The Shapley value, the Proper Shapley value, and sharing rules for cooperative ventures

van den Brink, René; Levínský, René; Zelený, Miroslav

published in
Operations Research Letters
2020

DOI (link to publisher)
10.1016/j.orl.2019.11.003

document version
Publisher's PDF, also known as Version of record

document license
Article 25fa Dutch Copyright Act

Operations Research Letters 48 (2020) 55–60

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

The Shapley value, the Proper Shapley value, and sharing rules for cooperative ventures

René van den Brink [a], René Levínský [b,*,1], Miroslav Zelený [c,1]

[a] Department of Econometrics and Tinbergen Institute, School of Business and Economics, VU University, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands
[b] Center for Economic Research and Graduate Education – Economics Institute, Politických vězňů, 936/7, 110 00 Praha, Czech Republic
[c] Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha, Czech Republic

ARTICLE INFO

Article history:
Received 7 June 2019
Received in revised form 1 October 2019
Accepted 15 November 2019
Available online 28 November 2019

Keywords:
Equity principle
Cooperative venture game
Shapley value
Proper Shapley value

ABSTRACT

In this note, we discuss two solutions for cooperative transferable utility games, namely the Shapley value and the Proper Shapley value. We characterize positive Proper Shapley values by affine invariance and by an axiom that requires proportional allocation of the surplus according to the individual singleton worths in generalized joint venture games. As a counterpart, we show that affine invariance and an axiom that requires equal allocation of the surplus in generalized joint venture games characterize the Shapley value.

1. Introduction

Situations where a set of agents can generate certain worth by cooperating can be described by a *cooperative transferable utility game* (*TU-game* for short). A TU-game consists of a nonempty finite set of *players* and a *characteristic function* that assigns to every subset of the player set, called a *coalition*, a real number. This number represents the (transferable) utility that is obtained by the players of the corresponding coalition. The main question is how to allocate the worths that the coalitions can gain among individual players.

The best known approach to this problem is to allocate the worths according to the *Shapley value* of the corresponding TU-game which distributes the so-called *Harsanyi dividends* equally among the players in the corresponding coalitions [14]. The Harsanyi dividend of a coalition can be seen as the "real value" added by this coalition which has not yet been realized by its subcoalitions, cf. [3]. The *weighted Shapley value* [13] is a modification of the Shapley value where exogenously given positive weights are attributed to players and each Harsanyi dividend is allocated among the players of the corresponding coalition in proportion to these weights. This approach can capture some external asymmetry between the players, e.g., their bargaining power or influence in a network. The *Proper Shapley value* introduced by Vorob'ev and Liapounov [15] internalizes this external asymmetry in the sense that it chooses the payoff allocation of one specific weighted Shapley value, namely the weighted Shapley value that allocates the worths of the grand coalition among individual players in proportion to their weights. Thus the Proper Shapley value is defined as a fixed point of a mapping related to the weighted Shapley value (see Section 2 for the definition).

The main aim of our paper is to characterize and to compare the Shapley value and the Proper Shapley value. Despite of the fact that the Shapley value is single-valued while the Proper Shapley value is not, we are able to introduce axioms that make a comparison between these two solutions possible. Another difference between these solutions is that the Shapley value is linear, therefore additive, while the Proper Shapley value is not. The first axiom we introduce is a weaker version of additivity which is satisfied by both solutions (taking account of the fact that one is single-valued and the other is set-valued, see Section 3 for more details). This axiom, called *affine invariance*, roughly says that if we allocate the worths of the grand coalition identically among individual players in two TU-games then we have to allocate the worths in this particular way in any affine combination of these TU-games.

We show that these solutions satisfy two different equity principles. Both are stated for a special type of a TU-game, called the *generalized joint venture game*. This class of TU-games generalizes the *joint venture games* introduced by Moulin [9]. A joint venture game is a TU-game such that non-zero Harsanyi dividends are only allowed for the grand coalition and the singletons. A generalized joint venture game is a TU-game such that there is no more than one nonsingleton coalition (not necessarily the grand

* Corresponding author.
E-mail address: rene.levinsky@cerge-ei.cz (R. Levínský).

1 Research supported by the grant GAČR, Czech Republic 17-19672S.

https://doi.org/10.1016/j.orl.2019.11.003

one) with a non-zero dividend. It turns out that these TU-games form a basis for the space of TU-games which we will use later on.

For the Shapley value we extend the equal sharing principle of [9] and introduce a *generalized equal sharing principle*, which requires that in any generalized joint venture game, every singleton receives its own worth and the dividend of the possible nonsingleton coalition with nonzero dividend is allocated equally among the players in this coalition. It turns out that this axiom and affine invariance characterize the Shapley value.

Finally, we extend the proportional sharing principle of [9] by introducing a *generalized proportional sharing principle* which requires that in any generalized joint venture game, every singleton receives its own worth and the dividend of the possible nonsingleton coalition with nonzero dividend is allocated in proportion to the individual worths. This axiom and affine invariance characterize the positive part of the Proper Shapley value.

In this note, we describe the essential difference between the Shapley value and the Proper Shapley value based on equality and proportionality. In our opinion, the equality is relevant in situations where players can freely manipulate their individual worths. R. van den BrinkIn such games, the worths of particular players are, in fact, non-informative and therefore irrelevant, and the equal split of the dividends is the only natural distribution. In contrast, in situations where individual worths cannot be misrepresented or their misrepresentation is costly, proportionality seems more appropriate.

Proportionality is an established principle in several theories of value allocation, but also in claims problems, cf. [5]. In claims problems, proportionality is rather obvious since the model is fully described by individual claims and one (monetary) estate, thus the obvious way to apply proportionality is to split the estate among the claimants in proportion to their individual claims. However, proportionality is much less studied and is much less obvious in TU-games. Recently, there has been a growing body of literature on proportionality in TU-games. Besides the Proper Shapley value, which is studied in this paper, proportionality in TU-games is studied, e.g., in [1,2,11].

