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Group of point transformations of time dependent harmonic oscillators

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GROUP OF POINT TRANSFORMATIONS OF TIME DEPENDENT HARMONIC OSCILLATORS

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INTRODUCTION

In general, a physical system has invariant quantities which are very often related to its symmetry and to the invariance of the equation that describe it. A detailed study of the invariance property of the differential equation will be helpful in understanding this relation.

The work is concerned with a preliminary investigation of the Lie-group which leaves invariant the Newtonian and Lagrangian equation of motion for a one-dimensional harmonic oscillator. A brief review of Ehrenfest's adiabatic principle and the later treatments on exact and adiabatic invariants will be presented.

The program starts with a historical survey of adiabatic and exact invariants in Chapter 1 and a review of the theory of continuous group transformations in Chapter 2.

In Chapter 3, the differential equation of the harmonic oscillator is considered and, by applying Lie's theory, infinitesimal transformations that leave it invariant are sought.

To obtain a better insight into the general characteristics of oscillator type equations, the following particular cases will be studied in detail:

For the equation

$$\frac{d^2x}{dt^2} + k(t)\ x = o,$$

where x is considered as a function of t, t being the independent variable:

Case 1	$k(t) = \omega^2$	ω = constant
Case 2	$k(t) = \omega_o^2 + \alpha t$	ω_o and α are constants
Case 3	$k(t) = \omega_o^2 e^{\alpha t}$	

Case 4. For the equation

$$\frac{\partial^2 x}{\partial t^2} + kx = o$$

where k and t are considered as independent variables and x as a function of k and t.

Case 5. For the system of equations

$$\frac{d^2x}{dt^2} + kx = o$$

$$\frac{dk}{dt} - \alpha k = o$$

t being the independent variable, x and k are functions of t.

The generators, labeled U_m, and their first and second extensions will be calculated in each case.

It must be mentioned that in Case 1, the set of generators had already been obtained by R. L. Anderson (1974) and their classification was developed by Wulfman and Wybourne[1] with the result that the global Lie-group for the harmonic oscillator is SL(3,R).

Once the generators are found, the program continues in Chapter 4 by calculating the sets of commutators and the metric tensors. It will be found that the Lie algebra closes under the commutation operation in each case. The local Lie-group is semi-simple and still SL(3,R) in Cases 2 and 3. Furthermore, in both Cases 4 and 5, the sub-set of generators obtained by excluding one generator obey the same Lie algebra as in the remaining cases.

Finally, Chapter 5 deals with the asymptotic behavior of the generators (as α approaches zero) for the Cases 2, 3 and 5, and it also will be shown that they converge to the generators of the Case 1.

A detailed calculation for the generators in the Case 5, given in the Appendix, closes this work.

NOTATIONS AND CONVENTIONS

In this work, a number of symbols are used to represent physical variables. Physical time is represented by the letter $\underline{t}$ and position by $\underline{x}$. Unless otherwise would be specifically stated, position's time-derivative is abbreviated by using an over-dot, i.e.

$$\dot{x} \equiv \frac{dx}{dt}$$

and the same treatment is followed for all variables which depend only on t; for instance, see $\dot{T}_1$ in Eq. (3.9); $\dot{A}i$ in Eq. (3.2-2); $\dot{Y}_o$ in Eq. (3.3-3), and so on. Similarly, the k-derivative is abbreviated by an over-star, i.e.

$$\overset{*}{J}_o(z) \equiv \frac{d}{dk}\,(J_o(z)), \quad z = z(k)$$

and a prime denotes the derivative of a function with respect to its argument, e.g.:

$$J_o'(z) \equiv \frac{d}{dz}\,(J_o(z))$$

It is then evident that

$$\dot{A}i(z) = Ai'(z)\dot{z}$$

$$\overset{*}{J}_o(z) = J_o'(z)\overset{*}{z}$$

When a function of several variables occurs its partial derivatives are denoted by attaching a subscript to the function, e.g., in Eq. (3.4-8):

$$\xi^{(1)}_{\overset{*}{x}} \equiv \frac{\partial \xi^{(1)}}{\partial \overset{*}{x}}$$

Numbered superscripts in parentheses are reserved to denote extensions of generators and their corresponding coefficients.

Numbered subscripts on a symbol denote several parameters of similar kind, e.g.: constants of integration, which in some cases are functions of another variable (see Appendix). When a set of variables is transformed into another set, the new variables will be denoted by symbols entirely distinct from the old ones. As usual, $0(\varepsilon)$ denotes any quantity which, as $\varepsilon \to o$, approaches zero at least as fast as ε.

1. HISTORICAL SURVEY OF ADIABATIC AND EXACT INVARIANTS

In the earlier years of the present century, physicists were involved in discussions[2] concerning the constitution of atoms and molecules (Bohr and Sommerfeld theories), the quantum conditions in Planck's treatment of the harmonic oscillator, the apparent paradox between the "classical" Wien's displacement law and "statistical" Planck's radiation law, and many other interesting problems in Physics. Among these Ehrenfest[3] stated his adiabatic principle[4] as

> "If a system be affected in a reversible adiabatic way, allowed motions are transformed into allowed motions";

this principle involves quantities called <u>adiabatic invariants</u>.* For instance, in problems of arbitrary periodic motions, $2\bar{T}/\nu$ is an adiabatic invariant, $\bar{T}$ being the mean kinetic energy and ν being the frequency. This adiabatic invariant reduces to E/ν in the case of one-dimensional harmonic motions, where E represents the total energy. Sommerfeld's quantum conditions are other examples of adiabatic invariants, when

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*If the rate of change of a parameter k, in the Hamiltonian of an oscillatory system, is infinitesimal the parameter is said to vary adiabatically.

infinitely slow changes of the parameters in the action or phase integrals are considered.

Many applications of Ehrenfest's adiabatic principle can be found in the literature[5]; perhaps the most outstanding is the explanation of the Stark effect by Schwarzchild and Epstein[6].

Although adiabatic invariants were recognized to be only approximate to the lowest significant order, the question of whether their constancy might be valid to higher orders seems not to have been considered at that time or in that connection.

Alfven[7] showed that the magnetic moment of a spiraling particle in a varying magnetic field is a constant to first order of infinitesimal changes in the parameters involved; this was later shown to be true in the next order by Helwig for a general field[8].

Kulsrud considered a vibrating harmonic oscillator (whose frequency ω is initially constant, then varies in an arbitrary fashion, and finally becomes constant) and found that E/ω is an adiabatic invariant to as many orders as ω has continuous derivatives[9]. Later, Kruskal proved the analogous result for the gyrating particle. Lenard[10] did the same for the anharmonic oscillator, and many others have settled this last by proving the invariance to all others.[11]

Kruskal[12] unified and simplified all the preceding attempts of finding adiabatic invariants to higher order for Hamiltonian systems whose solutions were periodic or nearly periodic. Since Hamilton's equations of motion for a system are a first-order autonomous system, he derived appropriate formal series solutions to such a system of differential equations and proved that the given system had exact solutions for a large range of the independent variable and that the formal series represented the exact solutions asymptotically.

Lewis[13] applied Kruskal's theory to classical and quantum time-dependent harmonic oscillators and found exact invariants in both problems. The results, without using an adiabatic invariance approach, become exact or nonasymptotic. The systems discussed have a Hamiltonian of the form

$$H = \frac{1}{2\varepsilon}\,(p^2 + \omega^2(t)q^2) \qquad (1\text{-}1)$$

where q is a canonical coordinate, p is its associated conjugate momentum, $\omega(t)$ is an arbitrary complex function of t, and ε is a positive real parameter. The exact invariants I are

$$I = \frac{1}{2}\,\{(q/\rho)^2 + (p\rho - \varepsilon\dot{\rho}q)^2\}\ , \qquad (1\text{-}2)$$

ρ being any function of t that satisfies

$$\varepsilon^2\ddot{\rho} + \omega^2(t)\rho - (1/\rho)^3 = 0 \qquad (1\text{-}3)$$

Several classical examples are given and a brief discussion of a quantum system, whose results reduce to the usual ones for the harmonic oscillator when ω is taken as a positive real constant, closes his application of Kruskal's theory.

A more detailed consideration of the quantum time-dependent harmonic oscillator and the problem of a charged particle in a time-dependent electromagnetic field was made by Lewis and Riesenfield.[14] Explicitly time-dependent invariants, their eigenvalues and eigenstates using operator methods, the connection between eigenstates, and solutions of the Schrödinger equation were also found. As a limiting case of the exact invariants, they deduced the sudden and adiabatic approximations.

Using a different approach Symon[15] investigated a general class of Hamiltonian systems which include both linear and non-linear classical harmonic oscillators. He also carefully defined the terms "exact invariant" and "adiabatic invariant" and calculated such invariants for several forms of the Hamiltonian.

In Classical Mechanics, knowledge of contact transformations, as well as point transformations, admitted by a system, is necessary for a full understanding of the relation between its symmetry properties and the quantities it leaves conserved. For a number of practical reasons I will, however, in this thesis confine my attention to the problem of determining the groups of point transformations that leave invariant the systems in Cases 1 through 5 indicated above.

2. CONTINUOUS TRANSFORMATIONS GROUPS

1. The transformation-group of one parameter[16]

Consider a transformation

$$T: \quad \begin{aligned} \bar{x} &= f(x,t) \\ \bar{t} &= g(x,t) \end{aligned} \qquad (2.1)$$

by means of which the point (x,t) is transferred to the new position $(\bar{x},\bar{t})$ in the same plane and referred to the same coordinate system. The *inverse* transformation, namely the operation of transferring the point $(\bar{x},\bar{t})$ back to its original position (x,t), is obtained by solving for x and t in terms of $\bar{x}$ and $\bar{t}$, thus

$$T^{-1}: \quad \begin{aligned} x &= \overset{-1}{f}(\bar{x},\bar{t}) \\ t &= \overset{-1}{g}(\bar{x},\bar{t}) \end{aligned} \qquad (2.2)$$

The result of performing the transformations T and T^{-1} in succession, in either order, is the *identity* transformation

$$\begin{aligned} \bar{x} &= x \\ \bar{t} &= t \end{aligned} \qquad (2.3)$$

Consider the aggregate of the transformations included in the family

$$\begin{aligned} \bar{x} &= f(x,t;\ \varepsilon) \\ \bar{t} &= g(x,t;\ \varepsilon) \end{aligned} \qquad (2.4)$$

where ε is a parameter which can vary continuously over a given range (assuming that f and g are analytic functions of x and t and differentiable with respect to ε in its range). These transformations are said to form a <u>finite continuous group</u> when any two successive transformations of the family are equivalent to a single transformation of the family. It has been assumed that every transformation in the group contains both the identity and the inverse transformations. Under these circumstances, the transformations given in Eq. (2.4) are called a <u>group of one parameter</u>.

