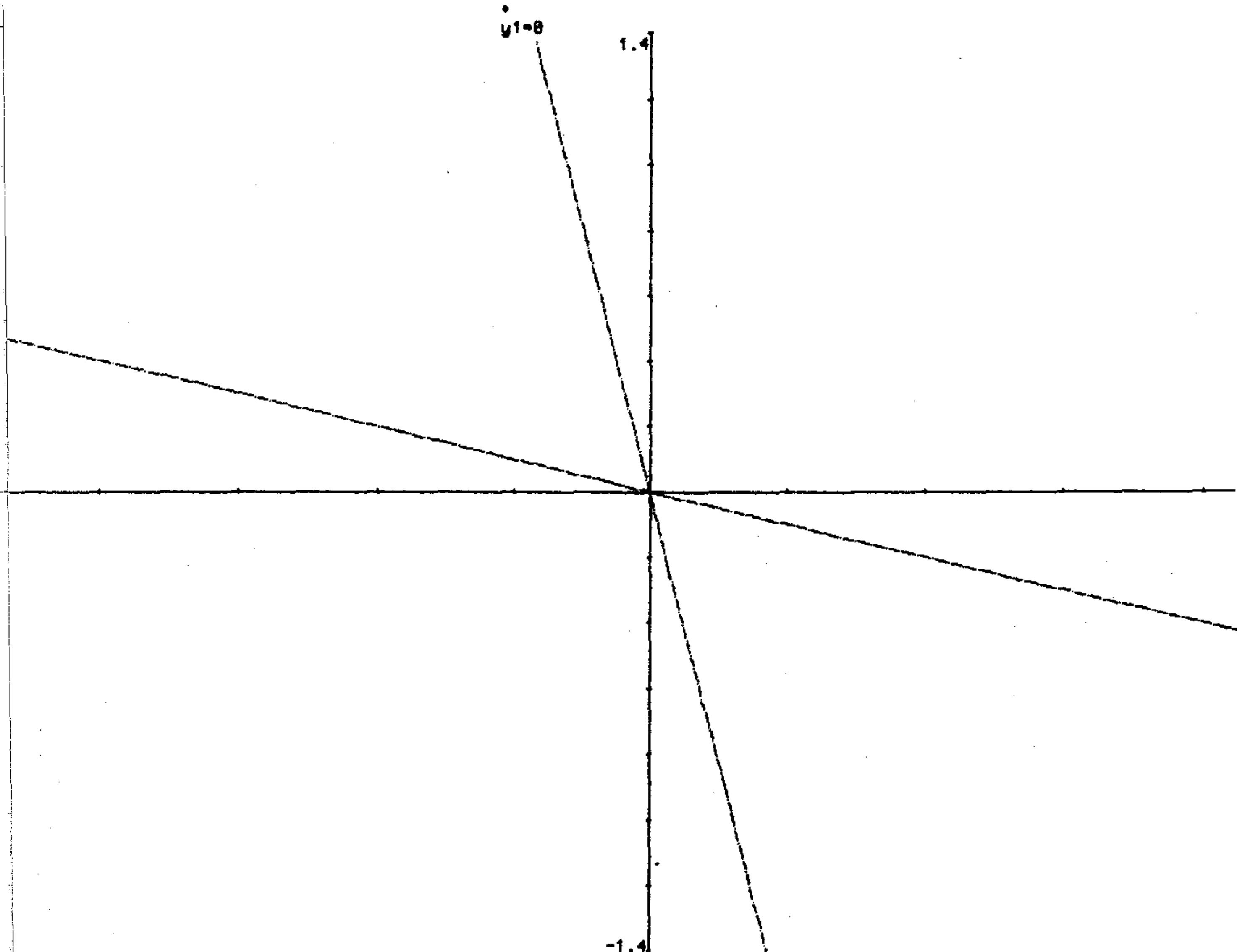


Fig.4.3.2 (circular- Circular Type)