Proportionality in economics and operations research is used, e.g., for the so-called 2-games (games where only coalitions of size 2 have a nonzero dividend). These games have many applications such as the Terrestrial Flight Telephone System games [10] or the queuing games [8].

The Proper Shapley solution for 2-games with three players splits each dividend of a coalition of two players taking into account also the opportunities each player of the coalition has with the player outside of the coalition. This principle can be applied to queuing games, where the players are jobs to be served on a machine such that the machine can only serve one job at a time and the jobs have different waiting costs. Neither the Proportional rule nor the Proportional Shapley value can be applied to this type of games since the individual worths are zero.

For further motivation and discussion of proportionality in the context of TU-games, we refer to the introduction of [4].

2. Preliminaries

A *cooperative transferable utility game* (*TU-game* for short) is a pair (N, v), where $N = \{1, \dots, n\}$ is a nonempty finite set consisting of n players and v is a *characteristic function* which assigns a real number $v(S)$ to any *coalition* $S \subseteq N$ and satisfies $v(\emptyset) = 0$. The real number $v(S)$ is the *worth* of S which the members of the coalition S can distribute among themselves. In certain applications, $v(S)$ can also be interpreted as the *cost* which needs to be split among members of S. The set of all TU-games is denoted by $\mathcal{G}$.

Let $(N, v) \in \mathcal{G}$. The *Harsanyi dividends* $\Delta_{N,v}(S)$, where $S \subseteq N$, are defined inductively by

$$\Delta_{N,v}(S) = \begin{cases} 0, & \text{for } S = \emptyset, \\ v(S) - \sum_{T \subsetneq S} \Delta_{N,v}(T), & \text{for } S \neq \emptyset, \end{cases}$$

see [6]. Let us note that $v(S) = \sum_{T \subseteq S} \Delta_{N,v}(T)$ for every $S \subseteq N$. This formula shows that dividends uniquely determine the characteristic function. A coalition $S \subseteq N$ in the TU-game (N, v) is called *essential* if $\Delta_{N,v}(S) \neq 0$, otherwise S is called *inessential*. Any TU-game where all the coalitions but singletons are inessential is called *inessential*.

We employ the following notation. Let $y \in \mathbb{R}^N$ and $S \subseteq N$. The symbol $|S|$ denotes the cardinality of S. The symbol y_S stands for $\sum_{i \in S} y_i$. By convention, the value of any empty sum of real numbers is zero, i.e., $y_\emptyset = 0$. A *payoff vector* in a TU-game (N, v) is an n-dimensional vector, where $n = |N|$, whose components are the payoffs of the corresponding players. A payoff vector $x \in \mathbb{R}^N$ for a TU-game (N, v) is *efficient* if it exactly distributes the worth $v(N)$ of the grand coalition N, i.e., if $x_N = v(N)$. The set of all efficient payoff vectors of (N, v) is denoted by $X(N, v)$, the set of all efficient payoff vectors with positive coordinates is denoted by $X_+(N, v)$, and the set of all efficient payoff vectors with nonnegative coordinates is denoted by $X_0(N, v)$.

Let $\mathcal{C} \subseteq \mathcal{G}$ be a subclass of TU-games. A *single-valued solution* on $\mathcal{C}$ is a function f that assigns to every TU-game $(N, v) \in \mathcal{C}$ a payoff vector $f(N, v) \in \mathbb{R}^N$. A *set-valued solution* F on $\mathcal{C}$ assigns a set of payoff vectors $F(N, v) \subseteq \mathbb{R}^N$ to every TU-game $(N, v) \in \mathcal{C}$.

The best known single-valued solution for TU-games is the *Shapley value* [13] which distributes the Harsanyi dividends of the TU-game *equally* among the players in the corresponding coalitions, i.e., the Shapley value is the function $\varphi\colon \mathcal{G} \to \mathbb{R}^N$ defined by $\varphi(N, v) = \big(\varphi_i(N, v)\big)_{i \in N}$, where

$$\varphi_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{1}{|S|} \Delta_{N,v}(S), \quad i \in N.$$

Given a weight vector $\omega \in \mathbb{R}^N$ with positive weights $\omega_i > 0$, $i \in N$, the corresponding *weighted Shapley value* [13] is the function $\varphi^\omega\colon \mathcal{G} \to \mathbb{R}^N$ defined by

$$\varphi_i^\omega(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{\omega_i}{\omega_S} \Delta_{N,v}(S), \quad i \in N.$$

The weighted Shapley value thus distributes the dividends of coalitions proportionally to the exogenously given weights of the players. Clearly, if all weights ω_i are equal to each other then the weighted Shapley value $\varphi^\omega(N, v)$ is equal to the Shapley value $\varphi(N, v)$.

Another solution, the Proper Shapley value, was introduced by Vorob'ev and Liapounov [15]. This solution is defined as follows. Let $(N, v) \in \mathcal{G}$. To simplify notation, we denote $h(x) = \varphi^x(N, v)$ for $x \in X_+(N, v)$. We define a multi-valued mapping H assigning a subset of $\mathbb{R}^N$ to each element x of $X_0(N, v)$ by

$$\begin{aligned} H(x) = \{\alpha \in \mathbb{R}^N \,|\, &\text{there exists a sequence } (x^j) \subseteq X_+(N, v) \\ &\text{such that } x^j \to x \text{ and } h(x^j) \to \alpha\}. \end{aligned}$$

Note that the set-valued function H depends on (N, v), but we will omit this parameter for the sake of simplicity. The graph of H is just the closure of the graph of the mapping h.

