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Axiomatizations of arithmetic and the first-order/second-order divide

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Abstract It is often remarked that first-order Peano Arithmetic is non-categorical but deductively well-behaved, while second-order Peano Arithmetic is categorical but deductively ill-behaved. This suggests that, when it comes to axiomatizations of mathematical theories, expressive power and deductive power may be orthogonal, mutually exclusive desiderata. In this paper, I turn to Hintikka's (Philos Top 17(2):69–90, 1989) distinction between descriptive and deductive approaches in the foundations of mathematics to discuss the implications of this observation for the first-order logic versus second-order logic divide. The descriptive approach is illustrated by Dedekind's 'discovery' of the need for second-order concepts to ensure categoricity in his axiomatization of arithmetic; the deductive approach is illustrated by Frege's *Begriffsschrift* project. I argue that, rather than suggesting that any use of logic in the foundations of mathematics is doomed to failure given the impossibility of combining the descriptive approach with the deductive approach, what this apparent predicament in fact indicates is that the first-order versus second-order divide may be too crude to investigate what an adequate axiomatization of arithmetic should look like. I also conclude that, insofar as there are different, equally legitimate projects one may engage in when working on the foundations of mathematics, there is no such thing as the One True Logic for this purpose; different logical systems may be adequate for different projects.

Keywords Axiomatizations of arithmetic · First-order logic · Second-order logic · Categoricity · Logical pluralism

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1 Introduction

It is a well-known fact that first-order Peano Arithmetic (PA1) is not categorical, i.e. it does not uniquely describe the sequence of the natural numbers that is typically viewed as the 'intended model' of arithmetic. Indeed, PA1 equally describes structures that strictly contain the sequence of the natural numbers but are not isomorphic to it, and these are known as the non-standard models of arithmetic. It is equally well known that second-order Peano Arithmetic (PA2) in turn, *is* categorical in that it is satisfied only by the intended model of arithmetic, namely the series of natural numbers, and by models isomorphic to the intended one.

However, what PA2 offers in terms of obtaining categoricity, it takes away in terms of deductive tractability. Because second-order logic has an 'ill-behaved' (non-axiomatizable) underlying notion of logical consequence, any second-order theory will presumably inherit its deductive shortcomings. Thus, apparently we cannot have our arithmetical cake and eat it: we can either have categoricity [with (PA2)] or a deductively well-behaved account of arithmetical theorems [with (PA1)], but not both. In fact, the conflict between the desiderata of expressive power and of deductive power with respect to axiomatizations of arithmetic is an instantiation of a more general phenomenon, namely the conflict between expressiveness and tractability (Levesque and Brachman 1987).

This situation has received a number of philosophical interpretations. Tennant (2000) pessimistically describes it as a 'pre-Gödelian predicament', and argues that it represents the impossibility of the project of 'monomathematics'. Read (1997) is less pessimistic and observes that, contrary to what many seem to think, Gödel's incompleteness results do not represent the total failure of Frege's logicist project because categoricity for arithmetic *can* be obtained with logical (albeit second-order) axiomatizations of arithmetic (as had been shown already by Dedekind). Hintikka (1989) draws on similar observations to distinguish two different uses of logic for the foundations of mathematics—the descriptive use and the deductive use—and three senses of completeness: semantic completeness, deductive completeness, and descriptive completeness (categoricity).

In what follows, I take Hintikka's distinction between descriptive and deductive uses of logic in (the foundations of) mathematics as my starting point to discuss what the apparent impossibility of having our arithmetical cake and eating it (i.e. of combining deductive power with expressive power to characterize arithmetic with logical tools) means for the first-order logic versus second-order logic debate. It is often argued [as discussed in (Rossberg 2004) and (Bueno 2010)] that the problematic status of the second-order consequence relation is sufficient to exclude second-order logic from the realm of what counts as 'logic'. However, this criticism presupposes that the deductive use must take precedence over the descriptive use, a claim that is both historically and philosophically contentious. I argue that, if logical systems are viewed first and foremost as *tools* to be applied for the investigation of different subject matters, and if different applications are prima facie equally legitimate (for example, Hintikka's descriptive and deductive uses), then the descriptive incompleteness of first-order logic with respect to arithmetic is just as serious as the deductive limitations of second-order logic (in this case, not restricted to arithmetic).[1] These observations suggest a modest form of logical pluralism (Russell 2013): in the foundations of mathematics, there is no such thing as the One True Logic, but only different logics appropriate for different applications.