Flip lines dy1/dt=0

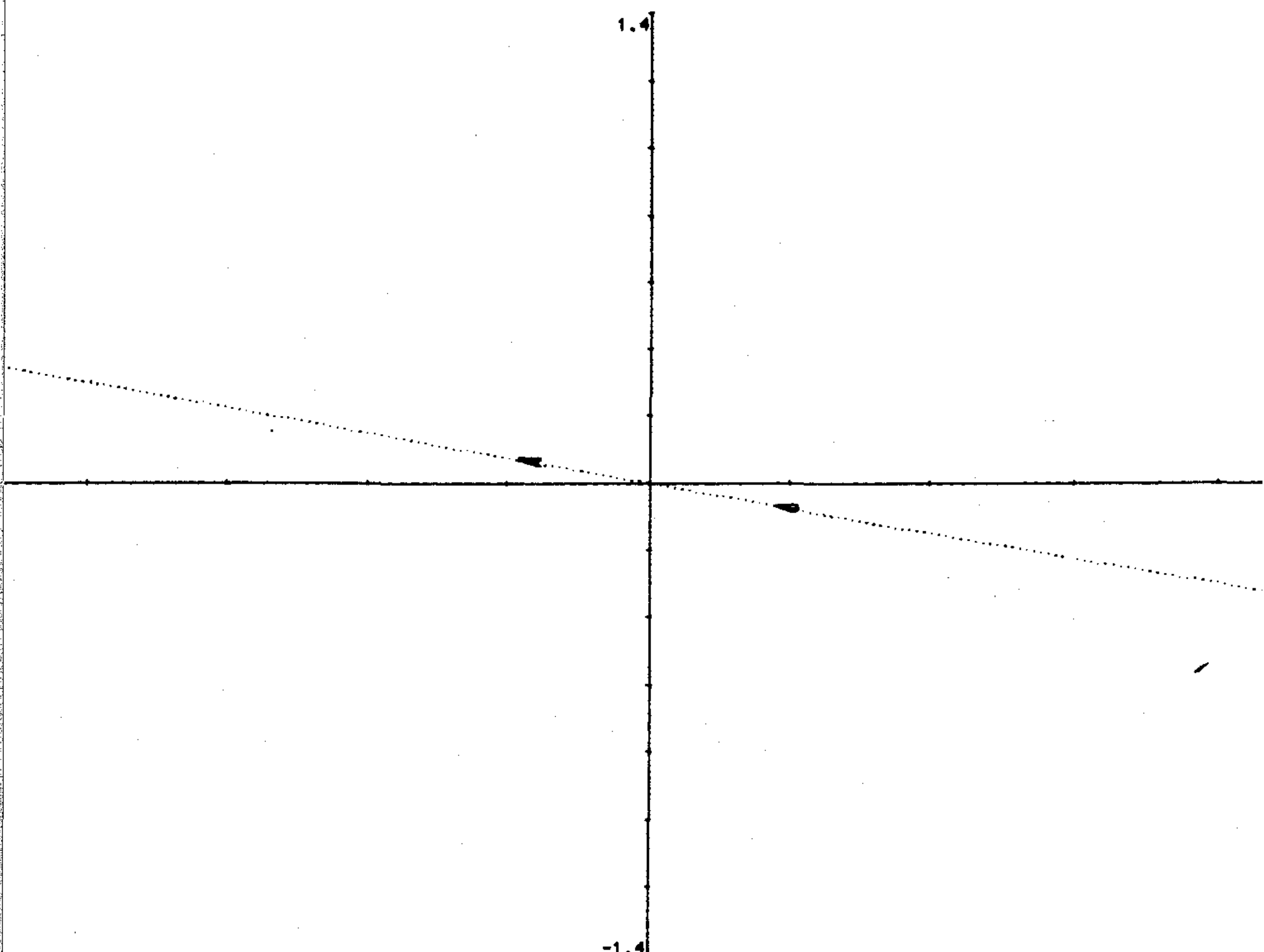