2. <u>Infinitesimal transformations</u>

Suppose that $\varepsilon=o$ is the value of the parameter which produces the identity transformation in the group-transformation law given in Eq. (2.4); thus,

$$f(x,t;\, o) = x$$
$$g(x,t;\, o) = t$$

If ε is an infinitesimal quantity, the transformations given in Eq. (2.4) will be such that $\bar{x}$ differs only infinitesimally from x, and $\bar{t}$ from t. This transformation will differ only infinitesimally from the identity transformation, and it is called an <u>infinitesimal transformation</u>. Besides, it can be shown[17] that each one-parameter group has only one infinitesimal transformation. Let

$$\bar{x} = x+\varepsilon\xi(x,t) \qquad (2.5)$$
$$\bar{t} = t+\varepsilon\eta(x,t)$$

be the equations of the infinitesimal transformations,

where

$$\xi(x,t) = (f_\varepsilon)_{\varepsilon=o} \tag{2.6}$$

$$\eta(x,t) = (g_\varepsilon)_{\varepsilon=o} \tag{2.7}$$

and consider the variation of any analytic function F(x,t) due to the change produced by the infinitesimal transformation (2.5) on x and t; thus

$$\begin{aligned} \Delta F(x,t) &= F(\bar{x},\bar{t}) - F(x,t) \\ &= F(x + \Delta x,\ t + \Delta t) - F(x,t) \\ &= \varepsilon\ [\xi F_x + \eta F_t]\ * \end{aligned}$$

or

$$F(\bar{x},\bar{t}) = F(x,t) + \varepsilon UF(x,t) \tag{2.8}$$

where

$$U = \xi(x,t)\frac{\partial}{\partial x} + \eta(x,t)\frac{\partial}{\partial t} \tag{2.9}$$

is called the <u>generator</u> of the infinitesimal transformation, since UF represents it. In particular

$$\begin{aligned} Ux &= \xi(x,t) \\ Ut &= \eta(x,t) \end{aligned} \tag{2.10}$$

so that

$$UF = Ux \cdot F_x + Ut \cdot F_t \tag{2.11}$$

---

*Retaining terms only of the first order in ε.

The function $F(\bar{x},\bar{t})$ given in Eq. (2.8) may be regarded as a function of x,t and ε; regard x and t as fixed and let the function be expanded as a Maclaurin series in ε. Thus

$$F(\bar{x},\bar{t}) = F_o + F_o' \varepsilon + \frac{1}{2} F_o'' \varepsilon^2 + \ldots, \tag{2.12}$$

where

$$F_o = [F(\bar{x},\bar{t})]_{\varepsilon=o} = F(x,t)$$

$$F_o' = \left[\frac{\partial F(\bar{x},\bar{t})}{\partial \varepsilon}\right]_{\varepsilon=o} = [F_{\bar{x}}\, \xi(\bar{x},\bar{t}) + F_{\bar{t}}\, \eta(\bar{x},\bar{t})]_{\varepsilon=o}$$

$$= \frac{\partial F}{\partial x} \xi(x,t) + \frac{\partial F}{\partial t} \eta(x,t) = UF(x,t)$$

and

$$F_o'' = \left[\frac{\partial^2 F(\bar{x},\bar{t})}{\partial \varepsilon^2}\right]_{\varepsilon=o} = [\{\xi(x,t)\partial_{\bar{x}} + $$

$$+ \eta(\bar{x},\bar{t})\partial_{\bar{t}}\}^2 F(\bar{x},\bar{t})]_{\varepsilon=o}$$

$$= U^2 F(x,t).$$

Therefore, Eq. (2.12) becomes

$$F(\bar{x},\bar{t}) = F(x,t) + \varepsilon UF + \frac{1}{2} \varepsilon^2 U^2 F + \ldots \tag{2.13}$$

or, written symbolically,

$$F(\bar{x},\bar{t}) = e^{\varepsilon U} F(x,t). \tag{2.14}$$

In particular, the equations for the infinitesimal transformations are

$$\bar{x} = x + \varepsilon\xi(x,t) \tag{2.15}$$
$$\bar{t} = t + \varepsilon\eta(x,t),$$

where Eq. (2.10) has been used and only terms up to the first order in ε have been retained.

A function F(x,t) is called <u>invariant</u> if, when $\bar{x}$ and $\bar{t}$ are obtained from Eq. (2.5) and for all values of ε

$$F(\bar{x},\bar{t}) = F(x,t) \tag{2.16}$$

Then, for F to be invariant under the transformation (2.15) it is necessary and sufficient that

$$UF = \xi\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial t} = 0 \tag{2.17}$$

3. <u>The extended group</u>

Consider a one-parameter group with transformations given as in Eq. (2.4) and let

$$\dot{\bar{x}} = \frac{d\bar{x}}{d\bar{t}} = \frac{df}{dg}$$

$$= \frac{f_t + f_x\dot{x}}{g_t + g_x\dot{x}}{}^{*} \equiv f^{(1)}(x,t,\dot{x};\ \varepsilon) \tag{2.18}$$

for a fixed value of the parameter ε. Thus in general the equations

$$\bar{x} = f(x,t;\ \varepsilon) \tag{2.19}$$

$$\bar{t} = g(x,t;\ \varepsilon)$$

$$\dot{\bar{x}} = f^{(1)}(x,t,\dot{x};\ \varepsilon)$$

form a group†, which is known as the <u>first-extended group</u>. Similarly, let

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*See notations and conventions.

†These results are easily extended in the case of more than 2 variables and higher differential coefficients.

$$\ddot{\bar{x}} = \frac{d\dot{\bar{x}}}{d\bar{t}} = \frac{df^{(1)}}{dg}$$

$$= \frac{f_t^{(1)} + f_x^{(1)}\dot{x} + f_{\dot{x}}^{(1)}\ddot{x}}{g_t + g_x\dot{x}} \equiv f^{(2)}(x,t,\dot{x},\ddot{x};\varepsilon) \quad (2.20)$$

also for a fixed value of ε. The set of equations

$$\begin{aligned} \bar{x} &= f(x,t;\ \varepsilon) \\ \bar{t} &= g(x,t;\ \varepsilon) \\ \dot{\bar{x}} &= f^{(1)}(x,t,\dot{x};\varepsilon) \\ \ddot{\bar{x}} &= f^{(2)}(x,t,\dot{x},\ddot{x};\varepsilon) \end{aligned} \quad (2.21)$$

form a group*, which is known as the <u>second-extended group</u>.

The infinitesimal form of these extensions can be written as

$$\dot{\bar{x}} = \dot{x} + \varepsilon\xi^{(1)}(x,t,\dot{x}) \quad (2.22)$$

$$\ddot{\bar{x}} = \ddot{x} + \varepsilon\xi^{(2)}(x,t,\dot{x},\ddot{x}) \quad (2.23)$$

retaining terms only of the first order in ε and where

$$\xi^{(1)}(x,t,\dot{x}) = \xi_t + \dot{x}(\xi_x - \eta_t) - \dot{x}^2\eta_x \quad (2.24)$$

and

$$\xi^{(2)}(x,t,\dot{x},\ddot{x}) = \xi_t^{(1)} + \dot{x}\xi_x^{(1)} - \ddot{x}(\eta_t + \dot{x}\eta_x - \xi_{\dot{x}}^{(1)})$$

$$= \xi_{tt} + \dot{x}(2\xi_{xt} - \eta_{tt}) + \dot{x}^2(\xi_{xx} - 2\eta_{xt})$$

$$- \dot{x}^3(\eta_{xx}) + \ddot{x}(\xi_x - 2\eta_t - 3\dot{x}\eta_x) \quad (2.25)$$

---

*These results are easily extended in the case of more than 2 variables and higher differential coefficients.

Then,

$$U^{(1)} = \xi\partial_x + \eta\partial_t + \xi^{(1)}\partial_{\dot{x}} \tag{2.26}$$

is the generator of the infinitesimal transformation in the first-extension, and

$$U^{(2)} = \xi\partial_x + \eta\partial_t + \xi^{(1)}\partial_{\dot{x}} + \xi^{(2)}\partial_{\ddot{x}} \tag{2.27}$$

is the generator of the infinitesimal transformation of the second extended group.

4. Lie Algebra

There are associated with any r-parameter Lie group r infinitesimal generators which are characterized by their commutation properties. The algebra of the r-dimensional vector space is defined by the requirement that the generators U_m satisfy the condition

$$U_mU_n - U_nU_m \equiv [U_m, U_n] = C^r_{mn}U_r . \tag{2.28}$$

Evidently,

$$C^r_{mn} = -C^r_{nm}, \tag{2.29}$$

where the C^r_{mn} are called the structure constants of the algebra. Furthermore, the Jacobi identity:

$$[[X_m, X_n], X_r] + [[X_n,X_r], X_m] + [[X_r,X_m], X_n] = 0 \tag{2.30}$$

in terms of the structure constants reads

$$C^p_{mn}\, C^q_{pr} + C^p_{nr}\, C^q_{pm} + C^p_{rm}\, C^q_{pn} = 0 \tag{2.31}$$

In the problems considered here, the generators have the general form

$$U_m = \xi^m \partial_x + \eta^m \partial_t + \zeta^m \partial_k \tag{2.32}$$

Then,

$$\begin{aligned}[U_m, U_n] = {} & (\xi^m\xi^n_x - \xi^n\xi^m_x + \eta^m\xi^n_t - \eta^n\xi^m_t + \zeta^m\xi^n_k - \zeta^n\xi^m_k)\partial_x + \\ & (\xi^m\eta^n_x - \xi^n\eta^m_x + \eta^m\eta^n_t - \eta^n\eta^m_t + \zeta^m\eta^n_k - \zeta^n\eta^m_k)\partial_t + \\ & (\xi^m\zeta^n_x - \xi^n\zeta^m_x + \eta^m\zeta^n_t - \eta^n\zeta^m_t + \zeta^m\zeta^n_k - \zeta^n\zeta^m_k)\partial_k.\end{aligned} \tag{2.33}$$

Since the extended generators obey the same commutation relations as the generators[18], it is only necessary to calculate the commutation relations for the generators.

5. <u>Semi-simple groups</u>

The symmetrical tensor of the second rank*

$$g_{ab} = C^n_{am} \; C^m_{bn} \tag{2.34}$$

constructed from structure constants is useful in providing

*Also known as the metric or Killing tensor.

a classification of groups. If the group is semi-simple[19], then

$$\det[g_{ab}] \neq 0 \tag{2.35}$$

ONE-DIMENSIONAL HARMONIC OSCILLATOR

Consider Newton's and Lagrange's equation of motion for a one-dimensional harmonic oscillator

$$S \equiv \ddot{x} + kx = 0 \tag{3.1}$$

where

$$k = k(t) \tag{3.2}$$

and let

$$U^{(2)} = \xi\partial_x + \eta\partial_t + \xi^{(1)}\,\partial_{\dot{x}} + \xi^{(2)}\,\partial_{\ddot{x}} \tag{3.3}$$

be the second-extended generator of a transformation-group obtained by seeking infinitesimal transformations of x and t that leave invariant the above equation of motion, i.e.

$$U^{(2)}\,S = 0 \qquad \text{whenever} \quad S = 0 \tag{3.4}$$

Explicitly, the action of this generator on S is

$$[\xi^{(2)} + k\xi \;\; + x\dot{k}\eta]_{S=0} = 0$$

or

$$\psi_1 + \psi_2\dot{x} + \psi_3\dot{x}^2 + \psi_4\dot{x}^3 = 0 \tag{3.5}$$

with

$$\begin{aligned}
\psi_1 &= \xi_{tt} + k\xi + (2k\eta_t - k\xi_x + \dot{k}\eta)x \\
\psi_2 &= 2\xi_{xt} - \eta_{tt} + 3kx\eta_x \\
\psi_3 &= \xi_{xx} - 2\eta_{xt} \\
\psi_4 &= -\eta_{xx}
\end{aligned} \tag{3.6}$$

where the condition S=0 (i.e.: $\ddot{x} = -kx$) has been used. Since $\psi_1, \psi_2, \psi_3, \psi_4$ are functions only of x and t, equation (3.5) holds for all values of the x, t, $\dot{x}$ variables if

$$\psi_1 = \psi_2 = \psi_3 = \psi_4 = 0 \tag{3.7}$$

Equations (3.6) under condition (3.7) are named <u>the determining equations</u>. The last two of them are easily integrated to give.

$$\eta(x,t) = xT_1 + T_2 \tag{3.8}$$

$$\xi(x,t) = x^2\dot{T}_1 + xT_3 + T_4 \tag{3.9}$$

where the T's are functions only of t. In order to obtain explicit forms of the generators, it is necessary to know k(t). Three special cases will be considered here.

