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Chapter 15
Mathematical Details of Specific Difference and Differential Equations and Mathematical Analysis of Emerging Network Behaviour

Abstract In this chapter, additional mathematical details are presented for many of the chapters concerning the specific difference and differential equations, and mathematical analysis of emerging behaviour. For modeling and analysis of practical applications, insight into these details may not be necessary, but they may deepen insight from the mathematical and technical angle.

Keywords Difference equation · Differential equation · Network behaviour · Mathematical analysis

15.1 Introduction

As also discussed at the end of Chap. 1, in many chapters, especially from Chaps. 1 to 9, mathematical and procedural details were kept at a minimum to obtain optimal readability for a wide group of readers with diverse multidisciplinary backgrounds. As the Network-Oriented Modeling approach based on reified temporal-causal networks presented in this book abstracts from specific implementation details, making use of the dedicated software environment, modeling can be done without having to design procedural or algorithmic specifications. Moreover, a modeler does not even need to explicitly specify difference or differential equations to get a simulation done, as these are already taken care for by the software environment, based on the modeler's input in the form of the role matrices specifying the conceptual representation of the network model. Therefore, in Chaps. 1–9 all underlying specific procedural elements and difference or differential equations were usually not discussed, although the underlying universal difference and differential equation were briefly mentioned and discussed more extensively in Chap. 10. The only mathematical details that were addressed for design of a network model concern the combination functions used, most of which are already given in the combination function library. For analysis of the emerging behaviour of a network model also these combination functions are central, as the equilibrium equations are

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based on them. Moreover, especially for Chaps. 11–14 in the current chapter (Sects. 15.6–15.9) details of the proofs are discussed that were omitted in those chapers.

However, for those readers who still want to see more mathematical details that are covered in the software environment, the current chapter presents these in more depth in different sections, as a kind of appendices to many of the chapters. These sections should be read in conjunction with the concerning chapter, since that chapter itself is not repeated here as a whole.

15.2 Two Different Formulations of Hebbian Learning Are Equivalent

In this section it is shown why the two forms of modeling adaptation (the hybrid form and the temporal-causal form) discussed in Chap. 1, Sects. 1.4.1 and 1.6.1 are mathematically equivalent. Recall from Chap. 1, Sect. 1.6.1 the Eqs. (1.1), (1.3) and (1.4), for network adaptation by Hebbian learning based on network reification, here renumbered to (15.1)–(15.3):

$$Y(t+\Delta t) = Y(t) + \boldsymbol{\eta}_Y[\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) - Y(t)]\Delta t \tag{15.1}$$

$$\mathbf{hebb}_{\boldsymbol{\mu}}(V_1, V_2, W) = V_1V_2(1-W) + \boldsymbol{\mu}W \tag{15.2}$$

$$\boldsymbol{\mu} = 1 - \boldsymbol{\zeta}/\boldsymbol{\eta} \text{ or } \boldsymbol{\zeta} = (1-\boldsymbol{\mu})\boldsymbol{\eta} \tag{15.3}$$

Moreover, in Fig. 1.2 in Chap. 1, Sect. 1.4.1 for the hybrid adaptation model the following equation is described:

$$\boldsymbol{\omega}_{X_i,Y}(t+\Delta t) = \boldsymbol{\omega}_{X_i,Y}(t) + (\boldsymbol{\eta}X_i(t)Y(t)(1-\boldsymbol{\omega}_{X_i,Y}(t)) - \boldsymbol{\zeta}\boldsymbol{\omega}_{X_i,Y}(t))\Delta t \tag{15.4}$$

Based on relations (15.1), (15.2) and (15.3) it can be verified that the difference equation shown in (15.4) actually is mathematically equivalent to the standard Eq. (15.1) for reified temporal-causal networks using the above combination function (15.2) for the reification state. However, the latter formulation in (15.2) provides a more transparent and more unified format than (15.4). The equivalence can be found through rewriting of the mathematical formulas by elementary mathematical rules, starting from (15.4). In Box 15.1 for readers with less mathematical background the steps have been explained in some detail.

Box 15.1 Modeling the equation from the hybrid approach in (15.4) from Chap. 1, Sect. 1.4.1, Fig. 1.2 in the standard format for a temporal-causal network

$$
\begin{aligned}
\boldsymbol{\omega}_{X_i,Y}(t+\Delta t) &= \boldsymbol{\omega}_{X_i,Y}(t) + (\boldsymbol{\eta} X_i(t)Y(t)(1-\boldsymbol{\omega}_{X_i,Y}(t)) - \boldsymbol{\zeta}\,\boldsymbol{\omega}_{X_i,Y}(t))\Delta t \\
&\quad \mathit{from}(4) \\
&= \boldsymbol{\omega}_{X_i,Y}(t) + (\boldsymbol{\eta} X_i(t)Y(t)(1-\boldsymbol{\omega}_{X_i,Y}(t)) - (1-\boldsymbol{\mu})\boldsymbol{\eta}\,\boldsymbol{\omega}_{X_i,Y}(t))\Delta t \\
&\quad \mathit{applying}(3) \\
&= \boldsymbol{\omega}_{X_i,Y}(t) + \boldsymbol{\eta}[X_i(t)Y(t)(1-\boldsymbol{\omega}_{X_i,Y}(t)) - (1-\boldsymbol{\mu})\,\boldsymbol{\omega}_{X_i,Y}(t)]\Delta t \\
&\quad (\mathit{anti})\mathit{distribution\,for}\ \boldsymbol{\eta} \\
&= \boldsymbol{\omega}_{X_i,Y}(t) + \boldsymbol{\eta}[X_i(t)Y(t)(1-\boldsymbol{\omega}_{X_i,Y}(t)) - \boldsymbol{\omega}_{X_i,Y}(t) + \boldsymbol{\mu} w_{X_i,Y}(t)]\Delta t \\
&\quad \mathit{distribution\,for}\ \boldsymbol{\omega}_{X_i,Y}(t) \\
&= \boldsymbol{\omega}_{X_i,Y}(t) + \boldsymbol{\eta}[X_i(t)Y(t)(1-\boldsymbol{\omega}_{X_i,Y}(t)) + \boldsymbol{\mu} w_{X_i,Y}(t) - \boldsymbol{\omega}_{X_i,Y}(t)]\Delta t \\
&\quad \mathit{commutation\ of} - \boldsymbol{\omega}_{X_i,Y}(t)\ \mathit{and}\ \boldsymbol{\mu}\boldsymbol{\omega}_{X_i,Y}(t) \\
&= \boldsymbol{\omega}_{X_i,Y}(t) + \boldsymbol{\eta}[\mathbf{hebb}_{\boldsymbol{\mu}}(X_i(t), Y(t), \boldsymbol{\omega}_{X_i,Y}(t)) - \boldsymbol{\omega}_{X_i,Y}(t)]\Delta t \quad \mathit{applying}(2)
\end{aligned}
$$

As shown in Box 15.1, Eq. (15.4) for $\boldsymbol{\omega}_{X_i,Y}$ displayed in Fig. 1.2 in Chap. 1 can be rewritten into the following mathematically equivalent equation:

$$\boldsymbol{\omega}_{X_i,Y}(t+\Delta t) = w_{X_i,Y}(t) + \boldsymbol{\eta}[\mathbf{hebb}_{\boldsymbol{\mu}}(X_i(t), Y(t), \boldsymbol{\omega}_{X_i,Y}(t)) - \boldsymbol{\omega}_{X_i,Y}(t)]\Delta t$$

In terms of the reification state $\mathbf{W}_{X_i,Y}$ substituted for $\boldsymbol{\omega}_{X_i,Y}$ this is

$$\mathbf{W}_{X_i,Y}(t+\Delta t) = \mathbf{W}_{X_i,Y}(t) + \boldsymbol{\eta}[\mathbf{hebb}_{\boldsymbol{\mu}}(X_i(t), Y(t), \mathbf{W}_{X_i,Y}(t)) - \mathbf{W}_{X_i,Y}(t)]\Delta t$$

and this form is indeed exactly the standard equation form for a temporal-causal network applied to the reification state $\mathbf{W}_{X_i,Y}$, with $k = 3$, incoming impacts defined by the two upward (blue) arrows in Fig. 1.4 in Chap. 1, Sect. 1.4.2 and a connection to $\mathbf{W}_{X_i,Y}$ itself, and combination function $\mathbf{hebb}_{\boldsymbol{\mu}}(V_1, V_2, W)$ defined by (15.2) above.