Definition 1. Let $(N, v) \in \mathcal{G}$. A vector $x \in X_0(N, v)$ is called a *Proper Shapley value* of (N, v) if $x \in H(x)$. We denote

$$\mathrm{PSV}(N, v) = \big\{x \in X_0(N, v) \,|\, x \text{ is a Proper Shapley value of } (N, v)\big\}$$

and

$$\mathcal{G}_P = \big\{(N, v) \in \mathcal{G} \,|\, \mathrm{PSV}(N, v) \neq \emptyset\big\}.$$

We refer to the solution that assigns to every $(N, v) \in \mathcal{G}_P$ the set of all Proper Shapley values $\mathrm{PSV}(N, v)$ as the *Proper Shapley solution*.

Remark 1. The definition of the Proper Shapley value is rather canonical for vectors having all coordinates strictly positive. If we want to include also vectors having some zero coordinates, then the situation becomes less straightforward. There are several possibilities how to define such an extension. In the above definition we use one of these possibilities which was discussed in detail in [4]. Another possibility is presented in Example 1. We do not want to open the discussion on which extension is "the right one" in this paper. Therefore, we focus just on the characterization of the "strictly positive part" of the Proper Shapely value where all the extensions coincide.

3. Axiomatization and main results

We start with the following definition.

Definition 2. We say that a class $\mathcal{C} \subseteq \mathcal{G}$ is *affine* if for any TU-games $(N, v_1), (N, v_2) \in \mathcal{C}$, $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 + \lambda_2 = 1$, we have $\lambda_1 v_1 + \lambda_2 v_2 \in \mathcal{C}$.

The next axiom is a weakening of Shapley's additivity axiom.

Axiom 1. *A single-valued solution f satisfies* affine invariance *on a class $\mathcal{C} \subseteq \mathcal{G}$ if for any TU-games $(N, v_1), (N, v_2) \in \mathcal{C}$ with $f(N, v_1) = f(N, v_2)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 + \lambda_2 = 1$, we have $f(N, \lambda_1 v_1 + \lambda_2 v_2) = f(N, v_1) = f(N, v_2)$ provided $\lambda_1 v_1 + \lambda_2 v_2 \in \mathcal{C}$.*

For set-valued solutions, affine invariance is defined as follows.

Axiom 2. *A set-valued solution F satisfies* affine invariance *on a class $\mathcal{C} \subseteq \mathcal{G}$ if for any TU-games $(N, v_1), (N, v_2) \in \mathcal{C}$, $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 + \lambda_2 = 1$, and $x \in \mathbb{R}^N$ with $x \in F(N, v_1) \cap F(N, v_2)$, we have $x \in F(N, \lambda_1 v_1 + \lambda_2 v_2)$ provided $\lambda_1 v_1 + \lambda_2 v_2 \in \mathcal{C}$.*

Remark 2. One can easily prove that a set-valued solution F satisfies affine invariance on an affine class $\mathcal{C}$ if and only if, for every $k \in \mathbb{N}$, $(N, v_1), \ldots, (N, v_k) \in \mathcal{C}$, $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ with $\sum_{i=1}^{k} \lambda_i = 1$, and $x \in \mathbb{R}^N$ with $x \in F(N, v_i)$ for every $i \in \{1, \ldots, k\}$, we have $x \in F(N, v)$, where $v = \sum_{i=1}^{k} \lambda_i v_i$. Note that by repeated application of the condition from Definition 2, it follows that $v \in \mathcal{C}$.

A TU-game (N, v) is a *joint venture game*, if $\Delta_{N,v}(S) = 0$ whenever $|S| \neq 1$ and $S \neq N$ [9]. For this class of TU-games, Moulin analyzes two sharing rules, namely the equal sharing rule and the proportional sharing rule. Let us note that Moulin's proportional sharing rule is structurally identical to an older concept introduced by Homans [7], cf. [12], who referred to it as to the *equity principle*. Further, since any two-player TU-game belongs to the class of joint venture games, the proportional sharing rule for two-player TU-games corresponds to *proportional standardness for two-player TU-games* as defined in [11]. We define the following generalization of joint venture games.

Definition 3. A TU-game (N, v) is called a *generalized joint venture game* if there exists a coalition $E \subseteq N$ such that $\Delta_{N,v}(T) = 0$ whenever $|T| \neq 1$ and $T \neq E$. The set of all generalized joint venture games is denoted by $\mathcal{G}_1$. For $(N, v) \in \mathcal{G}_1$, the symbol $E(N, v)$ denotes the unique nonsingleton essential coalition if such a coalition exists, otherwise $E(N, v) = \emptyset$.

Remark 3. The symbol $\mathcal{G}_1$ is chosen to stress the fact that any TU-game in $\mathcal{G}_1$ contains at most one nonsingleton essential coalition. The class $\mathcal{G}_1$ clearly contains all inessential TU-games as well as joint venture games.

Now, as Moulin for joint venture games, we define two possible sharing rules for generalized joint venture games. The generalized equal sharing principle requires that in generalized joint venture games, the surplus of the only essential nonsingleton coalition is allocated equally over the players in that coalition.

Axiom 3. *A single-valued solution f satisfies the* generalized equal sharing principle *if for every $(N, v) \in \mathcal{G}_1$, we have*

$$f_i(N, v) = \begin{cases} v(\{i\}) + \frac{1}{|E(N,v)|} \Delta_{N,v}\big(E(N, v)\big), & \text{for } i \in E(N, v), \\ v(\{i\}), & \text{for } i \in N \setminus E(N, v). \end{cases}$$

We formulate the generalized proportional sharing principle for set-valued solutions for any generalized joint venture game with the exception of some peculiar games (in fact, in all these games the worth of the grand coalition is zero). The surplus of the only essential nonsingleton coalition is allocated over the players in that coalition in proportion to their singleton worths.