The paper proceeds as follows. In the first two sections, I discuss Hintikka's descriptive use and deductive use of logic in mathematics: I illustrate the former with Dedekind's search for a categorical characterization of arithmetic, and the latter with Frege's search for a tool that would allow for gap-free formulations of mathematical proofs. I then argue that the difficulty with combining the descriptive use with the deductive use is in fact not as fundamental as it might seem. I conclude by offering some remarks on the so-called 'dispute' between first- and second-order logic, arguing that there are a number of reasons to think that there is no such real dispute after all, and that a form of pragmatic logical pluralism emerges as a plausible position.

2 The descriptive use

Hintikka describes the descriptive use of logic for investigations on the foundations of mathematics in the following terms:

> [This is] the use of logical notions … for the purpose of capturing certain structures, viz., the different structures studied in various mathematical theories. The pursuit of this task typically leads to the formulation of axiom systems which use the logical concepts just mentioned for different mathematical theories. (Hintikka 1989, p. 64)

In this sense, logical notions have a descriptive use in mathematics insofar as they are used to *describe* certain mathematical structures such as the sequence of the natural numbers, geometric systems of points and lines etc. Indeed, most of the early uses of axiomatization in the foundations of mathematics aimed at (complete) descriptions of portions of mathematical theory by means of accurate descriptions of the underlying mathematical *structures* (Awodey and Reck 2002a). Some examples are Dedekind on arithmetic, and Veblen on geometry.

In recent (still unpublished) work, Blanchette (2014) contrasts two approaches to logic in the early days of formal axiomatics, which she refers as the 'deductivist approach' (exemplified by Hilbert and Peano) and the 'model-centric' approach (exemplified by Dedekind and Veblen). The model-centric approach has models (structures) as the subject matter of investigation, and categoricity (i.e. uniquely characterizing the target domain up to isomorphism) as the main goal. Indeed, Blanchette's account of the 'model-centric' approach is essentially equivalent to Hintikka's account of the descriptive use of logic in mathematics (We will see later on that the same basically

[1] Naturally, PA1 is not that much of a success in terms of deductive power either. For starters, the underlying logic, first-order logic, is only semi-decidable, and of course, any (first-order or otherwise) system containing arithmetic (in a suitable sense of containing) is itself deductively incomplete. However, in comparison with the deductively hopeless PA2, PA1 may still seem more promising.

holds of Blanchette's deductivist approach and Hintikka's deductive use, but with some important qualifications).

Hintikka's distinction is of course also related to the modern model-theory versus proof-theory distinction. However, these two distinctions are conceptually not equivalent: Hintikka's distinction pertains foremost to the *goals* of the theorist, not to the technical apparatus used (though of course goals will to a great extent determine what apparatus is most suitable in each case). Indeed, a theorist adopting the deductive perspective may use models to investigate the non-deducibility of certain claims (as done e.g. by Hilbert in his work on geometry), but may still not be interested in the particular properties of these models as such: they are then just means to an end, not the end of the investigation themselves. So for the present purposes, we will stick to Hintikka's formulation of the distinction (which is nearly equivalent to Blanchette's distinction).

In the descriptive approach, the axiomatizer starts with a specific, presumably unique mathematical structure in mind, and then attempts to describe this structure accurately and completely by means of axioms formulated with logical notions. If such a description exhausts all of the relevant properties of the structure in question, then the axiomatization will not only describe the structure accurately, but also *uniquely*: it will be a description of nothing other than the intended mathematical structure in question (or structures identical to it according to the relevant parameters, e.g. isomorphism). An axiomatization that achieves this kind of completeness—which in turn yields uniqueness—is said to be *categorical*.

It is immediately apparent that the crucial feature of a system of logical notions for the descriptive use is *expressive power*: the more expressive the language is, the more fine-grained the description is likely to be, as the language will have the resources to express a greater number of the relevant properties of the structure. Note that the descriptive approach is compatible with (though it does not necessitate) a platonic conception of mathematical structures, according to which they have some sort of antecedent, independent and fully determined existence. The task of the mathematician is then to describe and investigate the pre-determined properties of these structures.[2] (The descriptive approach is also compatible with an anti-Platonist conception of mathematical objects, which are then viewed as 'created' by the mathematician herself, and then described by logical means).

