

# Automorphisms of partial endomorphism semigroups

By J. Araújo, V. H. Fernandes, M. M. Jesus, V. Maltcev and J. D. Mitchell

**Abstract.** In this paper we propose a general recipe for calculating the automorphism groups of semigroups consisting of partial endomorphisms of relational structures with a single  $m$ -ary relation for any  $m \in \mathbb{N}$  over a finite set.

We use this recipe to determine the automorphism groups of the following semigroups: the full transformation semigroup, the partial transformation semigroup, and the symmetric inverse semigroup, and their wreath products, partial endomorphisms of partially ordered sets, the full spectrum of semigroups of partial mappings preserving or reversing a linear or circular order. We also determine the automorphism groups of the so-called Madhavan semigroups as an application of the methods developed herein.

## 1. Introduction and the Main Result

A number of works in the literature are dedicated to calculating the automorphism groups of certain transformation semigroups. In an earlier paper [3], a method for calculating automorphism groups of some such objects is given. In this paper we prove a more general result and use it to find the automorphism group of several well-known transformation semigroups. In order to state our main result we must recall some definitions and introduce some notation.

We assume throughout the paper that  $\Omega$  is a finite set. We denote the semigroup of all partial mappings on  $\Omega$  under composition of functions by  $P_\Omega$ , the semigroup of total mappings on  $\Omega$  by  $T_\Omega$ , and the group of permutations on  $\Omega$  by  $S_\Omega$ . If  $\Omega = \{1, 2, \dots, m\}$ , then we abbreviate  $P_\Omega$ ,  $T_\Omega$ , and  $S_\Omega$  to  $P_m$ ,  $T_m$ , and  $S_m$ , respectively. If  $U$  is a subsemigroup of  $P_\Omega$ , then we let  $\text{Aut}(U)$  denote the group of automorphisms of  $U$ . The group of *inner automorphisms* of  $U$  is defined

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as

$$\text{Inn}(U) = \{ \phi_a : f \mapsto a^{-1}fa \mid a \in S_\Omega \text{ and } a^{-1}Ua = U \}.$$

The *image* of  $f \in P_\Omega$  is denoted by  $\text{im}(f)$  and the *domain* of  $f$  by  $\text{dom}(f)$ . A mapping  $f \in P_\Omega$  is called a *constant with value*  $\alpha$  if  $\beta f = \alpha$  for all  $\beta \in \text{dom}(f)$ . For the sake of convenience, we will assume that the empty mapping  $\emptyset$  is also a constant.

Let  $m \in \mathbb{N}$ . Then an  $m$ -ary relation  $\rho$  on  $\Omega$  is just a subset of

$$\Omega^m = \{ (\alpha_1, \alpha_2, \dots, \alpha_m) \mid \alpha_1, \alpha_2, \dots, \alpha_m \in \Omega \}.$$

If  $\rho$  is an  $m$ -ary relation, then define

$$\rho' = \{ (\alpha_1, \alpha_2, \dots, \alpha_m) \in \rho \mid \alpha_i \neq \alpha_j \text{ if } i \neq j \}.$$

Let  $\rho$  and  $\sigma$  be  $m$ -ary relations on  $\Omega$ . We say that a subsemigroup  $U$  of  $P_\Omega$  *acts transitively from*  $\rho'$  *to*  $\sigma'$  if for all  $(\alpha_1, \dots, \alpha_m) \in \rho'$  and  $(\beta_1, \dots, \beta_m) \in \sigma'$  there exists  $f \in U$  with

$$(\alpha_1 f, \dots, \alpha_m f) = (\beta_1, \dots, \beta_m).$$

If  $U$  is a group of permutations and  $\rho' = \sigma' = (\Omega^m)'$ , then our definition of transitivity is just the usual definition of  $m$ -transitivity for permutation groups. If  $(\alpha_1, \dots, \alpha_m) \in \Omega^m$  and  $f \in U$ , then we denote  $(\alpha_1 f, \dots, \alpha_m f)$  by  $(\alpha_1, \dots, \alpha_m)^f$ . The monoid of *partial endomorphisms of*  $\rho$  is

$$\text{PEnd}(\rho) = \{ f \in P_\Omega \mid (\alpha_1, \dots, \alpha_m) \in \rho \cap (\text{dom}(f))^m \text{ implies } (\alpha_1, \dots, \alpha_m)^f \in \rho \}.$$

If  $\rho$  is a binary relation on  $\Omega$ , then an *anti-automorphism* of  $\rho$  is an element  $f \in S_\Omega$  such that  $(\alpha, \beta) \in \rho$  implies  $(\beta, \alpha)^f \in \rho$ . Since  $\Omega$  is finite, the set of automorphisms and anti-automorphisms of  $\rho$  forms a group. The following definition can be thought of as a generalization of this notion to relations of higher arity. Let  $H$  be a subgroup of  $S_m$ . Then we define

$$\text{Aut}_H(\rho) = \{ f \in S_\Omega \mid (\forall (\alpha_1, \dots, \alpha_m) \in \rho)(\exists t \in H)((\alpha_{1t}, \dots, \alpha_{mt})^f \in \rho) \}.$$

Again since  $\Omega$  is finite,  $\text{Aut}_H(\rho)$  is a group. The group of automorphisms and anti-automorphisms of a binary relation  $\rho$  mentioned above is denoted  $\text{Aut}_{S_2}(\rho)$  using this notation. We also require the following definition:

$$N(\rho, H) = \{ (\alpha_1, \dots, \alpha_m) \in (\Omega^m)' \mid (\alpha_{1t}, \dots, \alpha_{mt}) \notin \rho \text{ for all } t \in H \}.$$

We are now ready to state the main result of this paper. Note that Theorem 1.1 is a generalization of [3, Theorem 2.1].

**Theorem 1.1.** *Let  $\rho$  be an  $m$ -ary relation on a finite set  $\Omega$  for some  $m \in \mathbb{N}$ , let  $H$  be a subgroup of  $S_m$ , and let  $U$  is a subsemigroup of  $\text{PEnd}(\rho)$  such that*

- (i)  $U$  contains a constant idempotent with value  $\alpha$  for all  $\alpha \in \Omega$ ;
- (ii)  $U$  acts transitively from  $N(\rho, H) \cup \rho'$  to  $\rho'$ .

*Then  $\text{Aut}(U) = \{ \phi_a : f \mapsto a^{-1}fa \mid a \in \text{Aut}_H(\rho') \text{ and } a^{-1}Ua = U \}$ . Moreover, if  $a, b \in \text{Aut}_H(\rho')$  such that  $a \neq b$  and  $\phi_a, \phi_b \in \text{Aut}(U)$ , then  $\phi_a \neq \phi_b$ .*

We prove Theorem 1.1 in Section 2. In Section 3, we derive several corollaries of Theorem 1.1, and discuss whether it is possible to weaken its hypothesis and still obtain its conclusion. In Sections 4 and 5, we use the main theorem and its corollaries to determine the automorphism groups of the following semigroups: the full transformation semigroup, the partial transformation semigroup, and the symmetric inverse semigroup, and their wreath products, the partial endomorphisms of finite partially ordered sets, the full spectrum of semigroups of partial mappings preserving or reversing a linear or circular order. In Section 6 we determine the automorphism groups of the Madhavan semigroups as an application of the methods developed in the proof of Theorem 1.1.

## 2. Proof of Theorem 1.1

We prove Theorem 1.1 in the following sequence of lemmas, some of them belonging to the folklore of this topic and included here for the sake of completeness.

**Lemma 2.1.** *Let  $V$  be a subsemigroup of constants in  $P_\Omega$  such that for all  $\alpha \in \Omega$  there exists a constant idempotent in  $V$  with value  $\alpha$ . Then  $\text{Aut}(V) = \text{Inn}(V)$ .*

PROOF. The inclusion  $\text{Inn}(V) \leq \text{Aut}(V)$  is obvious.

