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Estimating the parameters of a generalized lambda distribution

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Abstract

The method of moments is a popular technique for estimating the parameters of a generalized lambda distribution (GLD), but published results suggest that the percentile method gives superior results. However, the percentile method cannot be implemented in an automatic fashion, and automatic methods, like the *starship method*, can lead to prohibitive execution time with large sample sizes. A new estimation method is proposed that is automatic (it does not require the use of special tables or graphs), and it reduces the computational time. Based partly on the usual percentile method, this new method also requires choosing which quantile u to use when fitting a GLD to data. The choice for u is studied and it is found that the best choice depends on the final goal of the modeling process. The sampling distribution of the new estimator is studied and compared to the sampling distribution of estimators that have been proposed. Naturally, all estimators are biased and here it is found that the bias becomes negligible with sample sizes $n \geqslant 2 \times 10^3$. The .025 and .975 quantiles of the sampling distribution are investigated, and the difference between these quantiles is found to decrease proportionally to $1/\sqrt{n}$. The same results hold for the moment and percentile estimates. Finally, the influence of the sample size is studied when a normal distribution is modeled by a GLD. Both bounded and unbounded GLDs are used and the bounded GLD turns out to be the most accurate. Indeed it is shown that, up to $n = 10^6$, bounded GLD modeling cannot be rejected by usual goodness-of-fit tests.

Keywords: GLD; Estimating distributions; Goodness-of-fit; Simplex; Percentiles

1. Introduction

The family of four-parameter generalized lambda distributions (GLD) is known for its high flexibility. It provides an accurate approximation of most of the usual statistical distributions (e.g. Gaussian, uniform, lognormal, Weibull) as illustrated in Fig. 1 and detailed by Karian and Dudewicz (2000). GLD are used in many fields where precise data modeling is required such as finance (Tarsitano, 2004; Corrado, 2001), corrosion (Najjar et al., 2003), meteorology (Öztürk and Dale, 1982), fatigue of materials (Bigerelle et al., 2005), independent component analysis (Eriksson
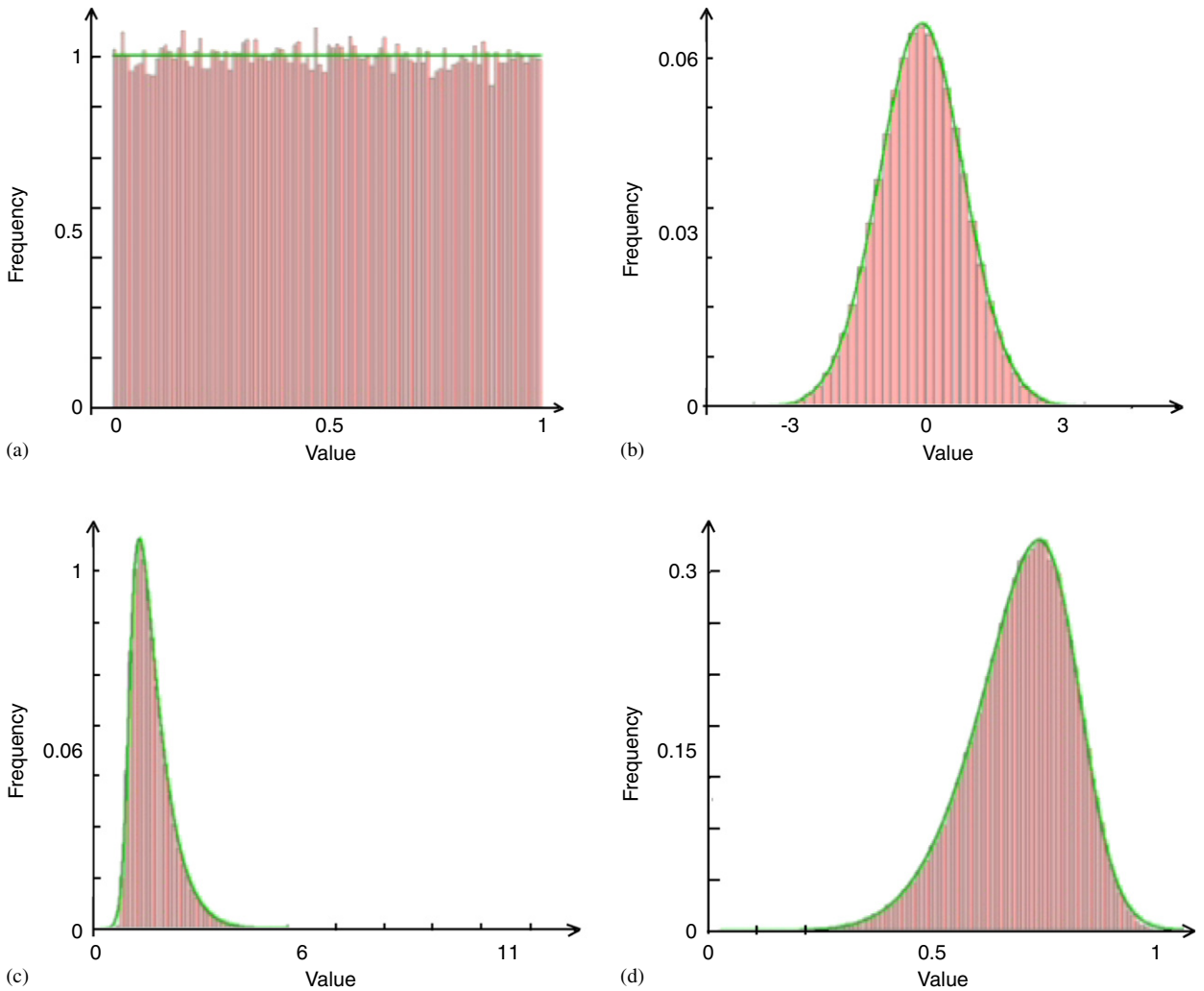

Fig. 1. Generalized lambda distribution modeling results obtained by using the LambdaFinder software (Bigerelle et al., 2005) in the cases of (a) a uniform distribution, (b) a standard Gaussian distribution, (c) a lognormal distribution and (d) a right skewed Weibull distribution.

et al., 2000; Karvanen et al., 2002), statistical process control (Pal, 2005; Fournier et al., 2006; Negiz and Cinar, 1997), simulation of queue systems (Dengiz, 1988) or for generating random numbers (Stengos and Wu, 2006; Wilcox, 2002).

GLDs are often used to model empirical data and several methods for estimating its parameters are available. Generally, these estimators use complex minimization techniques. Therefore, except for the results of King and MacGillivray (1999), which are limited to sample sizes up to 200, there is little or no information about the sampling distribution of any estimator that might be used. Several estimates of the parameters of a GLD have been devised (Karian and Dudewicz, 2000; King and MacGillivray, 1999; Lakhany and Mausser, 2000; Ramberg et al., 1979; Öztürk and Dale, 1985): the method of moments, the method of percentiles, the "least-square" method, and goodness-of-fit methods. A thorough study of the method of moments and the method of percentiles is provided in Karian and Dudewicz (2003), where it is concluded that the percentile method performs best over a wide range of situations. However, the method cannot be completely automated. Automatic methods based on goodness of fit become impractical with large sample sizes, due to high execution time. There are practical concerns about alternative estimators that are reviewed in the paper, and so a major goal in this paper is to suggest an alternative estimator and then compare it to extant techniques.

2. The GLD

Lambda distributions were introduced by Tukey (1962), which were subsequently generalized to GLDs, a family of four-parameter statistical distributions (Filliben, 1969, 1975; Joiner and Rosenblatt, 1971; Ramberg and Schmeiser, 1972, 1974). This section summarizes its central features, reviews extant estimation techniques, and then introduces a new estimation method.

2.1. Definitions

Here, GLDs are defined and some of their properties are briefly reviewed. For more details, see Karian and Dudewicz (2000). A Generalized Lambda Distribution, denoted by $GLD(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, can be described in terms of a percentile function, Q, the inverse of the cdf (cumulative distribution function), F:

$$Q(y) = Q(y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 + \frac{y^{\lambda_3} - (1-y)^{\lambda_4}}{\lambda_2}, \tag{1}$$

where $y \in [0, 1]$, and where λ_1 and λ_2 are, respectively, location, scale parameters, and λ_3 and λ_4 are, respectively, related to skewness and the kurtosis. The probability density function (pdf) of the $GLD(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is given by

$$f(x) = f(Q(y)) = \frac{\lambda_2}{\lambda_3 y^{\lambda_3 - 1} + \lambda_4 (1-y)^{\lambda_4 - 1}}. \tag{2}$$

Eq. (2) defines a pdf if and only if f meets the following conditions:

$$\forall x \in D \ f(x) \geqslant 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) \, dx = 1 \tag{3}$$

with D the domain of definition of f. These conditions lead to the specification of six regions of the (λ_3, λ_4) space in which Eq. (2) defines a valid pdf (Karian and Dudewicz, 2000):

$$\text{Region 1} = \{(\lambda_3, \lambda_4) \mid \lambda_3 \leqslant -1, \lambda_4 \geqslant 1\}, \tag{4}$$

$$\text{Region 2} = \{(\lambda_3, \lambda_4) \mid \lambda_3 \geqslant 1, \lambda_4 \leqslant -1\}, \tag{5}$$

$$\text{Region 3} = \{(\lambda_3, \lambda_4) \mid \lambda_3 \geqslant 0, \lambda_4 \geqslant 0\}, \tag{6}$$

$$\text{Region 4} = \{(\lambda_3, \lambda_4) \mid \lambda_3 \leqslant 0, \lambda_4 \leqslant 0\}, \tag{7}$$

$$\text{Region 5} = \left\{ (\lambda_3, \lambda_4) \mid -1 < \lambda_3 < 0, \lambda_4 > 1, \frac{(1-\lambda_3)^{1-\lambda_3}}{(\lambda_4 - \lambda_3)^{\lambda_4 - \lambda_3}} (\lambda_4 - 1)^{\lambda_4 - 1} < \frac{-\lambda_3}{\lambda_4} \right\} \tag{8}$$

and

$$\text{Region 6} = \left\{ (\lambda_3, \lambda_4) \mid \lambda_3 > 1, -1 < \lambda_4 < 0, \frac{(1-\lambda_4)^{1-\lambda_4}}{(\lambda_3 - \lambda_4)^{\lambda_3 - \lambda_4}} (\lambda_3 - 1)^{\lambda_3 - 1} < \frac{-\lambda_4}{\lambda_3} \right\} \tag{9}$$

in addition, the GLD defined by a set of parameters included in these six valid regions, can exhibit either bounded (they are defined on a finite support $[a, b]$) or unbounded densities either on both sides (they are defined on an infinite support $]-\infty, +\infty[$) or on one side (they are defined on $[a, +\infty[$ or $]-\infty, b]$). These latter cases will not be tackled in the present paper. These cases and the associated restrictions on the parameters are summarized in Table 1.