As mentioned above, what Hintikka presents as the descriptive use of logic in mathematics was arguably the predominant approach in the early days of formal axiomatics, in the second half of the 19th century (one notable exception being Frege, more on whom shortly). And indeed, Dedekind's famous letter to Keferstein is quite possibly one of the most vivid illustrations of the descriptive approach in action; because the axiomatic project was still something of a novelty at the time, Dedekind carefully explains his procedure when formulating an axiomatization for arithmetic (basically what is now known, by a twist of fate, as Peano Arithmetic).

[2] Naturally, how the mathematician has *epistemic access* to the properties of these abstract structures is a notoriously thorny epistemic problem for the Platonist—the well-known 'Benacerraf challenge' (Benacerraf 1973).

> How did my essay come to be written? Certainly not in one day; rather, it is a synthesis constructed after protracted labor, based upon a prior analysis of the sequence of natural numbers just as it presents itself, in experience, so to speak, for our consideration. What are the mutually independent fundamental properties of the sequence N, that is, those properties that are not derivable from one another but from which all others follow?[3] (Dedekind, Letter to Keferstein, p. 99/100)

Thus, a pre-existing structure, the sequence of natural numbers, presents itself 'for our consideration', so that we can attempt to determine what its basic properties are. Dedekind then lists what appear to be the key properties of this structure, such as that it is composed of individuals called numbers, which in turn stand in a very special kind of relation to one another (the successor relation). (At this point in the text, these are the properties that can be expressed in purely first-order terminology, so the list is not yet complete).

One might think that, by offering an (apparently) exhaustive list of properties which, taken together, seem to describe the basic facts about this structure, the axiomatization would be complete also in the sense of picking out a *unique referent*, namely the intended structure, the sequence of the natural numbers. But Dedekind quickly adds that this is (unfortunately) not the case:

> I have shown in my reply, however, that these facts are still far from being adequate for completely characterising the nature of the number sequence N. All these facts would hold also for every system S that, besides the number sequence N, contained a system T, of arbitrary additional elements t [satisfying certain conditions previously stated]. But such a system S is obviously something quite different from our number sequence N, and I could so choose it that scarcely a single theorem of arithmetic would be preserved in it. What, then, must we add to the facts above in order to cleanse our system S again of such alien intruders t as disturb every vestige of order and to restrict it to N? (Dedekind, Letter to Keferstein, p. 100)

In other words: while the properties he had just listed are all present in the sequence N, they do not seem to *exhaust* the relevant properties of this structure, because they are equally true of other structures which are demonstrably very different from N. So an axiomatization guided only by these properties is not categorical because it does not uniquely refer to the intended sequence N; indeed, it also refers to structures strictly containing N but also containing additional, disruptive elements. How doe we get rid of these alien intruders?

[3] However, the paragraph then continues (emphasis added): "And how should we divest these properties of their specifically arithmetic character so that they are subsumed under more general notions and under *activities of the understanding* without which no thinking is possible at all but with which a foundation is provided for the reliability and completeness of proofs and for the construction of consistent notions and definitions?" So there is a sense in which Dedekind is *also* interested in the deductive perspective. (I owe this point to an anonymous referee.) Nevertheless, the focus of his work on the natural numbers remains descriptive and 'model-centric', as also argued by Blanchette. (It is also interesting to notice the clear Kantian undertones of this passage).

For our purposes, it is important to notice again that the properties listed by Dedekind up to this point have in common the fact that they can all be expressed by means of (what we now refer to as) purely first-order terminology. To be sure, that there may be an important distinction between first- and higher-order logics is something that started to become more widely acknowledged only in the 1930s, after Gödel's 1929 completeness theorem.[4] Until then, first-order logic was not recognized as a privileged, particularly stable fragment of the logical system developed for the logicist project of Russell and Whitehead. So Dedekind had no reason to notice this peculiarity about these properties, or to make an effort to exclude higher-order terminology.