CASE 1. <u>$k(t) = \omega^2$ = constant.</u>*

This case reduces to the simple harmonic oscillator and a great simplification of the problem is obtained due to the fact that

$$\dot{k} = 0. \tag{3.1-1}$$

Using Eqs. (3.8), (3.9), (3.1-1) and the first two determining equations, solutions to the T's are obtained as

$$T_1(t) = a_1e^{i\omega t} + a_2e^{-i\omega t}$$

$$T_2(t) = a_5e^{2i\omega t} + a_6e^{-2i\omega t} + a_7 \tag{3.1-2}$$

$$T_3(t) = a_5 i\omega e^{2i\omega t} + a_6(-i\omega)e^{-2i\omega t} + a_8$$

$$T_4(t) = a_3e^{i\omega t} + a_4e^{-i\omega t},$$

---

*This case has already been solved by R. L. Anderson (1974)[1].

where $a_1, a_2, \ldots, a_8$ are constants. The generators of the infinitesimal transformations can be written as

$$U = \sum_{n=1}^{8} b_n U_n$$

with

$$
\begin{aligned}
U_1 &= ix^2 e^{i\omega t}\partial_x + \frac{1}{\omega}\, x e^{i\omega t}\partial_t \\
U_2 &= -ix^2 e^{-i\omega t}\partial_x + \frac{1}{\omega}\, x e^{-i\omega t}\partial_t \\
U_3 &= e^{i\omega t}\partial_x \\
U_4 &= e^{-i\omega t}\partial_x \\
U_5 &= ixe^{2i\omega t}\partial_x + \frac{1}{\omega}\, e^{2i\omega t}\partial_t \\
U_6 &= -ixe^{-2i\omega t}\partial_x + \frac{1}{\omega}\, e^{-2i\omega t}\partial_t \\
U_7 &= \frac{1}{\omega}\,\partial_t \\
U_8 &= x\partial_x
\end{aligned}
\qquad (3.1\text{-}3)
$$

and the b_n 's are new constants related directly to above a_n's.

It follows from these results that the coefficients of the first-extended generators are

$$
\begin{aligned}
\xi^{(1)}_{\substack{1\\2}} &= -(\omega x^2 \mp ix\dot{x} + \frac{1}{\omega}\,\dot{x}^2)e^{\pm i\omega t} \\
\xi^{(1)}_{\substack{3\\4}} &= \pm\, i\omega e^{\pm i\omega t}
\end{aligned}
\qquad (3.1\text{-}4)
$$

$$\xi^{(1)}_{\substack{5\\6}} = -(2\omega x \pm i\dot{x})e^{\pm 2i\omega t}$$

$$\xi^{(1)}_{7} = 0$$

$$\xi^{(1)}_{8} = \dot{x}$$

and for the second-extended generators are

$$\xi^{(2)}_{\substack{1\\2}} = -\{\pm i\omega^2 x^2 + \frac{3}{\omega}\dot{x}(\ddot{x} + \omega^2 x)\} e^{\pm i\omega t}$$

$$\xi^{(2)}_{\substack{3\\4}} = -\omega^2 e^{\pm i\omega t}$$

$$\xi^{(2)}_{\substack{5\\6}} = \pm i(4\omega^2 x + 3\ddot{x}) e^{\pm 2i\omega t} \qquad (3.1\text{-}5)$$

$$\xi^{(2)}_{7} = 0$$

$$\xi^{(2)}_{8} = \ddot{x} .$$

CASE 2. $k(t) = \omega_o^2 + \alpha t$. ω_o, α are constants

Here one obtains

$$\dot{k} = \alpha = \text{constant} \qquad (3.2\text{-}1)$$

and k's higher derivatives vanish. Using Eqs. (3.8), (3.9), (3.2-1) and the first two determining equations, solutions for the T's are obtained as

$$T_1(t) = a_1 Ai + a_2 Bi$$

$$T_2(t) = a_5 Ai^2 + a_6 Bi^2 + a_7 AiBi \qquad (3.2\text{-}2)$$

$$T_3(t) = a_5 Ai\dot{A}i + a_6 Bi\dot{B}i + a_7 Ai\dot{B}i + a_8$$

$$T_4(t) = a_3 Ai + a_4 Bi$$

where Ai, Bi are the Airy functions[20] with the argument

$$z = -\alpha^{-2/3}(\omega_o^2 + \alpha t). \qquad (3.2\text{-}3)$$

Hence,

$$\dot{z} = -\alpha^{1/3} \qquad (3.2\text{-}4)$$

and Ai^2, AiBi, Bi^2 are products of these functions. Again, the generators can be taken as

$$U = \sum_{n=1}^{8} b_n U_n$$

with

$$U_1 = \sqrt{\pi/\dot{z}}\,\{x^2\dot{A}i\;\partial_x + xAi\partial_t\}$$

$$U_2 = \sqrt{\pi/\dot{z}}\,\{x^2\dot{B}i\;\partial_x + xBi\partial_t\}$$

$$U_3 = \sqrt{\pi/\dot{z}}\,\{Ai\;\partial_x\}$$

$$U_4 = \sqrt{\pi/\dot{z}}\,\{Bi\;\partial_x\}$$

$$U_5 = \pi/\dot{z}\,\{xAi\dot{A}i\;\partial_x + Ai^2\;\partial_t\} \qquad (3.2\text{-}5)$$

$$U_6 = \pi/\dot{z}\,\{xBi\dot{B}i\;\partial_x + Bi^2\;\partial_t\}$$

$$U_7 = \pi/\dot{z}\,\{xAi\dot{B}i\;\partial_x + AiBi\;\partial_t\}$$

$$U_8 = x\partial_x .$$

The first extensions are

$$\xi_1^{(1)} = -\sqrt{\pi/\dot{z}}\,\{x^2kAi - x\dot{x}\dot{A}i + \dot{x}^2Ai\}$$

$$\xi_2^{(1)} = -\sqrt{\pi/\dot{z}}\,\{x^2kBi - x\dot{x}\dot{B}i + \dot{x}^2Bi\}$$

$$\xi_3^{(1)} = \sqrt{\pi/\dot{z}}\,\{\dot{A}i\}$$

$$\xi_4^{(1)} = \sqrt{\pi/\dot{z}}\,\{\dot{B}i\} \qquad (3.2\text{-}6)$$

$$\xi_5^{(1)} = -(\pi/\dot{z})\,\{x[kAi^2 - (\dot{A}i)^2] + \dot{x}Ai\dot{A}i\}$$

$$\xi_6^{(1)} = -(\pi/\dot{z})\,\{x[kBi^2 - (\dot{B}i)^2] + \dot{x}Bi\dot{B}i\}$$

$$\xi_7^{(1)} = -(\pi/\dot{z})\,\{x[kAiBi - \dot{A}i\dot{B}i] + \dot{x}Ai\dot{B}i\} + \dot{x}$$

$$\xi_8^{(1)} = \dot{x} .$$

The second extensions are

$$\xi_1^{(2)} = -\sqrt{\pi/\dot{z}}\ \{x^2\ (\alpha Ai + k\dot{A}i) + 3\dot{x}Ai\ (\ddot{x} + kx)\}$$

$$\xi_2^{(2)} = -\sqrt{\pi/\dot{z}}\ \{x^2\ (\alpha Bi + k\dot{B}i) + 3\dot{x}Bi\ (\ddot{x} + kx)\}$$

$$\xi_3^{(2)} = -\sqrt{\pi/\dot{z}}\ \{kAi\}$$

$$\xi_4^{(2)} = -\sqrt{\pi/\dot{z}}\ \{kBi\} \qquad (3.2\text{-}7)$$

$$\xi_5^{(2)} = -(\pi/\dot{z})\ \{xAi\ (\alpha Ai + k\dot{A}i) + 3Ai\dot{A}i\ (\ddot{x} + kx)\}$$

$$\xi_6^{(2)} = -(\pi/\dot{z})\ \{xBi\ (\alpha Bi + k\dot{B}i) + 3Bi\dot{B}i\ (\ddot{x} + kx)\}$$

$$\xi_7^{(2)} = -(\pi/\dot{z})\ \{xAi\ (\alpha Bi + k\dot{B}i) + 3Ai\dot{B}i\ (\ddot{x} + kx)\} + 2(\ddot{x} + kx)$$

$$\xi_8^{(2)} = \ddot{x}.$$

CASE 3. $k(t) = \omega_o^2 e^{\alpha t}$. ω_o and α are constants.

In this case,

$$\dot{k} = \alpha k \tag{3.3-1}$$

and

$$\ddot{k} = \alpha^2 k \tag{3.3-2}$$

The corresponding solutions for the T's become

$$\begin{aligned} T_1(t) &= a_1 J_o + a_2 Y_o \\ T_2(t) &= a_5 J_o^2 + a_6 Y_o^2 + a_7 J_o Y_o \\ T_3(t) &= a_5 J_o \dot{J}_o + a_6 Y_o \dot{Y}_o + a_7 J_o \dot{Y}_o + a_8 \\ T_4(t) &= a_3 J_o + a_4 Y_o \end{aligned} \tag{3.3-3}$$

where J_o, Y_o* are the Bessel and Hankel functions, respectively,[21] with the argument

$$z = \frac{2}{\alpha} \omega_o e^{\frac{1}{2}\alpha t} \tag{3.3-4}$$

Hence,

$$\begin{aligned} \dot{z} &= \omega_o e^{\frac{1}{2}\alpha t} \\ &= \sqrt{k} \end{aligned} \tag{3.3-5}$$

and J_o^2, $J_o Y_o$, Y_o^2 denote products of these functions.

The generators U are

$$U = \sum_{n=1}^{8} b_n U_n$$

*J_o and Y_o are two independent solutions to Bessel's D. E. of order $\nu = 0$.

with

$$U_1 = \frac{1}{\sqrt{\alpha}} \{x^2\dot{J}_o\partial_x + xJ_o\partial_t\}$$

$$U_2 = \frac{1}{\sqrt{\alpha}} \{x^2\dot{Y}_o\partial_x + xY_o\partial_t\}$$

$$U_3 = \frac{1}{\sqrt{\alpha}} \{J_o\partial_x\}$$

$$U_4 = \frac{1}{\sqrt{\alpha}} \{Y_o\partial_x\} \qquad (3.3\text{-}6)$$

$$U_5 = \frac{1}{\alpha} \{xJ_o\dot{J}_o\partial_x + J_o^2\partial_t\}$$

$$U_6 = \frac{1}{\alpha} \{xY_o\dot{Y}_o\partial_x + Y_o^2\partial_t\}$$

$$U_7 = \frac{1}{\alpha} \{xJ_o\dot{Y}_o\partial_x + J_oY_o\partial_t\}$$

$$U_8 = x\partial_x .$$

The first extensions of these generators are

$$\xi_1^{(1)} = -\frac{1}{\sqrt{\alpha}} \{x^2kJ_o - x\dot{x}\dot{J}_o + \dot{x}^2J_o\}$$

$$\xi_2^{(1)} = -\frac{1}{\sqrt{\alpha}} \{x^2kY_o - x\dot{x}\dot{Y}_o + \dot{x}^2Y_o\}$$

$$\xi_3^{(1)} = \frac{1}{\sqrt{\alpha}} \{\dot{J}_o\}$$

$$\xi_4^{(1)} = \frac{1}{\sqrt{\alpha}} \{\dot{Y}_o\}$$

$$\xi_5^{(1)} = -\frac{1}{\alpha} \{x[kJ_o^2 - (\dot{J}_o)^2] + \dot{x}J_o\dot{J}_o\}$$

$$\xi_6^{(1)} = -\frac{1}{\alpha}\{x[kY_o^2 - (\dot{Y}_o)^2] + \dot{x}Y_o\dot{Y}_o\}$$