2.2. Existing parameter estimation methods

As previously noted, several estimates of the parameters of a GLD have been devised. A thorough study of the method of moments and the method of percentiles is provided in Karian and Dudewicz (2003). Using a L^2 Norm the authors conclude that the *method of percentiles is superior to the method of moments over a broad range of* (α_3, α_4)-*space*

Table 1
Definition domains of the GLD for various combinations of λ_3 and λ_4 (from Karian and Dudewicz, 2000)

λ_3	λ_4	Support
$\lambda_3 < -1$	$\lambda_4 > 1$	$]-\infty, \lambda_1 + 1/\lambda_2]$
$\lambda_3 > 1$	$\lambda_4 < -1$	$[\lambda_1 - 1/\lambda_2, \infty[$
$-1 < \lambda_3 < 0$	$\lambda_4 > 1$	$]-\infty, \lambda_1 + 1/\lambda_2]$
$\lambda_3 > 1$	$-1 < \lambda_4 < 0$	$[\lambda_1 - 1/\lambda_2, \infty[$
$\lambda_3 > 0$	$\lambda_4 > 0$	$[\lambda_1 - 1/\lambda_2, \lambda_1 + 1/\lambda_2]$
$\lambda_3 > 0$	$\lambda_4 = 0$	$[\lambda_1 - 1/\lambda_2, \lambda_1]$
$\lambda_3 = 0$	$\lambda_4 > 0$	$[\lambda_1, \lambda_1 + 1/\lambda_2]$
$\lambda_3 < 0$	$\lambda_4 < 0$	$]-\infty, \infty[$
$\lambda_3 < 0$	$\lambda_4 = 0$	$]-\infty, \lambda_1]$
$\lambda_3 = 0$	$\lambda_4 < 0$	$[\lambda_1, \infty[$

(with α_3 and α_4 being, respectively, the third and fourth moments). Because the percentile method plays a role in the new estimator proposed here, the details of the method are briefly reviewed.

Let $\hat{\pi}_p$ be an estimate of the pth quantile. Estimates of the GLD parameters are based on

$$\hat{\rho}_1 = \hat{\pi}_{0.5}, \tag{10}$$

$$\hat{\rho}_2 = \hat{\pi}_{1-u} - \hat{\pi}_u, \tag{11}$$

$$\hat{\rho}_3 = \frac{\hat{\pi}_{0.5} - \hat{\pi}_u}{\hat{\pi}_{1-u} - \hat{\pi}_{0.5}} \tag{12}$$

and

$$\hat{\rho}_4 = \frac{\hat{\pi}_{0.75} - \hat{\pi}_{0.25}}{\hat{\rho}_2}, \tag{13}$$

where u is chosen by the investigator such that $u \in]0, 0.25[$.

Let

$$\rho_1 = Q\left(\frac{1}{2}\right) = \lambda_1 + \frac{\left(\frac{1}{2}\right)^{\lambda_3} - \left(\frac{1}{2}\right)^{\lambda_4}}{\lambda_2}, \tag{14}$$

$$\rho_2 = Q(1-u) - Q(u) = \frac{(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}}{\lambda_2}, \tag{15}$$

$$\rho_3 = \frac{Q\left(\frac{1}{2}\right) - Q(u)}{Q(1-u) - Q\left(\frac{1}{2}\right)} = \frac{(1-u)^{\lambda_4} - u^{\lambda_3} + \left(\frac{1}{2}\right)^{\lambda_3} - \left(\frac{1}{2}\right)^{\lambda_4}}{(1-u)^{\lambda_3} - u^{\lambda_4} + \left(\frac{1}{2}\right)^{\lambda_4} - \left(\frac{1}{2}\right)^{\lambda_3}} \tag{16}$$

and

$$\rho_4 = \frac{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}{\rho_2} = \frac{\left(\frac{3}{4}\right)^{\lambda_3} - \left(\frac{1}{4}\right)^{\lambda_4} + \left(\frac{3}{4}\right)^{\lambda_4} - \left(\frac{1}{4}\right)^{\lambda_3}}{(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}}. \tag{17}$$

Then estimates of the GLD parameters are obtained by solving the set of equations

$$\hat{\rho}_i = \rho_i \quad \text{for } i = 1, 2, 3, 4. \tag{18}$$

Naturally, the choice of u affects the estimates of the GLD parameters (Karian and Dudewicz, 2000). Moreover, for some values of u, there may not exist any acceptable estimates of the GLD parameters (in terms of a goodness-of-fit

test). For results on how the choice of u affects the standard error of the statistics $\hat{\rho}_i$ ($i \in [2, 4]$) see Gibbons and Chakraborti (2003). The choice of u affects the new estimator proposed here and is studied in Section 3.1.

Note that $\hat{\rho}_3 = \rho_3$ and $\hat{\rho}_4 = \rho_4$ involve only λ_3 and λ_4. Therefore, this subsystem can first be solved, and then the estimates of λ_3 and λ_4 can be used to estimate λ_2 and λ_1 from

$$\lambda_2 = \frac{(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}}{\hat{\rho}_2}, \tag{19}$$

$$\lambda_1 = \hat{\rho}_1 - \frac{\left(\frac{1}{2}\right)^{\lambda_3} - \left(\frac{1}{2}\right)^{\lambda_4}}{\lambda_2}. \tag{20}$$

There is no analytic solution to the subsystem $\hat{\rho}_3 = \rho_3$ and $\hat{\rho}_4 = \rho_4$, and so the first step consists in minimizing a function of two variables (u is a fixed parameter) such that

$$\phi(u, \lambda_3, \lambda_4) = (\hat{\rho}_3 - \rho_3)^2 + (\hat{\rho}_4 - \rho_4)^2 \tag{21}$$

and then one can estimate λ_2 and λ_1 with (19) and (20). So what is required is the minimization of a highly non-linear bivariate function with several local minima. As an iterative scheme must be used to carry out this minimization, an initial estimate of the parameters is required. Currently, these initial estimates are obtained either using tabulated values (Ramberg et al., 1979) or by using an interactive visualization of the surface to minimize (Bigerelle et al., 2005). Moreover, a goodness-of-fit test is typically performed to establish the validity of the resulting estimates, as suggested in Lakhany and Mausser (2000). Additionally, even though the choice of the starting values can be automated thanks to tabulated results, such a procedure cannot cover the whole range of possible cases.

Regarding the goodness-of-fit method of estimation, the "starship" method, proposed by King and MacGillivray (1999), can be used, which minimizes a goodness-of-fit criterion on a 4D (four dimensional) space (A direct minimization of a four-variable ($\lambda_1, \lambda_2, \lambda_3, \lambda_4$) function is carried out). This method can be tremendously time consuming (especially for large sample sizes). Indeed, due to the numerous local minima that appear in practice, a very precise meshing of the 4D space must be used. To reduce the computation time, Lakhany and Mausser (2000) proposed an interesting iterative method. Instead of calculating the goodness-of-fit estimator on a full 4D grid, they used successive simplex from random starting points until the goodness-of-fit test stops rejecting the model. This results in an acceptable estimate of the parameters, but not necessarily the values that provide the best fit. Moreover, compared to the estimates that provide the best fit, bias and standard errors can be relatively high.

2.3. A new combined method

In essence, there are two types of estimation strategies when dealing with a GLD. The first type estimates the parameters with something like the method of moments or the percentile method, and then checks on the results are performed with a goodness-of-fit test. This results in a quick estimate of the parameters but it is possible that the goodness-of-fit test will reject. The second type uses a goodness-of-fit criterion. More precisely, the strategy is to find the set of parameters ($\lambda_1, \lambda_2, \lambda_3, \lambda_4$) that give the lowest value of (e.g.) the Kolmogorov–Smirnov estimator E_{KS} defined by

$$E_{KS} = \max_{x \in D} |\hat{F}_n(x) - F(x)|, \tag{22}$$

where D and F have the same definitions as in Section 2.1, and $\hat{F}_n$ is the empirical distribution function (edf). However, this approach requires expensive computation times, particularly when the sample size is large. In order to deal with this problem, the authors resorted to a new method that combines the two types of estimation methods just described.

Briefly, to reduce execution time, rather than minimizing E_{KS} as a function of four parameters, minimization is performed on a two-dimension grid only. More precisely, the (λ_3, λ_4) space is discretized by a 2D square grid of $N \times N$ couples of values. Therefore one must define the support $[\lambda_3^{\min}; \lambda_4^{\min}] \times [\lambda_3^{\max}; \lambda_4^{\max}]$ and the step s of the grid. Many techniques exist to optimize the grid in order to minimize the computation time, but they require a certain amount of a priori knowledge about the GLD. King and MacGillivray (1999) present some guidelines to build such an efficient grid. Nonetheless, the values of λ_3 and λ_4 are not sufficient to calculate E_{KS}: a set of (λ_1, λ_2) parameters must be associated
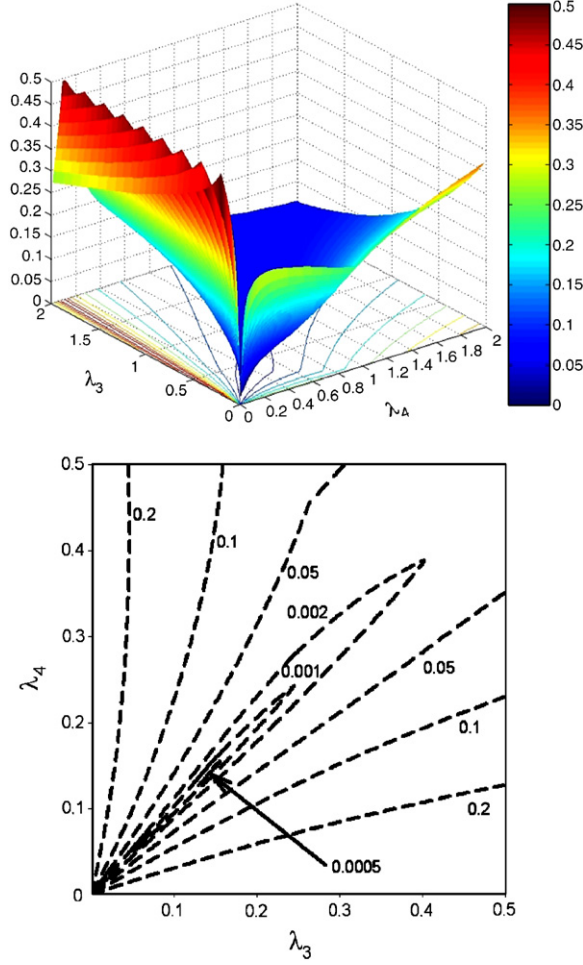
Fig. 2. Values of the E_{KS} distance in the (λ_3, λ_4) space obtained for the modeling of 10^4 standard Gaussian data.

with each set of the (λ_3, λ_4) parameters of the grid. The strategy here, to avoid a discretization of a full 4D space, only one pair of (λ_1, λ_2) parameters will be associated with each point of the grid, thanks to Eqs. (19) and (20). As detailed above, these equations give, respectively, the values of λ_2 and λ_1 as the solutions of $\hat{\rho}_2 = \rho_2$ and $\hat{\rho}_1 = \rho_1$.

For each point (i, j) of the grid, E_{KS} is calculated between the empirical data and the *GLD* with parameters $(\lambda_1(i, j), \lambda_2(i, j), \lambda_3(i, j), \lambda_4(i, j))$. The minimal value of E_{KS} is determined over the whole grid and then a simplex (Nelder and Mead, 1965) (this minimization method is used because it does not require the computation of complex derivatives and also to ensure some consistency with the work done by previous authors, Ramberg et al., 1979; Öztürk and Dale, 1985) is started from the grid point where this minimal value occurs. Fig. 2 presents the response surface obtained for a series of 10^4 standard Gaussian values. Notice that the surface is very smooth and that in this particular case, there is apparently a unique minimum in the range of (λ_3, λ_4) values tested.