15.3 Numerical Representation for an Example Reified Network Model

This section addresses the specific difference equations for the example reified network model described in Chap. 3, Sect. 3.7, as used in the software. From the specifications shown in Chap. 3, Sect. 3.7, Box 3.8, the difference equations are derived according to the format in (15.1) above, or (for the manager opinion state) according to the universal difference equation in Chap. 3, Sect. 3.5 (or in Chap. 10) as follows. Based on the role matrix specifications shown in Box 3.8 in Chap. 3, the difference equations for the three reification states are obtained as can be seen in Box 15.2.

Box 15.2 Difference equations for the reification states $\mathbf{H}_{\text{manageropinion}}$, $\mathbf{C}_{1,\text{manageropinion}}$, and $\mathbf{C}_{2,\text{manageropinion}}$

$$\mathbf{H}_{\text{manageropinion}}(t+\Delta t) = \mathbf{H}_{\text{manageropinion}}(t) + 0.5\,[\text{available time}(t) - \mathbf{H}_{\text{manageropinion}}(t)]\Delta t$$

$$\mathbf{C}_{1,\text{manageropinion}}(t+\Delta t) = \mathbf{C}_{1,\text{manageropinion}}(t)$$
$$+0.5[\mathbf{ssum}_{0.02}(0.01X_1(t), 0.01X_2(t), 0.01X_3(t), 0.01X_4(t), 0.01X_5(t), 0.01X_6(t), 0.01X_7(t),$$
$$-0.05\text{disappointment}(t)) - \mathbf{C}_{1,\text{manageropinion}}(t)]\Delta t$$

This can be rewritten into

$$\begin{aligned}\mathbf{C}_{1,\text{manageropinion}}(t+\Delta t) = \mathbf{C}_{1,\text{manageropinion}}(t) &+ 0.5[0.5X_1(t) + 0.5X_2(t) + 0.5X_3(t) + 0.5X_4(t)\\ &+ 0.5X_5(t) + 0.5X_6(t) + 0.5X_7(t) - 2.5\text{disappointment}(t)\\ &- \mathbf{C}_{1,\text{manageropinion}}(t)]\Delta t\end{aligned}$$

$$\mathbf{C}_{2,\text{manageropinion}}(t+\Delta t) = \mathbf{C}_{2,\text{manageropinion}}(t)$$
$$+0.5\,[\text{disappointment}(t) - \mathbf{C}_{2,\text{manageropinion}}(t)]\Delta t$$

The difference equation for the base state manager opinion is given by the universal difference equation described in Chap. 3, Sect. 3.5 or in Chap. 10; see Box 15.3. According to formula (15.1), and the specifications in Chap. 3, Box 3.8, the base states available time and disappointment get the difference equations as shown in Box 15.3.

Box 15.3 Difference equations for the base states
Substituting the manager opinion state for Y in the universal difference equation and using the role matrix specifications in the row for X_8 in Chap. 3, Box 3.8 provides:

$$\text{manageropinion}(t+\Delta t) = \text{manageropinion}(t) + \mathbf{H}_{\text{manageropinion}}(t)$$
$$\left[\frac{\mathbf{C}_{1,\text{manageropinion}}(t)\text{bcf}_1(\mathbf{W}_{X_1,\text{manageropinion}}(t)X_1(t),\ldots,\mathbf{W}_{X_7,\text{manageropinion}}(t)X_7(t)) + \mathbf{C}_{2,\text{manageropinion}}(t)\text{bcf}_2(\mathbf{W}_{X_1,\text{manageropinion}}(t)X_1(t),\ldots,\mathbf{W}_{X_7,\text{manageropinion}}(t)X_7(t))}{\mathbf{C}_{1,\text{manageropinion}}(t) + \mathbf{C}_{2,\text{manageropinion}}(t)} - \text{manageropinion}(t)\right]\Delta t$$

Using $\mathbf{W}_{X_i,\text{manageropinion}} = 1$ for all i (see the row for X_8 in **mcw** in Box 3.8), and $\text{bcf}_1(..) = \mathbf{ssum}_\lambda(..)$ and $\text{bcf}_2(..) = \mathbf{alogistic}_{\sigma,\tau}(..)$, this can be rewritten as

$$\text{manageropinion}(t+\Delta t) = \text{manageropinion}(t) + \mathbf{H}_{\text{manageropinion}}(t)$$
$$[\frac{\mathbf{C}_{1,\text{manageropinion}}(t)\,\mathbf{ssum}_7(X_1(t),\ldots,X_7(t)) + \mathbf{C}_{2,\text{manageropinion}}(t)\mathbf{alogistic}_{5,5.5}(X_1(t),\ldots,X_7(t))}{\mathbf{C}_{1,\text{manageropinion}}(t) + \mathbf{C}_{2,\text{manageropinion}}(t)}$$
$$-\ \text{manageropinion}(t)]\Delta t$$

According to temporal-causal format (1) above, and the role matrix specifications in Box 3.8, the base states available time and disappointment get the following difference equations:

$$\text{available time}(t+\Delta t) = \text{available time}(t) + 0.04[\mathbf{alogistic}_{18,0.2}(\text{available time}(t)) - \text{available time}(t)]\Delta t$$
$$\text{disappointment}(t+\Delta t) = \text{disappointment}(t) + 0.025[\mathbf{alogistic}_{18,0.2}(\text{disappointment}(t)) - \text{disappointment}(t)]\Delta t$$

Similarly, according to (1) and the role matrix specifications in Box 3.8 in Chap. 3, the group members X_i, i = 1, …, 7 get the following difference equations:

$$X_i(t+\Delta t) = X_i(t) + 0.005[\mathbf{ssum}_{\lambda_i}\left(\boldsymbol{\omega}_{X_1,X_i}X_1(t),\ldots,\boldsymbol{\omega}_{X_7,X_i}X_7(t)\right) - X_i(t)]\Delta t$$

where $\boldsymbol{\lambda}_i$ is the sum of the incoming weights $\boldsymbol{\omega}_{X_j,X_i}$ for X_i.

15.4 The Difference Equations for Combined Hebbian Learning and State-Connection Modulation

In Chap. 5, Sects. 5.3 and 5.4 an example reified network model was described in which the Hebbian learning adaptation principle is combined with the state-connection modulation adaptation principle. In the current section the underlying difference equations used in the implementation are shown in some detail. Recall that for the reification states of the connection weights the following combination functions were used. For *Hebbian learning* of a connection from state X_i to state X_j with connection weight reification state **W** the function described in (15.2) above where $\boldsymbol{\mu}$ is the persistence factor with 1 as full persistence. For *state-connection modulation* with control state cs_2 for connection weight reification state **W**:

$$\mathbf{scm}_{\boldsymbol{\alpha}}(V_1, V_2, W, V) = W + \boldsymbol{\alpha}\, VW(1-W) \qquad (15.5)$$

where $\boldsymbol{\alpha}$ is the modulation parameter for **W** from cs_2, V is the single impact from cs_2, and W is the value of **W**; the V_1 and V_2 are auxiliary variables allowing to (partly) separate the arguments used in the two functions. For Hebbian learning separately the difference equation is:

$$\mathbf{W}(t+\Delta t) = \mathbf{W}(t) + \boldsymbol{\eta}_{\mathbf{W}}\left[\mathbf{c}_{\mathbf{W}}(X_i(t), X_j(t), \mathbf{W}(t)) - \mathbf{W}(t)\right]\Delta t \tag{15.6}$$

with

$$\mathbf{c}_{\mathbf{W}}(V_1, V_2, W) = \mathbf{hebb}_{\boldsymbol{\mu}}(V_1, V_2, W) = V_1 V_2(1-W) + \boldsymbol{\mu}\, W \tag{15.7}$$