Let  $\phi \in \text{Aut}(V)$ . If  $\emptyset \in V$ , then clearly  $\emptyset\phi = \emptyset$ . If  $f \in V \setminus \{\emptyset\}$ , then denote by  $\alpha_f$  the unique element of  $\text{im}(f)$ . Let  $f, g \in V \setminus \{\emptyset\}$  with  $\alpha_f = \alpha_g$ . By assumption, there exists an idempotent constant  $h \in V$  with  $\alpha_h = \alpha_f = \alpha_g$ . Then  $fh = f$  and  $gh = g$  and so  $(fh)\phi = f\phi h\phi \neq \emptyset$ . It follows that

$$\alpha_{f\phi} = \alpha_{(fh)\phi} = \alpha_{f\phi h\phi} = \alpha_{h\phi} = \alpha_{g\phi h\phi} = \alpha_{(gh)\phi} = \alpha_{g\phi}.$$

Hence  $\alpha_{f\phi} = \alpha_{g\phi}$  for all  $f, g \in V \setminus \{\emptyset\}$  with  $\alpha_f = \alpha_g$ . Therefore the mapping  $a : \Omega \rightarrow \Omega$  defined by  $\alpha_f \mapsto \alpha_{f\phi}$  is a well-defined bijection, i.e.  $a \in S_\Omega$ .

Now,  $\alpha_g \in \text{dom}(f)$  for some  $g \in V$  if and only if  $gf \neq \emptyset$  if and only if  $(gf)\phi \neq \emptyset$  if and only if  $g\phi f\phi \neq \emptyset$  if and only if  $\alpha_g a = \alpha_{g\phi} \in \text{dom}(f\phi)$ . That is,  $\text{dom}(f\phi) = \text{dom}(f)a = \text{dom}(a^{-1}f) = \text{dom}(a^{-1}fa)$ .

Hence for all  $\alpha \in \text{dom}(f\phi) = \text{dom}(a^{-1}fa)$  we have

$$(\alpha)(f\phi) = \alpha_{f\phi} = (\alpha_f)a = (\alpha)a^{-1}fa.$$

Thus  $f\phi = a^{-1}fa$  for all  $f \in V$  and so  $\phi \in \text{Inn}(V)$ .  $\square$

**Lemma 2.2.** *Let  $U$  be a subsemigroup of  $P_\Omega$  such that for all  $\alpha \in \Omega$  there exists a constant idempotent in  $U$  with value  $\alpha$ . Then  $\text{Aut}(U) = \text{Inn}(U)$ .*

PROOF. Let  $V$  denote the set of all constants in  $U$  and let  $V_\alpha = \{f \in V \mid \text{im}(f) \subseteq \{\alpha\}\}$  for every  $\alpha \in \Omega$ . It is easy to check  $V_\alpha$  is a minimal left ideal of  $U$  and  $V_\alpha$  contains an idempotent. Consequently,  $V = \cup_{\alpha \in \Omega} V_\alpha$  is a subsemigroup of  $U$ . Conversely, let  $I \subseteq U$  be a minimal left ideal such that  $I$  contains an idempotent  $f$  where  $\alpha f = \alpha$  for some  $\alpha \in \Omega$ . Then  $V_\alpha = V_\alpha f \subseteq I$  and the minimality of  $I$  implies  $V_\alpha = I$ . As an automorphisms sends every minimal left ideal containing an idempotent into a minimal left ideal containing an idempotent, it follows that  $V\phi = V$  for all  $\phi \in \text{Aut}(U)$ .

Let  $\phi \in \text{Aut}(U)$ . Then we will prove that  $\phi \in \text{Inn}(U)$ . Since  $\text{Aut}(V) = \text{Inn}(V)$ , by Lemma 2.1 and the preceding paragraph, it follows that  $\phi|_V \in \text{Inn}(V)$ .

As in the proof of Lemma 2.1, for all  $f \in V \setminus \{\emptyset\}$ , we denote by  $\alpha_f$  the unique element in  $\text{im}(f)$ . Since  $\phi|_V \in \text{Aut}(V) = \text{Inn}(V)$ , there exists  $a \in S_\Omega$  such that

$$f\phi|_V = a^{-1}fa \text{ and } \alpha_f a = \alpha_{f\phi} = \alpha_{f\phi|_V}$$

for all  $f \in V$ .

Let  $f \in V$  and  $g \in U$  be arbitrary. Then, as in the proof of Lemma 2.1,  $\alpha_f \in \text{dom}(g)$  if and only if  $(\alpha_f)a = \alpha_{f\phi} \in \text{dom}(g\phi)$ . It follows that  $\text{dom}(g\phi)$  equals  $\text{dom}(a^{-1}ga)$ . Moreover,

$$(\alpha_f a)(g\phi) = (\alpha_{f\phi})(g\phi) = \alpha_{f\phi g\phi} = \alpha_{(fg)\phi} = (\alpha_{fg})a = (\alpha_{fg})a$$

and so  $g\phi : \alpha_f a \mapsto (\alpha_{fg})a$  for all  $\alpha_f \in \text{dom}(g)$ . That is,  $g\phi : \alpha \mapsto (\alpha)a^{-1}ga$  for all  $\alpha \in \text{dom}(g\phi)$ , as required. Finally,  $a^{-1}Ua = U$  follows by the assumption that  $\phi$  is an automorphism. Thus  $\phi$  is an inner automorphism.  $\square$

PROOF OF THEOREM 1.1. From Lemma 2.2, we have that  $\text{Aut}(U) = \text{Inn}(U)$ . It remains to prove that  $\text{Inn}(U) = \{\phi_a : f \mapsto a^{-1}fa \mid a \in \text{Aut}_H(\rho') \text{ and } a^{-1}Ua = U\}$ .

Let  $a \in S_\Omega \setminus \text{Aut}_H(\rho')$ . We will prove that  $a^{-1}Ua \neq U$ . From the definition of  $\text{Aut}_H(\rho')$ , we have that for every  $t \in H$  there exists  $(\alpha_1, \dots, \alpha_m) \in \rho'$  such that  $(\alpha_{1t}, \dots, \alpha_{mt})^a \notin \rho$ . There are two cases to consider.

**Case 1.** *there exist  $t \in H$  and  $(\beta_1, \dots, \beta_m) \in \rho'$  with  $(\beta_{1t}, \dots, \beta_{mt})^a \in \rho$ .*

Let  $(\alpha_1, \dots, \alpha_m) \in \rho'$  such that  $(\alpha_{1t}, \dots, \alpha_{mt})^a \notin \rho$ . Since  $U$  is transitive from  $\rho'$  to  $\rho'$ , there exists  $f \in U$  such that  $(\beta_1, \dots, \beta_m)^f = (\alpha_1, \dots, \alpha_m)$ . Then we have  $(\beta_{1t}, \dots, \beta_{mt})^f = (\alpha_{1t}, \dots, \alpha_{mt})$  and so

$$[(\beta_{1t}, \dots, \beta_{mt})^a]^{a^{-1}fa} = (\beta_{1t}, \dots, \beta_{mt})^{fa} = (\alpha_{1t}, \dots, \alpha_{mt})^a \notin \rho.$$

Hence  $a^{-1}fa \notin \text{PEnd}(\rho)$  and, in particular,  $a^{-1}fa \notin U$ , as required.

**Case 2.** *for all  $t \in H$  and for all  $(\beta_1, \dots, \beta_m) \in \rho'$  we have  $(\beta_{1t}, \dots, \beta_{mt})^a \notin \rho$ .*

By assumption,  $(\beta_1, \dots, \beta_m)^a \in N(\rho, H)$  for all  $(\beta_1, \dots, \beta_m) \in \rho'$ . Let  $t \in H$  and  $(\beta_1, \dots, \beta_m) \in \rho'$ . Assume that  $(\beta_{1t}, \dots, \beta_{mt})^{a^{-1}} \in \rho'$ . Then

$$[(\beta_{1t}, \dots, \beta_{mt})^{a^{-1}}]^a = (\beta_{1t}, \dots, \beta_{mt}) \in N(\rho, H).$$

Since  $t^{-1} \in H$ , it follows that  $(\beta_1, \dots, \beta_m) \notin \rho'$ , a contradiction. Hence for all  $t \in H$  and for all  $(\beta_1, \dots, \beta_m) \in \rho'$  we have that  $(\beta_{1t}, \dots, \beta_{mt})^{a^{-1}} \notin \rho'$ .