From 2.1, several considerations limit the range of admissible values of the GLD parameters. An additional issue is that the support of a GLD can be either bounded (it is defined on a finite support $[a, b] \subset \mathbb{R}$) or unbounded (it is defined on $\mathbb{R}$). Each case corresponds to different domains of (λ_3, λ_4) values as pointed out in Table 1. The choice between bounded or unbounded GLD in practical modeling will be discussed in Section 4. Here, if a bounded version is specified, the algorithm is restricted to explore only the values of λ_3 and λ_4 corresponding to a bounded GLD, which, from Table 1, leads to the conditions $\lambda_3 > 0$ and $\lambda_4 > 0$. Moreover, to ensure that the modeling will be defined at least
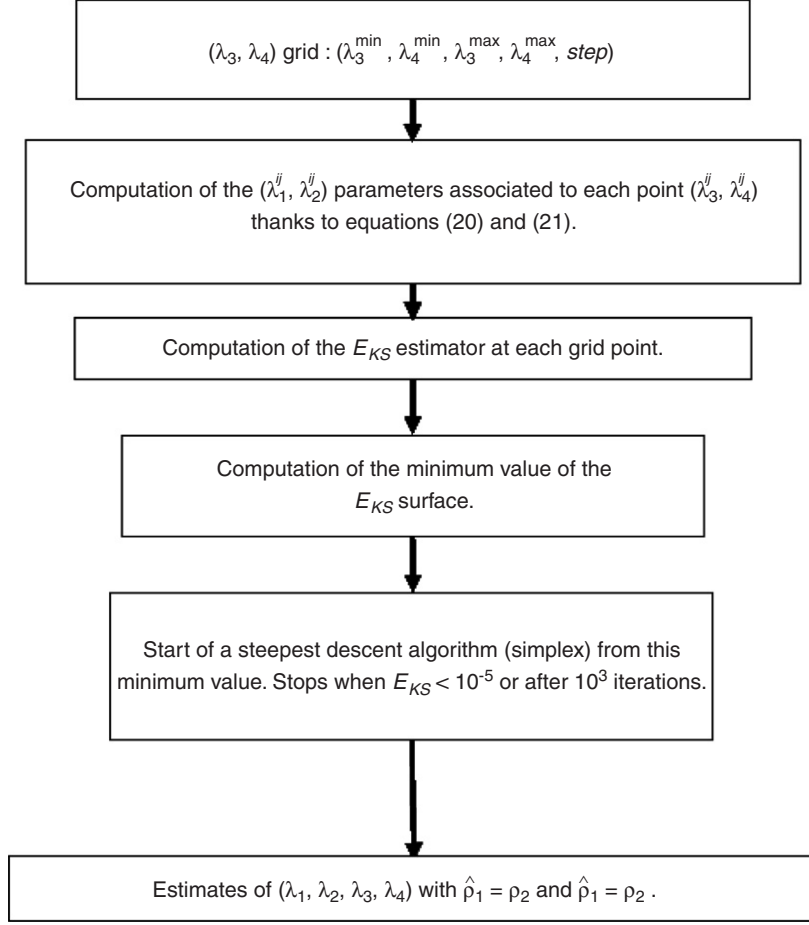
Fig. 3. Flowchart illustrating the algorithm proposed for the new method of calculation of the GLD parameters.

on all the empirical data (of the data set S) the two following conditions are imposed:

$$\lambda_1 - \frac{1}{\lambda_2} \leqslant \min(S), \tag{23}$$

$$\lambda_1 + \frac{1}{\lambda_2} \geqslant \max(S). \tag{24}$$

A flowchart of the proposed estimation method is shown in Fig. 3. Source code will be provided by the corresponding author via email upon request.

It should be noted that this new estimator minimizes E_{KS} given that the first two equations (18) of the percentile method are satisfied. However, the resulting estimate is not necessarily optimal in terms of minimizing the Kolmogorov–Smirnov distance because the explored values of λ_1 and λ_2 are constrained by the first two equations of system (18). But the adequacy obtained with this 2D method was compared with the starship method using a very fine 4D meshing in several simulations. In all cases, the values of the four parameters were only slightly different, and the differences in E_{KS} never exceeded 2% in the case of the 4D method. Additionally, when plotting the two GLD distributions obtained both by a 2D and a 4D minimization, no visible difference was seen. Therefore, these two methods appear to lead to very comparable GLDs, but the new 2D method is faster, as detailed in the Appendix, which also gives some additional information about how various estimators compare in terms of bias and standard errors.

3. GLD modeling of data coming from a GLD

In this section, the sampling distribution of the proposed estimator is studied. We begin with the situation where data are generated from a GLD. In reality, the actual distribution will not be exactly GLD, and some results when this is the case are presented as well.

3.1. Results on the choice for u

When using the percentile method, the choice for u is usually 0.1 (Karian and Dudewicz, 2000), which is said to be a "*good compromise*". Indeed for small sample sizes, u must not be too small in order to allow the calculation of $\hat{\pi}_u$ and $\hat{\pi}_{1-u}$ involved in Eqs. (11)–(13), and thus must satisfy $(n+1)u \geqslant 1$ (where n is the sample size). Nevertheless, for large sample sizes this leads to a wide range of possible values of u:

$$\frac{1}{n+1} \leqslant u < 0.25. \tag{25}$$

Evidently, a comprehensive study regarding the choice of u, has not been published, and so this issue is addressed here using simulations studies based on 10^4 replications and sample sizes n. We begin with the situation where data are generated from $GLD(0, 0.19, 0.14, 0.14)$ (these specific values of the GLD parameters were chosen to study a symmetric distribution close to the standard Gaussian one (see Fig. 4)). Of course, the generality of the results are limited, but this choice was made to illustrate tendencies in cases where the actual GLD parameters are *not too close* to the boundaries of the intervals given by Eqs. (4)–(9). Here, n is taken to be 10^4. Results on other sample sizes are reported in 3.2.

Fig. 5 shows a plot of the .95 confidence intervals (the .025 and .975 quantiles of the sampling distribution) and the mean values of the four estimators of the four parameters for different values of u. For these four estimators, the value of u has very little influence on their mean value. Moreover, it does not affect much the width of the 95% confidence interval for the parameters λ_3 and λ_4. However, the confidence interval of the location parameter λ_1 is found to widen substantially as the value of u decreases. In the case of the scale parameter λ_2, estimated with relatively high precision,
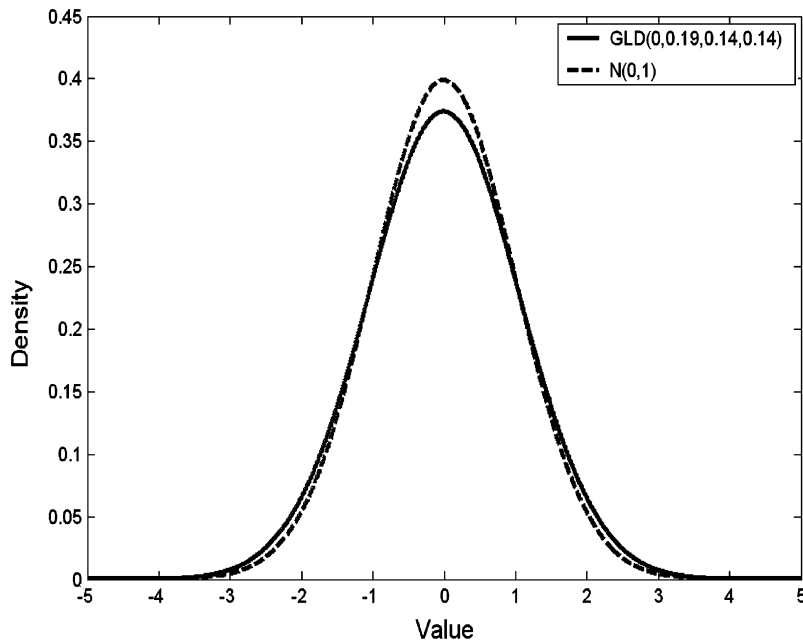

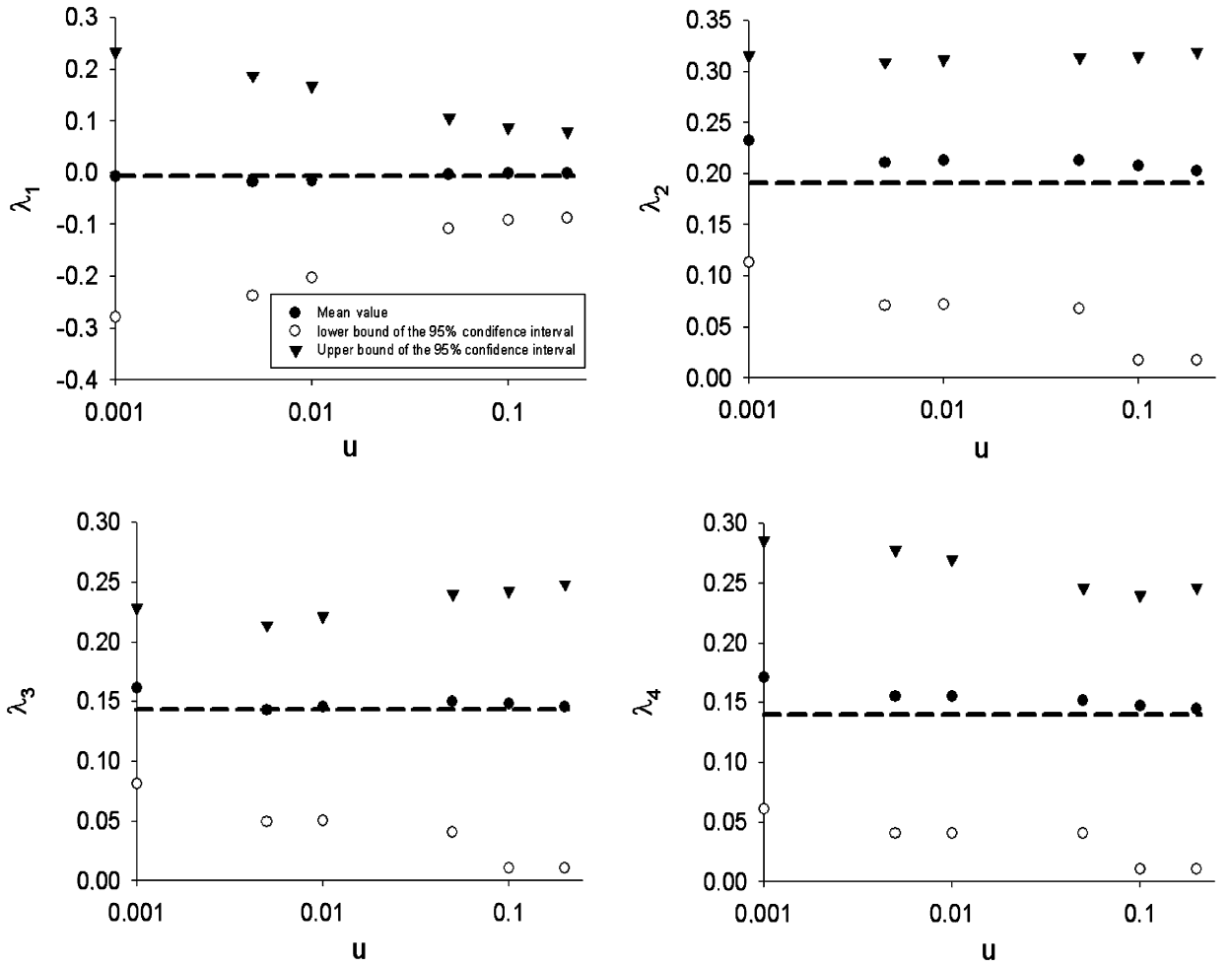Fig. 4. PDF of $GLD(0, 0.19, 0.14, 0.14)$ compared to the standard Gaussian one.