For *state-connection modulation* with control state cs_2 for connection weight reification state **W** the difference equation is:

$$\mathbf{W}(t+\Delta t) = \mathbf{W}(t) + \boldsymbol{\eta}_{\mathbf{W}}\left[\mathbf{c}_{\mathbf{W}}(X_i(t), X_j(t), \text{cs}_2(t), \mathbf{W}(t)) - \mathbf{W}(t)\right]\Delta t \tag{15.8}$$

with

$$\mathbf{c}_{\mathbf{W}}(V_1, V_2, W, V) = \mathbf{scm}_{\boldsymbol{\alpha}}(V_1, V_2, W, V) = W + \boldsymbol{\alpha} VW(1-W) \tag{15.9}$$

Note that the first two auxiliary variables of $\mathbf{scm}_{\boldsymbol{\alpha}}(V_1, V_2, W, V)$ are not used in the formula (15.9) for $\mathbf{scm}_{\boldsymbol{\alpha}}(V_1, V_2, W, V)$. These variables are included to be able to combine this function with the Hebbian learning function while using the same sequence of variables. More specifically, this combination is done as follows. These two adaptive combination functions are used as a weighted average with $\boldsymbol{\gamma}_{\mathbf{1}}$ and $\boldsymbol{\gamma}_{\mathbf{2}}$ the combination function weights for $\mathbf{hebb}_{\boldsymbol{\mu}}(V_1, V_2, W)$ and $\mathbf{scm}_{\boldsymbol{\alpha}}(V_1, V_2, W, V)$, respectively, as follows:

$$\mathbf{W}(t+\Delta t) = \mathbf{W}(t) + \boldsymbol{\eta}_{\mathbf{W}}\left[\mathbf{c}_{\mathbf{W}}(X_i(t), X_j(t), \mathbf{W}(t), \text{cs}_2(t)) - \mathbf{W}(t)\right]\Delta t \tag{15.10}$$

with

$$\mathbf{c}_{\mathbf{W}}(V_1, V_2, W, V) = \boldsymbol{\gamma}_{\mathbf{1}}\mathbf{hebb}_{\boldsymbol{\mu}}(V_1, V_2, W) + \boldsymbol{\gamma}_{\mathbf{2}}\mathbf{scm}_{\boldsymbol{\alpha}}(V_1, V_2, W, V) \tag{15.11}$$

So, basically the difference equation for the reification state **W** for the weight of the connection from X_i to X_j is:

$$\begin{aligned}
\mathbf{W}(t+\Delta t) &= \mathbf{W}(t) + \boldsymbol{\eta}_{\mathbf{W}}[\boldsymbol{\gamma}_{\mathbf{1}}\mathbf{hebb}_{\boldsymbol{\mu}}(V_1, V_2, W) + \boldsymbol{\gamma}_{\mathbf{2}}\mathbf{scm}_{\boldsymbol{\alpha}}(V_1, V_2, W, V) - \mathbf{W}(t)]\Delta t \\
&= \mathbf{W}(t) + \boldsymbol{\eta}_{\mathbf{W}}[\boldsymbol{\gamma}_{\mathbf{1}}[V_1 V_2(1-W) + \boldsymbol{\mu} W] + \boldsymbol{\gamma}_{\mathbf{2}}[W + \boldsymbol{\alpha} VW(1-W)] - \mathbf{W}(t)]\Delta t \\
&= \mathbf{W}(t) + \boldsymbol{\eta}_{\mathbf{W}}[\boldsymbol{\gamma}_{\mathbf{1}}\left[X_i(t)X_j(t)(1-\mathbf{W}(t)\right) + \boldsymbol{\mu}\mathbf{W}(t)] \\
&\quad + \boldsymbol{\gamma}_{\mathbf{2}}[\mathbf{W}(t) + \boldsymbol{\alpha}\text{cs}_2(t)\mathbf{W}(t)(1-\mathbf{W}(t))] - \mathbf{W}(t)]\Delta t
\end{aligned}$$

However, also taking into account that the speed factor of **W** is adaptive and represented by a reification state **H**, the equation becomes:

$$\begin{aligned}\mathbf{W}(t+\Delta t) = \mathbf{W}(t) + \mathbf{H}(t)[\boldsymbol{\gamma}_\mathbf{1}\left[X_i(t)X_j(t)(1-\mathbf{W}(t)\right) + \boldsymbol{\mu}\mathbf{W}(t)] \\ + \boldsymbol{\gamma}_\mathbf{2}[\mathbf{W}(t) + \boldsymbol{\alpha}\mathrm{cs}_2(t)\mathbf{W}(t)(1-\mathbf{W}(t))] - \mathbf{W}(t)]\Delta t\end{aligned}$$

This can also be rewritten into a correct temporal-causal format (with speed factor by default 1), based on the universal difference equation as shown in Chap. 10:

$$\begin{aligned}\mathbf{W}(t+\Delta t) &= \mathbf{W}(t) + [\mathbf{H}(t)[\boldsymbol{\gamma}_\mathbf{1}\left[X_i(t)X_j(t)(1-\mathbf{W}(t)\right) + \boldsymbol{\mu}\mathbf{W}(t)] \\ &\quad + \boldsymbol{\gamma}_\mathbf{2}[\mathbf{W}(t) + \boldsymbol{\alpha}\mathrm{cs}_2(t)\mathbf{W}(t)(1-\mathbf{W}(t))]] \\ &\quad + (1 - \mathbf{H}(t))\mathbf{W}(t) - \mathbf{W}(t)]\Delta t \\ &= \mathbf{W}(t) + [\mathbf{c}^*_\mathbf{W}(\mathbf{H}(t), X_i(t), X_j(t), \mathbf{W}(t), \mathrm{cs}_2(t)) - \mathbf{W}(t)]\Delta t\end{aligned}$$

where

$$\mathbf{c}^*_\mathbf{W}(H, V_1, V_2, W, V) = H[\boldsymbol{\gamma}_\mathbf{1}[V_1V_2(1-W) + \boldsymbol{\mu}W] + \boldsymbol{\gamma}_\mathbf{2}[W + \boldsymbol{\alpha}V\,W(1-W)]] + (1 - H)W$$

15.5 Difference and Differential Equations for Multilevel Connection Weight Reification States

In Chap. 6, Sect. 6.3 the combination functions for the example model were described. In the current section, the difference equations used in implementation will be added. The base level and first reification level states are addressed in Box 15.4.

Box 15.4 Combination functions and difference equations for the base level and first reification level
Base level:
Base state X_i combination function and difference equation
The combination function for the base states X_i is basically the advanced logistic sum function $\mathbf{alogistic}_{\sigma,\tau}(..)$.
However, as the connection weights are reified at the first reification level, based on the universal combination function format, the following adaptive form for the combination function for the base states X_i is needed here:

$$\begin{aligned}\mathbf{c}^*_Y(W_1, \ldots, W_k, V_1, \ldots, V_k,) &= \mathbf{alogistic}_{\sigma,\tau}(W_1V_1, \ldots, W_kV_k) \\ &= \left[\frac{1}{1+\mathrm{e}^{-\sigma(W_1V_1+\ldots+W_kV_k-\tau)}} - \frac{1}{1+\mathrm{e}^{\sigma\tau})}\right](1+\mathrm{e}^{-\sigma\tau})\end{aligned}$$