Dedekind then notices that solving the issue of the alien intruders t was the hardest part of his enterprise, as the problem is:

> How can I, without presupposing any arithmetic knowledge, give an unambiguous conceptual foundation to the distinction between the elements n [the legitimate numbers] and the elements t [the alien intruders]? (Dedekind, Letter to Keferstein, p. 101)

The solution to this conundrum is offered by the technical notion of *chains*, which he had introduced in previous work. He explains this notion in the following terms:

> … an element n of S belongs to the sequence N if and only if n is an element of *every* part K of S that possesses the following two properties: (i) the element 1 belongs to K and (ii) the image $\varphi(K)$ is a part of K. (Dedekind, Letter to Keferstein, p. 101, emphasis in the original)

(The function φ had been introduced previously). Dedekind's notion of chain can be glossed in modern terminology as "the minimal closure of a set A in a set B containing A under a function f on B (where being "minimal" is conceived of in terms of the general notion of intersection)." (Reck 2011, section 2.2) What matters for the present purposes is that the notion of chain thus understood, i.e. in terms of the key concept of *closure*, involves quantification over *sets* of elements n, and thus cannot be expressed solely with first-order terminology.

Dedekind correctly claims that the notion of chain thus formulated offers a satisfactory solution to the problem of the alien intruders t. One might of course consider the possibility of there being other solutions to the problem of alien intruders not requiring the move to second-order quantification, which he did not envisage. However, it is now well known that first-order axiomatizations of arithmetic are inherently non-categorical [as per the usual Löwenheim–Skolem and compactness considerations—(Read 1997)]. At least a modicum of second-order terminology is indeed required for an axiomatization to describe only the 'intended structures'—the sequence of the natural numbers and structures isomorphic to it—and to exclude what are known as the non-standard models of arithmetic.[5]

[4] See (Shapiro 1991, Chapter 7) for the history of the 'triumph' of first-order languages.

[5] Why the so-called intended models of arithmetic should have a privileged status vis-à-vis the so-called non-standard models is in itself an interesting philosophical question. Many researchers working on the

The need for second-order terminology to achieve categoricity in the case of axiomatizations of arithmetic is an illustration of the general point that the descriptive use of logic for mathematics, as defined by Hintikka, will generally require quite expressive logical languages. Arguably, first-order languages will systematically fail to deliver the expressive power required for the precise description of non-trivial, infinite mathematical structures, and this may be one of the reasons for the (purported) inadequacy of first-order logic to account for ‘ordinary’ mathematical practice (Shapiro 1985).

3 The deductive use

Hintikka describes the deductive use of logic for investigations in the foundations of mathematics in the following terms:

> In order to facilitate, systematize, and criticize mathematicians’ reasoning about the structures they are interested in, logicians have isolated various valid inference patterns, systematized them, and even constructed various ways of mechanically generating an infinity of such inference patterns. I shall call this the *deductive* use of logic in mathematics. (Hintikka 1989, p. 64)

So the main difference between the descriptive and the deductive uses, as Hintikka conceives of them, seems to be that the objects of the descriptive use are the mathematical structures themselves, whereas the object of the deductive use is the mathematician’s *reasoning* about these very structures. This is an important distinction, but it would be a mistake to view the deductive use merely as seeking to emulate the *actual* reasoning practices of mathematicians. Typically, the idea is to produce a *rational reconstruction* that does not necessarily mirror the actual inferential steps of an ordinary mathematical proof, but which shows that the theorem in question indeed follows from the assumptions of the proof, through evidently valid inferential steps.

Frege’s *Begriffsschrift* project is arguably the first example of the deductive use of logic in mathematics. One of his main goals was to create a tool to make explicit all presuppositions which would ‘sneak in unnoticed’ in ordinary mathematical proofs. Here is the famous passage from the preface of the *Begriffsschrift* where he presents this point:

> To prevent anything intuitive from penetrating here unnoticed, I had to bend every effort to keep the chain of inferences free of gaps. In attempting to comply with this requirement in the strictest possible way I found the inadequacy of language to be an obstacle; no matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more and more complex, to attain the precision that my purpose required. This deficiency led me to the idea of the present ideography. Its first purpose, therefore, is to provide

Footnote 5 continued
topic in fact see the non-standard models as fascinating structures in their own right, not as ‘alien intruders’ (Enayat and Kossak 2004). However, from the point of view of Dedekind’s project, he clearly started with a very specific structure in mind, and sought to characterize it completely and uniquely. From this point of view, non-categoricity is indeed a failure.