Fig. 5. Ninety-five percent confidence interval and mean values of (a) λ_1, (b) λ_2, (c) λ_3, (d) λ_4 for 10^4 data sets of 10^4 values coming from $GLD(0, 0.19, 0.14, 0.14)$ and approached by a GLD using different values of u (the theoretical GLD parameters are indicated by the horizontal dotted lines).

the 95% confidence interval width is slightly smaller for small values of u. More specifically, the lower .025 quantile of the sampling distribution, when estimating λ_2, is higher which has a direct and strong effect on the support of the obtained GLD as shown in Fig. 6a.

The accuracy of the estimated moments and percentiles is also an interesting indication of the modeling ability of a GLD. The first four GLD moments, denoted by α_i ($i = 1, 2, 3, 4$) can be calculated using the following equations:

$$\alpha_1 = \lambda_1 + \frac{A}{\lambda_2},$$
(26)

$$\alpha_2 = \frac{B - A^2}{\lambda_2^2},$$
(27)

$$\alpha_3 = \frac{C - 3AB + 2A^3}{\lambda_2^3 \alpha_2^{3/2}}$$
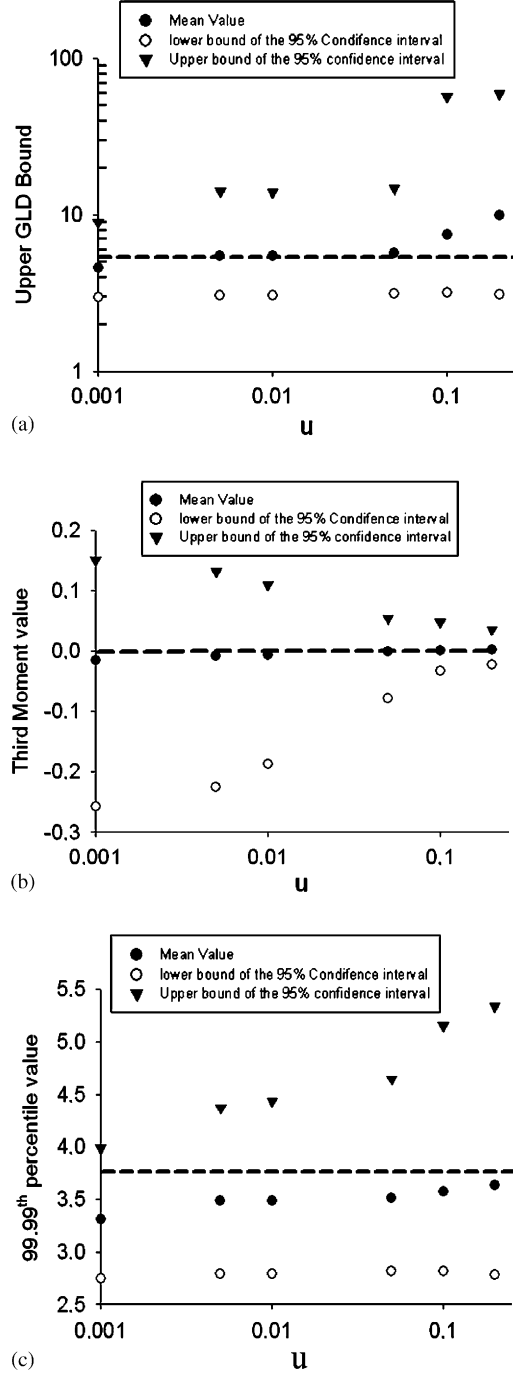(28)

Fig. 6. Ninety-five percent confidence interval of (a) the upper bound of the GLD support (as defined in Table 1), (b) the third moment of the modeled GLD, and (c) the 99.99th percentile of the modeled GLD, for 10^4 data sets of 10^4 values coming from $GLD(0, 0.19, 0.14, 0.14)$ and approached by a GLD using different values of u (the corresponding theoretical values of the $GLD(0, 0.19, 0.14, 0.14)$ are indicated by horizontal dotted lines.

and

$$\alpha_4 = \frac{D - 4AC + 6A^2B - 3A^4}{\lambda_2^4 \alpha_2^2},$$

(29)

where

$$A = \frac{1}{1 + \lambda_3} - \frac{1}{1 + \lambda_4}, \tag{30}$$

$$B = \frac{1}{1 + 2\lambda_3} + \frac{1}{1 + 2\lambda_4} - 2\beta(1 + \lambda_3, 1 + \lambda_4), \tag{31}$$

$$C = \frac{1}{1 + 3\lambda_3} - \frac{1}{1 + 3\lambda_4} - 3\beta(1 + 2\lambda_3, 1 + \lambda_4) + 3\beta(1 + \lambda_3, 1 + 2\lambda_4) \tag{32}$$

and

$$D = \frac{1}{1 + 4\lambda_3} + \frac{1}{1 + 4\lambda_4} - 4\beta(1 + 3\lambda_3, 1 + \lambda_4) + 6\beta(1 + 2\lambda_3, 1 + 2\lambda_4) - 4\beta(1 + \lambda_3, 1 + 3\lambda_4) \tag{33}$$

with β the Riemann function:

$$\forall (e, f) \in \mathbb{R}_+^2, \quad \beta(e, f) = \int_0^1 x^{e-1}(1 - x)^{f-1}\, \mathrm{d}x. \tag{34}$$

Concerning the first two moments and the fourth, the choice for u was found to have no significant influence. However, in the case of the third moment, the choice for u impacts the width of the 95% confidence interval (again meaning the difference between the .975 and .025 quantiles of the sampling distribution). This is illustrated by Fig. 6 in which it can also be noticed that for values of u lower than 0.01, even skewness is slightly underestimated. These results show that using very small values for u significantly decreases the modeling accuracy as far as properties of symmetry are concerned. Evidently, this result comes from the fact that when u is very small, the estimates of the percentiles $\hat{\pi}_u$ and $\hat{\pi}_{1-u}$ has a relatively large standard error. So in this case, a value of u sufficiently high should be used to ensure that the corresponding percentiles can be estimated with relatively high precision.

As for estimating the percentiles, u was found to have no significant influence on central percentiles (such as the 50th or 90th). However, for more extreme percentiles, the choice for u makes a difference as indicated in Fig. 6c, which deals with estimating the 99.99th percentile. For low values of u, this extreme percentile is significantly underestimated, whereas, as u increases, the bias decreases, but the confidence interval is larger.

These results illustrate the difficulty in choosing u. There is no absolute best choice since the accuracy of the result depends on the final goal of the modeling process. Indeed, the use of low u values (e.g. $\leqslant 0.01$) is desirable if the focus is on the bounds. However, low values of u result in a widening of some confidence intervals (e.g. for λ_1 and the third moment). On the other hand, if we need to approach the theoretical distribution from which the empirical data come while minimizing the noise due to the estimation, it seems more appropriate to use values of u larger than 0.02 (in the present simulation, this limit will decrease if the sample size increases), since in the range $0.02 \leqslant u < 0.25$ the confidence intervals of the location parameter λ_1 and the third moment decrease significantly. It is thus possible to optimize the value of u for a given criterion (e.g. the accuracy of the predicted bounds) but it may be at the expense of other properties.

Of course, the choice for u would benefit from a more comprehensive study, including the effect of the sample size. The next section reports results for a range of sample sizes that help provide perspective.

3.2. Influence of the sample size

Here, the value of u will be fixed at 0.1, as done in Karian and Dudewicz (2000). Simulations are performed as previously described, only now a wide range of sample sizes is used. In each case the values of the four GLD parameters and the level of adequacy (the value of E_{KS}) are recorded. Here, $10 \leqslant n \leqslant 10^7$ is considered.

Fig. 7 presents the mean values of the estimates of the GLD parameters and the 95% confidence intervals of their sampling distributions. As indicated in Fig. 7 the estimation of the GLD parameters, based on the new method, is approximately unbiased for $n \geqslant 10^3$, meaning that $(E(\hat{\lambda}_i) - \lambda_i \leqslant 10^{-3})$. Additionally for all four parameters, the sampling distribution of the estimators are almost symmetrically distributed when $n \geqslant 2 \times 10^3$.

For samples of size $n \geqslant 2 \times 10^3$, the .025 and .975 quantiles of the sampling distribution are approximately given by $\lambda_i = \lambda_i^0 + \gamma_{i,1}.n^{\gamma_{i,2}}$. The fitted coefficients and the corresponding correlations are given in Table 2. In all cases
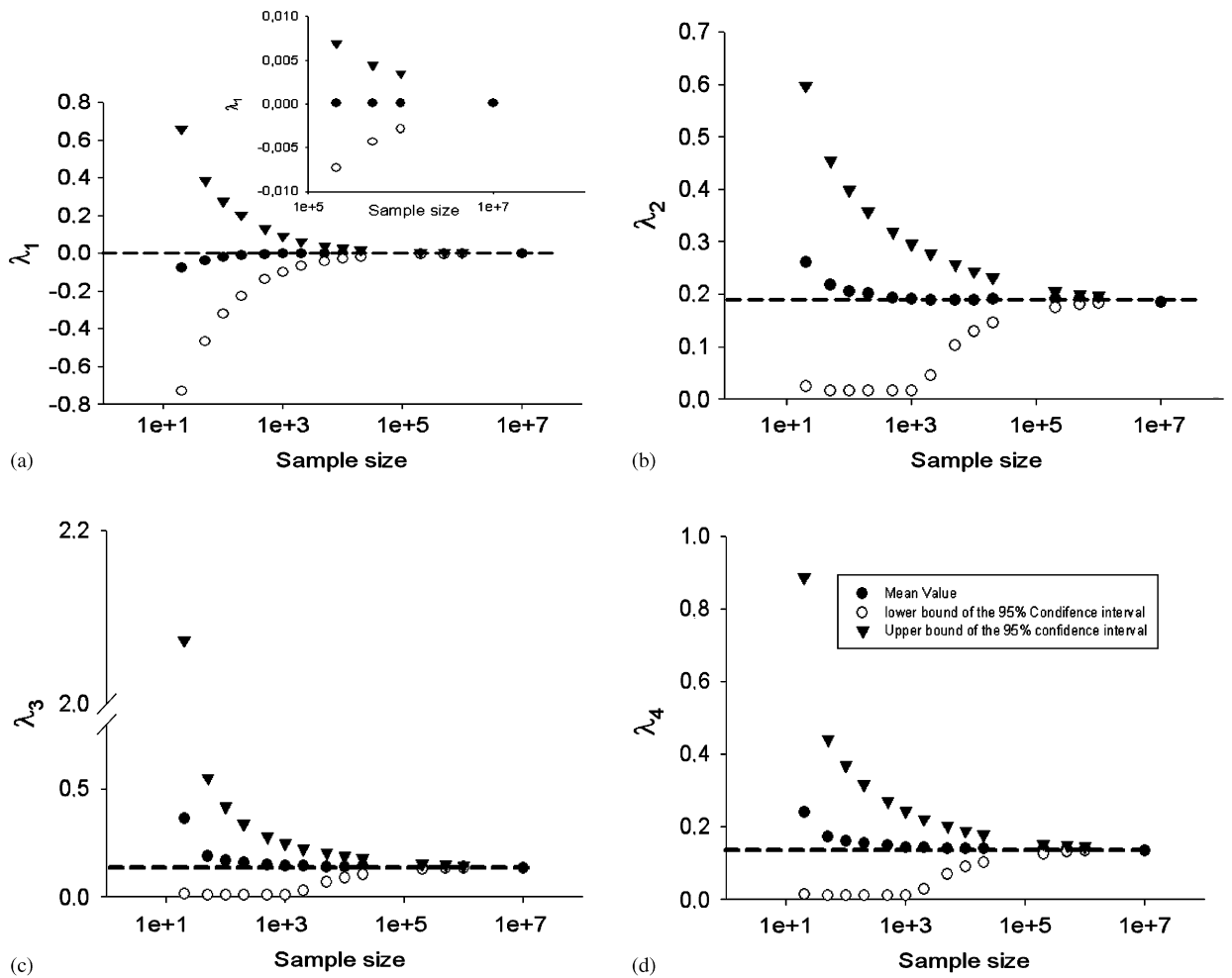
Fig. 7. Influence of the sample size on the 95% confidence interval and on the mean value of the estimates of the four GLD parameters for various sample sizes and when sampling coming from $GLD(0, 0.19, 0.14, 0.14)$. (Actual values are indicated by the horizontal dotted line.)