Axiom 4. *A set-valued solution F satisfies the* generalized proportional sharing principle *if for every $(N, v) \in \mathcal{G}_1$, which is inessential or satisfies $\sum_{j \in E(N,v)} v(\{j\}) \neq 0$ and $v(E(N, v)) \neq 0$, we have $F(N, v) = \{x\}$, where*

$$x_i = \begin{cases} v(\{i\}) + \frac{v(\{i\})}{\sum_{j \in E(N,v)} v(\{j\})} \Delta_{N,v}\big(E(N, v)\big), & \text{for } i \in E(N, v), \\ v(\{i\}), & \text{for } i \in N \setminus E(N, v). \end{cases}$$

Remark 4. Observe that if $(N, v) \in \mathcal{G}_1$ is inessential, then $E(N, v) = \emptyset$ and for any F that satisfies the generalized proportional sharing principle, we have $F(N, v) = \{x\}$, where $x_i = v(\{i\})$ for every $i \in N$. An analogous observation can be made also for the generalized equal sharing principle.

Remark 5. We discuss two crucial axioms, i.e., the affine invariance and the generalized proportional sharing principle in the light of two other prominent solution concepts based on the proportionality principle, namely the *Proportional rule* and *Proportional Shapley value* (see [1,2]).

The Proportional rule PR allocates the worth $v(N)$ of the "grand coalition" in proportion to the individual worths. Formally, we define

$$\mathrm{PR}_i(N, v) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N), \qquad i \in N,$$

for $(N, v) \in \mathcal{C}^*$, where $\mathcal{C}^*$ is the class containing all games with nonzero sum of individual worths. The Proportional rule satisfies affine invariance. Indeed, if $(N, v_1), (N, v_2) \in \mathcal{C}^*$ satisfy $\mathrm{PR}(N, v_1) = \mathrm{PR}(N, v_2)$, then $v_1(N) = v_2(N)$ and thus also either $v_1(\{i\}) = \alpha v_2(\{i\})$, $i \in N$, for some $\alpha \neq 0$ or $v_1(N) = v_2(N) = 0$. Both possibilities give $\mathrm{PR}(N, \lambda_1 v_1 + \lambda_2 v_2) = \mathrm{PR}(N, v_1)$, whenever $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 v_1 + \lambda_2 v_2 \in \mathcal{C}^*$.

However, the Proportional rule does not satisfy the generalized proportional sharing principle. Consider a game $(N, v) \in \mathcal{G}_1$ defined by $N = \{1, 2, 3\}$ and

$$\Delta_{N,v}(S) = \begin{cases} 1, & \text{for } |S| = 1, \\ 6, & \text{for } S = \{1, 2\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, clearly $v(N) = 9$ and $\mathrm{PR}(N, v) = (3, 3, 3)$ while the proportional sharing principle requires the payoff vector $(4, 4, 1)$.

The Proportional Shapley value PS allocates every dividend in proportion to the individual worths of its members. Formally, we define

$$\mathrm{PS}_i(N, v) = \sum_{S \subseteq N,\ i \in S} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_{N,v}(S), \qquad i \in N,$$

for $(N, v) \in \mathcal{C}^{**}$, where $\mathcal{C}^{**}$ is the class containing all games such that $\sum_{j\in S} v(\{j\}) \neq 0$ for every nonempty $S \subseteq N$. On contrary to PR, PS satisfies the generalized proportional sharing principle (by its definition), however, it does not satisfy affine invariance. To this end define games (N, v_1) and (N, v_2) by $N = \{1, 2, 3\}$,

$$\Delta_{N,v_1}(\{1\}) = 3,\ \ \Delta_{N,v_1}(\{2\}) = 9,\ \ \Delta_{N,v_1}(\{3\}) = 3,$$
$$\Delta_{N,v_1}(\{1, 3\}) = 12,$$
$$\Delta_{N,v_2}(\{1\}) = 3,\ \ \Delta_{N,v_2}(\{2\}) = 3,\ \ \Delta_{N,v_2}(\{3\}) = 9,$$
$$\Delta_{N,v_2}(\{1, 2\}) = 12,$$

and $\Delta_{N,v_1}(S) = \Delta_{N,v_2}(S) = 0$ otherwise. Clearly, $\mathrm{PS}(N, v_1) = \mathrm{PS}(N, v_2) = (9, 9, 9)$. Consider now $v = \frac{1}{2}v_1 + \frac{1}{2}v_2$, i.e.,

$$\Delta_{N,v}(\{1\}) = 3,$$
$$\Delta_{N,v}(\{2\}) = \Delta_{N,v}(\{3\}) = \Delta_{N,v}(\{1, 2\}) = \Delta_{N,v}(\{1, 3\}) = 6,$$

and $\Delta_{N,v}(S) = 0$ otherwise. Then $v \in \mathcal{C}^{**}$ and $\mathrm{PS}(N, v) = (7, 10, 10)$, although affine invariance requires the payoff vector $(9, 9, 9)$.

The next two results provide characterizations of the Shapley value and of the positive part of the Proper Shapley value using the equity principles defined above.