FIG. A1 Attractor/Repellor

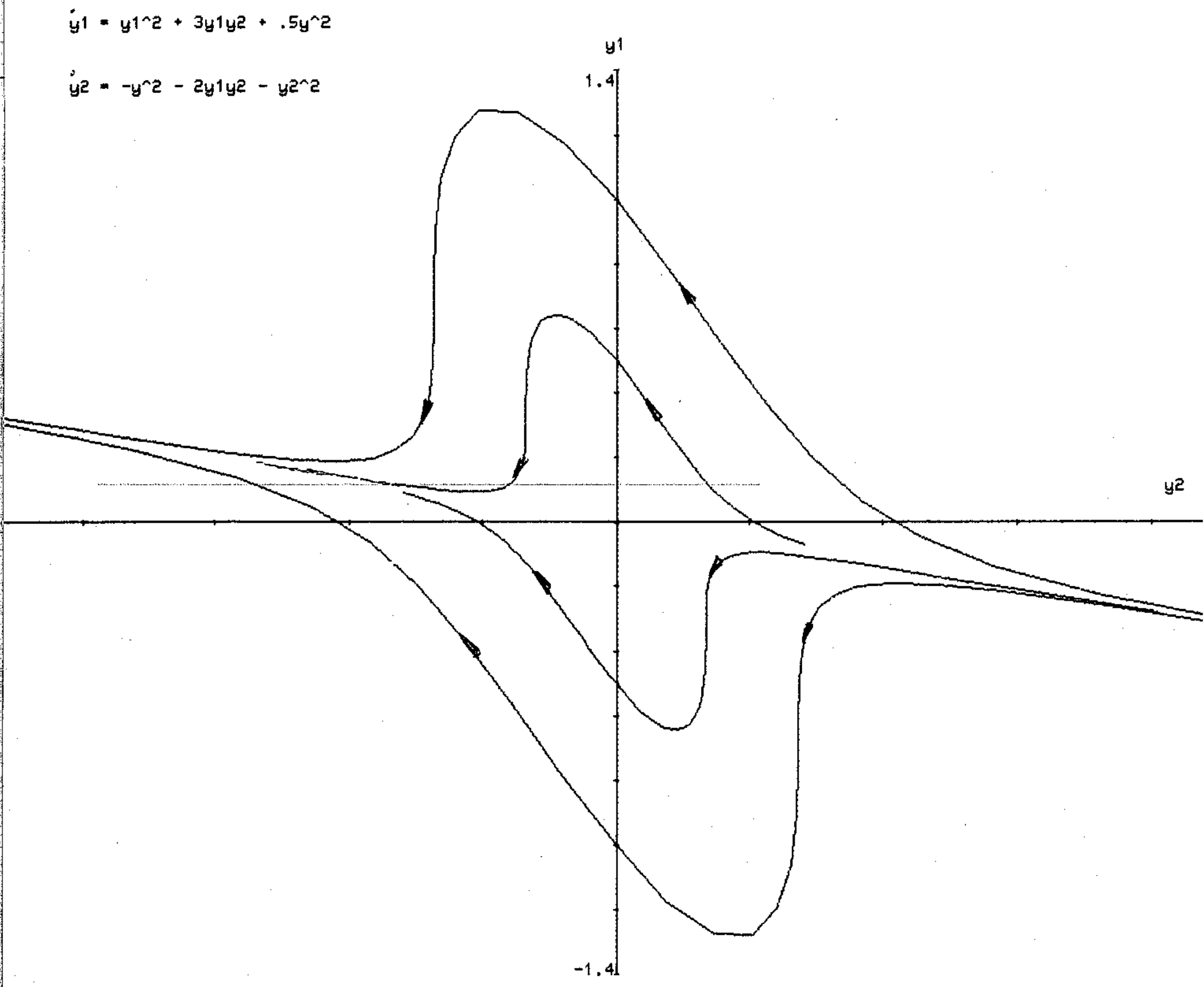