Here W_i refers to connection weight reification state value $\mathbf{W}_{X_i,Y}(t)$ and V_i to state value $X_i(t)$.
This combination function defines the following difference equation for Y (see Chap. 4, Sect. 4.2, Table 4.1):

$$Y(t+\Delta t) = Y(t) + \boldsymbol{\eta}_Y[\mathbf{alogistic}_{\boldsymbol{\sigma},\boldsymbol{\tau}}(\mathbf{W}_1(t)X_1(t), \ldots, \mathbf{W}_k(t)X_k(t)) - Y(t)]\Delta t$$

where

$$\mathbf{W}_i(t) = \mathbf{W}_{X_i,Y}(t)$$

First reification level:
Connection weight reification state $\mathbf{W}_{Y,Xi}$ combination function and difference equation
See Chap. 4, Sects. 4.3.2, and 4.2, or Chap. 3, Sect. 3.6.1, or (Treur 2016), Chap. 11, Sect. 11.7, the combination function $\mathbf{slhomo}_{\boldsymbol{\alpha}}$**(..)** for connection weight reification state $\mathbf{W}_{X_i,Y}$ is basically

$$\mathbf{slhomo}_{\boldsymbol{\alpha}}(V_1, V_2, W) = W + \boldsymbol{\alpha} W(1 - W)(\boldsymbol{\tau} - |V_1 - V_2|)$$

where

- W refers to connection weight reification state value $\mathbf{W}_{X_i,Y}(t)$
- V_1 to $X_1(t)$ and V_2 to $X_2(t)$
- $\boldsymbol{\alpha}$ is a homophily modulation factor for $\mathbf{W}_{X_i,Y}$
- $\boldsymbol{\tau}$ is a homophily tipping point for $\mathbf{W}_{X_i,Y}$.

However, as the speed factor and tipping point are reified at the second reification level, based on the universal combination function format, the following adaptive form for the combination function for connection weight reification state $\mathbf{W}_{X_i,Y}$ is needed here:

$$\mathbf{c}^*_{\mathbf{W}_{X_i,Y}}(H, V_1, V_2, T, W) = H(W + \boldsymbol{\alpha} W(1 - W)(T - |V_1 - V_2|)) + (1 - H)W$$

where

- H refers to the speed factor reification $\mathbf{H}_{\mathbf{W}_{X_i,Y}}(t)$ for $\mathbf{W}_{X_i,Y}$
- W to connection weight reification $\mathbf{W}_{X_i,Y}(t)$
- T to homophily tipping point reification state value $\mathbf{TP}_{\mathbf{W}_{X_i,Y}}(t)$ for $\mathbf{W}_{X_i,Y}$
- V_1 to $X_1(t)$ and V_2 to $X_2(t)$
- $\boldsymbol{\alpha}$ is a homophily modulation factor.

This combination function (together with connection weights and speed factor 1) defines the following difference equation for connection weight reification state $\mathbf{W} = \mathbf{W}_{X_i,Y}$ (see Sect. 4.2, Table 4.1):

$$\mathbf{W}(t+\Delta t) = \mathbf{W}(t) + [\mathbf{H}(t)(\mathbf{W}(t) + \boldsymbol{\alpha}\,\mathbf{W}(t)(1-\mathbf{W}(t))(\mathbf{TP}(t)|X_i(t) - Y(t)|)) \\ + (1-\mathbf{H}(t))\mathbf{W}(t) - \mathbf{W}]\Delta t$$

where

$$\mathbf{H}(t) = \mathbf{H}_{\mathbf{W}X_i,Y}(t) \quad \mathbf{TP}(t) = \mathbf{TP}_{\mathbf{W}\,X_i,Y}(t)$$

The reification states at the second reification level are addressed in Box 15.5 (homophily tipping point reification state), and Box 15.6 (connection weight speed factor reification state).

Box 15.5 Combination function and difference equation for the homophily tipping point reification state at the second reification level

Second reification level: tipping point reification state $\mathbf{TP}_{\mathbf{W}_{X_i,Y}}$ combination function and difference equation

The following combination function called *simple linear tipping point function* **sltip**$_{\mathbf{v},\boldsymbol{\alpha}}$**(..)** can be used for the second order reification state $\mathbf{TP}_{\mathbf{W}_{X_i,Y}}$ at the second reification level (upper, purple plane):

$$\mathbf{sltip}_{\mathbf{v},\boldsymbol{\alpha}}(W_1, \ldots, W_k, T) = T + \boldsymbol{\alpha}T(1-T)(\mathbf{v} - (W_1 + \ldots + W_k)/k)$$

where

- T refers to the homophily tipping point reification value $\mathbf{TP}_{\mathbf{W}_{X_i,Y}}(t)$ for $\mathbf{W}_{X_i,Y}$
- W_j to connection weight reification value $\mathbf{W}_{X_i,Y}(t)$
- $\boldsymbol{\alpha}$ is a modulation factor for the tipping point $\mathbf{TP}_{\mathbf{W}_{X_i,Y}}$
- $\mathbf{v}$ is a norm for Y for average connection weight $\mathbf{W}_{X_1,Y}$ to $\mathbf{W}_{X_k,Y}$

This function can be explained as follows. The norm parameter $\mathbf{v}$ indicates the preferred average level of the connection weights $\mathbf{W}_{X_i,Y}$ for person Y. The part $(\mathbf{v} - (W_1 + \ldots + W_k)/k)$ in the formula is positive when the current average connection weight $(W_1 + \ldots + W_k)/k$ is lower than this norm, and negative when it is higher that the norm. When T is not 0 or 1, in the first case, the combination function provides a value higher than T, which makes that the tipping point is increased, and as a consequence more connections are strengthened by the homophily adaptation, so the average connection weight will become more close to the norm $\mathbf{v}$. In the second case, the opposite takes place: the combination function provides a value lower than T, which makes that the tipping point is decreased, and as a consequence more connections are weakened by the homophily adaptation, so also now the average connection weight will become more close to the norm **v**. Together this makes that in principle (unless in the meantime other factors change) the average connection weight will approximate the norm **v**. The factor *T* (1 − *T*) in the formula takes care that the values for *T* stay within the [0, 1] interval.

Together with connection weights and speed factor 1, this combination function defines the following difference equation for tipping point reification state $\mathbf{TP} = \mathbf{TP}_{\mathbf{W}_{X_i,Y}}$ (see Sect. 4.2, Table 4.2):

$$\mathbf{TP}(t+\Delta t) = \mathbf{TP}(t) + \boldsymbol{\eta}[[\mathbf{TP}(t) + \boldsymbol{\alpha}\mathbf{TP}(t)(1-\mathbf{TP}(t))(\mathbf{v} - (\mathbf{W}_1(t) + \ldots + \mathbf{W}_k(t))/k)] - \mathbf{TP}(t)]\Delta t$$

where

$$\mathbf{W}_i(t) = \mathbf{W}_{X_i,Y}(t)$$

Box 15.6 Combination function and difference equation for the connection weight speed factor reification state at the second reification level

Second reification level:

speed factor reification state $\mathbf{H}_{\mathbf{W}_{Y,Xi}}$ combination function and difference equation

For the adaptive connection adaptation speed factor the following combination function called *simple linear speed function* $\mathbf{slspeed}_{\mathbf{v},\boldsymbol{\alpha}}$**(..)** can be considered making use of a similar mechanism using a norm for connection weights.

$$\mathbf{slspeed}_{\mathbf{v},\boldsymbol{\alpha}}(W_1, \ldots, W_k, , H) = H + \boldsymbol{\alpha} H(1-H)(\mathbf{v} - (W_1 + \ldots + W_k)/k)$$

where

- *H* refers to $\mathbf{W}_{Y,X_i}$ speed factor reification value $\mathbf{H}_{\mathbf{W}_{X_i,Y}}(t)$
- W_j to connection weight reification value $\mathbf{W}_{X_i,Y}(t)$
- $\boldsymbol{\alpha}$ is a modulation factor for $\mathbf{H}_{\mathbf{W}_{X_i,Y}}$
- **v** is a norm for average of (incoming) connection weights for *Y*

This function can be explained as follows. Also here the norm parameter **v** indicates the preferred average level of the connection weights $\mathbf{W}_{X_i,Y}$ for person *Y*. The part $(\mathbf{v} - (W_1 + \ldots + W_k)/k)$ in the formula is positive when the current average connection weight $(W_1 + \ldots + W_k)/k$ is lower than this norm, and negative when it is higher that the norm. When *H* is not 0 or 1, in the first case, the combination function provides a value higher than *H*, which makes that the speed factor is increased, and the connections are changing faster by the homophily adaptation. In the second case, the combination function provides a value lower than H, which makes that the speed factor is decreased, and as a consequence the homophily adaptation speed is lower. The factor $H(1 - H)$ in the formula takes care that the values for H stay within the [0, 1] interval.