> us with *the most reliable test of the validity of a chain of inferences* and to point out every presupposition that tries to sneak in unnoticed, so that its origin can be investigated. (Frege 1879/1977, pp. 5–6, emphasis added)

Again, it is important to bear in mind that Frege's project (and similar projects) is not that of describing the actual chains of inference of mathematicians in mathematical proofs. It is a *normative* project, even if he is not a revisionist who thinks that mathematicians make systematic mistakes in their practices (as Brouwer would later claim). He wants to formulate a tool that could put any given chain of inferences to test, and thus also to isolate presuppositions not made explicit in the proof. If these presuppositions happen to be true statements, then the proof is still sound, but we thereby become aware of all the premises that it in fact relies on.

For the success of this essentially *epistemic* project, the language in question should preferably operate on the basis of mechanical procedures, so that the test in question would always produce reliable results, i.e. ensuring that no hidden contentual considerations be incorporated into the application of rules (Sieg 1994, section 1.1). It is thus clear why Frege's project required a deductively well-behaved system, one with a precisely formulated underlying notion of deductive consequence. Indeed, in the *Grundgesetze* Frege criticizes Dedekind's lack of explicitness concerning inferential steps—incidentally, not an entirely fair criticism, given the different nature of Dedekind's project.

It is well known that Frege's deductive concerns were not particularly influential in the early days of formal axiomatics (and it is also well known that his own system in fact does not satisfy the deductive desiderata entirely). In effect, in the works of pioneers such as Dedekind, Peano, Hilbert etc., a precise and purely formal notion of deductive consequence was still missing (Awodey and Reck 2002a, section 3.1). It was only with Whitehead & Russell's *Principia Mathematica*, published in the 1910s, that the importance of this notion started to be recognized (among other reasons, because they were the first to take Frege's deductive project seriously). What this means for the present purposes is that Hintikka's notion of the deductive use of logic in the foundations of mathematics had little impact in the early days of applications of logic to mathematics, i.e. the final decades of the 19th century and the first decade of the 20th century—with the very notable exception of Frege, that is.

However, with the 'push' generated by the publication of *Principia Mathematica*, the deductive approach became increasingly pervasive in the 1910s, reaching its pinnacle in Hilbert's meta-mathematical program in the 1920s. Hilbert, whose earlier work on geometry also contained elements of the deductive/deductivist approach,[6] famously developed a new approach to the foundations of mathematics in the 1920s, one in which meta-mathematical questions were to be treated as mathematical questions themselves.

[6] As argued by Blanchette (2014). Her idea is that Hilbert's interest in models at that stage was essentially motivated by the idea of non-deducibility rather than by interest in those models as such. "Because models are a tool for showing non-deducibility, there is no interest in a 'range of models' or in claims about truth across 'all models'."

Hilbert's program was not a purely deductive program as Frege's had been. Indeed, the general idea was to treat axiomatizations/theories as mathematical objects in themselves so as to address meta-mathematical questions, but this required that not only the axioms but also the *rules of inference* within the theories be fully specified. Moreover, one of the key questions motivating Hilbert's program, the famous *Entscheidungsproblem*, and more generally the idea of a decision procedure for the whole of mathematics, has a very distinctive deductive flavor: is there a decision procedure which would allow us, for every mathematical statement, to ascertain whether it is or it is not a mathematical theorem?

So the golden era of the deductive use of logic in the foundations of mathematics started in the 1910s, after the publication of *Principia Mathematica*, and culminated in the 1920s, with Hilbert's program. Naturally, Gödel's discovery that there can be no complete and computable axiomatization of the first-order theory of the natural numbers in the early 1930s (and later on, Turing's and Church's negative answers to the *Entscheidungsproblem*) represented a real cold shower for these deductive aspirations. Indeed, the advent of model-theory in the late 1930s and 1940s might be viewed as a return to the predominance of the descriptive project at the expense of the deductive project, simply because the description of mathematical structures (models) becomes again the main (though not the only) focus of model-theorists.

Currently, both projects survive in different guises, but it is fair to say that the general optimism regarding the reach of each of them of the early days of formal axiomatics, especially the deductive project, has somewhat diminished. However, this is the case precisely because the limits of both have been investigated with the very formal tools developed within them, which in itself can be viewed as a remarkable feat.

