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Fusion rules for abelian extensions of Hopf algebras

Christopher Goff

We investigate the representation theory and fusion rules of a class of cocentral abelian (quasi-)Hopf extensions of Hopf algebras which includes twisted (generalized) quantum doubles of finite groups, and a certain quasi-Hopf algebra of Schauenburg associated to group-theoretical fusion categories. We then present a nontrivial example with noncommutative fusion rules.

1. Introduction

We present here a "ground-up" approach to attaining the fusion rules for a class of cocentral abelian extensions of Hopf algebras. Moreover, by not requiring strict coassociativity of the coproduct in the extension, our results are applicable not only to cocentral abelian (Hopf) extensions of Hopf algebras, but also to certain quasi-Hopf extensions as well. One such example, from [Schauenburg 2002], arises in the study of group-theoretical fusion categories (see also [Natale 2005]). (For a definition of group-theoretical fusion categories and basic properties, see [Etingof et al. 2005].) Another family of examples includes the twisted quantum double of a finite group, introduced in [Dijkgraaf et al. 1991], and the generalization which is defined in [Goff and Mason 2010].

In Section 2, we review definitions and notation, largely following [Kashina et al. 2002; Witherspoon 2004]. (For more information on extensions of Hopf algebras, consult [Andruskiewitsch 1996; Montgomery 1993], or, for quasi-Hopf extensions, [Masuoka 2002].) Then, Section 3 contains explicit formulas for irreducible characters and central idempotents for such extensions, as well as the inner product for which the irreducible characters form an orthonormal set. In Section 4, we write down the character of the tensor product representation and combine it with the inner product to deduce the fusion coefficients. The main result containing the fusion coefficients for irreducible representations, Theorem 4.5, is anticipated in [Witherspoon 2004] but is presented in this note without reference to Hochschild

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cohomology *per se*. Corollary 4.6 points out the connection to the K_0-ring of a group-theoretical fusion category. Then, in Section 5, we apply these formulas to a generalized twisted quantum double of a finite group [Goff and Mason 2010]. Indeed, our Section 5 supersedes [Goff and Mason 2010, Section 3]. Finally, Section 6 contains a nontrivial example of a cocentral abelian extension having noncommutative fusion rules.

2. Cocentral abelian extensions

We follow closely the notation of [Kashina et al. 2002] with a few exceptions. First, our action is a right action, consistent with [Andruskiewitsch and Natale 2003]. Second, our modules will be right modules rather than left.

Let L and G be finite groups and let $\mathbb{F}$ be an algebraically closed field of characteristic not dividing $|G||L|$. An abelian extension H is of the form

$$0 \to (\mathbb{F}G)^* \to H \to \mathbb{F}L \to 0,$$

where $H = (\mathbb{F}G)^*\#_\sigma^\tau \mathbb{F}L$, $\sigma : \mathbb{F}L \otimes \mathbb{F}L \to (\mathbb{F}G)^*$ is a group 2-cocycle, and $\tau : (\mathbb{F}G)^* \to \mathbb{F}L \otimes \mathbb{F}L$ is the dual of a group 2-cocycle. The condition on $\mathbb{F}$ assures that H is semisimple and cosemisimple. We specialize to a *cocentral* abelian extension, meaning that $(\mathbb{F}L)^* \subseteq Z(H^*)$, and thus that the coaction $\mathbb{F}L \to \mathbb{F}L \otimes (\mathbb{F}G)^*$ inherent in the extension is trivial. The cocentrality also has consequences for the tensor product structure on irreducible modules, as we will see in Section 4.

There is a right action of $\mathbb{F}L$ on $(\mathbb{F}G)^*$ which induces an action on $\mathbb{F}G$ via $(f \leftharpoonup \ell)(g) := f(g \leftharpoonup \ell^{-1})$ for $g \in G$, $\ell \in L$, and extended linearly. Since L acts as automorphisms of $(\mathbb{F}G)^*$, L permutes the idempotents of the dual basis. Thus, the action can be viewed as an action of L on G, also by automorphisms. For the basis $\{p_g \mid g \in G\}$, we have $p_g \leftharpoonup \ell = p_{g \leftharpoonup \ell}$. Moreover, let L_g be the stabilizer of g in L and $\mathbb{O}(g)$ the orbit of g under the action of L. That is,

$$L_g = \{\ell \in L \mid g \leftharpoonup \ell = g\} \quad \text{and} \quad \mathbb{O}(g) = \{g \leftharpoonup \ell \mid \ell \in L\}.$$

Let T_g be a complete set of right coset representatives for L_g in L. That is, $L = \bigcup_{y \in T_g} L_g y$. Note that $\mathbb{O}(g) = \{g \leftharpoonup y \mid y \in T_g\}$.

We can write σ and τ in terms of the dual basis via

$$\sigma(x, y) = \sum_{g \in G} \sigma_g(x, y) p_g \quad \text{and} \quad \tau(x) = \sum_{g,h \in G} \tau_{g,h}(x)(p_g \otimes p_h),$$

where $\sigma_g(x, y), \tau_{g,h}(x) \in \mathbb{F}$. There are many identities satisfied by σ and τ, such as

$$\sigma_{g \leftharpoonup z}(x, y)\sigma_g(z, xy) = \sigma_g(z, x)\sigma_g(zx, y) \tag{1}$$

and

$$\tau_{g,h}(x)\tau_{g\leftharpoonup x,h\leftharpoonup x}(y)\sigma_g(x,y)\sigma_h(x,y)=\tau_{g,h}(xy)\sigma_{gh}(x,y), \tag{2}$$

for all $g, h \in G$, $x, y, z \in L$.