This combination function defines the following difference equation for speed factor reification state $\mathbf{H} = \mathbf{H}_{\mathbf{W}_{X_i,Y}}$:

$$\mathbf{H}(t+\Delta t) = \mathbf{H}(t) + \boldsymbol{\eta}[[\mathbf{H}(t) + \boldsymbol{\alpha}\mathbf{H}(t)(1-\mathbf{H}(t))(\mathbf{v} \\ - (\mathbf{W}_1(t) + \ldots + \mathbf{W}_k(t))/k)] - \mathbf{H}(t)]\Delta t$$

where

$$\mathbf{W}_i(t) = \mathbf{W}_{X_i,Y}(t)$$

15.6 Emerging Behaviour for Types of Aggregation and Types of Connectivity

This section presents a number of proofs that were left out from Chap. 11.

Proposition 6 Suppose a network with nonnegative connections has normalised scalar-free combination functions.

(a) If X_1, …, X_k are the states from which Y gets its incoming connections, and $X_1(t) = \ldots = X_k(t) = V$ for some common value V, then also $\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) = V$.
(b) If, moreover, the combination functions are monotonic, and $X_1, \ldots, X_k$ are the states from which Y gets its incoming connections, and $V_1 \leq X_1(t), \ldots, X_k(t) \leq V_2$ for some values V_1 and V_2, then also $V_1 \leq \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) \leq V_2$ and if $\boldsymbol{\eta}_Y\Delta t \leq 1$ and $V_1 \leq Y(t) \leq V_2$ then $V_1 \leq Y(t+\Delta t) \leq V_2$.

Proof

(a) This follows from

$$\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) = \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}V, \ldots, \boldsymbol{\omega}_{X_k,Y}V) = V\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}) \\ = V$$

(b) This follows from $V_1 = V_1\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}) = \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}V_1, \ldots, \boldsymbol{\omega}_{X_k,Y}V_1) \leq \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) \leq \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}V_2, \ldots, \boldsymbol{\omega}_{X_k,Y}V_2) = V_2\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}) = V_2$

The last part follows from

$$\begin{aligned} Y(t+\Delta t) &= Y(t) + \boldsymbol{\eta}_Y[\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) - Y(t)]\Delta t \\ &= (1-\boldsymbol{\eta}_Y\Delta t)Y(t) + \boldsymbol{\eta}_Y\Delta t\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) \end{aligned}$$

So $Y(t + \Delta t)$ is a weighted average with weights between 0 and 1 of $Y(t)$ and $\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t))$ which both are in the interval $[V_1, V_2]$. Therefore $Y(t + \Delta t)$ itself also is in the interval $[V_1, V_2]$.

Theorem 1 (common state values provide equilibria) Suppose a network with nonnegative connections is based on normalised and scalar-free combination functions. Then the following hold.

(a) Whenever all states have the same value V, the network is in an equilibrium state.
(b) If for every state for its initial value V it holds $V_1 \leq V \leq V_2$, then for all t for every state Y it holds $V_1 \leq Y(t) \leq V_2$. In an achieved equilibrium for every state for its equilibrium value V it holds $V_1 \leq V \leq V_2$.

Proof

(a) It follows from Proposition 6(a) that the criterion of Lemma 1 is fulfilled.
(b) From Proposition 6(b) it follows by induction over the time steps that during a simulation for every state Y it holds $V_1 \leq Y(t) \leq V_2$ and therefore in a limit situation in an achieved equilibrium for every state for its equilibrium value V it holds $V_1 \leq V \leq V_2$.

■

Lemma 3 (Relating radical and max expressions) Suppose $a_1, \ldots, a_k$ are any nonnegative real numbers. Then

$$\lim_{n\to\infty} \sqrt[n]{a_1^n + \ldots + a_k^n} = \max(a_1, \ldots, a_k)$$

Proof First note that on the one hand

$$\max(a_1, \ldots, a_k) = \sqrt[n]{\max(a_1, \ldots, a_k)^n} \leq \sqrt[n]{\mathrm{a}_1^n + \ldots + \mathrm{a}_k^n}$$

and on the other hand

$$\begin{aligned} &\sqrt[n]{\mathrm{a}_1^n + \ldots + \mathrm{a}_k^n} \leq \sqrt[n]{\max(a_1, \ldots, a_k)^n + \ldots + \max(a_1, \ldots, a_k)^n} \\ &\quad = \max(a_1, \ldots, a_k)\sqrt[n]{k} \end{aligned}$$

So

$$\max(a_1, \ldots, a_k) \leq \sqrt[n]{a_1^n + \ldots + a_k^n} \leq \max(a_1, \ldots, a_k)\sqrt[n]{k}$$

Now

$$\lim_{n\to\infty} \ln\left(\sqrt[n]{k}\right) = \lim_{n\to\infty} \ln(k)/n = 0$$

and therefore

$$\lim_{n\to\infty} \sqrt[n]{k} = \lim_{n\to\infty} e^{\ln\left(\sqrt[n]{k}\right)} = e^0 = 1.$$

This proves that for any nonnegative real numbers $a_1, \ldots, a_k$ it holds

$$\lim_{n\to\infty} \sqrt[n]{a_1^n + \ldots + a_k^n} = \max(a_1, \ldots, a_k)$$

■

Theorem 8 Let for each n the normalised Euclidean combination function $\mathbf{eucl}_{n,\lambda(n)}(V_1, \ldots, V_k)$ be given with scaling factor $\lambda(n)$, and let the normalised scaled maximum combination function $\mathbf{smax}_{\lambda}(V_1, \ldots, V_k)$ be given with scaling factor λ. Then for all $V_1, \ldots, V_k$ it holds

$$\lim_{n\to\infty} \mathbf{eucl}_{n,\lambda(n)}(V_1, \ldots, V_k) = \mathbf{smax}_{\lambda}(V_1, \ldots, V_k)$$

where

$$\lambda(n) = \omega_{X_1,Y}^n + \ldots + \omega_{X_k,Y}^n$$

and

$$\lambda = \max\left(\omega_{X_1,Y}, \ldots, \omega_{X_k,Y}\right)$$

Proof Recal the normalised formulas described in Table 11.4 in Chap. 11:

$$\mathbf{eucl}_{n,\lambda(n)}(V_1, .., V_k) = \sqrt[n]{\frac{V_1^n + \ldots + V_k^n}{\omega_{X_1,Y^n} + \ldots + \omega_{X_k,Y^n}}}$$

$$\mathbf{smax}_{\lambda}(V_1, \ldots, V_k) = \max(V_1, \ldots, V_k)/\max(\omega_{X_1,Y}, \ldots, \omega_{X_k,Y})$$

where

$$\boldsymbol{\lambda}(n) = \boldsymbol{\omega}_{X_1,Y^n} + \ldots + \boldsymbol{\omega}_{X_k,Y^n}$$

and

$$\boldsymbol{\lambda} = \max\left(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}\right)$$

Apply Lemma 3 to both $\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}$ and $V_1, \ldots, V_k$ for $a_1, \ldots, a_k$ as follows

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{eucl}_{n,\boldsymbol{\lambda}(n)}(V_1, \ldots, V_k) &= \lim_{n\to\infty} \sqrt[n]{\frac{V_1^n + \ldots + V_k^n}{\boldsymbol{\lambda}(n)}} \\
&= \frac{\lim\limits_{n\to\infty} \sqrt[n]{\mathrm{V}_1^n + \ldots + \mathrm{V}_k^n}}{\lim\limits_{n\to\infty} \sqrt[n]{\boldsymbol{\omega}_{X_1,Y}^n + \ldots + {}_{X_k,Y}^n}} = \frac{\max(V_1, \ldots, V_k)}{\max\left(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}\right)} \\
&= \mathbf{smax}_{\boldsymbol{\lambda}}(V_1, \ldots, V_k).
\end{aligned}$$

15.7 Using Strongly Connected Components to Explore Emerging Behaviour for a Class of Combination Functions for Any Type of Network Connectivity

This section presents a number of proofs that were left out from Chap. 12.