4 The merely apparent incompatibility between the descriptive use and the deductive use

Given the (apparent) impossibility of tackling the descriptive and deductive projects at once with one and the same underlying logical system—what Tennant (2000) describes as 'the impossibility of monomathematics'—what should we conclude about the general project of using logic to investigate the foundations of mathematics? And what should we conclude about the first-order versus second-order divide? I now argue (*contra* Tennant) that the two projects are not entirely irreconcilable after all. My two pieces of evidence are the newly developed program of homotopy type theory, and the framework of ancestral arithmetic.

Recently, a new player joined the game: the research program known as homotopy type-theory (HoTT). It promises to bring in a whole new perspective to the foundations of mathematics. In particular, its base logic, Martin-Löf's constructive type-theory, is known to enjoy very favorable computational properties, but the focus on homotopy theory brings in a clear descriptive component as well. Thus, HoTT seems to represent new prospects for the possibility of unifying the descriptive perspective and the deductive perspective in virtue of the (surprising) interpretation of Martin-Löf's constructive type theory (a 'deductive project') into homotopy theory (a 'descriptive project'), which gave rise to HoTT (Awodey 2010).

HoTT draws inspiration from older developments in category theory; moreover, category theory can be studied successfully within the HoTT framework (Ahrens et al. 2013). Now, within category theory, different notions of categoricity have been defined, such as unique categoricity, variable categoricity and provable categoricity (Awodey and Reck 2002b). These relate in interesting ways to the concept of a *universal mapping property*, "which can be used to describe a particular mathematical structure" (Awodey and Reck 2002b, p. 92). Indeed, universal properties define mathematical objects uniquely up to isomorphism. Thus, alongside with the 'descriptive' focus of homotopy theory, HoTT includes a focus on the notion of categoricity via its tight connections with category theory.

Sociologically speaking, the HoTT community is somewhat divided with respect to the two approaches, descriptive and deductive. Some seem to focus predominantly on the former, while others, especially those focusing on the computational interpretation of HoTT, seem to prioritize the latter. But the fact that the two perspectives are represented in the community is all the more reason to think that HoTT may after all provide a platform for the unification of these two approaches, even if it is too early to tell whether HoTT will indeed change the terms of the game (as its proponents claim).

As for the first-order versus second-order divide, it is instructive to look in more detail into the idea of second-order extensions of first-order theories (or equivalently, subsystems of second-order logic), specifically with respect to arithmetic. Some of these proposals can be described as 'optimization projects', which seek to incorporate the least amount of second-order vocabulary so as to ensure categoricity, while producing a deductively well-behaved theory (Read 1997). In other words, the goal of an optimal tradeoff between expressiveness and tractability may not be entirely unreasonable after all.

One such example, which seems particularly instructive for the present purposes, is the framework of 'ancestral arithmetic' (Avron 2003; Cohen 2010).[7] Smith (2008) argues on plausible conceptual grounds that our basic intuitive grasp of arithmetic surely does not require the whole second-order conceptual apparatus, but only the concept of the *ancestral of a relation*, or the idea of the transitive closure under iterable operations (my parents had parents, who in turn had parents, who themselves had parents, and so on). Another way to arrive at a similar conclusion is to appreciate that what is needed to establish categoricity by extending a first-order theory is nothing more than the expressive power required to formulate the induction schema, or equivalently, the last, second-order axiom in the Dedekind/Peano axiomatization (the one needed to exclude 'alien intruders'). Here again, the concept of the ancestral of a relation is a plausible candidate (Smith 2008, section 3; Cohen 2010, section 5.3).

Extensions of first-order logic with the concept of the ancestral yield a number of interesting systems (Smith 2008, section 4; Cohen 2010, chapter 5). These systems, while not being fully axiomatizable (Smith 2008, section 4), enjoy a number of favor-

[7] (Cohen 2010, chap. 4) surveys some other systems between first-order logic and second-order logic that can also be (and have been) used for axiomatizations of arithmetic.

able proof-theoretical properties (Cohen 2010, chapter 5). Indeed, they are vastly 'better behaved' from a deductive point of view than full second-order logic—and of course, they are categorical.