If  $(\beta_1, \dots, \beta_m) \in \rho'$ , then since  $U$  is transitive from  $N(\rho, H)$  to  $\rho'$ , there exists  $f \in U$  such that  $[(\beta_1, \dots, \beta_m)^a]^f = (\beta_1, \dots, \beta_m)$ . Then

$$(\beta_1, \dots, \beta_m)^{afa^{-1}} = (\beta_1, \dots, \beta_m)^{a^{-1}} \notin \rho.$$

Thus  $afa^{-1} \notin U$  and so  $a^{-1}fa \notin U$ , as required.

Let  $a, b \in \text{Aut}_H(\rho')$  such that  $a \neq b$  and  $\phi_a, \phi_b \in \text{Aut}(U)$ . It remains to prove that  $\phi_a \neq \phi_b$ . But there exists  $\alpha \in \Omega$  such that  $(\alpha)a \neq (\alpha)b$  and so  $\phi_a$  and  $\phi_b$  differ on any constant idempotent in  $U$  with value  $\alpha$ .  $\square$

### 3. Corollaries and Examples

Note that if  $\rho$  in Theorem 1.1 satisfies  $\rho' = \emptyset$  and  $U$  is a subsemigroup of  $\text{PEnd}(\rho)$  satisfying (i) in Theorem 1.1, then Theorem 1.1 offers no new information regarding  $\text{Aut}(U)$ . That is, as there are no tuples of distinct elements in  $\rho$ ,  $U$  vacuously acts transitively from  $N(\rho, H) \cup \rho'$  to  $\rho'$ . Moreover, in this case  $\text{Aut}_H(\rho') = S_\Omega$  and so Theorem 1.1 reasserts that

$$\text{Aut}(U) = \{ \phi_a : f \mapsto a^{-1}fa \mid a \in S_\Omega \text{ and } a^{-1}Ua = U \} = \text{Inn}(U).$$

However, this conclusion can be derived from the much weaker Lemma 2.2.

The following is a useful corollary of Theorem 1.1, which we will apply when finding automorphisms of several transformation semigroups in Section 4.

**Corollary 3.1.** *Let  $\rho$  be an  $m$ -ary relation on a finite set  $\Omega$  for some  $m \in \mathbb{N}$ , let  $H$  a subgroup of  $S_m$ , and let  $U \in \{\text{PEnd}(\rho), \text{End}(\rho), \text{IEnd}(\rho)\}$  such that the following hold*

- (i)  $U$  contains a constant idempotent with value  $\alpha$  for all  $\alpha \in \Omega$ ;
- (ii)  $U$  acts transitively from  $\rho' \cup N(\rho, H)$  to  $\rho'$ ;
- (iii)  $\text{Aut}_H(\rho') = \text{Aut}_H(\rho)$ .

Then  $\text{Aut}(U) \cong \text{Aut}_H(\rho)$ .

PROOF. We prove the corollary in the case that  $U = \text{End}(\rho)$ , the remaining cases can be proved analogously.

It follows from Theorem 1.1 and (iii) in the hypothesis of the corollary that

$$\text{Aut}(U) = \{ \phi_a : f \mapsto a^{-1}fa \mid a \in \text{Aut}_H(\rho) \text{ and } a^{-1}Ua = U \}$$

and that distinct elements in  $\text{Aut}_H(\rho)$  induce distinct automorphisms of  $U$ .

Let  $a \in \text{Aut}_H(\rho)$ . We will prove that  $a^{-1}Ua = U$ . Let  $f \in \text{End}(\rho)$  be arbitrary and let  $(\alpha_1, \dots, \alpha_m) \in \rho$ . Since  $a \in \text{Aut}_H(\rho)$ , there exists  $t \in H$  such that  $(\alpha_{1t^{-1}}, \dots, \alpha_{mt^{-1}})^{a^{-1}} \in \rho$ . Hence, since  $f \in U$ ,  $(\alpha_{1t^{-1}}, \dots, \alpha_{mt^{-1}})^{a^{-1}f} \in \rho$  and so  $(\alpha_1, \dots, \alpha_m)^{a^{-1}fa} \in \rho$ . Hence  $a^{-1}fa \in U$  and so  $a^{-1}Ua = U$ , as required.

It follows that  $\text{Aut}(U) = \{ \phi_a : f \mapsto a^{-1}fa \mid a \in \text{Aut}_H(\rho) \}$ . Let  $F : \text{Aut}_H(\rho) \longrightarrow \text{Aut}(U)$  be defined by  $(a)F = \phi_a$ . Then, since distinct elements in  $\text{Aut}_H(\rho)$  induce distinct automorphisms of  $U$  and  $\text{Aut}(U)$  is finite,  $F$  is a bijection. It is straightforward to verify that  $F$  is a homomorphism, and the corollary follows.  $\square$

**Corollary 3.2.** *Let  $\rho$  be a reflexive binary relation on a finite set  $\Omega$  and let  $H$  be a subgroup of  $S_2$ . Then  $\text{Aut}_H(\rho') = \text{Aut}_H(\rho)$ .*

PROOF. Since  $\rho' \subseteq \rho$ , it follows that  $\text{Aut}_H(\rho) \leq \text{Aut}_H(\rho')$ . To prove the converse, let  $a \in \text{Aut}_H(\rho')$  and let  $(\alpha, \beta) \in \rho$  be arbitrary. If  $\alpha \neq \beta$ , then either  $(\alpha, \beta)^a \in \rho' \subseteq \rho$  or  $(\beta, \alpha)^a \in \rho' \subseteq \rho$ , as required. If  $\alpha = \beta$ , then, since  $\rho$  is reflexive,  $(\alpha, \beta)^a \in \rho$  and  $(\beta, \alpha)^a \in \rho$ . Hence  $a \in \text{Aut}_H(\rho)$  and so  $\text{Aut}_H(\rho') = \text{Aut}_H(\rho)$ .  $\square$

Let  $U, \rho$ , and  $H$  be as in Theorem 1.1. Then the following example demonstrates that there can exist  $a \in \text{Aut}_H(\rho')$  such that  $a^{-1}Ua \neq U$ .

**Example 3.3.** Let  $\Omega = \{1, 2, 3, 4, 5\}$ , let  $H = S_5$ , and let

$$\rho = \{(1, 2, 3, 4, 5), (2, 1, 3, 4, 5)\} \cup \{(\alpha_1, \alpha_2, \dots, \alpha_5) \mid \{\alpha_1, \alpha_2, \dots, \alpha_5\} = \{3, 4\}\} \\ \cup \{(\alpha, \alpha, \alpha, \alpha, \alpha) \mid \alpha \in \{1, \dots, 5\}\}.$$

Then  $N(\rho, H) = \emptyset$ ,  $\text{Aut}(\rho') = S_{\{1,2\}} = \text{Aut}(\rho)$ ,  $S_{\{1,2\}} \times S_{\{3,4,5\}}$  is a subgroup of  $\text{Aut}_H(\rho')$ , and every element of  $\text{Aut}_H(\rho)$  stabilizes  $\{3, 4\}$  setwise. In particular,  $\text{Aut}_H(\rho') \neq \text{Aut}_H(\rho)$ .

Let  $U = \text{Aut}(\rho) \cup \{f \in T_5 \mid \text{im}(f) = \{3, 4\} \text{ or } |\text{im}(f)| = 1\}$ . Then  $U$  is a subsemigroup of  $\text{PEnd}(\rho)$  and, since  $\text{Aut}(\rho)$  is transitive from  $\rho$  to  $\rho'$ ,  $U$  is also transitive from  $\rho'$  to  $\rho'$ . Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 3 & 4 \end{pmatrix} \in U.$$

Then  $(3, 4, 3, 4, 3)^{(45)f(45)} = (3, 5, 3, 5, 3) \notin \rho$ . Hence  $(45)f(45) \notin U$  and so the element  $(45) \in \text{Aut}_H(\rho')$  does not induce an inner automorphism of  $U$ .