Table 2
Coefficients of the relationship $\lambda_i = \lambda_i^0 + \gamma_{i,1} . n^{\gamma_{i,2}}$ for the bounds of the 95% confidence interval of the GLD parameters fitted from $n \geqslant 2 \times 10^3$ and the corresponding correlation coefficients R

	2.5th percentile				97.5th percentile			
	λ_i^0	$\gamma_{i,1}$	$\gamma_{i,2}$	R	λ_i^0	$\gamma_{i,1}$	$\gamma_{i,2}$	R
λ_1	0	-3.72	-0.53	0.9988	0	3.06	-0.51	0.9993
λ_2	0.19	-6.96	-0.51	0.9995	0.19	1.33	-0.35	0.9966
λ_3	0.14	-3.97	-0.47	0.9997	0.14	1.44	-0.37	0.9964
λ_4	0.14	-3.93	-0.47	0.9997	0.14	1.33	-0.36	0.9967

the correlation is quite high, and for the location parameter λ_1, both the lower and the upper bounds of its 95% confidence interval are proportional to $1/\sqrt{n}$. This result also holds for the .025 quantile of the sampling distribution when dealing with the estimates of the three other parameters. However, the .975 quantile shows an association closer to $n^{-1/3}$.
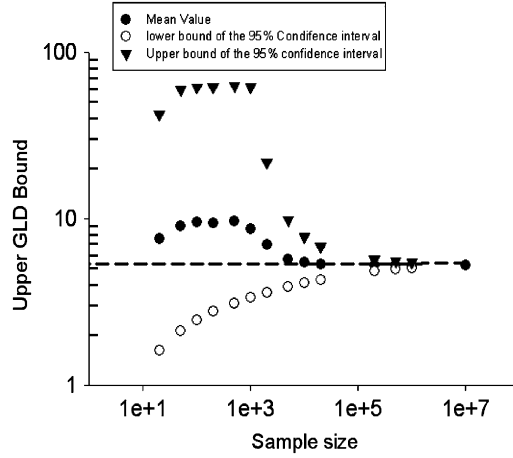
Fig. 8. Influence of the sample size on the 95% confidence interval and on the mean value of the upper bound of the GLD support calculated on samples coming from $GLD(0, 0.19, 0.14, 0.14)$ and approached by a GLD (the theoretical value is indicated by the horizontal dotted line).

Based on results shown in Fig. 7a and b, the upper bound of the GLD support, which can be calculated thanks to Table 1, as a function of the sample size, is shown in Fig. 8. As expected, for small sample sizes, the confidence interval is very wide and non-symmetrical. That is, the precision of the estimate is relatively poor.

As mentioned above, King and MacGillivray (1999) compared the standard error of several estimators of the GLD parameters when the sample size is small. The estimators were the starship method, the usual method of moments and the least-squares method developed by Öztürk and Dale (1985). The same situation was considered here, where observations were generated from $GLD(0, 0.1975, 0.1349, 0.1349)$, which is the GLD distribution used to approximate a standard Gaussian distribution. The results, which include the standard error of the new estimator, are shown in Fig. 9. This comparison highlights the fact that the new method performs very reasonably with respect to a standard error criterion.

3.2.1. Adequacy

Fig. 10 presents E_{KS} distance as a function of the sample size and also shows the .025 and .975 quantiles of the sampling distribution. The decrease is almost proportional to $1/\sqrt{n}$, with an excellent correlation, as shown in Table 3. The width of the 95% confidence interval decreases in the same way.

3.2.2. Confidence intervals of the first four moments

Like the GLD parameters, the GLD moments are not directly estimated, but result from the estimates of the parameters determined through the minimization of a global adequacy criterion. Fig. 11 shows the quantiles of the corresponding sampling distributions. For small sample sizes, the bias is more pronounced as the order of the moment increases. Here again the estimates of the GLD moments are reasonably unbiased for sample sizes larger than 10^3. Skewness of the sampling distribution becomes more pronounced as we move toward higher moments, and as expected, it decreases as the sample size increases. As shown in Table 4, as n gets large, the width of the confidence interval is not proportional to $1/\sqrt{n}$ when all the sample sizes are considered. However, if we only consider $n \geqslant 2 \times 10^3$ the previous proportionality still holds.

To add perspective, following Karian and Dudewicz (2000), estimates of $\hat{\alpha}_i - \alpha_i$ for $i = 1, 2, 3, 4$ are shown in Fig. 12. The same tendencies as those observed for GLD moments compared to their theoretical value (Fig. 11) are found. Indeed for small samples, the first and second empirical moments are globally underestimated, whereas the third and fourth are overestimated. Nevertheless, it shows that, even though no criterion related to the moment is used to obtain the values of the GLD parameters, both their empirical and theoretical values are correctly rendered by the GLD modeling using the new method presented here.
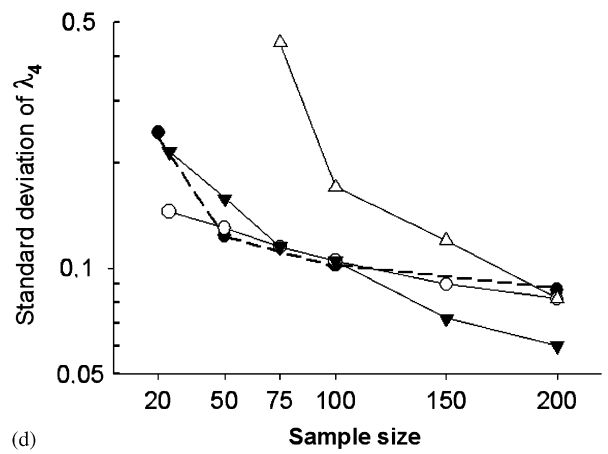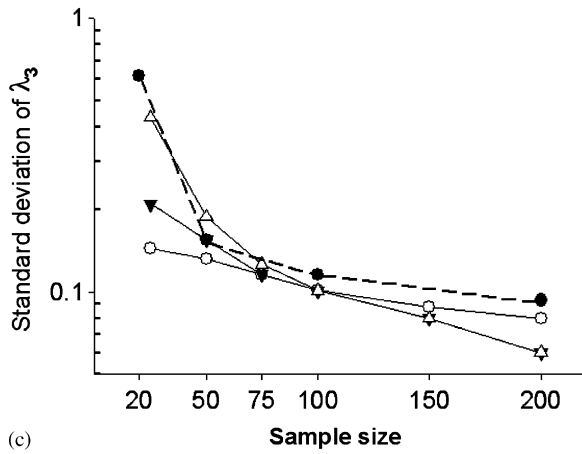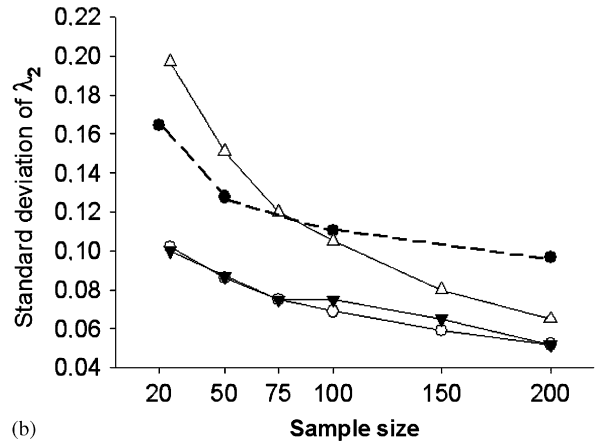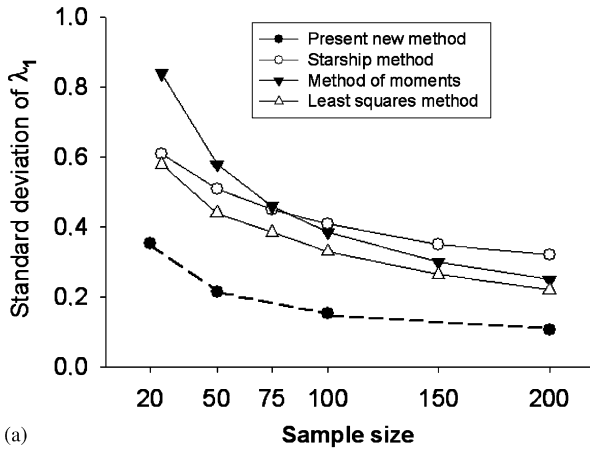
Fig. 9. Influence of the sample size on the standard deviations of the GLD parameters fitted on samples coming from $GLD(0, 0.1975, 0.1349, 0.1349)$ by three different methods (King and MacGillivray, 1999) and for samples coming from $GLD(0, 0.19, 0.14, 0.14)$ fitted with the new method (present study).
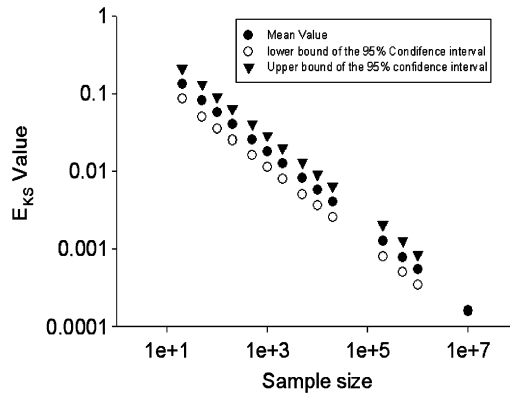

Fig. 10. Influence of the sample size on the 95% confidence interval and on the mean value of the E_{KS} distance calculated between the samples coming from $GLD(0, 0.19, 0.14, 0.14)$ and their GLD modeling.