Significant for our purposes is the status of the notion of the ancestral, straddled between first-order and second-order logic. Smith argues that the fact that this notion can be defined in second-order terms does not necessarily mean that it is an essentially higher-order notion:

> In sum, the claim is that the child who moves from a grasp of a relation to a grasp of the ancestral of that relation need not thereby manifest an understanding of second-order quantification interpreted as quantification over arbitrary sets. It seems, rather, that she has attained a distinct conceptual level here, something whose grasp requires going beyond a grasp of the fundamental logical constructions regimented in first-order logic, but which doesn't take us as far as an understanding of full second-order quantification. (Smith 2008, p. 4)

What this suggests is that the first-order versus second-order divide itself may be too coarse to describe adequately the conceptual building blocks of arithmetic. It is clear that purely first-order vocabulary will not yield categoricity, but it would be misguided to view the move to full second-order logic as the next 'natural' step. In effect, as argued by Smith, there is a sense in which the concept of the ancestral of a relation is essentially neither first-order nor second-order. So maybe the problem lies precisely with the coarse first-order versus second-order dichotomy when it comes to the key concepts at the foundations of arithmetic (such as the concept of the ancestral, or Dedekind's notion of chains, both instantiations of the general idea of closure).[8] We may need finer, intermediate categories to classify and analyze these concepts more accurately.[9]

5 A modest form of (instrumental) logical pluralism

Besides the possibility of establishing an optimal trade-off between expressivity and tractability, another consequence of the present analysis is that second-order logic and first-order logic may well both be equally legitimate as logical frameworks to be used in the foundations of mathematics. Naturally, I am not the first to defend the value of second-order logic for this purpose, despite the still prevailing general suspicion vis-à-vis second-order logic—in particular since Quine's (1970) influential criticism. Among others, Shapiro (1985, 1991) and Väänänen (2001, 2012) have both argued

[8] When it comes to providing foundations for the whole of mathematics, not only arithmetic, currently the 'dispute' is not so much between first-order and second-order logic (a rather different panorama with respect to when (Shapiro 1991) appeared), but between second-order logic and set theory (Väänänen 2001, 2012). However, insofar as set-theory is usually conceived of as a first-order theory, the dichotomy is still present.

[9] Read (1997, section 11) discusses a number of other 'logical solutions' to the issue of formulating intermediate systems (i.e. extensions of first-order logic) so as to ensure categoricity, such as so-called 'weak second-order logic' (which allows second-order quantification only over finite sets).

eloquently in favor of this framework. Rossberg (2004) has argued against the thesis that second-order logic cannot be logic proper in virtue of not having a complete deductive system. Bueno (2010) addresses five key objections to second-order logic, and concludes that each of them can be adequately dealt with by the second-order theorist.

In the present context, the main objection against the idea that second-order logic should be viewed as logic proper is its deficiency pertaining to the tractability of the calculus and the underlying consequence relation, which cannot be axiomatized. (This point is closely related to the completeness point discussed by Rossberg (2004), but tractability goes beyond completeness, including other meta-theoretical properties.) As formulated by Bueno:

> An important reason why second-order logic is viewed with suspicion derives from its non-axiomatizability. Since there is no complete proof procedure for this logic, this is used as an argument against it, assuming that something can be taken as a logic only if it is complete. But why is completeness taken to be a decisive feature of a logical system? Bluntly put, because it establishes that all valid sentences are derivable, and one does not want to face the predicament of *not being able to derive certain valid sentences*. (Bueno 2010, p. 368; emphasis added)

So the problem is not lack of completeness per se, but a more general deductive failure. Bueno goes on to argue that the completeness of first-order logic is in fact an upshot of its limited expressive power, which, in particular with respect to mathematical theories, can become a real limitation. This is reflected in particular in the failure of categoricity of a number of first-order theories such as PA1.

In terms of the present discussion, the observation that second-order logic is deductively 'ill-behaved' would be sufficient to disqualify it as logic proper *only* if it were granted that the *deductive* use (in particular, but not exclusively, in the foundations of mathematics) is the quintessential or perhaps even the only legitimate use of logic. I have here presented a historical argument against this claim: especially in the early days of formal axiomatic, but in fact throughout the (recent) history of logic, the *descriptive* use of logic in mathematics has been just as influential, and on many occasions more influential than the deductive use. Naturally, for the descriptive use, what is required is an expressive logical system, and this is precisely what second-order logic has to offer.