Proposition 1 Suppose the network is normalised.

(a) If the combination functions are scalar-free and X_1, …, X_k are the states from which Y gets its incoming connections, and $X_1(t)$ = … = $X_k(t)$ = V for some common value V, then also $\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) = V$.

(b) If the combination functions are scalar-free and X_1, …, X_k are the states from which Y gets its incoming connections, and for U_1, …, U_k, V_1, …, V_k and $\alpha \geq 0$ it holds $V_i = \boldsymbol{\alpha}\ U_i$, then $\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}V_1, \ldots, \boldsymbol{\omega}_{X_k,Y}V_k) = \boldsymbol{\alpha}\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}U_1, \ldots, \boldsymbol{\omega}_{X_k,Y}U_k)$ If in this situation in two different simulations, state values $X_i(t)$ and $X'_i(t)$ are generated then $X'_i(t) = \boldsymbol{\alpha}\, X_i(t) \Rightarrow X'_i(t + \Delta t) = \boldsymbol{\alpha}\, X_i(t + \Delta t)$

(c) If the combination functions are additive and X_1, …, X_k are the states with outgoing connections to Y, then for values $U_1, \ldots, U_k$, $V_1, \ldots, V_k$ it holds

$$\begin{aligned}
\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}(U_1 + V_1), \ldots, \boldsymbol{\omega}_{X_k,Y}(U_k + V_k)) &= \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}U_1, \ldots, \boldsymbol{\omega}_{X_k,Y}U_k) \\
&\quad + \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}V_1, \ldots, \boldsymbol{\omega}_{X_k,Y}V_k)
\end{aligned}$$

If in this situation in three different simulations, state values $X_i(t), X_i'(t)$ and $X_i''(t)$ are generated then

$$X_i''(t) = X_i(t) + X_i'(t) \Rightarrow X_i''(t+\Delta t) = X_i(t+\Delta t) + X_i'(t+\Delta t)$$

(d) If the combination functions are scalar-free and monotonically increasing, and X_1, …, X_k are the states with outgoing connections to Y, and $V_1 \le X_1(t), \ldots, X_k(t) \le V_2$ for some values V_1 and V_2, then also

$$V_1 \le \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) \le V_2$$

and if $\boldsymbol{\eta}_Y\Delta t \le 1$ and $V_1 \le Y(t) \le V_2$ then $V_1 \le Y(t+\Delta t) \le V_2$.

Proof

(a) This works as follows:

$$\begin{aligned}\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) &= \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}V, \ldots, \boldsymbol{\omega}_{X_k,Y}V) = V\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})\\ &= V\end{aligned}$$

(b) can easily be verified
(c) can easily be verified.
(d) This follows from

$$\begin{aligned}V_1 &= V_1\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}) = \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}V_1, \ldots, \boldsymbol{\omega}_{X_k,Y}V_1)\\ &\le \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) \le \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}V_2, \ldots, \boldsymbol{\omega}_{X_k,Y}V_2) = V_2\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}) = V_2\end{aligned}$$

and the second part from

$$\begin{aligned}Y(t+\Delta t) &= Y(t) + \boldsymbol{\eta}_Y[\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) - Y(t)]\Delta t\\ &= \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t))\boldsymbol{\eta}_Y\Delta t + Y(t)(1 - \boldsymbol{\eta}_Y\Delta t)\\ &\le V_2\boldsymbol{\eta}_Y\Delta t + V_2(1 - \boldsymbol{\eta}_Y\Delta t) = V_2\end{aligned}$$

and similarly for V_1

$$\begin{aligned}Y(t+\Delta t) &= \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t))\boldsymbol{\eta}_Y\Delta t + Y(t)(1 - \boldsymbol{\eta}_Y\Delta t)\\ &\ge V_1\boldsymbol{\eta}_Y\Delta t + V_1(1 - \boldsymbol{\eta}_Y\Delta t) = V_1\end{aligned}$$

■

Theorem 7 (equilibrium state values in relation to level 0 components in the linear case) Suppose the network N is normalised and the combination functions

are strictly monotonically increasing and linear. Assume that the states at level 0 that form a singleton component on their own are constant.
Then the following hold:

(a) For each state Y its equilibrium value is independent of the initial values of all states at some level $i > 0$. It is only dependent on the initial values for the states at level 0.
(b) More specifically, let B_1, …, B_p be the states in level 0 components. Then for each state Y its equilibrium value eq_Y is described by a linear function of the initial values $V_1, \ldots, V_p$ for $B_1, \ldots, B_p$, according to the following weighted average:

$$\mathrm{eq}_Y(V_1, \ldots, V_p) = d_{B_1,Y} V_1 + \ldots + d_{B_p,Y} V_p$$

Here the $d_{B_i,Y}$ are real numbers between 0 and 1 and the sum of them is 1:

$$d_{B_1,Y} + \ldots + d_{B_p,Y} = 1$$

(c) Each $d_{B_i,Y}$ is the equilibrium value for Y when the following initial values are used: $V_i = 1$ and all other initial values are 0:

$d_{Bi,Y} = eq_Y(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 as *ith* argument.

Proof From Proposition 1 it follows that he equilibrium value of Y is a linear function of the initial values of all states of N. Therefore the function is a linear combination of $e_i = \mathrm{eq}_Y(0, \ldots, 0, 1, 0, \ldots, 0)$ where only one state has initial value 1 and all other 0. An alternative, more theoretical linear algebra argument uses that the set of functions over time generated by the difference equations for different initial values forms an n-dimensional linear space with as basis the functions $d_i(t)$ generated for initial value 1 for state X_i and 0 for all other states. Therefore each generated function is a linear combination of such functions. By substituting $t = 0$ in them it is shown that the coefficients are the initial values and substituting t for an equilibrium shows that these initial values are the coefficients at that time point.

Now consider the different stratification levels. When all level 0 states have initial value 0, then by Theorem 5(a)(iii) they will have equilibrium value 0 as well. Then from Theorem 5(b)(ii) it follows that all states will have equilibrium value 0. In particular, this holds for cases that only one of the states at a level $i > 0$ have value 1 and all other states have initial value 0. This shows that from the linear combination the coefficient of these terms are 0. Therefore $\mathrm{eq}_Y(\ldots)$ is a function of $V_1, \ldots, V_p$ only.

■

Theorem 9 (equilibrium state values for components of level $i > 0$) Suppose the network is normalised, and consists of a strongly connected component plus a number of independent states $A_1, \ldots, A_p$ with outgoing connections to this strongly connected component. Then the following hold

(a) Suppose the combination functions are scalar-free and X_1, …, X_k are the states from which Y gets its incoming connections. If for U_1, …, U_k, V_1, …, V_k and $\alpha \geq 0$ it holds $V_i = \alpha\ U_i$ for all i, then $\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}V_1, \ldots, \boldsymbol{\omega}_{X_k,Y}V_k) = \alpha\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}U_1, \ldots, \boldsymbol{\omega}_{X_k,Y}U_k)$

(b) Suppose the combination functions are additive and X_1, …, X_k are the states from which Y gets its incoming connections. Then if for values U_1, …, U_k, V_1, …, V_k, W_1, …, W_k it holds $W_i = U_i + V_i$ for all i, then

$$\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}W_1, \ldots, \boldsymbol{\omega}_{X_k,Y}W_k) = \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}U_1, \ldots, \boldsymbol{\omega}_{X_k,Y}U_k) + \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}V_1, \ldots, \boldsymbol{\omega}_{X_k,Y}V_k)$$

(c) Suppose all combination functions of the network N are linear. Then for given connection weights and speed factors, for each state Y the achieved equilibrium value for Y only depends on the equilibrium values V_1, …, V_p of states A_1, …, A_p; the function $\text{eq}_Y(V_1, \ldots, V_p)$ denotes this achieved equilibrium value for Y.