Figure A.1

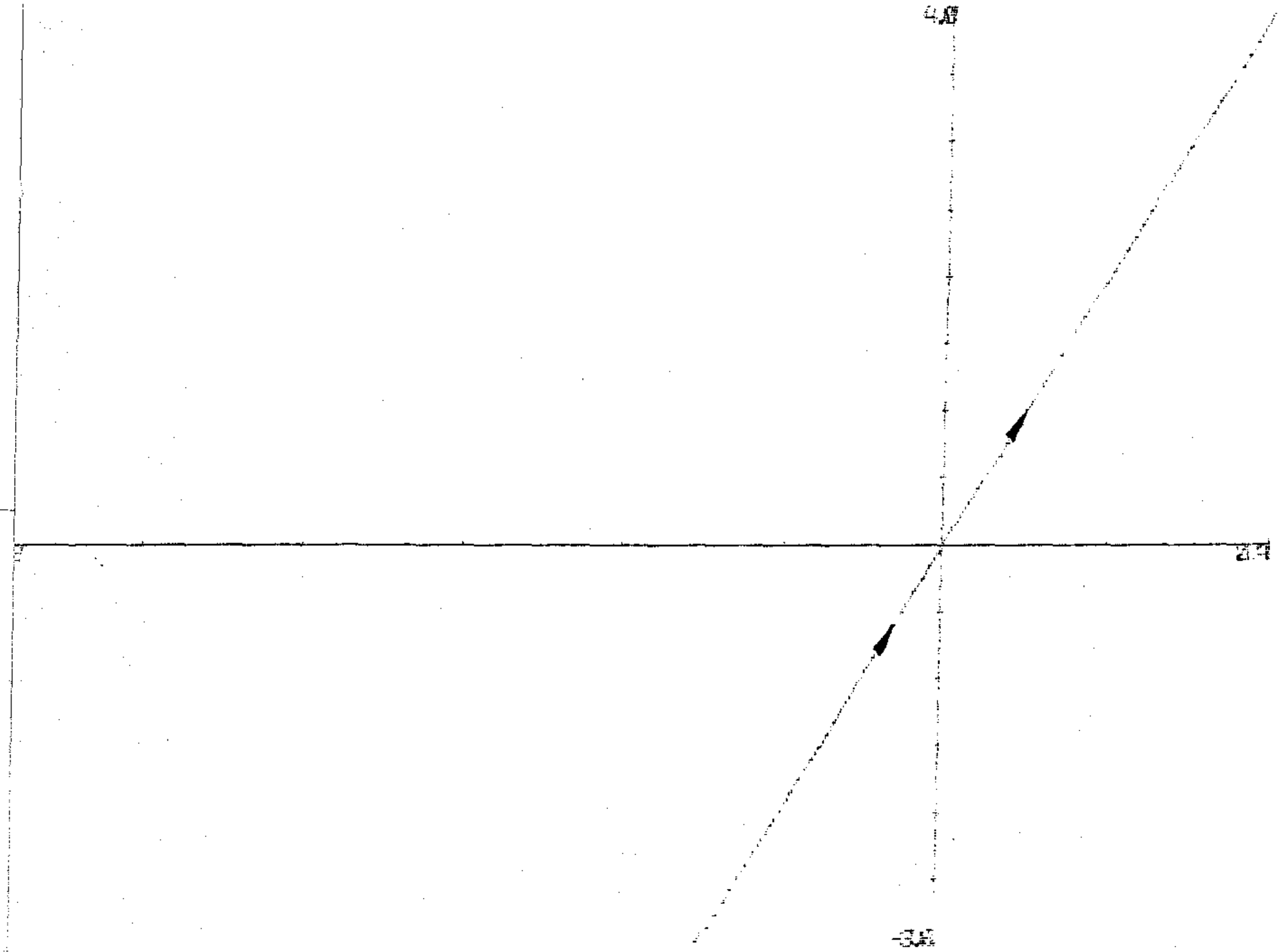