Writing $p_g\#x$ as $p_g\bar{x}$, we can write multiplication in H as

$$p_k\bar{z}p_h\bar{y}=\delta_{k\leftharpoonup z,h}\sigma_k(z,y)p_k\overline{zy},$$

for all $h, k \in G$, $y, z \in L$. We also occasionally write p_g for $p_g\bar{1}$ and $\bar{x}$ for $\sum_g p_g\bar{x}$, whence $\bar{x}p_g=p_{g\leftharpoonup x^{-1}}\bar{x}$. The unit element is $\bar{1}$.

For a cocentral abelian extension, the comultiplication is

$$\Delta(p_g\bar{x})=\sum_{h\in G}\tau_{h,h^{-1}g}(x)p_h\bar{x}\otimes p_{h^{-1}g}\bar{x},$$

for all $g \in G$, $x \in L$. The counit ϵ satisfies $\epsilon(p_g\bar{x})=\delta_{g,1}$. Finally, the antipode S is given by

$$S(p_g\bar{x})=\sigma_{g^{-1}\leftharpoonup x}(x^{-1},x)^{-1}\tau_{g^{-1},g}(x)^{-1}p_{g^{-1}\leftharpoonup x}\overline{x^{-1}}.$$

Remark 2.1. For H to be a Hopf algebra, Δ must be coassociative, which implies a certain condition on τ. We require only quasicoassociativity, which implies the existence of other structures, and a related condition on τ. We omit these details here, as all of our examples are proved elsewhere [Dijkgraaf et al. 1991; Natale 2005; Andruskiewitsch 1996] to be either coassociative or quasicoassociative.

3. Modules and characters

Irreducible modules for H are induced from irreducible modules for the group algebra of L_g, but twisted by the 2-cocycle σ_g. Select one g from each orbit under the action of L, then select T_g, a set of right coset representatives. Let

$$H_g := (\mathbb{F}G)^*\#_\sigma\mathbb{F}L_g.$$

If V is a right projective σ_g-representation space for L_g, then $V\otimes p_g$ is a right H_g-module via

$$(v\otimes p_g)\cdot(p_h\bar{x})=\delta_{g,h}(v\cdot x\otimes p_g)$$

for all $v \in V$, $h \in G$, $x \in L_g$.

The irreducible modules for H are induced from these. Let $\widehat{V}=(V\otimes p_g)\otimes_{H_g}H$, which is then a right H-module under right multiplication by H. In other words,

$$\widehat{V}=\sum_{y\in T_g}(v\otimes p_g)\otimes\bar{y},$$

with action given by

$$\begin{aligned}[(v\otimes p_g)\otimes \bar{y}]\cdot p_h\bar{x} &= (v\otimes p_g)\sigma_{h\leftharpoondown y^{-1}}(y,x)p_{h\leftharpoondown y^{-1}}\overline{yx}\\ &= (v\otimes p_g)\delta_{g,h\leftharpoondown y^{-1}}\sigma_g(y,x)p_g\overline{wy'}\\ &= (v\otimes p_g)\delta_{g,h\leftharpoondown y^{-1}}\sigma_g(y,x)\sigma_g(w,y')^{-1}(p_g\bar{w})(\overline{y'})\\ &= \delta_{g,h\leftharpoondown y^{-1}}\frac{\sigma_g(y,x)}{\sigma_g(w,y')}[(v\cdot w\otimes p_g)\otimes\overline{y'}],\end{aligned}$$

where $w\in L_g$, $y'\in T_g$ are chosen so that $wy'=yx$.

We introduce the notation $V_{(g,\varphi)}$ to represent the H-module induced from the projective σ_g-representation of L_g that has character φ, and we let $\rho_{(g,\varphi)}$ be the representation of $V_{(g,\varphi)}$, and $\chi_{(g,\varphi)}$ its character. Then one calculates

$$\chi_{(g,\varphi)}(p_h\bar{x}) = \delta_{g\leftharpoondown y,h}\delta_{yxy^{-1}\in L_g}\frac{\sigma_g(y,x)}{\sigma_g(yxy^{-1},y)}\varphi(yxy^{-1}), \tag{3}$$

where y is the unique element of T_g that maps g to h. We reiterate that $V_{(g,\varphi)}$ is irreducible if and only if φ is.

Remark 3.1. This can be seen as

$$\chi_{(g,\varphi)}(p_h\bar{x}) = \delta_{g\leftharpoondown y,h}\delta_{x\in L_h}\varphi^{(y)}(x), \tag{4}$$

where $\varphi^{(y)}$ is a projective representation of $L_h=L_g^y$ (*conjugate* to φ) with cocycle $\sigma_{g\leftharpoondown y}=\sigma_h$. See [Costache 2009, Lemma 59] for a similar calculation.