$$\xi_7^{(1)} = -\frac{1}{\alpha}\{x[kJ_oY_o - \dot{J}_o\dot{Y}_o] + \dot{x}J_o\dot{Y}_o\} + \dot{x}$$

$$\xi_8^{(1)} = \dot{x}$$

and the second extensions are found to be

$$\xi_1^{(2)} = -\frac{1}{\sqrt{\alpha}}\{x^2k(\alpha J_o + \dot{J}_o) + 3\dot{x}J_o(\ddot{x} + kx)\}$$

$$\xi_2^{(2)} = -\frac{1}{\sqrt{\alpha}}\{x^2k(\alpha Y_o + \dot{Y}_o) + 3\dot{x}Y_o(\ddot{x} + kx)\}$$

$$\xi_3^{(2)} = -\frac{1}{\sqrt{\alpha}}\{kJ_o\}$$

$$\xi_4^{(2)} = -\frac{1}{\sqrt{\alpha}}\{kY_o\} \qquad (3.3\text{-}8)$$

$$\xi_5^{(2)} = -\frac{1}{\alpha}\{xkJ_o(\alpha J_o + \dot{J}_o) + 3J_o\dot{J}_o(\ddot{x} + kx)\}$$

$$\xi_6^{(2)} = -\frac{1}{\alpha}\{xkY_o(\alpha Y_o + \dot{Y}_o) + 3Y_o\dot{Y}_o(\ddot{x} + kx)\}$$

$$\xi_7^{(2)} = -\frac{1}{\alpha}\{xkJ_o(\alpha Y_o + \dot{Y}_o) + 3J_o\dot{Y}_o(\ddot{x} + kx)\} + 2(\ddot{x} + kx)$$

$$\xi_8^{(2)} = \ddot{x}\,.$$

In the following two cases, the equation of motion for the harmonic oscillator is investigated from two different standpoints. The analyses are conceptually different from the previous three cases, and the re-establishment of determining equations is necessary.

CASE 4. k an independent variable.

Consider the equation of motion

$$S \equiv \frac{\partial^2 x}{\partial t^2} + kx = 0 \qquad (3.4\text{-}1)$$

where x is a function of (t,k). Let

$$\bar{x} = x + \varepsilon\xi$$
$$\bar{t} = t + \varepsilon\eta \qquad (3.4\text{-}2)$$
$$\bar{k} = k + \varepsilon\zeta$$

be infinitesimal transformations generated by

$$U = \xi\partial_x + \eta\partial_t + \zeta\partial_k \qquad (3.4\text{-}3)$$

(where ξ,η and ζ are functions of (x,t,k)) that leaves invariant S; i.e.

$$US = 0 \qquad \text{whenever} \qquad S = 0 \qquad (3.4\text{-}4)$$

Since the equation under study is of second-order, it is necessary to use the first and second extensions of the generators*

$$U^{(1)} = U + \xi^{(1)}\partial_{\dot{x}} \qquad (3.4\text{-}5)$$

$$U^{(2)} = U^{(1)} + \xi^{(2)}\partial_{\ddot{x}} \qquad (3.4\text{-}6)$$

$$= \xi\partial_x + \eta\partial_t + \zeta\partial_k + \xi^{(1)}\partial_{\dot{x}} + \xi^{(2)}\partial_{\ddot{x}}$$

where[22]

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*Here, $\dot{x} = x_t = \partial x/\partial t$, $\overset{*}{x} = x_k = \partial x/\partial k$, ... Since S does not involve k-derivative terms, then possible, ∂_{kk}, $\partial_{\overset{*}{x}}$, ..., terms need not be considered in these extensions.

$$\xi^{(1)}(x,t,k,\dot{x},\overset{*}{x}) = \xi_t + \dot{x}\,(\xi_x - \eta_t - \dot{x}\eta_x) - \overset{*}{x}(\dot{x}\zeta_x + \zeta_t) \qquad (3.4\text{-}7)$$

$$\xi^{(2)}(x,t,k,\dot{x},\overset{*}{x},\ddot{x},\dot{\overset{*}{x}}) = \xi_t^{(1)} + \dot{x}\xi_x^{(1)} + \ddot{x}(\xi_{\dot{x}}^{(1)} - \eta_t - \dot{x}\eta_x) + \dot{\overset{*}{x}}(\xi_{\overset{*}{x}}^{(1)} - \zeta_t - \dot{x}\zeta_x) \qquad (3.4\text{-}8)$$

$$= \xi_{tt} + \dot{x}(2\xi_{xt} - \eta_{tt}) - 2\dot{x}\dot{\overset{*}{x}}\zeta_{xt} - 2\dot{x}\dot{\overset{*}{x}}\zeta_x + \dot{x}^2(\xi_{xx} - 2\eta_{xt}) - \dot{x}^2\overset{*}{x}\zeta_{xx}$$

$$- \dot{x}^3\eta_{xx} - \overset{*}{x}\zeta_{tt} - 2\dot{\overset{*}{x}}\zeta_t + \ddot{x}(\xi_x - 2\eta_t - 3\dot{x}\zeta_x - \overset{*}{x}\zeta_x) \qquad (3.4\text{-}9)$$

The action of $U^{(2)}$ on S is then

$$\{\xi^{(2)} + k\xi + x\zeta\}_{S=0} = 0.$$

Using Eq. (3.4-9), one obtains

$$\psi_1 + \dot{x}\psi_2 + \overset{*}{x}\psi_3 + \dot{x}^2\psi_4 + \dot{x}^3\psi_5 + \dot{\overset{*}{x}}\psi_6 + \dot{x}\overset{*}{x}\psi_7 + \dot{x}^2\overset{*}{x}\psi_8 + \dot{x}\,\dot{\overset{*}{x}}\psi_9 = 0 \qquad (3.4\text{-}10)$$

with

$$\psi_1 = \xi_{tt} + k\xi + x\zeta - kx\,(\xi_x - 2\eta_t)$$

$$\psi_2 = 2\xi_{xt} - \eta_{tt} + 3kx\eta_x$$

$$\psi_3 = kx\zeta_x - \zeta_{tt}$$

$$\psi_4 = \xi_{xx} - 2\eta_{xt} \qquad (3.4\text{-}11)$$

$$\psi_5 = -\eta_{xx}$$

$$\psi_6 = -2\zeta_t$$

$$\psi_7 = -2\zeta_{xt}$$

$$\psi_8 = -\zeta_{xx}$$

$$\psi_9 = -2\zeta_x .$$

As a consequence of the independence of the variables, the general solution to Eq. (3.4-10) is obtained if

$$\psi_1 = \psi_2 = \dots = \psi_9 = 0, \qquad (3.4\text{-}12)$$

from which one finds that the generator is of the form

$$U = \sum_{n=1}^{9} K_n U_n, \qquad (3.4\text{-}13)$$

where K_n is any function only of k and

$$U_{\substack{1\\2}} = \pm\, i\, x^2 e^{\pm i\sqrt{k}t}\, \partial_x + \frac{1}{\sqrt{k}}\, x\, e^{\pm i\sqrt{k}t}\partial_t \qquad (3.4\text{-}14)$$

$$U_{\substack{3\\4}} = e^{\pm i\sqrt{k}t}\partial_x$$

$$U_{\substack{5\\6}} = \pm ixe^{\pm 2i\sqrt{k}t}\partial_x + \frac{1}{\sqrt{k}} e^{\pm 2i\sqrt{k}t}\partial_t$$

$$U_7 = \frac{1}{\sqrt{k}} \partial_t \tag{3.4-14}$$

$$U_8 = x\partial_x$$

$$U_9 = -t\partial_t + 2k\partial_k .$$

The first eight generators are the same as those obtained in S.H.O. problem (Case 1, Eq. (3.1-3)) with $k = \omega^2$. Their first and second extensions are, then, given in Eqs. (3.1-4), (3.1-5). For the remaining generator, U_9, these extensions read

$$\xi_9^{(1)} = \dot{x} \tag{3.4-15}$$

$$\xi_9^{(2)} = 2\ddot{x} \tag{3.4-16}$$

CASE 5. k a dependent variable.

Instead of

$$\ddot{x} + k(t)x = 0, \quad k(t) = \omega_o^2 e^{\alpha t},$$

let the following set of equations be considered:

$$\ddot{x} + kx = 0 \tag{3.5-1}$$

$$\dot{k} - \alpha k = 0 \tag{3.5-2}$$

where x and k are functions of t, α constant and k is treated as a dependent variable. Let

$$\begin{aligned} \bar{x} &= x + \varepsilon\xi \\ \bar{t} &= t + \varepsilon\eta \\ \bar{k} &= k + \varepsilon\zeta \end{aligned} \tag{3.5-3}$$

be infinitesimal transformations generated by

$$U = \xi\partial_x + \eta\partial_t + \zeta\partial_k \tag{3.5-4}$$

leaving invariant the system of equations (3.5-1), (3.5-2), i.e.

$$U(\ddot{x} + kx) = 0 \quad \text{whenever } \ddot{x} + kx = 0 \tag{3.5-5}$$

$$U(\dot{k} - \alpha k) = 0 \quad \text{whenever } \dot{k} - \alpha k = 0; \tag{3.5-6}$$

here ξ,η and ζ are functions of (x, t, k).

Because of the variables involved in the problem, it is necessary to consider first and second extensions of U such that

$$U^{(1)} = U + \xi^{(1)}\partial_{\dot{x}} + \zeta^{(1)}\partial_{\dot{k}} \tag{3.5-7}$$

and

$$U^{(2)} = U^{(1)} + \xi^{(2)}\partial_{\ddot{x}}$$

$$= \xi\partial_x + \eta\partial_t + \zeta\partial_k + \xi^{(1)}\partial_{\dot{x}}$$

$$+ \zeta^{(1)}\partial_{\dot{k}} + \xi^{(2)}\partial_{\ddot{x}} \qquad (3.5\text{-}8)$$

where

$$\xi^{(1)} = \xi_t + \dot{x}(\xi_x - \eta_t) + \dot{k}\xi_k - \dot{x}\dot{k}\eta_k - \dot{x}^2\eta_x \qquad (3.5\text{-}9)$$

$$\zeta^{(1)} = \zeta_t + \dot{k}(\zeta_k - \eta_t) + \dot{x}\zeta_x - \dot{x}\dot{k}\eta_x - \dot{k}^2\eta_k \qquad (3.5\text{-}10)$$

$$\xi^{(2)} = \xi_t^{(1)} + \dot{x}\xi_x^{(1)} + \dot{k}\xi_k^{(1)} + \ddot{k}\xi_{\dot{k}}^{(1)}$$

$$- \ddot{x}\,(\eta_t - \xi_{\dot{x}}^{(1)} + \dot{x}\eta_x + \dot{k}\eta_k) \qquad (3.5\text{-}11)$$

or

$$\xi^{(2)} = \xi_{tt} + \dot{x}(2\xi_{xt} - \eta_{tt}) + 2\dot{k}\xi_{kt} + 2\dot{x}\dot{k}\,(\xi_{xk} - \eta_{kt})$$

$$+ \dot{x}^2(\xi_{xx} - 2\eta_{xt}) - 2\dot{x}^2\dot{k}\eta_{xk} + \dot{k}^2\xi_{kk} - \dot{x}\dot{k}^2\eta_{kk}$$

$$- \dot{x}^3\eta_{xx} + \ddot{x}(\xi_x - 2\eta_t - 3\dot{x}\eta_x - 2\dot{k}\eta_k) + \ddot{k}\,(\xi_k - \dot{x}\eta_k)\,.$$