Proposition 1. *The Shapley value φ is the unique single-valued solution that satisfies affine invariance on $\mathcal{G}$ and the generalized equal sharing principle.*

Proposition 2. *Let F be a set-valued solution satisfying affine invariance on its affine domain* dom F *and the generalized proportional sharing principle. Then $F(N, v) \cap \mathbb{R}^N_{++} = \mathrm{PSV}(N, v) \cap \mathbb{R}^N_{++}$ for every $(N, v) \in \mathcal{G}_P$.*

Example 1. Proposition 2 would be meaningless if there did not exist a solution satisfying affine invariance and the generalized proportional sharing principle. It turns out that the following solution satisfies these axioms. For $(N, v) \in \mathcal{G}$, define $g_{N,v}: \mathbb{R}^N \to \mathbb{R}^N$ by

$$g_{N,v}(x)_i = v(\{i\}) + \sum_{\substack{S\subseteq N, i\in S\\ x_S\neq 0, |S|\geq 2}} \frac{x_i}{x_S}\Delta_{N,v}(S), \qquad i \in N,$$

and consider the set valued solution G defined by

$$G(N, v) = \{x \in \mathbb{R}^N \,|\, g_{N,v}(x) = x\}.$$

Note that $\mathrm{PSV}(N, v) \cap \mathbb{R}^N_{++} \subseteq G(N, v)$.

Proposition 3. *The set-valued solution G satisfies affine invariance and the generalized proportional sharing principle.*

4. Proofs

4.1. Proof of Proposition 1

Since the Shapley value obviously satisfies affine invariance and the generalized proportional sharing principle, it remains to prove uniqueness. Suppose that f is a single-valued solution which satisfies affine invariance and the generalized equal sharing principle.

Let $(N, v) \in \mathcal{G}$ and take any $x \in \mathbb{R}^N$. Now, for $S \subseteq N$, $|S| \geq 2$, we define the TU-game (N, v^x_S) by

$$\Delta_{N,v^x_S}(T) = \begin{cases} -\Delta_{N,v}(S), & \text{for } T = S,\\ x_i + \frac{1}{|S|}\Delta_{N,v}(S), & \text{for } T = \{i\},\ i \in S,\\ x_i, & \text{for } T = \{i\},\ i \in N \setminus S,\\ 0, & \text{otherwise.} \end{cases}$$

Clearly, we have $(N, v^x_S) \in \mathcal{G}_1$. Further, we define the inessential TU-game $(N, \nu^x) \in \mathcal{G}_1$ by

$$\Delta_{N,\nu^x}(T) = \begin{cases} x_i, & \text{for } T = \{i\},\ i \in N,\\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Finally, let us define the TU-game (N, w^x) by

$$w^x = v - (2^n - n - 1)\nu^x + \sum_{S\subseteq N, |S|\geq 2} v^x_S. \tag{2}$$

One can easily verify that

$$\Delta_{N,w^x}(T) = \begin{cases} 0, & \text{for } |T| \geq 2,\\ v(\{i\}) + \sum_{S\ni i, |S|\geq 2} \frac{1}{|S|}\Delta_{N,v}(S) = \sum_{S\ni i} \frac{1}{|S|}\Delta_{N,v}(S), & \\ \quad \text{for } T = \{i\}. & \end{cases}$$

The linear combination in (2) is affine, since we have

$$1 - (2^n - n - 1) + \sum_{S\subseteq N, |S|\geq 2} 1 = 1 - 2^n + \sum_{S\subseteq N} 1 = 1 - 2^n + 2^n = 1.$$

Now, we show that for $x \in \mathbb{R}^N$ we have $x = f(N, v)$ if and only if $x = \varphi(N, v)$. First, assume that $x = f(N, v)$. Since f satisfies the generalized equal sharing principle, we get $x = f(N, v^x_S)$ for every $S \subseteq N$, $|S| \geq 2$, and $x = f(N, \nu^x)$. Thus, by affine invariance of f we obtain $x = f(N, w^x)$. The TU-game (N, w^x) is inessential and therefore

$$x_i = w^x(\{i\}) = \sum_{S\ni i} \frac{1}{|S|}\Delta_{N,v}(S).$$

This shows that x is the Shapley value of (N, v).

Now suppose that x is the Shapley value of (N, v), i.e., $x = \varphi(N, v)$. Using (2) we can write

$$v = w^x + (2^n - n - 1)\nu^x - \sum_{S\subseteq N, |S|\geq 2} v^x_S.$$

Thus, v is an affine combination of TU-games from $\mathcal{G}_1$, $x = f(N, \nu^x)$, and $x = f(N, v^x_S)$ for $S \subseteq N$, $|S| \geq 2$, by the generalized equal sharing principle. Since x is the Shapley value of (N, v), we get $x = f(N, w^x)$ by the generalized equal sharing principle. Now, affine invariance of f yields $x = f(N, v)$ and we are done.

4.2. Proof of Proposition 2

Let F be a set-valued solution satisfying affine invariance on its affine domain dom F and the generalized proportional sharing principle.

Claim. *Let $(N, v) \in \mathcal{G}$, $x \in \mathbb{R}^N_{++}$. Then there exist TU-games $(N, v^0), (N, v^1), \ldots, (N, v^k) \in \mathcal{G}_1$ such that*

(a) *v is an affine combination of $v^0, v^1, \ldots, v^k$ of the form $v = v^0 + \sum_{j=1}^k \gamma_j v^j$ with $\sum_{j=1}^k \gamma_j = 0$,*
(b) *$x \in F(N, v^j)$, $j = 1, \ldots, k$, and*
(c) *we have*

$$\Delta_{N,v^0}(T) = \begin{cases} 0, & \text{for } T \subseteq N, |T| \geq 2,\\ \sum_{S\ni i} \frac{x_i}{x_S}\Delta_{N,v}(S), & \text{for } T = \{i\}, i \in N. \end{cases}$$

Proof of Claim. We set $\mathcal{A} = \{S \subseteq N \,|\, |S| \geq 2, \Delta_{N,v}(S) \neq 0\}$. For $S \in \mathcal{A}$ and $\alpha \in \mathbb{R}$, we define a TU-game $(N, w^{x,\alpha}_S)$ as follows:

$$\Delta_{N,w^{x,\alpha}_S}(T) = \begin{cases} -\alpha\Delta_{N,v}(S), & \text{for } T = S,\\ x_i + \frac{x_i}{x_S}\alpha\Delta_{N,v}(S), & \text{for } T = \{i\}, i \in S,\\ x_i, & \text{for } T = \{i\}, i \notin S,\\ 0, & \text{otherwise.} \end{cases}$$