Before writing down the central idempotents, we first note that the character χ_{reg} of the regular representation ρ_{reg} on H satisfies $\chi_{\text{reg}}(p_h\bar{x})=\delta_{x,1}|\mathbb{O}(h)||L_h|=\delta_{x,1}|L|$, and that, from the semisimplicity of H,

$$\rho_{\text{reg}} = \bigoplus_{(h,\psi)}\chi_{(h,\psi)}(1_H)\rho_{(h,\psi)},$$

where h ranges over the orbits and ψ ranges over the irreducible projective σ_h-representations of L_h. Let $z_{(g,\varphi)}$ denote the central idempotent corresponding to the representation $\rho_{(g,\varphi)}$. Then $\rho_{(h,\psi)}(z_{(g,\varphi)})=\delta_{g,h}\delta_{\varphi,\psi}(\dim\varphi)|L:L_g|\,\text{id}$.

Set $z_{(g,\varphi)}=\sum_{c\in G,\,d\in L}\alpha_{c,d}\;p_c\bar{d}$. We find the $\alpha_{c,d}$ by determining the value of the regular character on $S(p_{a^{-1}}\bar{b})z_{(g,\varphi)}$ two ways. First,

$$\begin{aligned}\chi_{\text{reg}}(S(p_{a^{-1}}\bar{b})z_{(g,\varphi)}) &= \sum_{c\in G,\,d\in L}\sigma_{a\leftharpoondown b}(b^{-1},b)^{-1}\tau_{a,a^{-1}}(b)^{-1}\alpha_{c,d}\chi_{\text{reg}}(p_{a\leftharpoondown b}\overline{b^{-1}}p_c\bar{d})\\ &= \tau_{a,a^{-1}}(b)^{-1}\alpha_{a,b}|L|.\end{aligned}$$

On the other hand, we have

$$\rho_{\text{reg}}(S(p_{a^{-1}}\bar{b})z_{(g,\varphi)}) = (\dim\varphi)|L:L_g|\rho_{(g,\varphi)}(S(p_{a^{-1}}\bar{b})),$$

which means

$$\chi_{\text{reg}}(S(p_{a^{-1}}\bar{b})z_{(g,\varphi)}) = (\dim\varphi)|L : L_g|\sigma_{a\leftharpoonup b}(b^{-1}, b)^{-1}\tau_{a,a^{-1}}(b)^{-1}\chi_{(g,\varphi)}(p_{a\leftharpoonup b}\overline{b^{-1}})$$

Solving for $\alpha_{a,b}$, we obtain

$$z_{(g,\varphi)} = \frac{(\dim\varphi)}{|L_g|}\sum_{a\in G,\, b\in L}\frac{1}{\sigma_{a\leftharpoonup b}(b^{-1}, b)}\chi_{(g,\varphi)}(p_{a\leftharpoonup b}\overline{b^{-1}})(p_a\bar{b}).$$

Simplifying somewhat using the delta functions within $\chi_{(g,\varphi)}$, we have:

Lemma 3.2. *The central idempotent of H corresponding to $V_{(g,\varphi)}$ is*

$$z_{(g,\varphi)} = \frac{(\dim\varphi)}{|L_g|}\sum_{a\in G}\sum_{b\in L_a}\frac{1}{\sigma_a(b^{-1}, b)}\chi_{(g,\varphi)}(p_a\overline{b^{-1}})(p_a\bar{b}). \qquad \square$$

Note that the first sum could be over $a \in \mathbb{O}(g)$, as $\chi = 0$ otherwise.

Proposition 3.3. *Letting*

$$\langle\alpha, \beta\rangle = \frac{1}{|L|}\sum_{a\in G}\sum_{b\in L_a}\frac{1}{\sigma_a(b^{-1}, b)}\alpha(p_a\overline{b^{-1}})\beta(p_a\bar{b}), \tag{5}$$

where α, β are characters of H, defines an inner product on the space of characters of H. The irreducible characters form an orthonormal basis with respect to this inner product.

We give three proofs to demonstrate the consistency with the character theory of projective representations of finite groups, and to demonstrate the relationship between certain conjugates of projective representations.

First proof. Clearly, (5) is linear in each component. The symmetry of (5) follows from (1) because $b \in L_a$. Using Lemma 3.2, we have

$$\begin{aligned}\left\langle\chi_{(g,\varphi)}, \chi_{(h,\psi)}\right\rangle &= \frac{1}{|L|}\sum_{a\in G}\sum_{b\in L_a}\frac{1}{\sigma_a(b^{-1}, b)}\chi_{(g,\varphi)}(p_a\overline{b^{-1}})\chi_{(h,\psi)}(p_a\bar{b})\\ &= \frac{1}{|L|}\chi_{(h,\psi)}\left(\frac{|L_g|}{\dim\varphi}z_{(g,\varphi)}\right)\\ &= \left(\frac{1}{|L|}\,\frac{|L_g|}{\dim\varphi}\right)(\dim\varphi)|L : L_g|\;\cdot\delta_{g,h}\delta_{\varphi,\psi} = \delta_{g,h}\delta_{\varphi,\psi}.\end{aligned} \qquad \square$$