(d) Suppose the combination functions of the network N are linear. For the given connection weights and speed factors for each i let $d_{i,Y}$ be the achieved equilibrium value for state Y in a situation with equilibrium values $A_i = 1$ *and* $A_j = 0$ for all $j \neq i$, *i.e.,* $d_{i,Y} = eq_Y(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 as *ith* argument. Then in the general case for these given connection weights and speed factors, for each Y in the strongly connected component its equilibrium value is a linear, monotonically increasing, continuous and differentiable function $\text{eq}_Y(\ldots)$ of the equilibrium values V_1, …, V_p of A_1, …, A_p satisfying the following linear relation: $\text{eq}_Y(V_1, \ldots, V_p) = d_{1,Y}\ V_1 + \ldots + d_{p,Y}\ V_p$. Here the sum of the $d_{i,Y}$ *is 1:* $d_{1,Y} + \ldots + d_{p,Y} = 1$. In particular, the equilibrium values are independent of the initial values for all states Y different from A_1, …, A_p. If the combination functions of N are strictly increasing, then $d_{i,Y} > 0$ for all i, and $\text{eq}_Y(..)$ is also strictly increasing.

Proof (a) and (b) follow from Proposition 1
(c) From (a) and (b) it follows that the equilibrium value of Y is a linear function of the initial values of all states of N. Therefore the function is a linear combination of $e_i = \text{eq}_Y(0, \ldots, 0, 1, 0, \ldots, 0)$ where only one state has initial value 1 and all other 0. However, when all independent states have (constant) value 0, from Theorem 5(b)(ii) it follows that all states will have equilibrium value 0. In particular, this holds for cases that only one of the states that are not independent have initial value 1 and all other states have initial value 0. This shows that from the linear combination the coefficient e_i of these terms are 0. Therefore $\text{eq}_Y(\ldots)$ is a function of V_1, …, V_p only. From a) and b) it follows that $\text{eq}_Y(V_1, \ldots, V_p)$ is linear, as indicated above. Therefore

$$\begin{aligned}\text{eq}_Y(V_1, \ldots, V_p) &= \text{eq}_Y(V_1, 0, \ldots, 0) + \ldots + \text{eq}_Y(0, \ldots, 0, V_i, 0, .., 0) + \ldots + \text{eq}_Y(0, \ldots, V_p) \\ &= \text{eq}_Y(1, 0, \ldots, 0)V_1 + \ldots + \text{eq}_Y(0, \ldots, 0, 1, 0, .., 0)V_i + \ldots + \text{eq}_Y(0, \ldots, 1)V_p \\ &= d_{1,Y}V_1 + \ldots + d_{i,Y}V_i + \ldots + d_{p,Y}V_p\end{aligned}$$

■

15.8 Analysis of Emerging Behaviour for Classes of Homophily Functions

This section presents a number of proofs that were left out from Chap. 13.

Proposition 1 Suppose for any function $\mathrm{d}(\tau, D)$ it holds

$$\mathrm{d}(\tau, D) > 0 \text{ iff } D < \tau$$
$$\mathrm{d}(\tau, D) < 0 \text{ iff } D > \tau$$

Then the following hold:

(a) For any $\alpha > 0$ the function

$$\mathrm{c}(V_1, V_2, W) = W + \alpha W(1 - W)\mathrm{d}(\tau, |V_1 - V_2|)$$

satisfies the tipping point condition, but not strict.

(b) For any $\alpha > 0$ the function

$$\mathrm{c}'(V_1, V_2, W) = W + \alpha \,\mathrm{Pos}(\mathrm{d}(\tau, |V_1 - V_2|))\,(1 - W) - \alpha \,\mathrm{Pos}(-\mathrm{d}(\tau, |V_1 - V_2|))W$$

satisfies the strict tipping point condition.

Proof

(a) The proof is mainly based on some algebraic rewriting.

Here as a first step it has to be proven that for any W with $0 < W < 1$ and all V_1, V_2 it holds

$$|V_1 - V_2| < \tau \Leftrightarrow \mathrm{c}(V_1, V_2, W) > W$$

This follows from

$$|V_1 - V_2| < \tau \Leftrightarrow \mathrm{d}(\tau, |V_1 - V_2|) > 0 \Leftrightarrow \alpha W(1 - W)\mathrm{d}(\tau, |V_1 - V_2|) > 0 \Leftrightarrow \mathrm{c}(V_1, V_2, W) > W.$$

Similarly the other two cases for $|V_1 - V_2| > \tau$ and $|V_1 - V_2| = \tau$ can be verified. From $\mathrm{c}(V_1, V_2, 0) = 0$ for all V_1, V_2 it follows that the strict tipping point requirement is not fulfilled.

(b) Also this proof is mainly based on some algebraic rewriting, thereby using Lemma 1.

First, $\mathrm{c}'(V_1, V_2, W)$ satisfies the tipping point condition; for any W with $0 < W < 1$ and all V_1, V_2 by Lemma 1 it holds:

$$
\begin{aligned}
|V_1 - V_2| < \tau &\Leftrightarrow \mathrm{d}(\tau, |V_1 - V_2|) > 0 \\
&\Leftrightarrow \mathrm{Pos}(\mathrm{d}(\tau, |V_1 - V_2|)) > 0 \,\text{and}\, \mathrm{Pos}(-\mathrm{d}(\tau, |V_1 - V_2|)) = 0
\end{aligned}
$$

Similarly the other two conditions.
It is strict because

$$|V_1 - V_2| < \tau \Rightarrow \mathrm{d}(\tau, |V_1 - V_2|) > 0 \Rightarrow \mathrm{c}'(V_1, V_2, 0) = \alpha\, \mathrm{Pos}(\mathrm{d}(\tau, |V_1 - V_2|) > 0$$

and

$$
\begin{aligned}
|V_1 - V_2| > \tau \Rightarrow \mathrm{d}(\tau, |V_1 - V_2|) < 0 &\Rightarrow \mathrm{c}'(V_1, V_2, 1) \\
&= 1 - \alpha\, \mathrm{Pos}(-\mathrm{d}(\tau, |V_1 - V_2|) < 1
\end{aligned}
$$

■

Proposition 2

(a) $\mathbf{log1hom}_{\tau,\alpha}(V_1, V_2, W)$ has tipping point τ, and is not strict
(b) $\mathbf{slog2hom}_{\tau,\alpha}(V_1, V_2, W)$ has tipping point τ, and is not strict
(c) $\mathbf{alog2hom}_{\tau,\alpha}(V_1, V_2, W)$ has a strict tipping point τ
(d) $\mathbf{exphomo}_{\tau,\sigma}(V_1, V_2, W)$ has a tipping point τ and is not strict

Proof

(a) This is based on some algebraic rewriting.