The following example shows that it is not true that if  $a \in \text{Aut}_H(\rho')$  and  $a^{-1}Ua = U$ , then  $a \in \text{Aut}_H(\rho)$ .

**Example 3.4.** Let  $\rho$  be the relation from Example 3.3 and let  $V = \text{Aut}(\rho) \cup \{f \in T_5 \mid f \text{ is constant}\}$ . Then, as in Example 3.3,  $V$  is transitive on  $\rho'$ . However,  $(45) \in \text{Aut}_H(\rho')$  and  $(45)V(45) = V$  but  $(45) \notin \text{Aut}_H(\rho)$ , as required.

#### 4. Applications I - Transformation semigroups

In this section we apply Theorem 1.1 to determine the automorphism groups of several well-known transformation semigroups, defined below. Some of the results contained in this section are well-known and included here only to illustrate how Theorem 1.1 can be used. Recall that  $T_\Omega$ ,  $P_\Omega$ , and  $I_\Omega$  denote the monoids of all total mappings, all partial mappings, and all partial injective mappings of the finite set  $\Omega$ , respectively. As above, if  $\Omega = \{1, 2, \dots, m\}$ , then we may write  $T_m$ ,  $P_m$ , or  $I_m$  instead of  $T_\Omega$ ,  $P_\Omega$ , or  $I_\Omega$ , respectively.

**Corollary 4.1.** Let  $U \in \{P_\Omega, T_\Omega, I_\Omega\}$  where  $\Omega$  is a finite set. Then  $\text{Aut}(U) \cong S_\Omega$ .

PROOF. Let  $\rho = \Omega \times \Omega$  and let  $H \leq S_2$  be arbitrary. Then

$$\text{PEnd}(\rho) = P_\Omega, \text{End}(\rho) = T_\Omega, \text{ and } \text{IEnd}(\rho) = I_\Omega.$$

In any case,  $U$  contains a constant idempotent with value  $\alpha$  for all  $\alpha \in \Omega$  and so part (i) of the hypothesis of Corollary 3.1 is satisfied. Now,  $N(\rho, H) = \emptyset$  and  $\rho' = (\Omega \times \Omega) \setminus \{(\alpha, \alpha) \mid \alpha \in \Omega\}$ . It is clear that  $U$  is transitive from  $\rho'$  to  $\rho'$  and so part (ii) of Corollary 3.1 is satisfied. Finally,  $\rho$  is a reflexive binary relation and so by Corollary 3.2,  $\text{Aut}_H(\rho') = \text{Aut}_H(\rho)$ . Thus, by Corollary 3.1,  $\text{Aut}(U) \cong \text{Aut}_H(\rho') = S_\Omega$ , as required.  $\square$

Next, we find the automorphisms of some wreath products of transformation semigroups. If  $S$  and  $T$  are semigroups acting or partially acting on sets  $\Omega$  and  $\Sigma$ , then the *wreath product* of  $S$  and  $T$  denoted  $S \wr T$  is the set  $S \times T^\Omega$  where  $T^\Omega$  denotes the set of mappings or partial mappings from  $\Omega$  to  $T$ , respectively, with multiplication

$$(s, f)(t, g) = (st, f \star^s g)$$

where  $(\alpha)^s g = (\alpha s)g$  for all  $\alpha \in \Omega$  and where  $(\alpha)f \star^s g = (\alpha)f \cdot (\alpha)^s g$ . The semigroup  $S \wr T$  acts on  $\Omega \times \Sigma$  as follows:  $(\alpha, \sigma)^{(s, f)} = (\alpha s, (\sigma)(\alpha f))$ . For further details about wreath products see [15] or [18].

**Corollary 4.2.** *Let  $U \in \{P_m \wr P_n, T_m \wr T_n, I_m \wr I_n\}$ . Then  $\text{Aut}(U) \cong S_m \wr S_n$ .*

PROOF. Let  $\Omega$  be a set of size  $mn$  and let  $\rho$  be an equivalence relation with  $m$  classes each of size  $n$ . Then  $\text{PEnd}(\rho) = P_m \wr P_n$ ,  $\text{End}(\rho) = T_m \wr T_n$ , and  $\text{IEnd}(\rho) = I_m \wr I_n$  (see [4, Lemma 2.1] for a proof). It is clear that  $U$  contains a constant idempotent with value  $\alpha$  for all  $\alpha \in \Omega$ .

Let  $H$  be the trivial subgroup of  $S_2$ . Then, since  $\rho$  is symmetric,  $N(\rho, H)$  contains all the pairs of distinct elements in  $\Omega^2 \setminus \rho'$ . We must prove that  $U$  is transitive from the pairs of distinct entries in  $\Omega^2$  to  $\rho'$ .

Let  $(\alpha, \beta) \in \Omega^2$  and  $(\gamma, \delta) \in \rho$  such that  $\alpha \neq \beta$  and  $\gamma \neq \delta$ . Then

$$f = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is an element of  $\text{IEnd}(\rho)$ , and so of  $\text{PEnd}(\rho)$ , such that  $(\alpha, \beta)^f = (\gamma, \delta)$ . Let  $g : \Omega \rightarrow \Omega$  be any total mapping with image  $\{\gamma, \delta\}$  such that  $\alpha g = \gamma$  and  $\beta g = \delta$ . Then  $g \in \text{End}(\rho)$  and again  $(\alpha, \beta)^g = (\gamma, \delta)$ .

It follows, by Corollaries 3.1 and 3.2, that  $\text{Aut}(U) \cong \text{Aut}(\rho) = S_m \wr S_n$ .  $\square$



### 5. Applications II – Ordered sets

In this section, we consider the automorphisms groups of semigroups of order-preserving partial mappings of a partially ordered set, and some related semigroups.

**Theorem 5.1.** *Let  $\Omega$  be a finite set, let  $\rho$  be partial order on  $\Omega$ , and let  $U \in \{\text{PEnd}(\rho), \text{End}(\rho), \text{IEnd}(\rho)\}$ . Then  $\text{Aut}(U)$  is isomorphic to the group of automorphisms and anti-automorphisms of  $\leq$ .*

PROOF. If  $\alpha, \beta \in \Omega$  such that  $(\alpha, \beta) \notin \rho$  and  $(\beta, \alpha) \notin \rho$ , then we will write  $\alpha \parallel \beta$ . An *anti-chain* is any subset  $\Sigma$  of  $\Omega$  where  $\alpha \parallel \beta$  for all  $\alpha, \beta \in \Sigma$ .

Since  $\rho$  is reflexive,  $\text{PEnd}(\rho)$ ,  $\text{End}(\rho)$ , and  $\text{IEnd}(\rho)$  contain a constant idempotent with value  $\alpha$  for all  $\alpha \in \Omega$ . If  $H = S_2$ , then

$$N(\rho, H) = \{ (\alpha, \beta) \in \Omega^2 \mid \alpha \parallel \beta \}.$$

We will prove that  $U$  is transitive from  $N(\rho, H) \cup \rho'$  to  $\rho'$ .

If  $\alpha, \beta, \gamma, \delta \in \Omega$  such that  $(\beta, \alpha) \notin \rho$  and  $(\gamma, \delta) \in \rho$ , then the mapping  $f$  from the proof of Corollary 4.2 is an element of  $\text{IEnd}(\rho)$  and  $\text{PEnd}(\rho)$  such that  $(\alpha, \beta)^f = (\gamma, \delta)$ .

It remains to prove that  $U$  is transitive from  $N(\rho, H) \cup \rho'$  to  $\rho'$  when  $U = \text{End}(\rho)$ . Let  $\Sigma$  be a subset of  $\Omega$ . Then we define

$$\Sigma^\wedge = \{ \beta \in \Omega \mid (\forall \alpha \in \Sigma) (\alpha, \beta) \in \rho' \text{ or } \beta \parallel \alpha \}$$

and

$$\Sigma^\vee = \{ \beta \in \Omega \mid (\forall \alpha \in \Sigma) (\beta, \alpha) \in \rho' \text{ or } \beta \parallel \alpha \}$$

If  $\Sigma$  is a maximal (with respect to containment) anti-chain in  $\rho$ , then the sets  $\Sigma^\wedge$ ,  $\Sigma$ , and  $\Sigma^\vee$  are disjoint and their union is the whole of  $\Omega$ .