Second proof. From (3), we obtain that $a \in \mathbb{O}(g) \cap \mathbb{O}(h)$ and thus $g = h$ or else the inner product is zero. Thus

$$\begin{aligned}
&\left\langle \chi_{(g,\varphi)}, \chi_{(h,\psi)} \right\rangle \\
&= \frac{\delta_{g,h}}{|L|} \sum_{\substack{a\in\mathbb{O}(g)\\ [a=g\leftharpoonup y]}} \sum_{b\in L_a} \frac{\sigma_g(y,b^{-1})\sigma_g(y,b)}{\sigma_a(b^{-1},b)\sigma_g(yb^{-1}y^{-1},y)\sigma_g(yby^{-1},y)} \varphi(yb^{-1}y^{-1})\psi(yby^{-1}) \\
&= \frac{\delta_{g,h}}{|L|} \sum_{\substack{a\in\mathbb{O}(g)\\ [a=g\leftharpoonup y]}} \sum_{b\in L_a} \frac{1}{\sigma_g(yb^{-1}y^{-1},yby^{-1})} \varphi(yb^{-1}y^{-1})\psi(yby^{-1})
\end{aligned}$$

by repeated application of (1). Hence

$$\left\langle \chi_{(g,\varphi)}, \chi_{(h,\psi)} \right\rangle = \frac{\delta_{g,h}}{|L_g|} \sum_{c\in L_g} \frac{1}{\sigma_g(c^{-1},c)} \varphi(c^{-1})\psi(c) = \delta_{g,h}\langle \varphi, \psi\rangle_{L_g} = \delta_{g,h}\delta_{\varphi,\psi}.$$

Here, $\langle\cdot,\cdot\rangle_{L_g}$ denotes the usual inner product for projective σ_g-representations of L_g. See [Nauwelaerts and Van Oystaeyen 1991, Proposition 2.8], for instance. □

Third proof. Using Remark 3.1,

$$\begin{aligned}
\left\langle \chi_{(g,\varphi)}, \chi_{(h,\psi)} \right\rangle &= \frac{\delta_{g,h}}{|L|} \sum_{\substack{a\in\mathbb{O}(g)\\ [a=g\leftharpoonup y]}} \sum_{b\in L_a} \frac{1}{\sigma_a(b^{-1},b)} \varphi^{(y)}(b^{-1})\psi^{(y)}(b) \\
&= \frac{\delta_{g,h}}{|L|} \sum_{a\in\mathbb{O}(g)} |L_a|\langle \varphi^{(y)}, \psi^{(y)}\rangle_{L_a} = \delta_{g,h}\delta_{\varphi,\psi}.
\end{aligned}$$

It is clear that $\varphi^{(y)} = \psi^{(y)}$ if and only if $\varphi = \psi$. □

4. Fusion rules

The character of the tensor product representation (via Δ) is

$$\begin{aligned}
&\chi_{(g,\varphi)\otimes(h,\psi)}(p_a\bar{b}) \\
&= \sum_{\substack{f\in G\\ [f\in\mathbb{O}(g),\ f^{-1}a\in\mathbb{O}(h)]\\ [f=g\leftharpoonup y,\ f^{-1}a=h\leftharpoonup w]}} \delta_{b\in L_f\cap L_{f^{-1}a}} \varphi(yby^{-1})\psi(wbw^{-1}) \frac{\tau_{f,f^{-1}a}(b)\sigma_g(y,b)\sigma_g(w,b)}{\sigma_g(yby^{-1},y)\sigma_g(wbw^{-1},b)} \\
&= \sum_{\substack{f\in G\\ [f\in\mathbb{O}(g),\ f^{-1}a\in\mathbb{O}(h)]\\ [f=g\leftharpoonup y,\ f^{-1}a=h\leftharpoonup w]}} \delta_{b\in L_f\cap L_{f^{-1}a}} \tau_{f,f^{-1}a}(b)\varphi^{(y)}(b)\psi^{(w)}(b) \\
&= \sum_{\substack{f\in G\\ [f\in\mathbb{O}(g),\ f^{-1}a\in\mathbb{O}(h)]\\ [f=g\leftharpoonup y,\ f^{-1}a=h\leftharpoonup w]}} \delta_{b\in L_f\cap L_{f^{-1}a}} \left[\varphi^{(y)}\otimes\psi^{(w)}\tau_{f,f^{-1}a}\right](b),
\end{aligned}$$

where $\left[\varphi^{(y)} \otimes \psi^{(w)} \tau_{f,f^{-1}a}\right]$ is a projective representation (of $L_f \cap L_{f^{-1}a} \leq L_a$) with cocycle σ_a. As explained in [Witherspoon 2004, (4.7)], the cocentrality of the extension, and the fact that the coproduct Δ is an algebra map, together imply that σ_a is cohomologous to $\sigma_f \cdot \sigma_{f^{-1}a}$ on $L_f \cap L_{f^{-1}a}$ via $\tau_{f,f^{-1}a}$. This is the content of Equation (2), which depends on the assumption of cocentrality.

Remark 4.1. If $h = 1$, then $\chi_{(g,\varphi)\otimes(1,\psi)} = \chi_{(g,\varphi\otimes\psi\downarrow_{L_g})}$. If $g = 1$ the result is similar. Hence, the irreducible representations induced from $1 \in G$ are in the center of the fusion algebra and their tensor products with other modules can be reduced to a calculation in the appropriate stabilizer. This generalizes a similar result in [Goff and Mason 2010].

We need two lemmas before calculating the fusion coefficients.