Thus, the (brief) historical analysis presented above provides grounds for a reflection on what it is that we expect of a system for it to count as logic proper: deductive tractability is certainly a desirable feature, but it may not be the decisive feature. (Bueno (2010) makes a similar point.) And if this is the case, then second-order logic is not inherently defective; it is not even necessarily less respectable than first-order logic. They simply have different strengths and weaknesses.

More generally, one of the conclusions I wish to draw from these observations is that there is no such thing as a unique 'correct' project in the foundations of mathematics. Here we focused on two distinct projects, the descriptive and the deductive project, which are both very influential and prima facie equally legitimate, both philosophically and historically.[10] The picture that emerges is thus of a multifaceted, pluralistic enterprise, not a uniquely defined project, and thus one allowing for multiple, equally legitimate perspectives.

Moreover, the choice of logical framework will depend on the exact goals of the formalization/axiomatization. Here, the focus has been on the expressiveness-tractability axis: if the goal is to capture chains of inferential steps leading from premises to conclusions in mathematical proofs, then tractability is to be prioritized; if the goal is to describe the underlying mathematical structures themselves, then expressivity is to be prioritized. And so, if there is no unique correct project in the foundations of mathematics (i.e. if there may be more than one legitimate theoretical goal when approaching mathematics with logical tools), and if each project is better served by a specific logical framework responding to its particular needs, then we seem to arrive at a modest form of *pragmatic, goal-oriented logical pluralism*. That is, given the plurality of legitimate goals, we may conclude that there is no such thing as the One True Logic when it comes to the foundations of mathematics.[11]

In particular, from the general pluralist point, we may also conclude that the 'dispute' between first-order and second-order logic as the most suitable underlying logical system for the foundations of mathematics is no real dispute; they each respond to different needs the theorist may have. (In fact, nothing rules out the possibility that other logical systems besides these two may also be fruitfully applied in the foundations of mathematics).

6 Conclusions

My starting point was the observation that first-order Peano Arithmetic is non-categorical but deductively well-behaved, while second-order Peano Arithmetic is categorical but deductively ill-behaved. I then turned to Hintikka's distinction between descriptive and deductive approaches in the foundations of mathematics. Both approaches were represented in the early days of formal axiomatics at the end of the 19th century, but the descriptive approach was arguably the predominant one at that stage.

Given the (apparent) impossibility of combining both approaches in virtue of the orthogonal desiderata of expressiveness and tractability, one might conclude (as Tennant (2000) seems to argue) that the project of providing logical foundations for mathematics itself is misguided from the start.

However, while it may seem that these two perspectives are incompatible, there is both the possibility of 'optimization projects', i.e. the search for the best trade-off between expressive and deductive power (e.g. ancestral arithmetic), and the possibility that an entirely new approach (maybe homotopy type-theory?) may even dissolve the

[10] To be sure, other legitimate, interesting projects in the foundations of mathematics may well arise in the future. Indeed, there is no reason to be sure that at this point we have exhausted the interesting connections between logic and mathematics for the foundations of mathematics.

[11] Another argument supporting a form of logical pluralism is the gap between logical theory and mathematical practice (Corcoran 1973; Shapiro 1985), which would lead to what Russell (2013, section 3.3) describes as 'pluralism about modeling'.

apparent incompatibility between the two projects. It is perhaps due to an excessive focus on the first-order versus second-order divide that we came to think that the two projects are incompatible. I have also argued that an investigation of the conceptual foundations of arithmetic seems to suggest that the first-order versus second-order dichotomy is in fact too coarse, as some key concepts (such as the concept of the ancestral of a relation) seem to inhabit a 'limbo' between the two realms.

Finally, the existence of more than one legitimate, distinct project in the foundations of mathematics (such as the descriptive and the deductive projects), and the idea that each project requires a formal framework with specific characteristics (in this case, expressibility and tractability), suggest a modest form of (instrumental) logical pluralism. In particular, many of the objections against the respectability of second-order logic as 'logic proper' seem tied to the presuppositions of the deductive approach, and thus become much less cogent once the legitimacy of the descriptive approach is fully recognized.

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