$$(3.5\text{-}12)$$

The action of $U^{(2)}$ on Eqs. (3.5-1), (3.5-2) implies

$$\{\xi^{(2)} + x\zeta + k\xi\}_{\ddot{x}=-kx} = 0 \qquad (3.5\text{-}13)$$

$$\{\zeta^{(1)} - \alpha\zeta\}_{\dot{k}=\alpha k} = 0.$$

Use of Eqs. (3.5-10), (3.5-12) in Eqs. (3.5-13) gives

$$\psi_1 + \dot{x}\psi_2 + \dot{x}^2\psi_3 + \dot{x}^3\psi_4 = 0 \qquad (3.5\text{-}14)$$

$$\psi_5 + \dot{x}\psi_6 = 0$$

with

$$\psi_1 = \xi_{tt} + k(\xi+2\alpha\xi_{tk}) + \alpha^2 k(\xi_k+k\xi_{kk})$$

$$+ x\{\zeta+k(2\eta_t-\xi_x) + 2\alpha k^2\eta_k\}$$

$$\psi_2 = 2\xi_{xt} - \eta_{tt} + 2\alpha k(\xi_{xk}-\eta_{tk}) - \alpha^2 k(\eta_k+k\eta_{kk})$$

$$+ 3kx\eta_x$$

$$(3.5\text{-}15)$$

$$\psi_3 = \xi_{xx} - 2\eta_{xt} - 2\alpha k\eta_{xk}$$

$$\psi_4 = -\eta_{xx}$$

$$\psi_5 = \zeta_t - \alpha\zeta + \alpha k(\zeta_k-\eta_t) - \alpha^2 k^2\eta_k$$

$$\psi_6 = \zeta_x - \alpha k\eta_x.$$

Again, as a consequence of the independence of the variables, the general solution to Eqs. (3.5-14) is found if

$$\psi_1 = \psi_2 = \ldots = \psi_6 = 0, \qquad (3.5\text{-}16)$$

from which the generator is*

$$U = \sum_{n=1}^{9} \Omega_n U_n \,, \qquad (3.5\text{-}17)$$

where Ω_n is any function with argument $y = \frac{1}{2}\,(\alpha t - \ln k)$.

The U_n are

$$U_1 = \frac{1}{\sqrt{\alpha}}\,\{\alpha x^2 k\,\overset{*}{J}_o \partial_x + x J_o \partial_t + \alpha x k J_o \partial_k\}$$

$$U_2 = \frac{1}{\sqrt{\alpha}}\,\{\alpha x^2 k\,\overset{*}{Y}_o \partial_x + x Y_o \partial_t + \alpha x k Y_o \partial_k\}$$

$$U_3 = \frac{1}{\sqrt{\alpha}}\,\{J_o \partial_x\}$$

$$U_4 = \frac{1}{\sqrt{\alpha}}\,\{Y_o \partial_x\} \qquad (3.5\text{-}18)$$

$$U_5 = \frac{1}{\alpha}\{\alpha x k J_o \overset{*}{J}_o \partial_x + J_o^2 \partial_t + \alpha k J_o^2 \partial_k\}$$

$$U_6 = \frac{1}{\alpha}\{\alpha x k Y_o \overset{*}{Y}_o \partial_x + Y_o^2 \partial_t + \alpha k Y_o^2 \partial_k\}$$

$$U_7 = \frac{1}{\alpha}\{\alpha x k J_o \overset{*}{Y}_o \partial_x + J_o Y_o \partial_t + \alpha k J_o Y_o \partial_k\}$$

$$U_8 = \{x \partial_x\}$$

$$U_9 = \{\partial_t\} \,,$$

J_o and Y_o being the Bessel and Hankel functions (of order 0), respectively, with the argument

$$z = \frac{2}{\alpha}\sqrt{k} \qquad (3.5\text{-}19)$$

and the over-star means their derivative with respect to k.

*Further details are given in Appendix.

The coefficients of the first extensions are

$$\xi^{(1)}_{\substack{1\\2}} = -\alpha^{-3/2}\left\{(\alpha\dot{x}^2 + \dot{k}x^2)\begin{pmatrix}J_o\\Y_o\end{pmatrix} + \alpha(\dot{k} - 2\alpha k)\begin{pmatrix}\overset{*}{J}_o\\\overset{*}{Y}_o\end{pmatrix}\right\}$$

$$\xi^{(1)}_{\substack{3\\4}} = \alpha^{-1/2}\left\{\dot{k}\begin{pmatrix}\overset{*}{J}_o\\\overset{*}{Y}_o\end{pmatrix}\right\}$$

$$\xi^{(1)}_{\substack{5\\6}} = -\alpha^{-2}\left\{x\dot{k}\begin{pmatrix}J_o^2\\Y_o^2\end{pmatrix} - \alpha^2 xk\dot{k}\begin{pmatrix}\overset{*}{J}{}_o^2\\\overset{*}{Y}{}_o^2\end{pmatrix} + \alpha(2\dot{k}-\alpha k)\begin{pmatrix}J_o\overset{*}{J}_o\\Y_o\overset{*}{Y}_o\end{pmatrix}\right\} \qquad (3.5\text{-}20)$$

$$\xi^{(1)}_{7} = -\alpha^{-2}\left\{x\dot{k}J_oY_o - \alpha^2 xk\dot{k}\overset{*}{J}_o\overset{*}{Y}_o + \alpha(2\dot{k}-\alpha k)\,J_o\overset{*}{Y}_o + \frac{\alpha}{k}\dot{k}\,\dot{x}\right\}$$

$$\xi^{(1)}_{8} = \dot{x}$$

$$\xi^{(1)}_{9} = 0$$

and

$$\zeta^{(1)}_{\substack{1\\2}} = \alpha^{-1/2}\left\{[\alpha x\dot{k} - (\dot{k}-\alpha k)\dot{x}]\begin{pmatrix}J_o\\Y_o\end{pmatrix} - (\dot{k}-\alpha k)\,x\dot{k}\begin{pmatrix}\overset{*}{J}_o\\\overset{*}{Y}_o\end{pmatrix}\right\}$$

$$\zeta^{(1)}_{3} = \zeta^{(1)}_{4} = \zeta^{(1)}_{8} = \zeta^{(1)}_{9} = 0 \qquad (3.5\text{-}21)$$

$$\zeta^{(1)}_{\substack{5\\6}} = \alpha^{-1}\left\{\alpha\dot{k}\begin{pmatrix}J_o^2\\Y_o^2\end{pmatrix} - 2(\dot{k}-\alpha k)\,\dot{k}\begin{pmatrix}J_o\overset{*}{J}_o\\Y_o\overset{*}{Y}_o\end{pmatrix}\right\}$$

$$\zeta^{(1)}_{7} = \alpha^{-1}\left\{\alpha\dot{k}J_oY_o - 2(\dot{k}-\alpha k)\dot{k}J_o\overset{*}{Y}_o + \frac{1}{k}(\dot{k}-\alpha k)\dot{k}\right\}.$$

The result of calculating the coefficients of the second extensions is

$$\xi_{\substack{1\\2}}^{(2)} = \frac{1}{\sqrt{\alpha}} \left\{ \frac{1}{\alpha^2 k} \left[(\dot{k}-\alpha k)x\dot{x}\dot{k} - \alpha x^2 k\ddot{k} - 3\alpha kx(\alpha\ddot{x} + x\dot{k}) \right] \binom{J_o}{Y_o} \right.$$

$$\left. - \frac{1}{\alpha k} \left[2\alpha k(\dot{k}-\alpha k)(x\ddot{x} + \dot{x}^2) - (\alpha\dot{x}-kx)x\dot{k}^2 + x\dot{x}k\ddot{k} \right] \binom{\overset{*}{J}_o}{\overset{*}{Y}_o} \right\}$$

$$\xi_{\substack{3\\4}}^{(2)} = -\frac{1}{\sqrt{\alpha}} \left\{ \frac{1}{\alpha^2 k} \left[\dot{k}^2 \binom{J_o}{Y_o} - \alpha^2(k\ddot{k} - \dot{k}^2)\binom{\overset{*}{J}_o}{\overset{*}{Y}_o} \right] \right\}$$

$$\xi_{\substack{5\\6}}^{(2)} = -\frac{1}{\alpha} \left\{ \frac{1}{\alpha^2 k} \left[2(\alpha k-\dot{k})\dot{x}\dot{k} + \alpha xk\ddot{k} \right] \binom{J_o^2}{Y_o^2} \right. \qquad (3.5\text{-}22)$$

$$- \left[2(\alpha k-\dot{k})\dot{x}\dot{k} + \alpha x(k\ddot{k} - \dot{k}^2) \right] \binom{\overset{*}{J}{}_o^2}{\overset{*}{Y}{}_o^2}$$

$$\left. + \frac{1}{\alpha k} \left[4xk\dot{k}^2 + 2\alpha\dot{x}(k\ddot{k}-\dot{k}^2) + \alpha k\ddot{x}(4\dot{k}-\alpha k) \right] \binom{J_o\overset{*}{J}_o}{Y_o\overset{*}{Y}_o} \right\}$$

$$\xi_7^{(2)} = -\frac{1}{\alpha} \left\{ \frac{1}{\alpha^2 k} \left[2(\alpha k-\dot{k})\dot{x}\dot{k} + \alpha xk\ddot{k} \right] J_oY_o \right.$$

$$- \left[2(\alpha k-\dot{k})\dot{x}\dot{k} + \alpha x(k\ddot{k} - \dot{k}^2) \right] \overset{*}{J}_o\overset{*}{Y}_o$$

$$+ \frac{1}{\alpha k} \left[4xk\dot{k}^2 + 2\alpha\dot{x}(k\ddot{k} - \dot{k}^2) + \alpha k\ddot{x}(4\dot{k}-\alpha k) \right] J_o\overset{*}{Y}_o$$

$$\left. - \frac{1}{\alpha k^2} \left[\alpha\dot{x}(k\ddot{k}-\dot{k}^2) + 2k\dot{k}(\alpha\ddot{x} + x\dot{k}) \right] \right\}.$$

$$\xi_8^{(2)} = \ddot{x}$$

$$\xi_9^{(2)} = 0.$$

4. LIE ALGEBRA

In order to obtain the classification of the Lie Algebra involved in the above problems, a study of commutators and metric tensors must be developed.

The set of commutation relations for generators U_m listed in Eqs. (3.1-3), (3.2-5), (3.3-6), (3.4-14), (3.5-18), are given in Tables 1 and 2. Cases 1 and 4 have the same commutator table between $U_1, U_2, \ldots, U_8$ in each case, and the extra generator U_9 (Case 4) commutes with every one as is seen in Table 1. The same situation occurs with U_9 (Case 5) and the set of commutators for Cases 2, 3, 5 are written in Table 2. These results show that the algebra is closed in each Case.

The metric tensors are constructed, using the structure constants deduced from these tables, and their forms are given below the corresponding Table.

Furthermore, in Cases 1 and 4,

$$\det [g_{ab}] = -\frac{(12)^9}{9} \neq 0 \quad a, b = 1, 2, \ldots, 8, \tag{4.1}$$

and in Cases 2, 3 and 5,

$$\det [g_{ab}] = -12(6)^6 \neq 0, \quad a, b = 1,2, \ldots, 8, \tag{4.2}$$

which indicate, according to Cartan's criterion, that the eight-parameter group is semi-simple in each Case.

As is noticed in Matrices 1 and 2, the metric tensors are non-diagonal. Linear combinations of generators can be taken, which will reduce them to diagonal form and a solution to an eigenvalue problem provides the standard form of the generators. Each case is studied separately.