Let  $\alpha, \beta, \gamma, \delta \in \Omega$  such that  $\alpha \neq \beta$ ,  $(\beta, \alpha) \notin \rho$ , and  $(\gamma, \delta) \in \rho'$ . If  $(\alpha, \beta) \in \rho$ , then since  $\Omega$  is finite, there exists a maximal antichain  $\Sigma$  in  $\rho$  such that  $\alpha \in \Sigma$ . It follows that  $\beta \in \Sigma^\wedge$ . Let  $f : \Omega \rightarrow \Omega$  be defined by

$$\epsilon f = \begin{cases} \gamma & \epsilon \in \Sigma \cup \Sigma^\vee \\ \delta & \epsilon \in \Sigma^\wedge. \end{cases}$$

Then  $f \in \text{End}(\rho)$  and  $(\alpha, \beta)^f = (\gamma, \delta)$ .

If  $(\alpha, \beta) \notin \rho$ , then as in the previous case, since  $\Omega$  is finite, there exists a maximal antichain  $\Sigma$  in  $\rho$  such that  $\alpha, \beta \in \Sigma$ . Let  $f : \Omega \rightarrow \Omega$  be defined by

$$\epsilon f = \begin{cases} \gamma & \epsilon \in (\Sigma \setminus \{\beta\}) \cup \Sigma^\vee \\ \delta & \epsilon \in \Sigma^\wedge \cup \{\beta\}. \end{cases}$$

Then  $f \in \text{End}(\rho)$  and  $(\alpha, \beta)^f = (\gamma, \delta)$ .

In any case,  $U$  is transitive from  $N(\rho, H) \cup \rho'$  to  $\rho'$ . Thus, by Corollaries 3.1 and 3.2,  $\text{Aut}(U) \cong \text{Aut}_{S_2}(\rho)$ , as required.  $\square$

If  $\rho$  is the usual total order of  $\{1, \dots, n\}$ , then  $\text{End}(\rho)$ ,  $\text{PEnd}(\rho)$ , and  $\text{IEnd}(\rho)$  are usually denoted  $\mathcal{O}_n$ ,  $\mathcal{PO}_n$ ,  $\mathcal{POI}_n$  (the semigroups of total, partial, and partial injective order-preserving mappings of the chain, respectively). These monoids have been extensively studied, for example see [1, 2, 8, 13, 14].

The following is an immediate corollary of Theorem 5.1.

**Corollary 5.2.** *If  $U \in \{\mathcal{PO}_n, \mathcal{O}_n, \mathcal{POI}_n\}$ , then*

$$\text{Aut}(U) = \langle f \mapsto f^{(1n)(2n-1)\cdots(\lfloor n/2 \rfloor \lfloor n/2 \rfloor + 1)} \rangle \cong C_2$$

where  $\lfloor n/2 \rfloor$  is the greatest integer less than  $n/2$ ,  $\lceil n/2 \rceil$  is the least integer greater than  $n/2$  and  $C_2$  denotes the cyclic group of order 2.  $\square$

Let  $\mathcal{OD}_n$ , let  $\mathcal{POD}_n$ , and let  $\mathcal{PODI}_n$  be the monoids of all total, partial, and partial injective order-preserving and order-reversing mappings of the chain  $1 \leq 2 \leq \dots \leq n$ , respectively. Again these monoids appear in several papers in the literature, for example see [9, 10, 11]. The automorphism groups of these monoids are given in the theorem below. However, the theorem is not a direct corollary of any of the preceding theorems, as these semigroups are not defined as the partial endomorphisms of a relation.

We require the following well-known combinatorial fact in order to find the automorphism groups of  $\mathcal{OD}_n$ ,  $\mathcal{POD}_n$ , and  $\mathcal{PODI}_n$ .

**Lemma 5.3.** *If  $p, q \in \mathbb{N}$ , then in any sequence of distinct natural numbers of length  $pq + 1$ , there exists a strictly increasing subsequence of length  $p$  or a strictly decreasing subsequence of length  $q$ .*

**Theorem 5.4.** *Let  $n \geq 10$  and  $U \in \{\mathcal{POD}_n, \mathcal{OD}_n, \mathcal{PODI}_n\}$ . Then  $\text{Aut}(U) = \langle s \mapsto s^{(1n)(2n-1)\cdots(\lfloor n/2 \rfloor \lfloor n/2 \rfloor + 1)} \rangle \cong C_2$ .*

PROOF. We prove the theorem in the case that  $U = \mathcal{OD}_n$ , the other cases follow by an analogous argument.

Let  $\rho$  be the ternary relation on  $\{1, \dots, n\}$  defined by

$$\rho = \{ (\alpha_1, \alpha_2, \alpha_3) \mid \alpha_1 \leq \alpha_2 \leq \alpha_3 \text{ or } \alpha_1 \geq \alpha_2 \geq \alpha_3 \}.$$

Then  $\mathcal{OD}_n \leq \text{End}(\rho)$ .

It is straightforward to verify that  $U$  is transitive from  $\rho' \cup N(\rho, S_3) = \rho'$  to  $\rho'$ , and  $U$  contains a constant idempotent with value  $\alpha$  for all  $\alpha \in \Omega$ .

We will prove that  $\text{Aut}(U) \cong \text{Aut}(\rho)$  and

$$\text{Aut}(\rho) = \langle (1n)(2n-1) \cdots (\lfloor n/2 \rfloor \lceil n/2 \rceil + 1) \rangle \cong C_2.$$

We begin by showing that  $\text{Aut}_{S_3}(\rho') = \text{Aut}_{S_3}(\rho)$ . Since  $\text{Aut}_{S_3}(\rho) \leq \text{Aut}_{S_3}(\rho')$  always holds, it suffices to show that  $\text{Aut}_{S_3}(\rho') \leq \text{Aut}_{S_3}(\rho)$ .

Let  $a \in \text{Aut}_{S_3}(\rho')$  and let  $(\alpha_1, \alpha_2, \alpha_3) \in \rho$ . Then there exists  $(\beta_1, \beta_2, \beta_3) \in \rho'$  such that  $\{\alpha_1, \alpha_2, \alpha_3\} \subseteq \{\beta_1, \beta_2, \beta_3\}$ . Since  $a \in \text{Aut}_{S_3}(\rho')$ , there exists  $t \in S_3$  such that  $(\beta_{1t}, \beta_{2t}, \beta_{3t})^a \in \rho$ . Hence  $(\alpha_{1t}, \alpha_{2t}, \alpha_{3t})^a \in \rho$  and so  $a \in \text{Aut}_{S_3}(\rho)$ .

It follows from Corollary 3.1 that  $\text{Aut}(U) \cong \text{Aut}_{S_3}(\rho)$ . The next step is to prove that  $\text{Aut}_{S_3}(\rho) = \text{Aut}(\rho)$ .

Clearly,  $\text{Aut}(\rho) \leq \text{Aut}_{S_3}(\rho)$ . To prove the converse inclusion, take arbitrary  $f \in \text{Aut}_{S_3}(\rho)$ . Then there exists  $t \in S_3$  such that  $(\alpha_{1t}, \alpha_{2t}, \alpha_{3t})^f \in \rho$  for all  $(\alpha_1, \alpha_2, \alpha_3) \in \rho$ . By Lemma 5.3, the sequence  $1f, 2f, \dots, 10f$  contains a subsequence  $\alpha_1 f, \alpha_2 f, \alpha_3 f$  where  $\alpha_1 < \alpha_2 < \alpha_3$  and  $\alpha_1 f < \alpha_2 f < \alpha_3 f$  or  $\alpha_1 f > \alpha_2 f > \alpha_3 f$ . Hence either  $(\alpha_1, \alpha_2, \alpha_3)^f \in \rho$  or  $(\alpha_3, \alpha_2, \alpha_1)^f \in \rho$ . This implies that  $t$  is the identity permutation  $\text{id}$  or  $t = (13)$ . In either case, by the definition of  $\rho$ ,  $f \in \text{Aut}(\rho)$ . Thus  $\text{Aut}(\rho) = \text{Aut}_{S_3}(\rho)$ .