Lemma 4.2. *Let a, $f \in G$, $y \in L$.*

(1) *Let α and β be projective σ_f-representations of L_f. Then*

$$\langle \alpha, \beta \rangle_{L_f} = \langle \alpha^{(y)}, \beta^{(y)} \rangle_{L_f^y}.$$

Note that $\alpha^{(y)}$ and $\beta^{(y)}$ are $\sigma_{f \leftharpoonup y}$-representations of $L_{f \leftharpoonup y} = L_f^y$.

(2) *Let α be a σ_f-representation of L_f and let β be a $\sigma_{f^{-1}a}$-representation of $L_{f^{-1}a}$. Then*

$$\left[\alpha \otimes \beta \tau_{f,f^{-1}a}\right]^{(y)} = \left[\alpha^{(y)} \otimes \beta^{(y)} \tau_{f \leftharpoonup y, f^{-1}a \leftharpoonup y}\right]$$

as $\sigma_{a \leftharpoonup y}$-representations of $L_f^y \cap L_{f^{-1}a}^y \leq L_a^y$.

Proof. The proof is straightforward, using (4), (1), and (2). □

We need a way to calculate products of L-orbits in $\mathbb{C}G$. The following formula appears in [Witherspoon 2004, Proof of Theorem 4.8], where the author relies on standard trace map properties of the L-algebra $\mathbb{Z}G$, citing general results of [Thévenaz 1995]. Our proof is specific to group actions on sets. Recall that if L acts on G, then L also acts on $G \times G$ *diagonally*: $(g_1, g_2) \leftharpoonup \ell = (g_1 \leftharpoonup \ell, g_2 \leftharpoonup \ell)$ for $\ell \in L$, $g_1, g_2 \in G$.

Lemma 4.3. *Let $g, h \in G$. Then*

$$\mathbb{O}(g)\mathbb{O}(h) = \sum_{x \in D} |L_{(g \leftharpoonup x)h} : L_{g \leftharpoonup x} \cap L_h| \mathbb{O}((g \leftharpoonup x)h),$$

where D is a complete set of $L_g \backslash L / L_h$ double coset representatives.

Proof. Consider the orbits of the diagonal action of L on $G \times G$. Evidently, $y \in L_g x L_h$ if and only if $\mathbb{O}_L((g \leftharpoonup x, h)) = \mathbb{O}_L((g \leftharpoonup y, h))$. Now pick $x \in D$ and consider the image of $\mathbb{O}_L((g \leftharpoonup x, h))$ in G under the product map. Clearly, the product $(g \leftharpoonup x)h$ is fixed by $L_{(g \leftharpoonup x)h}$ but also each component is fixed by

$L_g^x \cap L_h \le L_{(g \leftharpoondown x)h}$. So, the number of distinct ordered pairs $(g \leftharpoondown xw, h \leftharpoondown w)$ such that $(g \leftharpoondown xw)(h \leftharpoondown w) = (g \leftharpoondown x)h$ is $|L_{(g \leftharpoondown x)h} : L_g^x \cap L_h|$. Since L acts by automorphisms, this is also the number of times $\mathbb{O}((g \leftharpoondown x)h)$ appears in this term of the sum. □

Remark 4.4. The right hand side in Lemma 4.3 cannot generally be interpreted as a summation over distinct orbits. There may be $y \notin L_g x L_h$ for which $\mathbb{O}((g \leftharpoondown x)h) = \mathbb{O}((g \leftharpoondown y)h)$.

Anticipated in [Witherspoon 2004, Theorem 4.8], the following theorem gives the fusion coefficients for irreducible representations of H.

Theorem 4.5. *Let $g, h, k \in G$ and let φ be a σ_g-representation of L_g, ψ a σ_h-representation of L_h, and γ a σ_k-representation of L_k and consider the corresponding induced modules of H. Then*

$$\langle \chi_{(k,\gamma)}, \chi_{(g,\varphi)\otimes(h,\psi)} \rangle = \sum_{\substack{x \in D \\ (g \leftharpoondown x)h \in \mathbb{O}(k) \\ [(g \leftharpoondown xw')(h \leftharpoondown w') = k]}} \left\langle \gamma, \left[\varphi^{(xw')} \otimes \psi^{(w')} \tau_{g \leftharpoondown xw', h \leftharpoondown w'}\right]\right\rangle_{L_g^{xw'} \cap L_h^{w'}}$$

where D is a set of those $L_g \backslash L / L_h$ double coset representatives x satisfying

$$(g \leftharpoondown x)h \in \mathbb{O}(k),$$

and the inner product on $L_g^{xw'} \cap L_h^{w'} \le L_k$ is of projective σ_k-representations.