We have clearly $(N, w_S^{x,\alpha}) \in \mathcal{G}_1$. We define a TU-game (N, v^0) as the following sum:

$$v^0 = v - 2|\mathcal{A}|\nu^x + \sum_{S\in\mathcal{A}} (w_S^{x,\alpha^S} + w_S^{x,\beta^S}), \tag{3}$$

where ν^x is defined by (1), α^S and β^S are chosen in such a way that $\alpha^S \neq 0$, $\beta^S \neq 0$, $\alpha^S + \beta^S = 1$, $x_S + \alpha^S \Delta_{N,v}(S) \neq 0$, and $x_S + \beta^S \Delta_{N,v}(S) \neq 0$. Now we verify (a)–(c).

(a) Using (3) it is easy to choose $\gamma_1, \dots, \gamma_k$ and to denote appropriately the TU-games ν^x, w_S^{x,α^S}, w_S^{x,β^S}, $S \in \mathcal{A}$, by $v^1, \dots, v^k$ to satisfy (a).

(b) Because of the choice of α^S and β^S, $S \in \mathcal{A}$, the corresponding TU-games satisfy $E(N, w_S^{x,\alpha^S}) = E(N, w_S^{x,\beta^S}) = S$,

$$\sum_{i\in S} w_S^{x,\alpha^S}(\{i\}) = x_S + \alpha^S \Delta_{N,v}(S) \neq 0,$$

$$\begin{aligned} w_S^{x,\alpha^S}(S) &= \sum_{i\in S} \Delta_{N,w_S^{x,\alpha^S}}(\{i\}) + \Delta_{N,w_S^{x,\alpha^S}}(S) \\ &= x_S + \alpha^S \Delta_{N,v}(S) - \alpha^S \Delta_{N,v}(S) = x_S \neq 0, \end{aligned}$$

$$\sum_{i\in S} w_S^{x,\beta^S}(\{i\}) = x_S + \beta^S \Delta_{N,v}(S) \neq 0,$$

$$\begin{aligned} w_S^{x,\beta^S}(S) &= \sum_{i\in S} \Delta_{N,w_S^{x,\beta^S}}(\{i\}) + \Delta_{N,w_S^{x,\beta^S}}(S) \\ &= x_S + \beta^S \Delta_{N,v}(S) - \beta^S \Delta_{N,v}(S) = x_S \neq 0. \end{aligned}$$

Thus, we can apply the generalized proportional sharing principle for F to infer $x \in F(N, w_S^{x,\alpha^S})$ and $x \in F(N, w_S^{x,\beta^S})$. To see this, note that obviously $w_S^{x,\alpha^S}(\{i\}) = x_i$ for every $i \in N \setminus S$ and, for every $i \in S$, we have

$$\begin{aligned} w_S^{x,\alpha^S}(\{i\}) &+ \frac{w_S^{x,\alpha^S}(\{i\})}{\sum_{j\in S} w_S^{x,\alpha^S}(\{j\})} \cdot \left(-\alpha^S \Delta_{N,v}(S)\right) \\ &= x_i + \frac{x_i}{x_S}\alpha^S \Delta_{N,v}(S) - \frac{x_i + \frac{x_i}{x_S}\alpha^S \Delta_{N,v}(S)}{x_S + \alpha^S \Delta_{N,v}(S)} \alpha^S \Delta_{N,v}(S) \\ &= x_i + \frac{x_i}{x_S}\alpha^S \Delta_{N,v}(S) - \frac{x_i}{x_S}\alpha^S \Delta_{N,v}(S) = x_i. \end{aligned}$$

Similarly one can infer $x \in F(N, w_S^{x,\beta^S})$. Since ν^x is inessential, we get $x \in F(N, \nu^x)$ by applying Remark 4. This completes the proof of (b).

(c) Consider $T \subseteq N$ such that $|T| \geq 2$ and $T \in \mathcal{A}$. Then we have

$$\begin{aligned} \Delta_{N,v^0}(T) &= \Delta_{N,v}(T) - 2|\mathcal{A}|\Delta_{N,\nu^x}(T) \\ &\quad + \sum_{S\in\mathcal{A}} \left(\Delta_{N,w_S^{x,\alpha^S}}(T) + \Delta_{N,w_S^{x,\beta^S}}(T)\right) \\ &= \Delta_{N,v}(T) + \Delta_{N,w_T^{x,\alpha^T}}(T) + \Delta_{N,w_T^{x,\beta^T}}(T) \\ &= \Delta_{N,v}(T) - \alpha^T \Delta_{N,v}(T) - \beta^T \Delta_{N,v}(T) \\ &= \Delta_{N,v}(T)(1 - \alpha^T - \beta^T) = 0. \end{aligned}$$

If $T \subseteq N$, $|T| \geq 2$, and $T \notin \mathcal{A}$, then $\Delta_{N,v^0}(T) = \Delta_{N,v}(T) = 0$.