The final step in the proof is to show that

$$\text{Aut}(\rho) = \langle (1n)(2n-1) \cdots (\lfloor n/2 \rfloor \lceil n/2 \rceil + 1) \rangle.$$

Let  $f \in \text{Aut}(\rho)$ . We start by proving that  $1f \in \{1, n\}$ . Assume the contrary, that is,  $1f \notin \{1, n\}$ . If  $nf \neq 1$ , then  $(1, 1f^{-1}, n) \in \rho$  but  $(1, 1f^{-1}, n)^f = (1f, 1, nf) \notin \rho$ , a contradiction. Therefore  $nf = 1$ . But  $(1, nf^{-1}, n) \in \rho$  and  $(1, nf^{-1}, n)^f = (1f, n, nf) = (1f, n, 1) \notin \rho$  since  $1f \neq n$ , a contradiction. Hence we have that  $1f \in \{1, n\}$ .

If  $1f = 1$  and  $\alpha, \beta \in \{1, \dots, n\}$  such that  $\alpha < \beta$ , then  $(1, \alpha, \beta) \in \rho$  and so  $(1, \alpha f, \beta f) \in \rho$ . Hence  $\alpha f < \beta f$  and so  $f$  is the identity on  $\{1, 2, \dots, n\}$ .

If  $1f = n$  and  $\alpha, \beta \in \{1, \dots, n\}$  such that  $\alpha < \beta$ , then  $(1, \alpha, \beta) \in \rho$  and so  $(n, \alpha f, \beta f) \in \rho$ . Thus  $\alpha f > \beta f$  and so  $f = (1n)(2n-1) \cdots (\lfloor n/2 \rfloor \lceil n/2 \rceil + 1)$ .  $\square$

A finite sequence  $(\alpha_1, \alpha_2, \dots, \alpha_m)$  of natural numbers is called *cyclic* if there exists  $k \geq 1$  such that  $\alpha_{1g^k} \leq \alpha_{2g^k} \leq \dots \leq \alpha_{mg^k}$ , where  $g = (12 \dots m)$ . A partial mapping with domain and range contained in  $\{1, 2, \dots, n\}$  is *orientation-preserving* if the image of every cyclic sequence in  $\text{dom}(f)$  is a cyclic sequence.

Let  $\mathcal{OP}_n$ , let  $\mathcal{POP}_n$ , and let  $\mathcal{POPI}_n$  be the monoids of all total, partial, and partial injective orientation-preserving mappings, respectively, of the set  $\{1, 2, \dots, n\}$ . Some references from the literature concerning these monoids are [6, 7, 9, 10, 12, 17].

**Theorem 5.5.** *Let  $U \in \{\mathcal{POP}_n, \mathcal{OP}_n, \mathcal{POPI}_n\}$ . Then  $\text{Aut}(U) = \langle f \mapsto f^{(12 \dots n)}, f \mapsto f^{(1n)(2n-1) \dots ([n/2] \lceil n/2 \rceil + 1)} \rangle \cong D_{2n}$  where  $D_{2n}$  denotes the dihedral group with  $2n$  elements.*

PROOF. We prove the theorem only in the case that  $U = \mathcal{OP}_n$ , the other cases follow by analogous arguments. Let  $\rho$  denote the set of all cyclic sequences of length 3 over  $\{1, 2, \dots, n\}$  and let  $H = \langle (13) \rangle \leq S_3$ . Then  $\mathcal{OP}_n \leq \text{End}(\rho)$  and  $N(\rho, H) = \emptyset$ . We will prove that  $\mathcal{OP}_n$  acts transitively from  $\rho'$  to  $\rho'$ . Let  $(i, j, k), (i', j', k') \in \rho'$  be arbitrary. Then either  $i < j < k, j < k < i$  or  $k < i < j$ . If  $i < j < k$ , then define

$$f = \begin{pmatrix} 1 & \dots & i-1 & i & i+1 & \dots & j-1 & j & j+1 & \dots & n \\ i' & \dots & i' & i' & j' & \dots & j' & j' & k' & \dots & k' \end{pmatrix}.$$

If  $j < k < i$ , then define

$$f = \begin{pmatrix} 1 & \dots & j-1 & j & j+1 & \dots & k-1 & k & k+1 & \dots & n \\ j' & \dots & j' & j' & k' & \dots & k' & k' & i' & \dots & i' \end{pmatrix}.$$

If  $k < i < j$ , then define

$$f = \begin{pmatrix} 1 & \dots & k-1 & k & k+1 & \dots & i-1 & i & i+1 & \dots & n \\ k' & \dots & k' & k' & i' & \dots & i' & i' & j' & \dots & j' \end{pmatrix}.$$

In any case,  $f \in \mathcal{OP}_n$  and  $(i, j, k)^f = (i', j', k')$ .

Next, we verify that  $\text{Aut}_H(\rho') = \text{Aut}_H(\rho)$ . As always, we have  $\text{Aut}_H(\rho) \leq \text{Aut}_H(\rho')$ . Let  $f \in \text{Aut}_H(\rho')$  and let  $(\alpha_1, \alpha_2, \alpha_3) \in \rho$  be arbitrary. Then there exists  $t \in H$  such that  $(\alpha_{1t}, \alpha_{2t}, \alpha_{3t})^f \in \rho'$ . If  $(\alpha_1, \alpha_2, \alpha_3) \in \rho \setminus \rho'$ , then  $(\alpha_3 f, \alpha_2 f, \alpha_1 f)$  and  $(\alpha_1 f, \alpha_2 f, \alpha_3 f)$  lie in  $\rho \setminus \rho'$ , as  $\rho$  contains all triples with at most two distinct elements. On the other hand, if  $(\alpha_1, \alpha_2, \alpha_3) \in \rho'$ , then, by definition we have  $(\alpha_{1t}, \alpha_{2t}, \alpha_{3t})^f \in \rho$ . Hence  $\text{Aut}_H(\rho') = \text{Aut}_H(\rho)$ .

Therefore, by Corollary 3.1,  $\text{Aut}(\mathcal{OP}_n) \cong \text{Aut}_H(\rho)$  and it remains to prove that  $\text{Aut}_H(\rho) = \langle (1\ 2 \cdots n), (1\ n)(2\ n-1) \cdots (\lfloor n/2 \rfloor \lceil n/2 \rceil + 1) \rangle$ . First of all  $(1\ 2 \cdots n)$  and  $(1\ n)(2\ n-1) \cdots (\lfloor n/2 \rfloor \lceil n/2 \rceil + 1)$  are in  $\text{Aut}_H(\rho)$ .

Let  $f \in \text{Aut}_H(\rho)$ . Then there exists  $t \in \{\text{id}, (1\ 3)\}$  such that  $(\alpha_1, \alpha_2, \alpha_3) \in \rho$  implies  $(\alpha_{1t}, \alpha_{2t}, \alpha_{3t})^f \in \rho$ . By postmultiplying by the appropriate power of  $(1\ 2 \cdots n)$ , if necessary, we may assume that  $1f = 1$ .

Let  $t = \text{id}$ . Then for any  $\alpha$  and  $\beta$  with  $1 < \alpha < \beta \leq n$  we have  $(1, \alpha, \beta) \in \rho$  and so  $(1, \alpha f, \beta f) \in \rho$  which implies that  $\alpha f < \beta f$ . Hence  $f$  is the identity automorphism from  $\text{Aut}_H(\rho)$ .