CASES 1, 4.* Let

$$
\begin{aligned}
Q_1 &= -\frac{1}{2}\{U_1 + U_2 + i(U_3 - U_4)\} \\
Q_2 &= \frac{1}{2}\{U_1 + U_2 - i(U_3 - U_4)\} \\
Q_3 &= \frac{1}{2}\{-i(U_1 - U_2) + U_3 + U_4\} \\
Q_4 &= \frac{1}{2}\{i(U_1 - U_2) + U_3 + U_4\} \\
Q_5 &= U_7 \\
Q_6 &= U_8 \\
Q_7 &= \frac{1}{2} i(U_6 - U_5) \\
Q_8 &= \frac{1}{2}(U_5 + U_6)
\end{aligned}
\qquad (4.3)
$$

be a linear combination of generators listed in Eq. (3.1-3). The "new" set of generators is†

$$
\begin{aligned}
Q_{\substack{1\\2}} &= (1 \pm x^2)\sin\omega t\, \partial_x \mp \frac{1}{\omega} x \cos\omega t\, \partial_t \\
Q_{\substack{3\\4}} &= (1 \pm x^2)\cos\omega t\, \partial_x \pm \frac{1}{\omega} x \sin\omega t\, \partial_t \\
Q_5 &= \frac{1}{\omega}\partial_t
\end{aligned}
$$

---

*Generators U_9 are excluded in Cases 4 and 5, in order to keep a semi-simple group.

†These generators were known before.[1] For a direct comparison, set $\omega=1$ in Eq. (4.4).

$$Q_6 = x\partial_x \tag{4.4}$$

$$Q_7 = x\cos(2\omega t)\partial_x + \frac{1}{\omega}\sin(2\omega t)\partial_t$$

$$Q_8 = -x\sin(2\omega t)\partial_x + \frac{1}{\omega}\cos(2\omega t)\partial_t$$

The commutation relations of these generators are listed in Table 3, and the metric tensor corresponding to these generators is also evaluated there. Furthermore,

$$\det[g_{ab}] = -\frac{(12)^8}{3} \neq 0 \quad a,b = 1,2,\ \ldots,\ 8.$$

Wulfman and Wybourne[1] found the standard form of these generators, in the case $\omega=1$, and the comparison of the present results with their work shows that $U_1, U_2, \ldots, U_8$ are proportional to their $E_{\pm\alpha}$, $E_{\pm\beta}$, $E_{\pm\gamma}$, H_1 and H_2.

Case 2. Let

$$Q_{\substack{1\\2}} = \pm(i/\sqrt{\omega_o})\ \{\cos\lambda(U_1 \pm \omega_o U_4) - \sin\lambda(U_2 \pm \omega_o U_3)\}$$

$$Q_{\substack{3\\4}} = \pm(i/\sqrt{\omega_o})\ \{\sin\lambda(U_1 \pm \omega_o U_4) + \cos\lambda(U_2 \mp \omega_o U_3)\}$$

$$\begin{aligned} Q_5 &= -\{U_5 + U_6\} \\ Q_6 &= U_8 \\ Q_7 &= \sin 2\lambda\ (U_5 - U_6) + \cos 2\lambda\ (2U_7 - U_8) \\ Q_8 &= -\cos 2\lambda\ (U_5 - U_6) + \sin 2\lambda\ (2U_7 - U_8) \end{aligned} \tag{4-5}$$

be a linear combination of generators* given in Eq. (3.2-5),

---

*Explicit forms of the Q_m's are too long to be presented here, but their calculation are straight-forward using Eqs. (3.2-5) and (4-5).

where λ is a function of α and ω_o.*

This set of generators is found to obey the same commutation relations as those Ω_m's in the above Cases 1 and 4, and Table 3 summarizes them.

As a matter of fact, the special form of Eqs. (4-5) is suggested by looking at the limiting behavior, as α approaches zero, of the corresponding U_m's. This will be developed in the next Chapter.

Cases 3, 5. Let†

$$\Omega_{\substack{1\\2}} = \mp (\pi\omega_o)^{-1/2} \{\cos\mu(\pi U_1 \mp \omega_o U_4) + \sin\mu\ (U_2 \pm \pi\omega_o U_3)\}$$

$$\Omega_{\substack{3\\4}} = \mp (\pi\omega_o)^{-1/2} \{\sin\mu\ (\pi U_1 \mp \omega_o U_4) - \cos\mu\ (U_2 \pm \pi\omega_o U_3)\} \tag{4-6}$$

$$\Omega_5 = \pi U_5 + U_6/\pi$$

$$\Omega_6 = U_8$$

$$\Omega_7 = -\sin 2\mu\ (\pi U_5 - U_6/\pi) + \cos 2\mu\ (2U_7 - U_8)$$

$$\Omega_8 = \cos 2\mu\ (\pi U_5 - U_6/\pi) + \sin 2\mu\ (2U_7 - U_8)$$

be a linear combination of generators¶ given in Eqs. (3.3-6),

---

*The dependence of λ on α and ω_o is given in Chapter 5, Case. 2.

†Generator U_9 is excluded in Case 5.

¶Explicit forms of the Ω_m's can be obtained using Eqs. (3.3-6), (3.5-18) and (4.6).

(3.5-18), with μ being a function of α and ω_o.*

As in the above cases, the commutation relations of these generators are listed in Table 3.

As is known[1], the local Lie group of Newton's and Lagrange's equation for the oscillator is SL(3,R) for the case k=1, and since the generators Q_m, in all problems considered here, have the same set of commutation relations, it is concluded that the local Lie group is still SL(3,R) and also that the generators Q_1, Q_3, Q_5 form a compact subgroup SO(3).

*The dependence of μ on α and ω_o is given in Chapter 5, Case 3.

*Table 1. Commutator table for generators Um in Cases 1, 4.**

$U_m \backslash U_n$	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9
U_1	0	0	$-U_5$	$-U_7-3iU_8$	0	$-2iU_2$	$-iU_1$	$-U_1$	0
U_2	0	0	$-U_7+3iU_8$	$-U_6$	$2iU_1$	0	iU_2	$-U_2$	0
U_3	U_5	U_7-3iU_8	0	0	0	$-2iU_4$	$-iU_3$	U_3	0
U_4	U_7+3iU_8	U_6	0	0	$2iU_3$	0	iU_4	U_4	0
U_5	0	$-2iU_1$	0	$-2iU_3$	0	$-4iU_7$	$-2iU_5$	0	0
U_6	$2iU_2$	0	$2iU_4$	0	$4iU_7$	0	$2iU_6$	0	0
U_7	iU_1	$-iU_2$	iU_3	$-iU_4$	$2iU_5$	$-2iU_6$	0	0	0
U_8	U_1	U_2	$-U_3$	$-U_4$	0	0	0	0	0
U_9	0	0	0	0	0	0	0	0	0

*The extra column and row for U_9 do not count in Case 1.

*Table 2. Commutator table for generators Um in Cases 2, 3, 5.**

U_m \ U_n	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9
U_1	0	0	$-U_5$	$-U_7+2U_8$	0	U_2	0	$-U_1$	0
U_2	0	0	$-U_7-U_8$	$-U_6$	$-U_1$	0	$-U_2$	$-U_2$	0
U_3	U_5	U_7+U_8	0	0	0	U_4	U_3	U_3	0
U_4	U_7-2U_8	U_6	0	0	$-U_3$	0	0	U_4	0
U_5	0	U_1	0	U_3	0	$2U_7-U_8$	U_5	0	0
U_6	$-U_2$	0	$-U_4$	0	$-2U_7+U_8$	0	$-U_6$	0	0
U_7	0	U_2	$-U_3$	0	$-U_5$	U_6	0	0	0
U_8	U_1	U_2	$-U_3$	$-U_4$	0	0	0	0	0
U_9	0	0	0	0	0	0	0	0	0

*The extra column and row for U_9 count <u>only</u> in Case 5.

Table 3. Commutator table for generators Q_m.

$Q_m \backslash Q_n$	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8
Q_1	0	Q_7-3Q_6	Q_5	Q_8	$-Q_3$	Q_2	$-Q_2$	$-Q_4$
Q_2	$-Q_7+3Q_6$	0	$-Q_8$	$-Q_5$	$-Q_4$	Q_1	$-Q_1$	$-Q_3$
Q_3	$-Q_5$	Q_8	0	$-Q_7-3Q_6$	Q_1	Q_4	Q_4	$-Q_2$
Q_4	$-Q_8$	Q_5	Q_7+3Q_6	0	Q_2	Q_3	Q_3	$-Q_1$
Q_5	Q_3	Q_4	$-Q_1$	$-Q_2$	0	0	$2Q_8$	$-2Q_7$
Q_6	$-Q_2$	$-Q_1$	$-Q_4$	$-Q_3$	0	0	0	0
Q_7	Q_2	Q_1	$-Q_4$	$-Q_3$	$-2Q_8$	0	0	$-2Q_5$
Q_8	Q_4	Q_3	Q_2	Q_1	$2Q_7$	0	$2Q_5$	0

$$
\left[g_{ab}\right] = 4 \begin{bmatrix}
0 & 0 & 0 & -3i & 0 & 0 & 0 & 0 \\
0 & 0 & 3i & 0 & 0 & 0 & 0 & 0 \\
0 & 3i & 0 & 0 & 0 & 0 & 0 & 0 \\
-3i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

Matrix 1. Metric Tensor for Cases 1, 4.*

$$
\left[g_{ab}\right] = 2 \begin{bmatrix}
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2
\end{bmatrix}
$$

Matrix 2. Metric Tensor for Cases 2, 3, 5.*

$$
\left[g_{ab}\right] = \begin{bmatrix}
-12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -12 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -12 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 12
\end{bmatrix}
$$

Matrix 3. Metric tensor for the generators Q_m.

*The generators U_9 are excluded in Cases 4 and 5.

5. LIMITING BEHAVIOR

The purpose of this chapter is to show that the generators found in the problems where k is a function of time reduce to those in which k is constant.

Initially,

Case 1 $\quad k = \omega^2$

Case 2 $\quad k = \omega_o^2 + \alpha t$

Case 3 $\quad k = \omega_o^2 e^{\alpha t}$

Case 5 $\quad \dot{k} = \alpha k$

It is evident that in Cases 2, 3 and 5, as α approaches zero, k approaches a constant.

CASE 2. The generators found in Eq. (3.2-5) involve Airy functions and products of these functions and their derivatives. The argument of these functions is

$$- u \equiv z = -\alpha^{-2/3}(\omega_o^2 + \alpha t). \tag{5-1}$$

Thus

$$\dot{u} = -\dot{z} = \alpha^{1/3}. \tag{5-2}$$

As α approaches zero, z goes to $-\infty$ and u tends to $+\infty$; hence the asymptotic representation of Airy's functions with large argument is required.