Table 3
Coefficients of the power relationship: $E_{KS} = \gamma_{i,1}.n^{\gamma_{i,2}}$, and the corresponding correlation

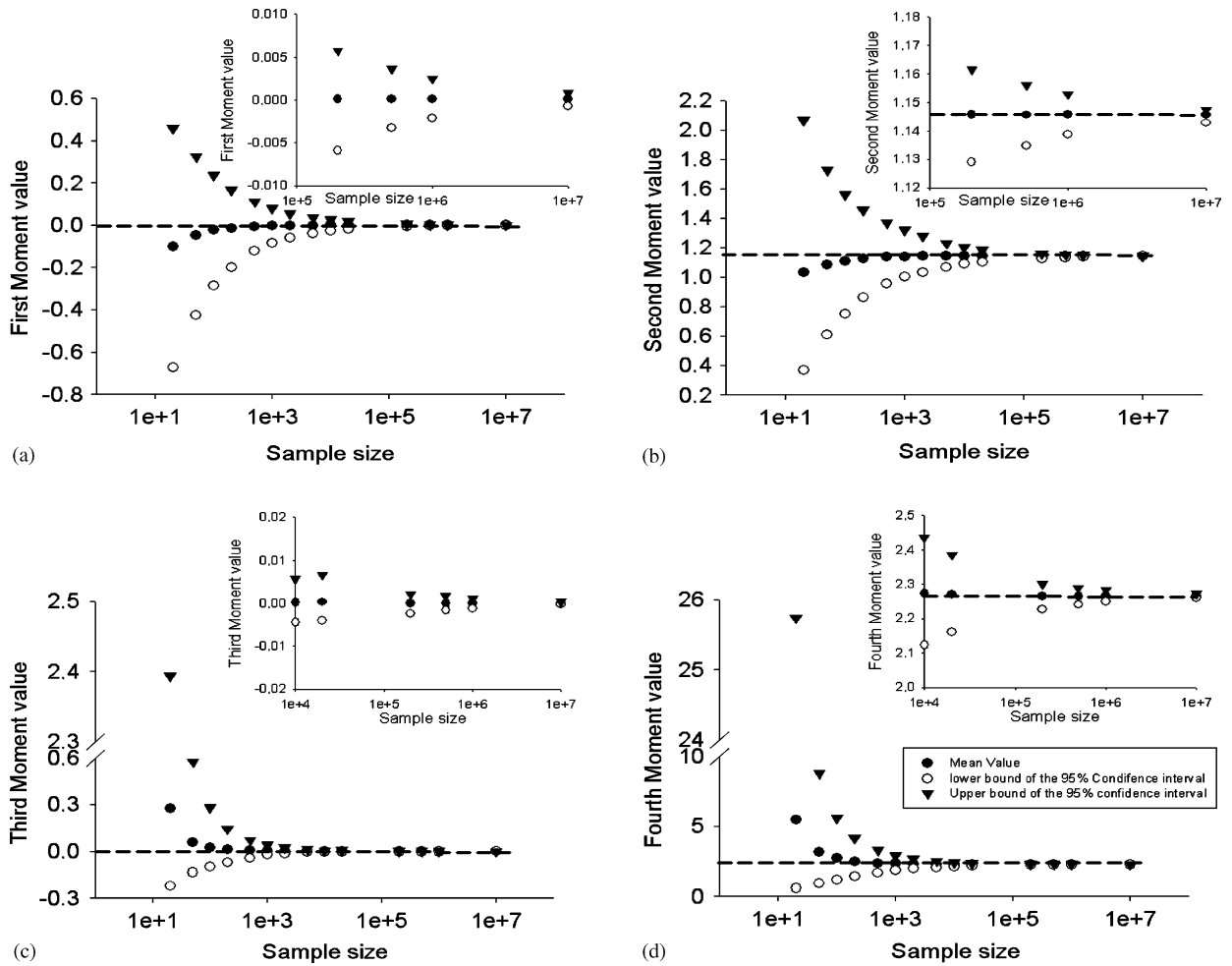|	$\gamma_{i,1}$	$\gamma_{i,2}$	R
Mean value	0.62	−0.51	$\simeq 1$
2.5th percentile	0.42	−0.53	0.9995
97.5th percentile	0.97	−0.51	$\simeq 1$

Fig. 11. Influence of the sample size on the 95% confidence interval and on the mean value of the first four GLD moments obtained by the GLD modeling of samples coming from $GLD(0, 0.19, 0.14, 0.14)$ (their theoretical values are indicated by the horizontal dotted line).

3.2.3. Confidence intervals when estimating percentiles

A third way to study the estimation process is in terms of estimating particular percentiles. Fig. 13 shows the results when the goal is to estimate the .9, .95, .99, .999 and .9999 quantiles. As in previous situations, the estimates are reasonably unbiased for $n \geqslant 10^3$ and the difference between the .975 and .025 quantiles of the sampling distributions decreases approximately like $1/\sqrt{n}$ for $n \geqslant 2 \times 10^3$.

Table 4
Coefficients of the modeling $\alpha_i = \alpha_i^0 + \gamma_{i,1} \cdot n^{\gamma_{i,2}}$ of the bounds of the 95% confidence interval of the GLD moments fitted for the whole n range and the corresponding correlation coefficients R

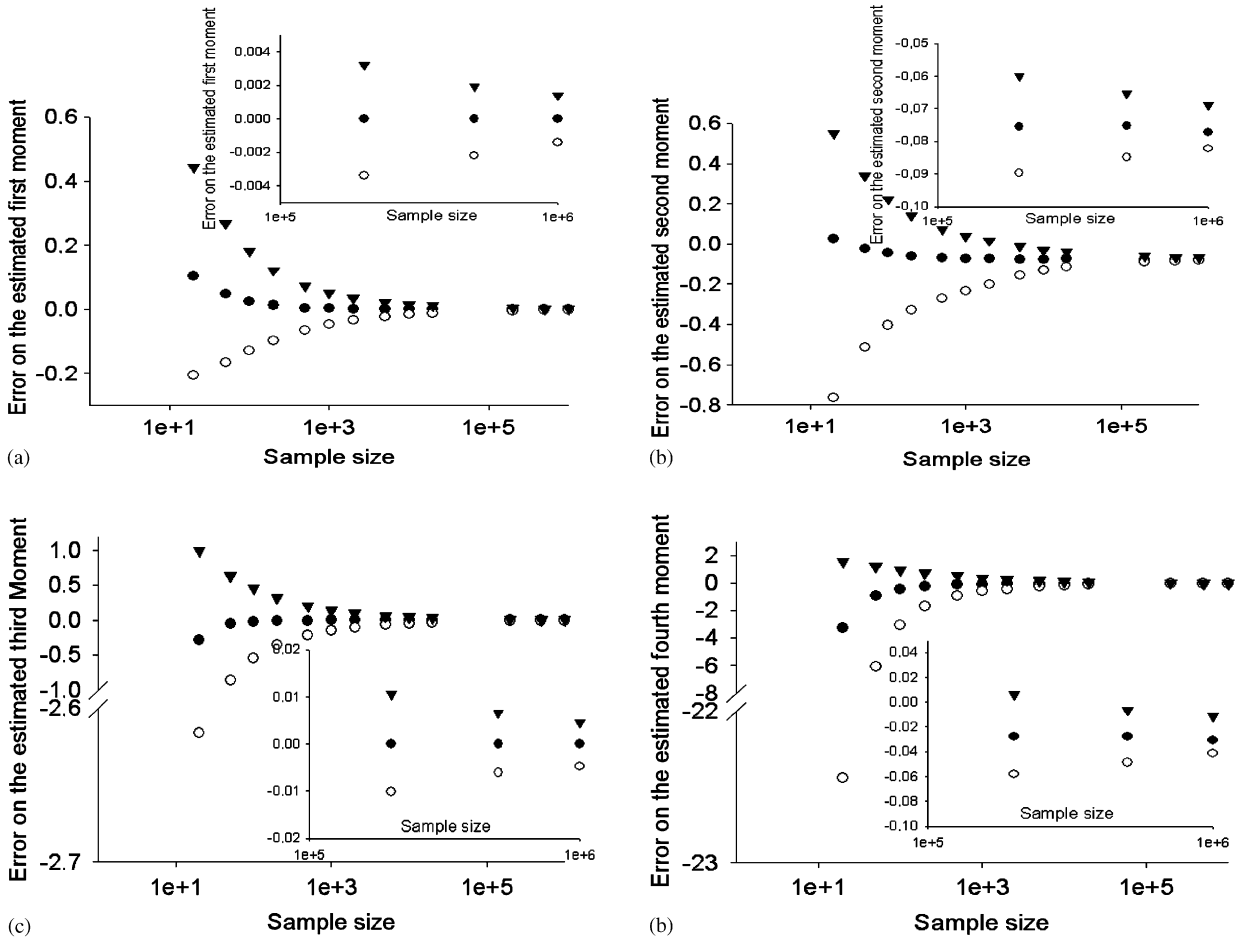|	2.5th percentile				97.5th percentile			
	α_i^0	$\gamma_{i,1}$	$\gamma_{i,2}$	R	α_i^0	$\gamma_{i,1}$	$\gamma_{i,2}$	R
Mean	0	−3.29	−0.53	0.9999	0	1.77	−0.44	0.9993
Standard deviation	1.146	−2.80	−0.43	0.9999	1.146	3.44	−0.45	0.9986
Skewness	0	−1.08	−0.53	0.9988	0	177.9	−1.44	0.9987
Kurtosis	2.265	−5.06	−0.35	0.9954	2.265	1066.7	−1.28	0.9981

Fig. 12. Ninety-five percent confidence interval and mean value of the difference between the empirical moments of a data set and the predicted GLD moment obtained by the corresponding modeling $\hat{\alpha}_i - \alpha_i$ for (a) $i = 1$, (b) $i = 2$, (c) $i = 3$, (d) $i = 4$.