Let  $t = (1\ 3)$ . Then for any  $\alpha$  and  $\beta$  with  $1 < \alpha < \beta \leq n$  we have  $(1, \alpha, \beta) \in \rho$  and so  $(\beta f, \alpha f, 1) \in \rho$  which implies that  $\beta f < \alpha f$ . Therefore  $f = (1\ n)(2\ n-1) \cdots (\lfloor n/2 \rfloor \lceil n/2 \rceil + 1) \in \text{Aut}_H(\rho)$  and the proof is concluded.  $\square$

A finite sequence  $(\alpha_1, \alpha_2, \dots, \alpha_m)$  of natural numbers is called *anti-cyclic* if there exists  $k \geq 1$  such that  $\alpha_{1g^k} \geq \alpha_{2g^k} \geq \dots \geq \alpha_{mg^k}$ , where  $g = (1\ 2 \cdots m)$ . A partial mapping with domain and range contained in  $\{1, 2, \dots, n\}$  is *orientation-reversing* if the image of every cyclic sequence in  $\text{dom}(f)$  is an anti-cyclic sequence.

Let  $\mathcal{OR}_n$ , let  $\mathcal{POR}_n$ , and let  $\mathcal{PORI}_n$  be the monoids of all total, partial, and partial injective orientation-preserving and reversing mappings, respectively, of the set  $\{1, 2, \dots, n\}$ . For further details of the known results concerning these monoids see [5, 9, 10, 12].

**Theorem 5.6.** *Let  $n \geq 17$  and  $U \in \{\mathcal{POR}_n, \mathcal{OR}_n, \mathcal{PORI}_n\}$ . Then  $\text{Aut}(U) = \langle f \mapsto f^{(1\ 2 \cdots n)}, f \mapsto f^{(1\ n)(2\ n-1) \cdots (\lfloor n/2 \rfloor \lceil n/2 \rceil + 1)} \rangle \cong D_{2n}$ .*

PROOF. We prove the theorem in the case that  $U = \mathcal{OR}_n$ , the remaining cases can be proved analogously. Let  $\rho$  be the set of cyclic sequencers of length 3 and let  $\sigma$  be the set of quadruples  $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  such that there exists  $t \in \{\text{id}, (1\ 3)\}$  and for all subsequences  $(\beta_1, \beta_2, \beta_3)$  of  $\vec{\alpha}$  we have that

$$(\beta_{1t}, \beta_{2t}, \beta_{3t}) \in \rho.$$

Now we will prove that  $\mathcal{OR}_n \leq \text{End}(\sigma)$ . Let  $f \in \mathcal{OR}_n$  and  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \sigma$ . Since  $f \in \mathcal{OR}_n$ , there exists  $s \in \{\text{id}, (1\ 3)\}$  such that  $(\gamma_{1s}, \gamma_{2s}, \gamma_{3s})^f \in \rho$  for all  $(\gamma_1, \gamma_2, \gamma_3) \in \rho$ . By definition of  $\sigma$ , there exists  $t \in \{\text{id}, (1\ 3)\}$  such that for all subsequences  $(\beta_1, \beta_2, \beta_3)$  of  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  we have that  $(\beta_{1t}, \beta_{2t}, \beta_{3t}) \in \rho$ . Therefore by definition of  $\sigma$ , we have that  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^f$  lies in  $\sigma$  with the correspondent permutation  $st \in \{\text{id}, (1\ 3)\}$ .

Next we will prove that  $\sigma'$  consists of all cyclic and anti-cyclic sequences of length 4 with 4 distinct elements. In one direction this is obvious – all the

cyclic and anti-cyclic sequences of length 4 lie in  $\sigma$ . Let  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \sigma'$ . Let the correspondent permutation  $t$  for this sequence be  $\text{id}$ . Then any subsequence  $(\beta_1, \beta_2, \beta_3)$  from  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  is cyclic. This means that if  $\alpha_i$  is the minimum among  $\alpha_1, \alpha_2, \alpha_3$  and  $\alpha_4$ , then since  $(\alpha_i, \alpha_{i+1}, \alpha_{i+2})$  and  $(\alpha_i, \alpha_{i+1}, \alpha_{i+3})$  are cyclic, we have  $\alpha_i < \alpha_{i+1} < \alpha_{i+2} < \alpha_{i+3}$  (here we worked modulo 4). Therefore  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  is cyclic. Analogously, it can be proved that in the case when  $t = (13)$ ,  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  must be anti-cyclic.

Let  $H = S_4$ . Then  $N(\sigma, H) = \emptyset$ . Now we will prove that  $U$  acts transitively from  $\sigma'$  to  $\sigma'$ . Let  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$  and  $(\beta_1, \beta_2, \beta_3, \beta_4)$  be any from  $\sigma'$ , i.e. any cyclic or anti-cyclic sequences of length 4 each of 4 distinct elements. There exist  $p, q \geq 1$  such that  $\alpha_1 g^p < \alpha_2 g^p < \alpha_3 g^p < \alpha_4 g^p$  and  $\beta_1 g^q < \beta_2 g^q < \beta_3 g^q < \beta_4 g^q$ , where  $g = (12 \cdots n)$ . There exists an order-preserving function  $f \in \mathcal{O}_n$  which extends  $\alpha_i g^p \mapsto \beta_i g^q$  for all  $i \in \{1, 2, 3, 4\}$ . Then  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^{g^p f g^{-q}} = (\beta_1, \beta_2, \beta_3, \beta_4)$  and since  $f, g \in \mathcal{OR}_n$ ,  $g^p f g^{-q} \in \mathcal{OR}_n$ .

Next, we will prove that  $\text{Aut}_H(\sigma') = \text{Aut}_H(\sigma) = \text{Aut}(\sigma)$ .

The first step is to show that  $\text{Aut}_H(\sigma') = \text{Aut}(\sigma')$ . Let  $f \in \text{Aut}_H(\sigma')$ . Then there exists  $t \in S_4$  such that  $(\alpha_{1t}, \alpha_{2t}, \alpha_{3t}, \alpha_{4t})^f \in \sigma'$  for all  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \sigma'$ . By Lemma 5.3, the sequence  $1f, \dots, 17f$  contains a subsequence  $\beta_1 f, \beta_2 f, \beta_3 f, \beta_4 f$  where either  $\beta_1 f < \beta_2 f < \beta_3 f < \beta_4 f$  or  $\beta_1 f > \beta_2 f > \beta_3 f > \beta_4 f$ . Therefore  $t \in \{\text{id}, (14)(23)\}$ . Since  $\sigma'$  consists of all cyclic and anti-cyclic sequences of length 4 of distinct 4 elements, we have that  $f \in \text{Aut}(\sigma')$ . Hence  $\text{Aut}_H(\sigma') \leq \text{Aut}(\sigma')$ . As observed above the converse inequality always holds, and so this step is concluded. That  $\text{Aut}_H(\sigma) = \text{Aut}(\sigma)$  follows by a similar argument.

Next, we prove that  $\text{Aut}(\sigma') = \text{Aut}(\sigma)$ . Again it suffices to prove that  $\text{Aut}(\sigma') \leq \text{Aut}(\sigma)$ . So, let  $f \in \text{Aut}(\sigma')$  and  $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \sigma$ . If  $|\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}| = 4$ , then  $\vec{\alpha}^f \in \sigma$  since  $f \in \text{Aut}(\sigma')$ . If  $|\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}| < 3$ , then  $\vec{\alpha}^f \in \sigma$ , as  $\sigma$  contains all quadruples containing at most 2 distinct elements. Let, finally,  $|\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}| = 3$ . Among all 4 subsequences of  $\vec{\alpha}$  of length 3 there will be two subsequences of distinct elements and these two subsequences must be equal. If this common sequence of distinct elements is  $(\beta_1, \beta_2, \beta_3)$  then it is cyclic or anti-cyclic. Then there exists a cyclic or anti-cyclic sequence  $\vec{\beta}$  with 4 distinct elements  $\vec{\beta}$  with subsequence  $(\beta_1, \beta_2, \beta_3)$ . As we showed earlier, it follows that  $\vec{\beta} \in \sigma'$ . Hence by definition  $(\beta_1, \beta_2, \beta_3)^f \in \rho$  or  $(\beta_3, \beta_2, \beta_1)^f \in \rho$ . Let  $(\gamma_1, \gamma_2, \gamma_3)$  be any of the 2 remaining subsequences of  $\vec{\alpha}$  of length 3. Then  $|\{\gamma_1, \gamma_2, \gamma_3\}| \leq 2$  and so  $(\gamma_1, \gamma_2, \gamma_3)^f \in \rho$  and  $(\gamma_3, \gamma_2, \gamma_1)^f \in \rho$ . Therefore again  $\vec{\alpha}^f \in \sigma$ . Thus, we proved that  $f \in \text{Aut}(\sigma)$ , as required.