Proof. Using the inner product (5), we have

$$\begin{aligned}
&\langle \chi_{(k,\gamma)}, \chi_{(g,\varphi)\otimes(h,\psi)} \rangle \\
&= \frac{1}{|L|} \sum_{\substack{a \in \mathbb{O}(k) \\ [a = k \leftharpoondown z]}} \sum_{\substack{f \in \mathbb{O}(g) \\ f^{-1}a \in \mathbb{O}(h) \\ [f = g \leftharpoondown y] \\ [f^{-1}a = h \leftharpoondown w]}} \sum_{b \in L_a \cap L_f} \gamma(zb^{-1}z^{-1})\varphi(yby^{-1})\psi(wbw^{-1}) \\
&\qquad \cdot \frac{\tau_{f,f^{-1}a}(b)\sigma_k(z,b^{-1})\sigma_g(y,b)\sigma_h(w,b)}{\sigma_a(b^{-1},b)\sigma_k(zb^{-1}z^{-1},z)\sigma_g(yby^{-1},y)\sigma_h(wbw^{-1},w)} \\
&= \frac{1}{|L|} \sum_{\substack{a \in \mathbb{O}(k) \\ [a = k \leftharpoondown z]}} \sum_{\substack{f \in \mathbb{O}(g) \\ f^{-1}a \in \mathbb{O}(h) \\ [f = g \leftharpoondown y] \\ [f^{-1}a = h \leftharpoondown w]}} \sum_{b \in L_a \cap L_f} \frac{\tau_{f,f^{-1}a}(b)}{\sigma_a(b^{-1},b)} \gamma^{(z)}(b^{-1})\varphi^{(y)}(b)\psi^{(w)}(b) \\
&= \frac{1}{|L|} \sum_{\substack{a \in \mathbb{O}(k) \\ [a = k \leftharpoondown z]}} \sum_{\substack{f \in \mathbb{O}(g) \\ f^{-1}a \in \mathbb{O}(h) \\ [f = g \leftharpoondown y] \\ [f^{-1}a = h \leftharpoondown w]}} |L_f \cap L_{f^{-1}a}| \left\langle \gamma^{(z)}, \left[\varphi^{(y)} \otimes \psi^{(w)} \tau_{f,f^{-1}a}\right]\right\rangle_{L_f \cap L_{f^{-1}a}}.
\end{aligned}$$

By Lemma 4.2 this is equal to

$$= \frac{1}{|L|} \sum_{\substack{a\in\mathbb{O}(k)\\ [a=k\leftharpoonup z]}} \sum_{\substack{f\in\mathbb{O}(g)\\ f^{-1}a\in\mathbb{O}(h)\\ [f=g\leftharpoonup y]\\ [f^{-1}a=h\leftharpoonup w]}} |L_f \cap L_{f^{-1}a}| \cdot \big\langle \gamma, \big[\varphi^{(yz^{-1})} \otimes \psi^{(wz^{-1})} \tau_{f\leftharpoonup z^{-1}, f^{-1}a\leftharpoonup z^{-1}}\big]\big\rangle_{L_f^{z^{-1}} \cap L_{f^{-1}a}^{z^{-1}}}$$

$$= \frac{1}{|L_k|} \sum_{\substack{f\in\mathbb{O}(g)\\ f^{-1}k\in\mathbb{O}(h)\\ [f=g\leftharpoonup y']\\ [f^{-1}k=h\leftharpoonup w']}} |L_f \cap L_{f^{-1}k}| \big\langle \gamma, \big[\varphi^{(y')} \otimes \psi^{(w')} \tau_{f, f^{-1}k}\big]\big\rangle_{L_f \cap L_{f^{-1}k}},$$

and by Lemma 4.3 this can further be written as

$$= \frac{1}{|L_k|} \sum_{\substack{x\in D\\ (g\leftharpoonup x)h\in\mathbb{O}(k)\\ [(g\leftharpoonup xw')(h\leftharpoonup w')=k]}} \big|L_g^{xw'} \cap L_h^{w'}\big| \, \big|L_{(g\leftharpoonup x)h} : L_g^x \cap L_h\big| \cdot \big\langle \gamma, \big[\varphi^{(xw')} \otimes \psi^{(w')} \tau_{f, f^{-1}k}\big]\big\rangle_{L_g^{xw'} \cap L_h^{w'}}$$

$$= \sum_{\substack{x\in D\\ (g\leftharpoonup x)h\in\mathbb{O}(k)\\ [(g\leftharpoonup xw')(h\leftharpoonup w')=k]}} \big\langle \gamma, \big[\varphi^{(xw')} \otimes \psi^{(w')} \tau_{g\leftharpoonup xw', h\leftharpoonup w'}\big]\big\rangle_{L_g^{xw'} \cap L_h^{w'}},$$

where D is a set of $L_g \backslash L / L_h$ double coset representatives with $(g \leftharpoonup x)h \in \mathbb{O}(k)$. Thus, the fusion rules for H modules can be determined from the fusion rules for projective σ_k-representations restricted to certain subgroups of L_k. □

As stated before, the theorem holds for certain quasi-Hopf extensions, including the examples in the following corollary and the next section.