The functions Ai(-u) and Bi(-u) are defined as[23]

$$Ai(-u) = \frac{1}{3}\sqrt{u}\ \{J_{1/3}(\tfrac{2}{3}u^{3/2}) + J_{-1/3}(\tfrac{2}{3}u^{3/2})\}$$

(5-3)

$$Bi(-u) = \sqrt{u/3}\ \{J_{-1/3}(\tfrac{2}{3}u^{3/2}) - J_{1/3}(\tfrac{2}{3}u^{3/2})\},$$

and since the Bessel functions included have a large argument and fixed order, the "Hankel" type expansion[24] is applied; i.e.

$$J_\nu(u) \sim \sqrt{2/\pi u}\ \cos(u - \pi\nu/2 - \pi/4) + O(|u|^{-3/2}) \quad (5\text{-}4)$$

The argument of the Bessel functions is

$$\frac{2}{3}u^{3/2} = \frac{2\omega_o^3}{3\alpha}(1 + \alpha t/\omega_o^2)^{3/2}.$$

Using the Binomial expansion, as α approaches zero

(5-5)

$$\frac{2}{3}u^{3/2} \to \omega_o t + \sigma,$$

where

(5-6)

$$\sigma = 2\omega_o^3/3\alpha + O(\alpha).$$

Hence, Eqs. (5.4) and (5.5) give

$$J_{1/3}(\tfrac{2}{3}u^{3/2}) \sim \sqrt{3/\pi u^{3/2}}\ \cos(\omega_o t + \lambda_1)$$

and

$$J_{-1/3}(\tfrac{2}{3}u^{3/2}) \sim \sqrt{3/\pi u^{3/2}}\ \cos(\omega_o t + \lambda_2)$$

with

$$\lambda_1 = \sigma - 5\pi/12; \qquad \lambda_2 = \sigma - \pi/12.$$

Thus

(5-7)

$$J_{1/3} + J_{-1/3} \sim (3/\sqrt{\pi u^{3/2}})\ \cos(\omega_o t + \lambda)$$

$$J_{-1/3} - J_{1/3} \sim -\sqrt{3/\pi u^{3/2}}\ \sin(\omega_o t + \lambda),$$

where

$$\lambda = \sigma - \pi/4 = 2\omega_o^3/3\alpha - \pi/4 + O(\alpha). \tag{5-8}$$

Two of the basic factors in the generators, in terms of u, are

$$\sqrt{\pi/-\dot{u}}\ \mathrm{Ai}(-u) = \frac{i}{3}\sqrt{\pi u/\dot{u}}\ (J_{1/3} + J_{-1/3})$$

$$\sqrt{\pi/-\dot{u}}\ \mathrm{Bi}(-u) = i\sqrt{\pi u/3\dot{u}}\ (J_{-1/3} - J_{1/3})$$

and using Eqs. (5-7), their limiting forms as α goes to zero are

$$\sqrt{\pi/\dot{z}}\ \mathrm{Ai}(z) = (i/\sqrt{\omega_o})\cos(\omega_o t + \lambda)$$

$$\sqrt{\pi/\dot{z}}\ \mathrm{Bi}(z) = -(i/\sqrt{\omega_o})\sin(\omega_o t + \lambda), \tag{5-9}$$

where use has been made of the fact that

$$\lim_{\alpha\to o}(\dot{u}\sqrt{u}) = \lim_{\alpha\to o}\sqrt{\omega_o^2 + \alpha t} = \omega_o.$$

Similarly, it is found that

$$\sqrt{\pi/\dot{z}}\ \dot{\mathrm{A}}\mathrm{i}(z) = -i\sqrt{\omega_o}\sin(\omega_o t + \lambda) \tag{5-10}$$

$$\sqrt{\pi/\dot{z}}\ \dot{\mathrm{B}}\mathrm{i}(z) = i\sqrt{\omega_o}\cos(\omega_o t + \lambda)$$

One may use the same procedure to obtain the remaining expressions contained in the generators U_m. Let

$$\tilde{U}_m = \lim_{\alpha\to o} U_m \qquad m = 1, 2, \ldots, 8,$$

represent the limiting behavior of the generators U_m. Then, by using the above expressions, one can obtain

$$\tilde{U}_1 = -(i\sqrt{\omega_o})\{x^2 \sin(\omega_o t+\lambda)\partial_x - \frac{1}{\omega_o} x \cos(\omega_o t+\lambda)\partial_t\}$$

$$\tilde{U}_2 = -(i\sqrt{\omega_o})\{x^2 \cos(\omega_o t+\lambda)\partial_x + \frac{1}{\omega_o} x \sin(\omega_o t+\lambda)\partial_t\}$$

$$\tilde{U}_3 = (i/\sqrt{\omega_o})\{\cos(\omega_o t+\lambda)\partial_x\}$$

$$\tilde{U}_4 = -(i/\sqrt{\omega_o})\{\sin(\omega_o t+\lambda)\partial_x\} \qquad (5\text{-}11)$$

$$\tilde{U}_5 = -\{-\frac{1}{2} x \sin(2\omega_o t+2\lambda)\partial_x + \frac{1}{\omega_o} \cos^2(\omega_o t+\lambda)\partial_t\}$$

$$\tilde{U}_6 = -\{\frac{1}{2} x \sin(2\omega_o t+2\lambda)\partial_x + \frac{1}{\omega_o} \sin^2(\omega_o t+\lambda)\partial_t\}$$

$$\tilde{U}_7 = \frac{1}{2}\{x\partial_x + x\cos(2\omega_o t+2\lambda)\partial_x + \frac{1}{\omega_o} \sin(2\omega_o t+2\lambda)\partial_t\}$$

$$\tilde{U}_8 = x\partial_x ,$$

from which the following linear combinations

$$\tilde{Q}_{\substack{1\\2}} = \pm (i/\sqrt{\omega_o})\{\cos\lambda(\tilde{U}_1 \pm \omega_o \tilde{U}_4) - \sin\lambda(\tilde{U}_2 \pm \omega_o \tilde{U}_3)\}$$

$$\tilde{Q}_{\substack{3\\4}} = \pm (i/\sqrt{\omega_o})\{\sin\lambda(\tilde{U}_1 \pm \omega_o \tilde{U}_4) + \cos\lambda(\tilde{U}_2 \pm \omega_o \tilde{U}_3)\} \qquad (5\text{-}12)$$

$$\tilde{Q}_5 = -\{\tilde{U}_5 + \tilde{U}_6\}$$

$$\tilde{Q}_6 = \tilde{U}_8$$

$$\tilde{Q}_7 = \sin 2\lambda\ (\tilde{U}_5 - \tilde{U}_6) + \cos 2\lambda\ (2\tilde{U}_7 - \tilde{U}_8)$$

$$\tilde{Q}_8 = -\cos 2\lambda\ (\tilde{U}_5 - \tilde{U}_6) + \sin 2\lambda\ (2\tilde{U}_7 - \tilde{U}_8),$$

produce identical generators to those in Eq. (4-4).

CASE 3. The generators listed in Eq. (3.3-6) involve Bessel and Hankel functions (of order 0) and products of these functions and their derivatives with the argument given in Eq. (3.3-4).

As α goes to zero, z approaches $+\infty$, so that the "Hankel" type expansion is used to find the asymptotic behavior of the generators; i.e., Eq. (5-4) and*

$$Y_n(z) \sim \sqrt{2\pi/z}\,\sin(z - n\pi/2 - \pi/4) + O\left(|z|^{-3/2}\right). \qquad (5\text{-}13)$$

Expanding Eq. (3.3-4) in Taylor's Series and taking the limit as α goes to zero,

$$z \to \omega_o t + \bar{\sigma} \qquad (5\text{-}14)$$

where

$$\bar{\sigma} = 2\omega_o/\alpha + O(\alpha)$$

Using Eqs. (5-4), (5-13), (5-14), it is found that

$$\lim_{\alpha\to 0} \frac{J_o(z)}{\sqrt{\alpha}} = (\pi\omega_o)^{-1/2}\cos(\omega_o t + \mu) \qquad (5\text{-}15)$$

$$\lim_{\alpha\to 0} \frac{Y_o(z)}{\sqrt{\alpha}} = (\pi/\omega_o)^{1/2}\sin(\omega_o t + \mu), \qquad (5\text{-}16)$$

where

$$\mu = \bar{\sigma} - \pi/4 = 2\omega_o/\alpha - \pi/4 + O(\alpha).$$

---

*There are ambiguities about the name and symbol for Hankel functions. I follow the definition given in Watson[25], No. 7.21, Eq. (5) p. 199.

Furthermore,

$$\lim_{\alpha\to o} \frac{\dot{J}_o(z)}{\sqrt{\alpha}} = -(\omega_o/\pi)^{1/2} \sin(\omega_o t + \mu) \tag{5-17}$$

$$\lim_{\alpha\to o} \frac{\dot{Y}_o(z)}{\sqrt{\alpha}} = (\pi\omega_o)^{1/2} \cos(\omega_o t + \mu)$$

and so on. The use of Eqs. (5-15), (5-17) and a similar procedure provide the remaining expressions such that the limiting behavior of the generators is found to be

$$\tilde{U}_m = \lim_{\alpha\to o} U_m \qquad m = 1, 2, \ldots, 8,$$

where

$$\begin{aligned}
\tilde{U}_1 &= (\omega_o/\pi)^{1/2}\{-x^2\sin(\omega_o t+\mu)\partial_x + \frac{1}{\omega_o} x \cos(\omega_o t+\mu)\partial_t\} \\
\tilde{U}_2 &= (\pi\omega_o)^{1/2}\{x^2\cos(\omega_o t+\mu)\partial_x + \frac{1}{\omega_o} x \sin(\omega_o t+\mu)\partial_t\} \\
\tilde{U}_3 &= (\pi\omega_o)^{-1/2}\{\cos(\omega_o t+\mu)\partial_x\} \\
\tilde{U}_4 &= (\pi/\omega_o)^{1/2}\{\sin(\omega_o t+\mu)\partial_x\} \\
\tilde{U}_5 &= (1/\pi)\{-\frac{1}{2} x \sin(2\omega_o t+2\mu)\partial_x + \frac{1}{\omega_o} \cos^2(\omega_o t+\mu)\partial_t\} \\
\tilde{U}_6 &= \pi\{\frac{1}{2} x \sin(2\omega_o t+2\mu)\partial_x + \frac{1}{\omega_o} \sin^2(\omega_o t+\mu)\partial_t\} \\
\tilde{U}_7 &= \frac{1}{2}\{x\partial_x + x\cos(2\omega_o t+2\mu)\partial_x + \frac{1}{\omega_o} \sin(2\omega_o t+2\mu)\partial_t\} \\
\tilde{U}_8 &= x\partial_x .
\end{aligned} \tag{5-18}$$

The following linear combinations

$$\tilde{Q}_{\substack{1\\2}} = \mp (\pi\omega_o)^{-1/2}\{\cos\mu(\pi\tilde{U}_1 \mp \omega_o \tilde{U}_4)$$

$$+ \sin\mu\ (\tilde{U}_2 \pm \pi\omega_o \tilde{U}_3)\}$$

$$\tilde{Q}_{\substack{3\\4}} = \mp (\pi\omega_o)^{-1/2}\{\sin\mu\ (\pi\tilde{U}_1 \mp \omega_o \tilde{U}_4)$$

$$- \cos\mu\ (\tilde{U}_2 \pm \pi\omega_o \tilde{U}_3)\}$$

$$\tilde{Q}_5 = \pi\tilde{U}_5 + \frac{1}{\pi}\tilde{U}_6 \qquad (5\text{-}19)$$

$$\tilde{Q}_6 = \tilde{U}_8$$

$$\tilde{Q}_7 = -\sin 2\mu\ (\pi\tilde{U}_5 - \frac{1}{\pi}\tilde{U}_6) + \cos 2\mu(2\tilde{U}_7 - \tilde{U}_8)$$

$$\tilde{Q}_8 = \cos 2\mu\ (\pi\ \tilde{U}_5 - \frac{1}{\pi}\tilde{U}_6) + \sin 2\mu\ (2\tilde{U}_7 - \tilde{U}_8)$$

give the same generators as those found in Case 1.