Now choose $T \subseteq N$ with $T = \{i\}$, $i \in N$. Then we have

$$\begin{aligned} \Delta_{N,v^0}(\{i\}) &= \Delta_{N,v}(\{i\}) - 2|\mathcal{A}|\Delta_{N,\nu^x}(\{i\}) \\ &\quad + \sum_{S\in\mathcal{A}} \left(\Delta_{N,w_S^{x,\alpha^S}}(\{i\}) + \Delta_{N,w_S^{x,\beta^S}}(\{i\})\right) \\ &= \Delta_{N,v}(\{i\}) - 2|\mathcal{A}|x_i \\ &\quad + \sum_{S\in\mathcal{A}, i\in S} \left(x_i + \frac{x_i}{x_S}\alpha^S \Delta_{N,v}(S) + x_i + \frac{x_i}{x_S}\beta^S \Delta_{N,v}(S)\right) \\ &\quad + \sum_{S\in\mathcal{A}, i\notin S} (x_i + x_i) \\ &= \Delta_{N,v}(\{i\}) + \sum_{S\in\mathcal{A}, i\in S} \frac{x_i}{x_S}\Delta_{N,v}(S) \\ &= \sum_{S\ni i} \frac{x_i}{x_S}\Delta_{N,v}(S). \qquad \text{(by the definition of } \mathcal{A}) \end{aligned}$$

This completes the proof of the claim. □

Now, we show that $F(N, v) \cap \mathbb{R}^N_{++} = \mathrm{PSV}(N, v) \cap \mathbb{R}^N_{++}$ for $(N, v) \in \mathcal{G}_P$.

Proof of the inclusion $F(N, v) \cap \mathbb{R}^N_{++} \subseteq \mathrm{PSV}(N, v)$**.** Suppose that $x \in F(N, v) \cap \mathbb{R}^N_{++}$. Take $v^0, \dots, v^k$ for (N, v) and x according to Claim. By (b), we have $x \in F(N, v^j)$ for $j \in \{1, \dots, k\}$ and (N, v^0) is an affine combination of (N, v) and $(N, v^1), \dots, (N, v^k)$. By affine invariance of F, we get $x \in F(N, v^0)$. Since the TU-game (N, v^0) is inessential, we have

$$x_i = v^0(\{i\}) = \sum_{S\ni i} \frac{x_i}{x_S}\Delta_{N,v}(S), \qquad i \in N.$$

Thus, we have $x = h(x)$ and, consequently, $x \in H(x)$. This shows that $x \in \mathrm{PSV}(N, v)$. □

Proof of the inclusion $\mathrm{PSV}(N, v) \cap \mathbb{R}^N_{++} \subseteq F(N, v)$**.** Consider $x \in \mathrm{PSV}(N, v) \cap \mathbb{R}^N_{++}$. Again, take $v^0, \dots, v^k$ for (N, v) and x according to Claim. Since $x \in \mathrm{PSV}(N, v) \cap \mathbb{R}^N_{++}$, we have $x = h(x)$, i.e.,

$$x_i = \sum_{S\ni i} \frac{x_i}{x_S}\Delta_{N,v}(S), \qquad i \in N.$$

This shows that $x \in F(N, v^0)$. Since $x \in F(N, v^j)$, $j = 1, \dots, k$, by (b) of Claim, we obtain $x \in F(N, v)$ using affine invariance of F. □

4.3. Proof of Proposition 3

Affine invariance on $\mathcal{G}$. Let $(N, v^1), (N, v^2) \in \mathcal{G}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 + \lambda_2 = 1$. Denote $v = \lambda_1 v^1 + \lambda_2 v^2$. Suppose that $x \in \mathbb{R}^N$ satisfies $x \in G(N, v^1) \cap G(N, v^2)$. Then, we have

$$x_i = g_{N,v^l}(x)_i = v^l(\{i\}) + \sum_{\substack{S\ni i \\ x_S\neq 0, |S|\geq 2}} \frac{x_i}{x_S}\Delta_{N,v^l}(S)$$

for $l \in \{1, 2\}$, $i \in N$.

This implies that

$$\begin{aligned} x_i &= \lambda_1 x_i + \lambda_2 x_i = \lambda_1 g_{N,v^1}(x)_i + \lambda_2 g_{N,v^2}(x)_i \\ &= \lambda_1 v^1(\{i\}) + \sum_{\substack{S\ni i \\ x_S\neq 0, |S|\geq 2}} \frac{\lambda_1 x_i}{x_S}\Delta_{N,v^1}(S) + \lambda_2 v^2(\{i\}) \\ &\quad + \sum_{\substack{S\ni i \\ x_S\neq 0, |S|\geq 2}} \frac{\lambda_2 x_i}{x_S}\Delta_{N,v^2}(S) \\ &= v(\{i\}) + \sum_{\substack{S\ni i \\ x_S\neq 0, |S|\geq 2}} \frac{x_i}{x_S}\left(\lambda_1 \Delta_{N,v^1}(S) + \lambda_2 \Delta_{N,v^2}(S)\right) \\ &= v(\{i\}) + \sum_{\substack{S\ni i \\ x_S\neq 0, |S|\geq 2}} \frac{x_i}{x_S}\Delta_{N,v}(S) = g_{N,v}(x)_i, \qquad i \in N. \end{aligned}$$

Thus, we have $x \in G(N, v)$. This shows that G satisfies affine invariance.

Generalized proportional sharing principle. Suppose that $(N, v) \in \mathcal{G}_1$ is inessential or satisfies $V := \sum_{j\in E(N,v)} v(\{j\}) \neq 0$ and $v(E(N, v)) \neq 0$. To simplify the notation we denote $E = E(N, v)$. Let $\tilde{x} \in \mathbb{R}^N$ be given by

$$\tilde{x}_i = \begin{cases} v(\{i\}) + \frac{v(\{i\})}{V}\Delta_{N,v}(E), & \text{for } i \in E, \\ v(\{i\}), & \text{for } i \in N \setminus E. \end{cases}$$