Corollary 4.6. *The fusion rules in Theorem 4.5 describe the* K_0*-ring for the group-theoretical module category* $\mathscr{C}(G \rtimes L, \omega, L, 1)$, *where* $\omega \in H^3(G \rtimes L, \mathbb{F}^*)$ *is the* 3*-cocycle associated to* $[\sigma, \tau]$ *in the relevant Kac exact sequence. See* [Schauenburg 2002; Natale 2003; Masuoka 2002] *for further cohomological details.*

Proof. Indeed, the theorem holds whenever the structure maps and (1) and (2) hold, even if H is a quasi-Hopf algebra (with coassociator Φ), because the fusion rules for H do not depend on the associativity constraint (determined by Φ) in the category of right H-modules, Mod-H. Thus these fusion rules hold for a certain quasi-Hopf algebra of Schauenburg, denoted (A^{op}, Φ) by Natale [2005], in the case when $A = (\mathbb{F}G)^* \#_\sigma^\tau \mathbb{F}L$, and the left action $\triangleright$ of G on L is trivial; i.e., when $GL = G \rtimes L$. (In this case, the structure maps and cocycles are exactly as in Section 2.) Natale, in the proof of her Theorem 4.4, cites [Schauenburg 2002] to demonstrate that (A^{op}, Φ)-Mod is tensor-equivalent to $\mathscr{C}(G \rtimes L, \omega, L, 1)$, where $\omega \in H^3(G \rtimes L, \mathbb{F}^*)$ is the 3-cocycle associated to $[\sigma, \tau]$ in the Kac exact sequence. □

5. Example: generalized twisted quantum doubles of finite groups

Other examples of abelian extensions satisfying the structure maps of Section 2 (and hence having fusion rules determined by Theorem 4.5) include twisted quantum doubles of finite groups [Dijkgraaf et al. 1991] and generalized twisted doubles of finite groups [Goff and Mason 2010]. We expand on the latter, but using right modules here. As mentioned earlier, this section supersedes [Goff and Mason 2010, Section 3].

Let G be a finite group, N a normal subgroup, and $\overline{G} := G/N$. We use the bar notation for elements in $\bar{G}$, i.e., if $g \in G$ then $\bar{g} = gN \in \bar{G}$. Then G acts naturally on $\overline{G}$ via conjugation, namely $\bar{g} \leftharpoonup x := x^{-1}\bar{g}x = \bar{g}^x = \overline{g^x} = \bar{x}^{-1}\bar{g}\bar{x}$, for all $x \in G$, $\bar{g} \in \bar{G}$.

In addition, let $\omega \in H^3(\overline{G}, \mathbb{F}^*)$, and let $\omega' := \mathrm{Infl}_{\overline{G}}^{G}\,\omega$. In analogy with σ and τ, define $\theta : \mathbb{F}G \otimes \mathbb{F}G \to \mathbb{F}\bar{G}^*$ and $\gamma : \mathbb{F}\bar{G}^* \to \mathbb{F}G \otimes \mathbb{F}G$ via

$$\theta = \sum_{\bar{g}\in\bar{G}} \theta_{\bar{g}} \;\text{ and }\; \gamma = \sum_{x,y\in\bar{G}} \gamma_{()}(x, y),$$

where

$$\theta_{\bar{g}}(x, y) = \frac{\omega(\bar{g}, \bar{x}, \bar{y})\omega(\bar{x}, \bar{y}, \bar{g}^{xy})}{\omega(\bar{x}, \bar{g}^{x}, \bar{y})}, \quad \gamma_{\bar{g}}(x, y) = \frac{\omega(\bar{x}, \bar{y}, \bar{g})\omega(\bar{g}, \bar{x}^{g}, \bar{y}^{g})}{\omega(\bar{x}, \bar{g}, \bar{y}^{g})}.$$

Notice that $\theta_{\bar{g}}$ and $\gamma_{\bar{g}}$ could be thought of as functions from $\mathbb{F}\bar{G} \otimes \mathbb{F}\bar{G}$ to $\mathbb{F}^*$ since they pass to the quotient $\bar{G}$. The generalized twisted double is then $D^{\omega}(G, \overline{G}) = (\mathbb{F}\overline{G})^* \#_{\theta}^{\gamma} (\mathbb{F}G)$. The maps θ and γ satisfy (1) and (2), *mutatis mutandis* [Dijkgraaf et al. 1991].

The irreducible (right) modules of $D^{\omega}(G, \bar{G})$ are induced from irreducible projective representations of centralizers. In particular, the character of the irreducible projective $\theta_{\bar{g}}$-representation φ of $C_G(\bar{g})$ is given by

$$\begin{aligned}\widehat{\chi}_{(\bar{g},\varphi)}(e(\bar{h}) \bowtie x) &= \delta_{\bar{g}^{y},\bar{h}}\delta_{yxy^{-1}\in C_G(\bar{g})} \frac{\theta_{\bar{g}}(y, x)}{\theta_{\bar{g}}(yxy^{-1}, y)}\varphi(yxy^{-1}) \\ &= \delta_{\bar{g}^{y},\bar{h}}\delta_{x\in C_G(\bar{h})}\varphi^{(y)}(x).\end{aligned}$$

Consistent with (5), the inner product on characters is given by

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{\bar{k}\in\bar{G}} \sum_{x\in C_G(\bar{k})} \frac{1}{\theta_{\bar{k}}(\bar{x}^{-1}, \bar{x})} \alpha(e(\bar{k}) \bowtie x^{-1})\beta(e(\bar{k}) \bowtie x),$$

and thus the fusion coefficients are given by

$$\big\langle \widehat{\chi}_{(\bar{k},\lambda)}, \widehat{\chi}_{(\bar{g},\varphi)\otimes(\bar{h},\psi)} \big\rangle = \sum_{\substack{x\in D \\ [\bar{g}^{xw'}\bar{h}^{w'}=\bar{k}]}} \big\langle \lambda, \big[\varphi^{(xw')} \otimes \psi^{(w')}\gamma_{()}(\bar{g}^{xw'}, \bar{h}^{w'})\big]\big\rangle_{C_G(\bar{g}^{xw'})\cap C_G(\bar{h}^{w'})}$$

where D is a set of $C_G(\bar{g})\backslash G/C_G(\bar{h})$ double coset representatives with $\bar{g}^x\bar{h} \in \mathbb{O}(\bar{k})$, and the inner product on $C_G(\bar{g}^{xw'}) \cap C_G(\bar{h}^{w'}) \leq C_G(\bar{k})$ is of $\theta_{\bar{k}}$-representations.