3.2.4. Some limits and perspectives

As indicated by simulations, GLD estimates are biased but consistent when data are generated from a GLD model. A partial explanation for this bias lays in the various constraints on the estimates used in the algorithm. However, the intrinsically high versatility of GLDs is also partly responsible (e.g. several different sets of parameters can define result in GLDs whose shapes will be close to each other, Karian and Dudewicz, 2000).
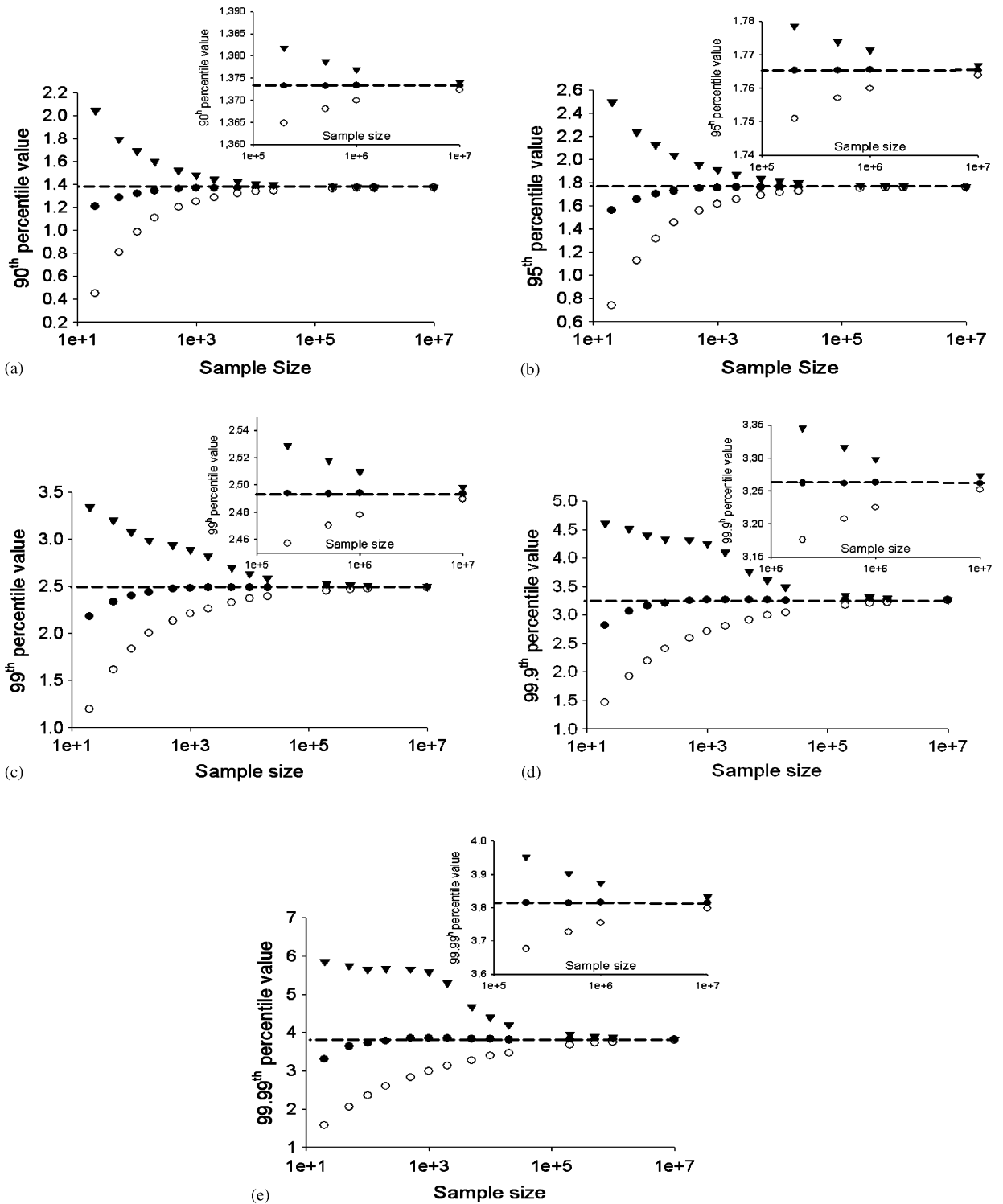
Fig. 13. Influence of the sample size on the 95% confidence interval and on the mean value of the some order statistics predicted by the GLD modeling of samples coming from $GLD(0, 0.19, 0.14, 0.14)$ (their theoretical values are indicated by the horizontal dotted lines).
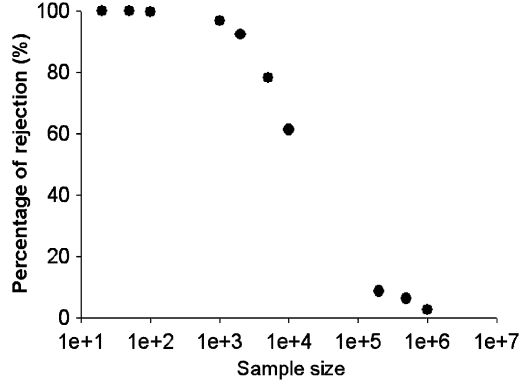
Fig. 14. Percentage of rejected GLD modeling when compared to the "*true*" GLD from which the samples are selected.

Bootstrap methods might reduce bias to a reasonable degree, but the extent to which this is true, and how it affects the standard errors of the estimates, remains to be determined.

The new method minimizes the distance between an empirical data set and the GLD model. The obtained E_{KS} distances were plotted in Fig. 10 and the Kolmogorov–Smirnov goodness-of-fit test never rejected the adequacy (between the empirical data set and the modeling) at the 5% confidence level. Of course, because the true underlying distribution is known ($GLD(0, 0.19, 0.14, 0.14)$), it is possible to test the modeling adequacy with this true distribution.

For each set of GLD estimates, a large number (10^6) of data coming from the modeled GLD were compared to the estimated fit using a two-sample Kolmogorov–Smirnov test. The results of the Kolmogorov–Smirnov test at a 5% confidence level are plotted in Fig. 14 and indicate that, for small sample sizes, the goodness-of-fit test always rejected.

4. Gaussian data approached by a GLD

In summary, a new method of estimating the parameters of a GLD has been proposed that is based on the minimization of the Kolmogorov–Smirnov distance in a two-dimension space. The new approach provides a quick and completely automated estimate of the GLD parameters while ensuring a relatively good fit to the data (which is not the case with usual percentile or moment-based methods for which an a posteriori goodness-of-fit test must be performed). The new method is based in part on the percentile method and extensive simulations provide some information relevant to choosing the quantile to be used, u. These partial results indicate that there is no trivial rule for choosing u. Indeed, depending on the final modeling goal, either a large or a small value may be more suitable.

So here, each sample is modeled by both a bounded and an unbounded GLD by the respective constraints ($\lambda_3 > 0$, $\lambda_4 > 0$) and ($\lambda_3 \leqslant 0$, $\lambda_4 \leqslant 0$).

4.1. Confidence intervals

Fig. 15 shows results on the size of the confidence intervals, again meaning the .025 and .975 quantiles of the sampling distribution of the four GLD estimators. As expected, the results are very similar to those presented in Fig. 7. It is interesting to note, however, that here the values of the four estimators stabilize around the values (0, 0.21, 0.142, 0.142) for sample sizes up to 10^5. However, the value of the last three estimators decrease again for larger sample sizes (see the details inserted in Fig. 15). This feature stems from Eqs. (23) and (24). When $\lambda_2 = 0.21$, which is the value around which the distribution is centered for $n \leqslant 10^5$, the GLD is defined on $[-4.76, 4.76]$. As the sample size increases, the probability of generating very large or very small values (i.e. outside of the previous interval) in a given data set increases, and thus the bounds of the GLD must become wider. This implies that λ_2 must decrease (cf. Eqs. (23) and (24)). As λ_2 is linked to λ_3 and λ_4 by Eq. (19) it also implies a decrease in these two parameters.

Results on the estimates obtained in the unbounded case are plotted in Fig. 16. First, notice that the confidence interval for the last three parameters is very narrow for $n \geqslant 10^3$. The location parameter has a very similar dependence
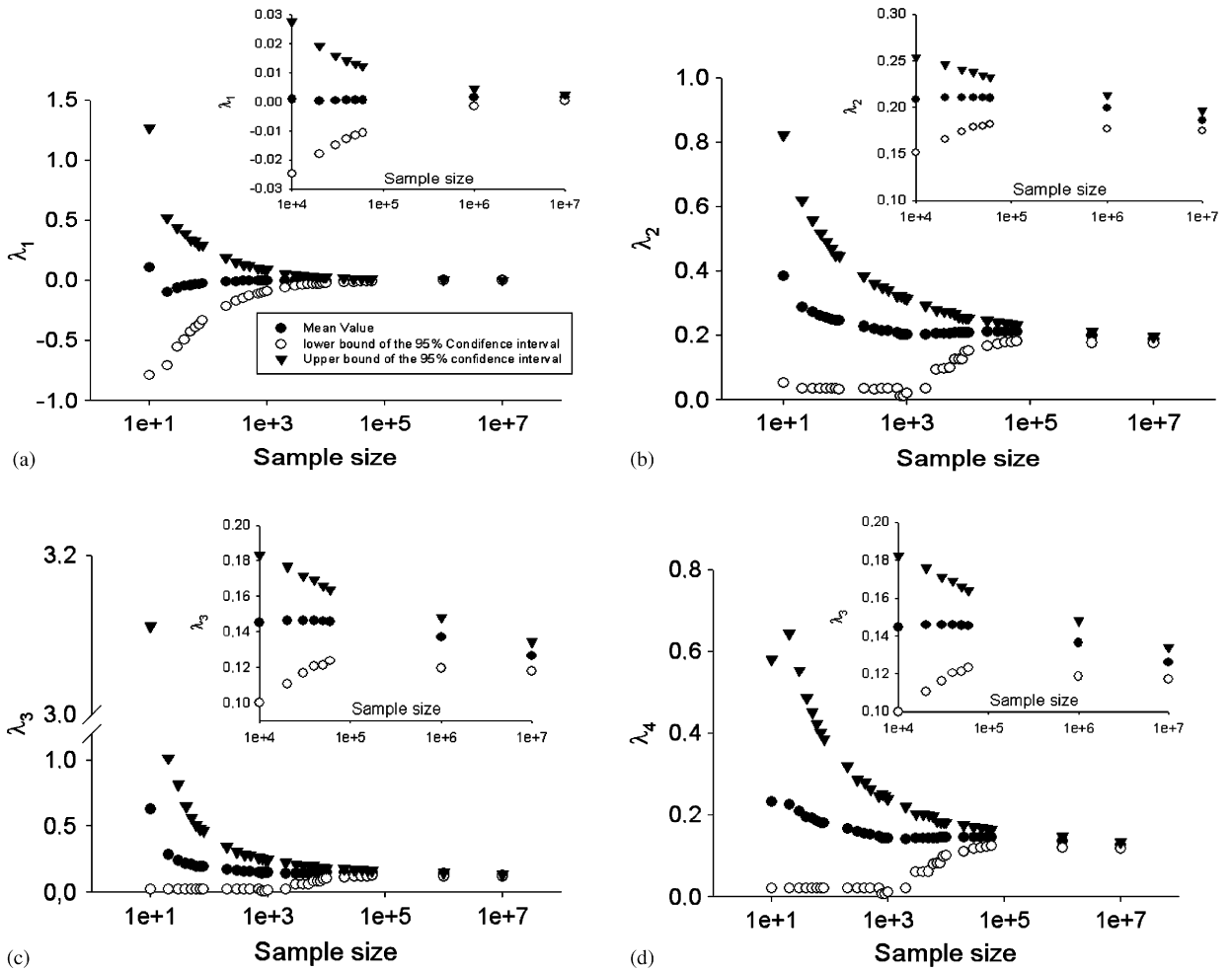
Fig. 15. The sample size versus the .025 and .975 quantiles of the sampling distribution and the mean value of the four GLD estimators calculated on samples coming from $N(0, 1)$ and based on a bounded GLD.

on the sample size as in the bounded case and also tends to 0 (or a slightly positive value). Also, the mean values continue to change up to $n = 10^6$.

4.2. Modeling adequacy

Fig. 17 shows that the mean value of E_{KS}, and the .025 and .975 quantiles of its sampling distribution for both bounded and the unbounded models, up to $n = 200$ approximately. For larger sample sizes, the distance between the unbounded modeling and the Gaussian sample stops decreasing with $E_{KS} = 0.01$, whereas the bounded GLD still continues to improve its accuracy up to $n = 10^6$. Nevertheless, for larger sample sizes the E_{KS} distance also ceases to decrease. These results indicate that for $n > 200$ the unbounded GLD is less adequate than the bounded one. Indeed, for sample sizes lower than 10^6, the GLD modeling is closer to the sample, in terms of E_{KS}, than the Gaussian one. Therefore, if there is no theoretical reason for the data to actually be Gaussian the use of a usual goodness-of-fit criterion suggests that a bounded GLD modeling is more satisfactory.