Fig.A.2 Attraxtor/Repellor

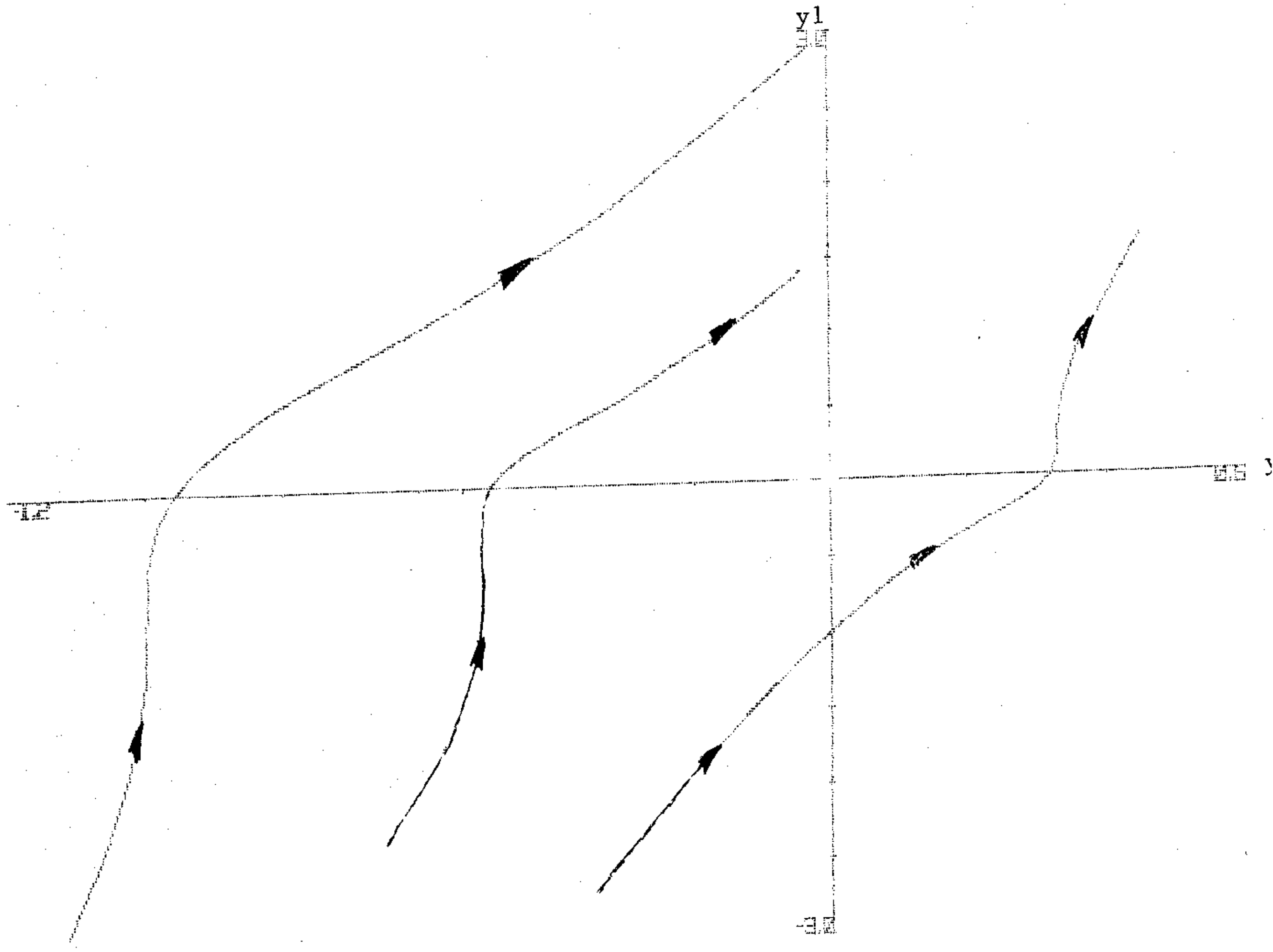