6. Example: noncommutative fusion rules

Noncommutative fusion rules for cocentral abelian extensions are not rare: choose $L = 1$, σ and τ trivial, and any nonabelian G, for instance. Also, see [Kosaki et al. 1997; Nikshych 1998; Zhu 2001]. We give here an example with σ, τ trivial, but nontrivial action of L. Let G be the dihedral group of order 18, and let

$$L \leq \operatorname{Aut} D_9 \cong \mathbb{Z}_9 \rtimes \mathbb{Z}_9^*.$$

Namely, $L = \langle 3\rangle \times \langle 4\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. [The first factor of L is with respect to addition, the second, multiplication.] If we let $G = \langle x, y \mid x^2 = y^9 = e,\ yx = xy^{-1}\rangle$, then

$$(x^c y^d) \leftharpoonup (a, b) := x^c y^{ac+bd}.$$

We choose L-orbit representatives $S = \{e, y^3, y^6, y, y^2, x, xy, xy^2\}$ with their respective stabilizers. Since L is abelian, $\chi^{(\ell)}_{(g,\varphi)} = \chi_{(g,\varphi)}$ for all $\ell \in L$. Note that in the decomposition of the product of orbits, we have

$$\mathbb{O}(x)\mathbb{O}(y) = 3\mathbb{O}(xy) \ \text{ and } \ \mathbb{O}(y)\mathbb{O}(x) = 3\mathbb{O}(xy^2),$$

which suffices to guarantee noncommutative fusion rules.

Theorem 6.1. *Let $M(g, \alpha)$ denote the irreducible representation of $(\mathbb{F}G)^*\#(\mathbb{F}L)$ induced from α, an irreducible representation of L_g for $g \in S$. The first five rules are commutative.*

i. $M(s, \alpha) \otimes M(t, \beta) = M(st, \alpha \otimes \beta)$ *for* $s, t \in \langle y^3\rangle$.

ii. $M(s, \alpha) \otimes M(g, \beta) = M(g, \alpha\downarrow_{L_g} \otimes \beta)$ *for* $s \in \langle y^3\rangle$, $g \in \{y, y^2, x, xy, xy^2\}$.

iii. $M(g, \alpha) \otimes M(g, \beta) = 3M(h, \alpha \otimes \beta)$ *if* $\{g, h\} = \{y, y^2\}$.

iv. $$M(y, \alpha) \otimes M(y^2, \beta) = \bigoplus_{s\in\langle y^3\rangle} \bigoplus_{\gamma\downarrow_{L_y}=\alpha\otimes\beta} M(s, \gamma).$$

v. $$M(g, \alpha) \otimes M(g, \beta) = \bigoplus_{s\in\langle y^3\rangle} \bigoplus_{\gamma\downarrow_{L_g}=\alpha\otimes\beta} M(s, \gamma) \text{ for } g \in \{x, xy, xy^2\}.$$

The rest of the list holds for all $\alpha, \beta, \delta, \epsilon, \zeta, \eta, \mu, \nu$.

vi. $$\begin{aligned} M(y, \alpha) \otimes M(x, \beta) &= \bigoplus_{\text{all }\gamma} M(xy^2, \gamma) = M(x, \delta) \otimes M(y^2, \epsilon) \\ &= M(y^2, \zeta) \otimes M(xy, \eta) = M(xy, \mu) \otimes M(y, \nu). \end{aligned}$$

vii. $$M(y,\alpha)\otimes M(xy,\beta)=\bigoplus_{\text{all }\gamma} M(x,\gamma)=M(xy,\delta)\otimes M(y^2,\epsilon)$$
$$=M(y^2,\zeta)\otimes M(xy^2,\eta)=M(xy^2,\mu)\otimes M(y,\nu).$$

viii. $$M(y,\alpha)\otimes M(xy^2,\beta)=\bigoplus_{\text{all }\gamma} M(xy,\gamma)=M(xy^2,\delta)\otimes M(y^2,\epsilon)$$
$$=M(y^2,\zeta)\otimes M(x,\eta)=M(x,\mu)\otimes M(y,\nu).$$

ix. $$M(x,\alpha)\otimes M(xy,\beta)=\bigoplus_{\text{all }\gamma} M(y,\gamma)=M(xy,\delta)\otimes M(xy^2,\epsilon)$$
$$=M(xy^2,\zeta)\otimes M(x,\eta).$$

x. $$M(x,\alpha)\otimes M(xy^2,\beta)=\bigoplus_{\text{all }\gamma} M(y^2,\gamma)=M(xy^2,\delta)\otimes M(xy,\epsilon)$$
$$=M(xy,\zeta)\otimes M(x,\eta).$$

Proof. Straightforward. □

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