First we show that $\tilde{x} \in G(N, v)$. If (N, v) is inessential, then clearly $g_{N,v}(\tilde{x}) = \tilde{x}$ and, consequently, $\tilde{x} \in G(N, v)$. So suppose that $V \neq 0$

and $v(E) \neq 0$. Observe that $\tilde{x}_E = V + \Delta_{N,v}(E) = v(E)$. Thus $\tilde{x}_E \neq 0$. Then we have for $i \in E$

$$\begin{aligned} g_{N,v}(\tilde{x})_i &= v(\{i\}) + \frac{\tilde{x}_i}{\tilde{x}_E}\Delta_{N,v}(E) \\ &= v(\{i\}) + \frac{v(\{i\}) + \frac{v(\{i\})}{V}\Delta_{N,v}(E)}{V + \Delta_{N,v}(E)}\Delta_{N,v}(E) \\ &= v(\{i\}) + \frac{v(\{i\})}{V}\Delta_{N,v}(E) = \tilde{x}_i. \end{aligned}$$

If $i \in N \setminus E$, then clearly $g_{N,v}(\tilde{x})_i = v(\{i\}) = \tilde{x}_i$. Thus we have $\tilde{x} \in G(N, v)$.

It remains to prove that $G(N, v)$ contains no point but $\tilde{x}$. Suppose that $x \in G(N, v)$, i.e., $g_{N,v}(x) = x$. We distinguish several possibilities.

(a) Assume that (N, v) is inessential. Then from the equality $x = g_{N,v}(x)$ we get $x_i = v(\{i\}) = \tilde{x}_i$ for every $i \in N$.

(b) Assume that $V \neq 0$, $v(E) \neq 0$, and moreover $x_E \neq 0$. Then the equality $x = g_{N,v}(x)$ can be written as

$$x_i = \begin{cases} v(\{i\}) + \frac{x_i}{x_E}\Delta_{N,v}(E), & \text{for } i \in E, \\ v(\{i\}), & \text{for } i \in N \setminus E. \end{cases} \tag{4}$$

For $i \in N \setminus E$, we have the desired equality $x_i = v(\{i\}) = \tilde{x}_i$. Summing up x_i over $i \in E$, we infer from (4)

$$x_E = \sum_{i \in E} v(\{i\}) + \Delta_{N,v}(E) = V + \Delta_{N,v}(E). \tag{5}$$

Thus we have $x_E \neq \Delta_{N,v}(E)$ and using (4), we infer for $i \in E$

$$x_i = \frac{v(\{i\})}{1 - \frac{\Delta_{N,v}(E)}{x_E}}.$$

Using (5) we get

$$x_i = \frac{v(\{i\})}{1 - \frac{\Delta_{N,v}(E)}{x_E}} = \frac{v(\{i\})}{1 - \frac{\Delta_{N,v}(E)}{V + \Delta_{N,v}(E)}} = v(\{i\}) + \frac{v(\{i\})}{V}\Delta_{N,v}(E) = \tilde{x}_i.$$

(c) Finally assume that $V \neq 0$, $v(E) \neq 0$, and $x_E = 0$. Then the equality $x = g_{N,v}(x)$ can be written as $x_i = v(\{i\})$ for $i \in N$. Summing up x_i over $i \in E$, we infer $0 = x_E = \sum_{i \in E} v(\{i\}) = V \neq 0$, a contradiction. This shows G satisfies the generalized proportionality principle, and completes the proof.

References

[1] S. Béal, S. Ferrières, E. Rémila, P. Solal, The proportional Shapley value and applications, Games Econom. Behav. 108 (2018) 93–112.

[2] M. Besner, Axiomatizations of the proportional Shapley value, Theory and Decision 86 (2) (2019) 161–183.

[3] A. Billot, J.-F. Thisse, How to share when context matters: the Möbius value as a generalized solution for cooperative games, J. Math. Econom. 41 (8) (2005) 1007–1029.

[4] R. van den Brink, R. Levínský, M. Zelený, On proper Shapley values for monotone TU-games, Internat. J. Game Theory 44 (2) (2015) 449–471.

[5] Y. Chun, The proportional solution for rights problems, Math. Social Sci. 15 (3) (1988) 231–246.

[6] J.C. Harsanyi, A bargaining model for the cooperative n-person game, in: Contributions To the Theory of Games, Vol. IV, in: Annals of Mathematics Studies, vol. 40, Princeton University Press, Princeton, N.J., 1959, pp. 325–355.

[7] C.G. Homans, Social Behavior: Its Elementary Forms, Routledge & Paul, London, 1961.

[8] F. Maniquet, A characterization of the Shapley value in queueing problems, J. Econom. Theory 109 (1) (2003) 90–103.

[9] H. Moulin, Equal or proportional division of a surplus, and other methods, Internat. J. Game Theory 16 (3) (1987) 161–186.

[10] A. van den Nouweland, P. Borm, W. van Golstein Brouwers, R. Groot Bruinderink, S. Tijs, A game theoretic approach to problems in telecommunication, Manage. Sci. 42 (1996) 294–303.

[11] K.M. Ortmann, The proportional value for positive cooperative games, Math. Methods Oper. Res. 51 (2) (2000) 235–248.

[12] R. Selten, The equity principle in economic behavior, in: H. Gottinger, W. Leinfellner (Eds.), Decision Theory and Social Ethics: Issues in Social Choice, D. Reidel Publishing Company, Dordrecht, Holland, 1978, pp. 289–301.

[13] L.S. Shapley, Additive and non-additive set functions (Ph.D. thesis), ProQuest LLC, Ann Arbor, MI, 1953, p. 73, Princeton University.

[14] L.S. Shapley, A value for n-person games, in: Contributions To the Theory of Games, Vol. 2, in: Annals of Mathematics Studies, vol. 28, Princeton University Press, Princeton, N. J., 1953, pp. 307–317.

[15] N.N. Vorob'ev, A.N. Liapounov, The proper Shapley value, Int. J. Math. Game Theory Algebra 10 (2) (2000) 115–120.