CASE 5. A different approach is followed here and the results of the above Case 3 will be used. The use of Eq. (I-22)* in the generators found in Eq. (3.5-18) enables one to write

$$U_1 = \frac{1}{\sqrt{\alpha}}\{\sqrt{k}\ x^2 J_o'\ \partial_x + xJ_o\partial_t + \alpha xkJ_o\partial_k\}$$

$$U_2 = \frac{1}{\sqrt{\alpha}}\{\sqrt{k}\ x^2 Y_o'\partial_x + xY_o\partial_t + \alpha xkY_o\partial_k\} \qquad (5\text{-}20)$$

$$U_3 = \frac{1}{\sqrt{\alpha}}\ J_o\partial_x$$

*See Appendix.

$$U_4 = \frac{1}{\sqrt{\alpha}} Y_o \partial_x$$

$$U_5 = \frac{1}{\alpha} \{\sqrt{k}\ x\ J_o J_o' \partial_x + J_o^2 \partial_t + \alpha k J_o^2 \partial_k\}$$

$$U_6 = \frac{1}{\alpha} \{\sqrt{k}\ x\ Y_o Y_o' \partial_x + Y_o^2 \partial_t + \alpha k Y_o^2 \partial_k\}$$

$$U_7 = \frac{1}{\alpha} \{\sqrt{k}\ x\ J_o Y_o' \partial_x + J_o Y_o \partial_t + \alpha k J_o Y_o \partial_k\}$$

$$U_8 = x\partial_x$$

and the Bessel and Hankel functions all have the argument

$$z = \frac{2}{\alpha} \sqrt{k} \qquad (5\text{-}21)$$

with the restriction

$$\dot{k} = \alpha k \qquad (5\text{-}22)$$

As α tends to zero, z goes to $+\infty$, $\dot{k}$ approaches zero, and $J_o/\sqrt{\alpha}$, $Y_o/\sqrt{\alpha}$, $J_o'/\sqrt{\alpha}$, $Y_o'/\sqrt{\alpha}$ and their products have a finite behavior, which imply that the coefficients of ∂_x and ∂_t converge. However, the coefficients of ∂_k approach zero faster than the others because of their α factor. This fact reduces the generators to those in Case 3 if Eq. (5-22) is used for it implies that

$$k = \omega_o^2\ e^{\alpha t} \qquad (5\text{-}23)$$

ω_o^2 being constant.

CONCLUSION

The results of a study carried out as an attempt to obtain some general characteristics of the invariance groups for differential equations describing time-dependent harmonic oscillators, show that oscillators with quite different time dependencies admit identical or very similar local Lie groups of point transformations. In Cases 2 and 3 the system has the same SL(3,R) Lie-algebra as the simple harmonic oscillator or Case 1. In Cases 4 and 5, the SL(3,R) is a sub-algebra, there being an additional generator which commutes with the eight SL(3,R) generators.

The local Lie groups, and their first and second extensions obtained for Cases 2 through 5 in this work, do not seem to have appeared before. The asymptotic connections established between these groups are also new.

This study was undertaken as part of an investigation of the relationship between invariance under point transformations and adiabatic invariance. Attaining this goal might give a new approach to Ehrenfest's adiabatic invariance principle. In addition to generalizing the work in this direction, the results of this thesis suggest further avenues for investigation, among which a study of finite transformations, path curves, and invariant families of curves should illuminate the physical significance of the results obtained here.

This work also suggests that for a wide variety of cases in where k is allowed to vary the equation,

$$\ddot{x} + kx = 0$$

admits the local Lie group of point transformations SL(3,R). It would evidently be of great interest to establish just how generalizable this result is.

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APPENDIX

AN EXAMPLE OF SOLVING THE DETERMINING EQUATIONS

Case 5. k a dependent variable

Reordering of Eqs. (3.5-15) under the condition of Eq. (3.5-16) gives

$$\zeta_x - \alpha k \eta_x = 0$$

$$\zeta_t - \alpha\zeta + \alpha k\ (\zeta_k - \eta_t) - \alpha^2 k^2 \eta_k = 0$$

$$\eta_{xx} = 0$$

$$\xi_{xx} - 2\eta_{xt} - 2\alpha k \eta_{xk} = 0$$

$$2\xi_{xt} - \eta_{tt} + 2\alpha k\ (\xi_{xk} - \eta_{kt}) - \alpha^2 k(\eta_k + k\eta_{kk})$$

$$+ 3kx\eta_x = 0$$

$$\xi_{tt} + k(\xi + 2\alpha\xi_{tk}) + \alpha^2 k\ (\xi_k + k\xi_{kk})$$

$$+ x\ \{\zeta + k\ (2\eta_t - \xi_x) + 2\alpha k^2 \eta_k\} = 0.$$

Integration of the above equations gives

(I-1)

$$\zeta = \alpha k x \gamma + \alpha k \delta + \beta$$

$$\eta = x\gamma + \delta$$

$$\xi = x^2(\gamma_t + \alpha k \gamma_k) + x\phi + \theta,$$

where β, γ, δ, ϕ and θ are functions of (t,k) obeying the conditions

$$\beta_t + \alpha k \beta_k - \alpha\beta = 0$$

$$\gamma_{tt} + 2\alpha k \gamma_{tk} + \alpha^2 k^2 \gamma_{kk} + \alpha^2 k \gamma_k + k\gamma = 0$$

(t,k)
$$\theta_{tt} + 2\alpha k \theta_{tk} + \alpha^2 k^2 \theta_{kk} + \alpha^2 k \theta_k + k\theta = 0 \qquad \text{(I-2)}$$

$$\delta_{tt} + 2\alpha k \delta_{tk} + \alpha^2 k^2 \delta_{kk} + \alpha^2 k \delta_k - 2(\phi_t + \alpha k \phi_k) = 0$$

$$\beta + \phi_{tt} + 2\alpha k \phi_{tk} + \alpha^2 k^2 \phi_{kk} + \alpha^2 k \phi_k + 2k\delta_t + 2\alpha k^2 \delta_k + \alpha k \delta = 0.$$

The transformations (I-3)

$$T = \alpha t$$

$$K = \ln k$$

reduce Eqs. (I-2) to the form

$$(e^{-K}\beta)_T + (e^{-K}\beta)_K = 0$$

$$\gamma_{TT} + 2\gamma_{KT} + \gamma_{KK} + \frac{1}{\alpha^2} e^K \gamma = 0$$

(T,K). (I-4)
$$\theta_{TT} + 2\theta_{KT} + \theta_{KK} + \frac{1}{\alpha^2} e^K \theta = 0$$

$$\delta_{TT} + 2\delta_{KT} + \delta_{KK} - \frac{2}{\alpha}(\phi_T + \phi_K) = 0$$

$$\frac{1}{\alpha^2}\beta + \phi_{TT} + 2\phi_{KT} + \phi_{KK} + \frac{1}{\alpha} e^K (2\delta_T + 2\delta_K + \delta) = 0.$$

The transformations[26] (I-5)

$$y = \frac{1}{2}(T-K)$$

$$r = \frac{1}{2}(T+K)$$

on Eq. (I-4) give

$$(r,y)\quad \begin{aligned} &(e^{y-r}\beta)_r = 0 \\ &\gamma_{rr} + \frac{1}{\alpha^2} e^{r-y}\gamma = 0 \\ &\theta_{rr} + \frac{1}{\alpha^2} e^{r-y}\theta = 0 \\ &\delta_{rr} - \frac{2}{\alpha}\phi_r = 0 \\ &\frac{1}{\alpha^2}\beta + \phi_{rr} + \frac{1}{\alpha} e^{r-y}(2\delta_r + \delta) = 0. \end{aligned} \tag{I-6}$$

Solution of the first equation of the above set yields

$$\beta(r,y) = C_o e^{r-y}, \tag{I-7}$$

where C_o is function of the variable y.

The remaining equations in (I-6) are transformed to

$$(\rho,y)\quad \begin{aligned} &\gamma_{\rho\rho} + e^{2\rho}\gamma = 0 \\ &\theta_{\rho\rho} + e^{2\rho}\theta = 0 \\ &\delta_{\rho\rho} - \frac{4}{\alpha}\phi_\rho = 0 \\ &\frac{1}{4}\alpha^2 C_o e^{2\rho} + \frac{1}{4}\phi_{\rho\rho} + \frac{1}{4}\alpha e^{2\rho}(\delta_\rho + \delta) = 0, \end{aligned} \tag{I-8}$$

where

$$\rho = \frac{1}{2}(r - y) - \ln(\alpha/2) \tag{I-9}$$

and Eq. (I-7) has been used in the last equation of (I-8).

The solution to γ and θ is written as

$$\gamma(\rho,y) = C_1J_o + C_2Y_o \qquad \text{(I-10)}$$

$$\theta(\rho,y) = C_3J_o + C_4Y_o \ ,$$

where C_1, C_2, C_3, C_4 are functions of y and J_o, Y_o are Bessel and Hankel functions[27] with the argument e^{ρ}.

Combination of the last two equations in (I-8) gives

$$\Lambda_{\rho\rho\rho} + 4I\Lambda_{\rho} + 2I'\Lambda = 0 \ , \qquad \text{(I-11)}$$

with

$$\Lambda(\rho,y) = \delta(\rho,y) + \frac{1}{\alpha} C_o \qquad \text{(I-12)}$$

and

$$I(\rho) = e^{2\rho}. \qquad \text{(I-13)}$$

Thus

$$I' = 2e^{2\rho}. \qquad \text{(I-14)}$$

The solution to Eq. (I-11) reads[28]

$$\Lambda = C_6J_o^2 + C_7J_oY_o + C_8Y_o^2 \ , \qquad \text{(I-15)}$$

where C_6, C_7, C_8 are functions of y. Using (I-15) in (I-12) gives

$$\delta(\rho,y) = C_6J_o^2 + C_7J_oY_o + C_8Y_o - \frac{1}{\alpha} C_o. \qquad \text{(I-16)}$$

Integration of the third equation of (I-8) leads to

$$\phi(\rho,y) = \frac{\alpha}{2} e^{\rho}(C_6J_oJ_o' + C_7J_oY_o' + C_8Y_oY_o') + C_5 \ , \qquad \text{(I-17)}$$

where J_o' and Y_o' are the derivatives of these functions with respect to their arguments, C_5 is function of y and the Wronskian between J_o and Y_o has been used.

Going back to the original variables (t,k), the solutions to Eq. (I-2) read

$$\beta = C_o k$$

$$\gamma = C_1 J_o + C_2 Y_o \tag{I-18}$$

$$\theta = C_3 J_o + C_4 Y_o$$

$$\delta = C_6 J_o^2 + C_7 J_o Y_o + C_8 Y_o^2 - \frac{1}{\alpha} C_o$$

$$\phi = \sqrt{k}\ \{C_6 J_o J_o' + C_7 J_o Y_o' + C_8 Y_o Y_o'\} + C_5,$$

where the argument of J_o, Y_o and their derivatives is (I-19)

$$z = \frac{2}{\alpha} \sqrt{k}$$

and (I-20)

$$y = \frac{1}{2} (\alpha t - \ln k)$$

for those C_n's.

Then, Eq. (I-1) becomes

$$\zeta = \Omega_1 \alpha k x J_o + \Omega_2 \alpha k x Y_o + \Omega_5 \alpha k J_o^2 + \Omega_6 \alpha k Y_o^2$$

$$+ \Omega_7 \alpha k J_o Y_o$$

$$\eta = \Omega_1 x J_o + \Omega_2 x Y_o + \Omega_5 J_o^2 + \Omega_6 Y_o^2 + \Omega_7 J_o Y_o + \Omega_9$$

$$\xi = \Omega_1 \alpha x^2 k \overset{*}{J}_o + \Omega_2 \alpha x^2 k \overset{*}{Y}_o + \Omega_5 \alpha x k J_o \overset{*}{J}_o + \Omega_6 \alpha x k Y_o \overset{*}{Y}_o \tag{I-21}$$

$$+ \Omega_7 \alpha x k J_o \overset{*}{Y}_o + \Omega_3 J_o + \Omega_4 Y_o + \Omega_8 x,$$

where $\overset{*}{J}_o$, $\overset{*}{Y}_o$ are the derivatives with respect to k; i.e.

$$\overset{*}{J}_o = \frac{1}{\alpha\sqrt{k}} J_o' \tag{I-22}$$

and the Ω_n's have been relabeled to agree with Eq. (3.5-17). From Eq. (I-21), the generators U_n in Eq. (3.5-18) are immediately obtained. The extra factor α, which appears in some of them, is chosen for a proper convergence of generators when α approaches zero (see Chapter 5).

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