Fig.A.2

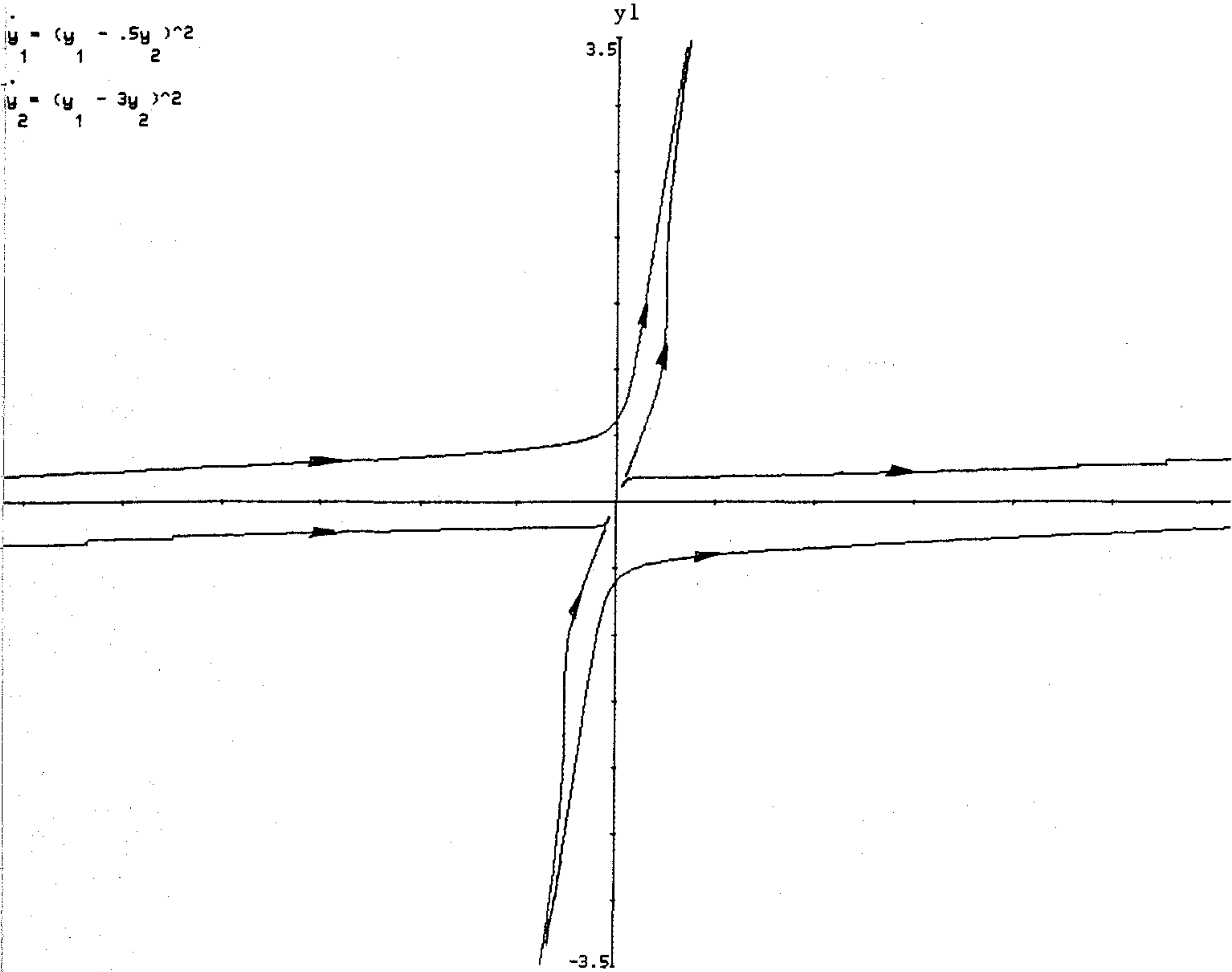


Fig. A.3