So, by Corollary 3.1,  $\text{Aut}(U) \cong \text{Aut}_H(\sigma) = \text{Aut}(\sigma)$  and we are left to prove that  $\text{Aut}(\sigma) = \langle (12 \cdots n), (1n)(2n-1) \cdots \rangle$ .

The inclusion  $\langle (1\ 2\ \cdots\ n), (1\ n)(2\ n-1)\ \cdots \rangle \leq \text{Aut}(\sigma)$  is obvious. Let  $f \in \text{Aut}(\sigma)$ . By post multiplying by a power of  $(1\ 2\ \cdots\ n)$ , if necessary, we may assume that  $1f = 1$ . We will prove that  $f \in \{\text{id}, (2\ n)(3\ n-1)\ \cdots\}$ , this will complete the proof.

First we prove that  $2f \in \{2, n\}$ . Assume that this is not the case. Then either  $nf \neq n$  or  $nf = n$ . If the former is true, then  $(1, 2, nf^{-1}, n) \in \sigma$ . But

$$(1, 2, nf^{-1}, n)^f = (1, 2f, n, nf) \notin \sigma,$$

a contradiction. In the latter case,  $(1, 2, 2f^{-1}, n) \in \sigma$  but

$$(1, 2, 2f^{-1}, n)^f = (1, 2f, 2, n) \notin \sigma,$$

a contradiction.

Hence  $2f \in \{2, n\}$  and by symmetry  $nf \in \{2, n\}$ . By post multiplying by  $(2\ n)(3\ n-1)\ \cdots$ , if necessary, we deduce that  $2f = n$  and  $nf = 2$ . Now, if  $2 < \alpha < \beta < n$ , then  $(2, \alpha, \beta, n)^f = (n, \alpha f, \beta f, 2) \in \sigma$  and so  $\alpha f > \beta f$ . This implies that  $f = (2\ n)(3\ n-1)\ \cdots$  concluding the proof.  $\square$

## 6. Further remarks

So far we applied our main result, Theorem 1.1, to describe the automorphisms of semigroups satisfying the conditions of the theorem. In this section we provide an example of a transformation semigroup which does not satisfy the conditions of Theorem 1.1, but still, using the developed machinery, we are able to calculate its automorphism group. This semigroup was first introduced by Madhavan [16] and is constructed as follows. Let  $\rho$  be an equivalence relation on a finite set  $\Omega$ . A partial mapping  $f \in \text{PEnd}(\rho)$  is named a  $\rho$ -mapping if the following hold:

- (1)  $(x, y) \in \rho$  implies  $xf = yf$  for all  $x, y \in \text{dom}(f)$ ;
- (2)  $(xf, yf) \in \rho$  implies  $(x, y) \in \rho$  for all  $x, y \in \text{dom}(f)$ ;
- (3) if  $x\rho$  denotes the equivalence class of  $x$  in  $\rho$ , then  $x\rho \cap \text{dom}(f) \neq \emptyset$  implies  $x\rho \subseteq \text{dom}(f)$  for all  $x \in \Omega$ .

The collection of all  $\rho$ -mappings forms a semigroup denoted by  $F_\rho(\Omega)$ . This semigroup is regular and satisfies the law

$$(e^2 = e \wedge f^2 = f \wedge g^2 = g) \longrightarrow (efg = feg).$$

Moreover, the semigroups  $P_\rho(\Omega)$  play a role analogous to the role of  $S_\Omega$  for groups, for the class of regular semigroups which satisfy this law.

Obviously  $P_\rho(\Omega) \leq \text{PEnd}(\rho)$  satisfies condition (i) of Theorem 1.1. However condition (ii) is not satisfied. In fact, for any  $(\alpha, \beta) \in \rho'$  there are no  $(\delta, \gamma) \in \Omega \times \Omega$  and  $f \in P_\rho(\Omega)$  with  $(\delta, \gamma)^f = (\alpha, \beta)$ . Thus we cannot apply Theorem 1.1 to determine the automorphism group of  $P_\rho(\Omega)$ . Nonetheless, we can determine the automorphism group of  $P_\rho(\Omega)$ .

**Theorem 6.1.** *Let  $\rho$  be an equivalence relation on a finite set  $\Omega$  and let  $P_\rho(\Omega)$  be the Madhavan semigroup of  $\rho$ . Then  $\text{Aut}(P_\rho(\Omega)) \cong \text{Aut}(\rho)$ .*

PROOF. By Lemma 2.2, it follows that

$$\text{Aut}(P_\rho(\Omega)) = \{ \phi_a : f \mapsto a^{-1}fa \mid a \in S_\Omega \text{ and } a^{-1}P_\rho(\Omega)a = P_\rho(\Omega) \}.$$

We prove that the permutations  $a \in S_\Omega$  such that  $a^{-1}P_\rho(\Omega)a = P_\rho(\Omega)$  are exactly the elements of  $\text{Aut}(\rho)$ . It is straightforward to verify that if  $f \in \text{Aut}(\rho)$ , then  $f^{-1}P_\rho(\Omega)f = P_\rho(\Omega)$ . Let  $f \in S_\Omega \setminus \text{Aut}(\rho)$  and suppose that  $f^{-1}P_\rho(\Omega)f = P_\rho(\Omega)$ . Then there exists  $(\alpha, \beta) \in \rho$  such that  $(\alpha, \beta)^f \notin \rho$ .

Let  $g$  be a partial mapping which sends  $\alpha\rho$  to  $\alpha f$  and where  $\text{dom}(g) = \alpha\rho$ . Then  $g \in P_\rho(\Omega)$  and so  $f^{-1}gf \in P_\rho(\Omega)$ . But  $(\alpha f)f^{-1}gf = (\alpha g)f = \alpha f^2$  and  $(\beta f)f^{-1}gf = (\beta g)f = \alpha f^2$ , a contradiction since  $(\alpha f, \beta f) \notin \rho$ .

We have shown that  $\text{Aut}(P_\rho(\Omega)) = \{ \psi_a : f \mapsto a^{-1}fa \mid a \in \text{Aut}(\rho) \}$ . Since  $P_\rho(\Omega)$  contains a constant idempotent with value  $\alpha$  for all  $\alpha \in \Omega$ , it follows, by the arguments at the end of the proofs of Theorem 1.1 and Corollary 3.1, that  $\text{Aut}(P_\rho(\Omega)) \cong \text{Aut}(\rho)$ .  $\square$

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DEPARTAMENTO DE MATEMÁTICA DA UNIVERSIDADE ABERTA RUA DA ESCOLA POLITÉCNICA,  
147, 1269-001 LISBOA, PORTUGAL

*E-mail:* mjoao@ptmat.fc.ul.pt

CENTRO DE LÁGEBRA DA UNIVERSIDADE DE LISBOA AV. PROF. GAMA PINTO, 2, 1649-003 LISBOA,  
PORTUGAL

*E-mail:* vhf@fct.unl.pt

DEPARTAMENTO DE MATEMÁTICA DA FACULDADE DE CIÊNCIAS E TECNOLOGIA DA UNIVERSI-  
DADE NOVA DE LISBOA QUINTA DA TORRE, 2829-516 CAPARICA, PORTUGAL

*E-mail:* messias@ptmat.fc.ul.pt

MATHEMATICAL INSTITUTE, ST ANDREWS UNIVERSITY, NORTH HAUGH, ST ANDREWS, FIFE,  
UK, KY16 9SS

*E-mail:* victor@mcs.st-and.ac.uk

MATHEMATICAL INSTITUTE, ST ANDREWS UNIVERSITY, NORTH HAUGH, ST ANDREWS, FIFE,  
UK, KY16 9SS

*E-mail:* jdm3@st-and.ac.uk