It is noted that when applying the Kolmogorov–Smirnov test at the 5% level, Fig. 18 shows that neither the Gaussian nor the bounded GLD is ever rejected until $n = 10^6$. On the contrary, the unbounded GLD is rejected in more and more cases as soon as $n = 6 \times 10^3$ and is always rejected for $n \geqslant 3 \times 10^4$.
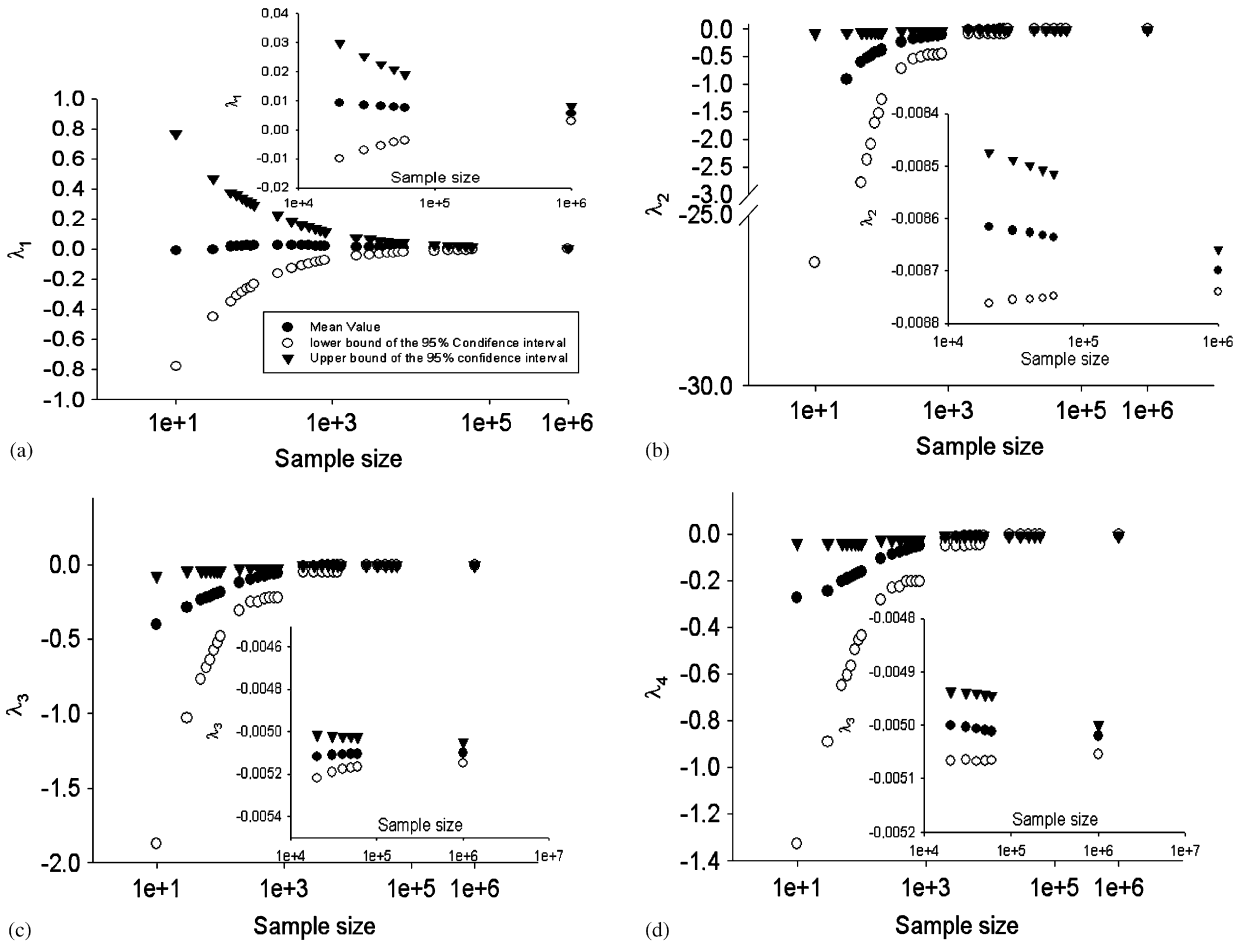
Fig. 16. Influence of the sample size on the 95% confidence interval and on the mean value of the four GLD estimators sample sizes of 10^4 from $N(0, 1)$, using an unbounded GLD.

5. Conclusions

In summary, a new method of estimating the parameters of a GLD has been proposed that is based on the minimization of the Kolmogorov–Smirnov distance in a two-dimension space. The new approach provides a quick and completely automated estimate of the GLD parameters while ensuring a relatively good fit to the data (which is not the case with usual percentile or moment-based methods for which an a posteriori goodness-of-fit test must be performed). The new method is based in part on the percentile method and extensive simulations provide some information relevant to choosing the quantile to be used, u. These partial results indicate that there is no trivial rule for choosing u. Indeed, depending on the final modeling goal, either a large or a small value may be more suitable.

Secondly, the sampling distribution of the new estimator was studied and compared to extant techniques. In particular, the standard error of the new estimator was found to compare favorably. The bias of the estimators might be corrected by the use of resampling techniques, but the extent to which this helps remains to be seen. All indications that the estimates are reasonably unbiased with a sample size that exceeds 10^3 and it was found that the difference between the .025 and .975 quantiles of the sampling distribution are proportional to $1/\sqrt{n}$.

Finally, both bounded and unbounded GLDs were used to model Gaussian data and were found valuable for sample sizes up to $n = 200$. However, for larger sample sizes, the bounded GLD definitely leads to a better fit. The bounded GLD is found to give a very accurate approximation of Gaussian data for sample sizes as large as $n = 10^6$. For larger
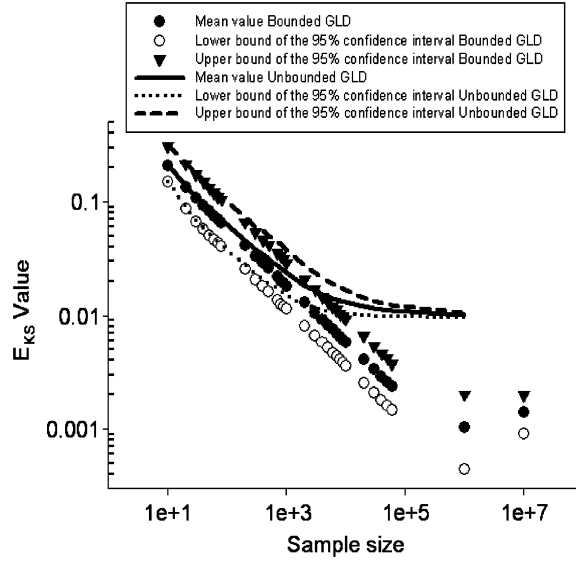
Fig. 17. Influence of the sample size on the 95% confidence interval and on the mean value of the E_{KS} distance obtained for both a bounded and an unbounded modeling of samples coming from $N(0, 1)$.
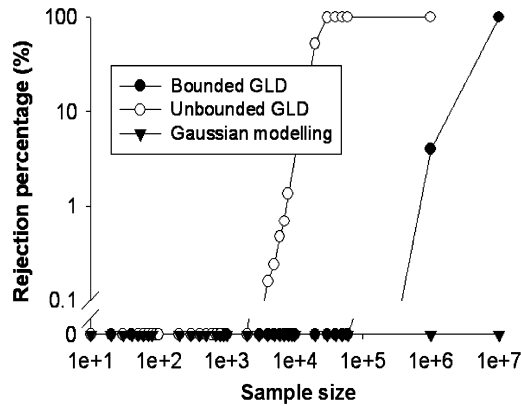

Fig. 18. Influence of the sample size on the percentage of GLD modeling rejected by the Kolmogorov–Smirnov test at the 5% confidence level.

sample sizes the GLD bounds must be widened in order to model the whole range of empirical data, which eventually leads to a loss of global adequacy.

Acknowledgments

Appendix

The present hybrid fitting method described is claimed to allow a fast and automatic approximation of a set of experimental data. To elaborate, as suggested by reviewers, this appendix provides a brief comparison of several automatic fitting methods in terms of time, adequacy and scatter. Of course, a more thorough study would be desirable. The results given here are meant to be suggestive. Results are presented in Table 5 based on simulations with 100

Table 5
Comparison of five estimators of the GLD parameters

Quantity		Present method	L&M moments method	L&M KS method	Starship KS method	Starship AD method
λ_1	Mean	−0.0003	−0.0576	−0.0042	0.2501	0.0072
	Standard error	0.0501	0.4090	0.0547	0.1000	0.0730
λ_2	Mean	0.1996	0.2989	0.2651	0.2002	0.1940
	Standard error	0.0565	0.0818	0.1298	0.0476	0.0302
λ_3	Mean	0.1511	0.2838	0.2378	0.1553	0.1460
	Standard error	0.0515	0.1769	0.1698	0.0463	0.0317
λ_4	Mean	0.1506	0.3101	0.2415	0.1475	0.1435
	Standard error	0.0503	0.1736	0.1735	0.0451	0.0286
Time (s)	Mean	3.1901	0.4412	2.5741	206.99	213.47
	Standard error	0.1502	0.3610	1.5216	40.4201	42.1521
E_{KS}	Mean	0.0185	0.0331	0.0241	0.0168	0.01856
	Standard error	0.0043	0.0111	0.0096	0.0027	0.0036

The values given are the mean and the estimated standard error of the estimators based on 100 sets of data coming from GLD(0, 0.19, 0.14, 0.14). L&M moments and KS methods are related to the method proposed by Lakhany and Mausser (2000), using automatic restarts of simplex for, respectively, a moments or Kolmogorov–Smirnov-based response surface. Starship KS and AD methods correspond to the starship method proposed by King and MacGillivray (1999) using either a Kolmogorov–Smirnov or an Anderson–Darling distance.

replications and $n = 1000$, with data generated from GLD(0, 0.19, 0.14, 0.14). Table 5 reports the mean values and the standard errors of the GLD estimators, the computation time of the fitting process, and the obtained adequacy in terms of E_{KS} distance. These calculations were made with the same amount of a priori knowledge on the GLD parameters. The range of the search grids and the size of their steps are the same. These results were obtained on a personal computer with a 2 GHz pentium IV processor, and 512Mo RAM.

The mean values of the four estimators indicate that the new hybrid method is the least biased. Moreover, only the starship methods provide slightly narrower confidence intervals. In terms of time, the fastest methods are those proposed by Lakhany and Mausser. The new hybrid method is only slightly slower, whereas the starship methods are around 100 times slower. Nevertheless, as the methods of Lakhany and Mausser stop as soon as an "*acceptable*" GLD is found (i.e. the Kolmogorov–Smirnov test does not reject at the 5% confidence level), the obtained GLD parameters are much more biased and have larger standard errors. This was expected because multiple minimas exist on the response surfaces. These problems could be addressed by the choice of a more severe test criterion (testing at the .01 level or even lower), which would however increase the computation time. Finally in terms of fit, the new hybrid method is found to lead to a fit that is as good as the starship method, whereas the Lakhany and Mausser's methods are less adequate as explained above.

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