For $\mathbf{log1hom}_{\tau,\alpha}(V_1, V_2, W)$ suppose $0 < W < 1$, then for $D = |V_1 - V_2|$ it holds

$$
\begin{aligned}
D < \tau \Leftrightarrow \mathrm{e}^{\sigma(D-\tau)} < 1 &\Leftrightarrow (1 - W)\mathrm{e}^{\sigma(D-\tau)} < 1 - W \\
&\Leftrightarrow W + (1 - W)\mathrm{e}^{\sigma(D-\tau)} < W + 1 - W = 1 \\
&\Leftrightarrow \frac{W}{W + (1 - W)\mathrm{e}^{\sigma(D-\tau)}} > W
\end{aligned}
$$

Similarly the other conditions can be verified:

$$
\begin{aligned}
D = \tau \Leftrightarrow \mathrm{e}^{\sigma(D-\tau)} = 1 &\Leftrightarrow (1 - W)\mathrm{e}^{\sigma(D-\tau)} = 1 - W \\
&\Leftrightarrow W + (1 - W)\mathrm{e}^{\sigma(D-\tau)} = W + 1 - W = 1 \\
&\Leftrightarrow \frac{W}{W + (1 - W)\mathrm{e}^{\sigma(D-\tau)}} = W
\end{aligned}
$$

To verify that it has no strict tipping point τ: for any V_1, V_2 it holds

$$\mathbf{log1hom}_{\tau,\alpha}(V_1, V_2, 0) = 0$$

(b) This proof is based on Proposition 1. Applying Proposition 1(a) to $\mathbf{slog2hom}_{\tau,\alpha}(V_1, V_2, W)$ consider

$$\mathrm{d}(\tau, D) = 0.5 - \frac{1}{1 + \mathrm{e}^{-\sigma(D-\tau)}}$$

This function indeed satisfies the conditions of Proposition 1; therefore from Proposition 1(a) it follows that it has tipping point τ but not strict tipping point τ.

(c) Also this proof is based on Proposition 1. For $\mathbf{alog2hom}_{\tau,\alpha}(V_1, V_2, W)$ it follows from Proposition 1(b) with the same d(τ, D) as above that it has strict tipping point τ.

(d) This is based on some algebraic rewriting.
For W with $0 < W < 1$ it holds

$$\begin{aligned} D < \tau &\Leftrightarrow \mathrm{e}^{\sigma(D-\tau)} < 1 \Leftrightarrow (1-W)\mathrm{e}^{\sigma(D-\tau)} < (1-W) \\ &\Leftrightarrow 1-(1-W)\mathrm{e}^{\sigma(D-\tau)} > 1-(1-W) = W \Leftrightarrow \mathbf{exphomo}_{\tau\sigma}(V_1, V_2, W) > W \end{aligned}$$

$$\begin{aligned} D = \tau &\Leftrightarrow \mathrm{e}^{\sigma(D-\tau)} = 1 \Leftrightarrow (1-W)\mathrm{e}^{\sigma(D-\tau)} = (1-W) \\ &\Leftrightarrow 1-(1-W)\mathrm{e}^{\sigma(D-\tau)} = 1-(1-W) = W \Leftrightarrow \mathbf{exphomo}_{\mathbf{t,s}}(V_1, V_2, W) = W \end{aligned}$$

$$\begin{aligned} D > \tau &\Leftrightarrow \mathrm{e}^{\sigma(D-\tau)} > 1 \Leftrightarrow (1-W)\mathrm{e}^{\sigma(D-\tau)} > (1-W) \\ &\Leftrightarrow 1-(1-W)\mathrm{e}^{\sigma(D-\tau)} < 1-(1-W) = W \Leftrightarrow \mathbf{exphomo}_{\tau,\sigma}(V_1, V_2, W) < W \end{aligned}$$

This shows it has tipping point τ. It has no strict tipping point, as

$$\mathbf{exphomo}_{\tau,\sigma}(V_1, V_2, 1) = 1 \text{ for all } V_1, V_2.$$

The following proposition shows that weighted averages of functions with tipping point τ also have a tipping point τ, and the same for having a strict tipping point.

Proposition 3 A weighted average (with positive weights) of homophily combination functions with tipping point τ also has tipping point τ, and with strict tipping point τ, also has strict tipping point τ.

Proof This can be verified in a straightforward manner.

Suppose

$$c(V_1, V_2, W) = \gamma_1 \mathrm{c}_1(V_1, V_2, W) + \ldots + \gamma_m \mathrm{c}_m(V_1, V_2, W)$$

with $\gamma_1 + \ldots + \gamma_m = 1$. Suppose $0 < W < 1$. Then $|V_1 - V_2| < \tau \Rightarrow \mathrm{c}_i(V_1, V_2, W) > W$ for all i, and therefore

$$\mathrm{c}(V_1, V_2, W) > \gamma_1 W + \ldots + \gamma_m W = W$$

Similarly

$$|V_1 - V_2| = \tau \Rightarrow \mathrm{c}(V_1, V_2, W) = W$$
$$|V_1 - V_2| > \tau \Rightarrow \mathrm{c}(V_1, V_2, W) < W$$

Now suppose $\mathrm{c}(V_1, V_2, W) > W$, then $|V_1 - V_2| = \tau$ or $|V_1 - V_2| > \tau$ cannot hold as they imply $\mathrm{c}(V_1, V_2, W) = W$ or $\mathrm{c}(V_1, V_2, W) < W$, therefore $|V_1 - V_2| < \tau$. The same for the other clauses. Moreover, suppose that the functions $\mathrm{c}_i(V_1, V_2, W)$ all have strict tipping point τ. Then

If $|V_1 - V_2| < \tau$ then

$$\mathrm{c}(V_1, V_2, 0) = \gamma_1 \mathrm{c}_1(V_1, V_2, 0) + \ldots + \gamma_m \mathrm{c}_m(V_1, V_2, 0) > 0$$

If $|V_1 - V_2| < \tau$ then

$$\mathrm{c}(V_1, V_2, 1) = \gamma_1 \mathrm{c}_1(V_1, V_2, 1) + \ldots + \gamma_m \mathrm{c}_m(V_1, V_2, 1) < \gamma_1 + \ldots + \gamma_m = 1$$

Therefore also $\mathrm{c}(V_1, V_2, W)$ has a strict tipping point.

■

Proposition 4

(a) When the homophily combination function $\mathrm{c}(V_1, V_2, W)$ is symmetric, and initially the network is fully symmetric, then the network is continually fully symmetric.
(b) For every $n > 0$ a Euclidean combination function of **n**th degree is strictly monotonically increasing, scalar-free, and symmetric.

Proof

(a) This follows from the fact that in this case the difference equation for reification state $\mathbf{W}_{X,Y}$ for $\omega_{X,Y}$ is symmetric in X and Y.
(b) A Euclidean combination function is composed of strictly monotonic functions as each function $V_i \rightarrow V_i^{\boldsymbol{n}}$ is monotonic for positive $\boldsymbol{n}$ and positive values V_i, and so are $W \rightarrow W/\lambda$ and $W \rightarrow W^{1/\boldsymbol{n}}$. From

$$\begin{aligned}
\mathbf{eucl}_{n,\lambda}(\alpha V_1, \ldots, \alpha V_k) &= \sqrt[n]{\frac{(\alpha V_1)^{\boldsymbol{n}} + \ldots + (\alpha V_k)^{\boldsymbol{n}}}{\lambda}} \\
&= \sqrt[n]{\frac{\alpha^{\boldsymbol{n}} V_1^{\boldsymbol{n}} + \ldots + \alpha^{\boldsymbol{n}} V_k^{\boldsymbol{n}}}{\lambda}} \\
&= \alpha \sqrt[n]{\frac{V_1^{\boldsymbol{n}} + \ldots + V_k^{\boldsymbol{n}}}{\lambda}} \\
&= \alpha\, \mathbf{eucl}_{n,\lambda}(V_1, \ldots, V_k)
\end{aligned}$$

it follows that it is scalar-free. The rest directly follows.

■

Lemma 3 Suppose the function c(V_1, V_2, W) has tipping point τ for V_1 and V_2. Then

(i) The value 0 for $W_{X,Y}$ can only be reached from $\mathbf{W}_{X,Y}(t)$ with $0 < \mathbf{W}_{X,Y}(t) < 1$ if $|X(t) - Y(t)| > \tau$
(ii) The value 1 for $\mathbf{W}_{X,Y}$ can only be reached from $\mathbf{W}_{X,Y}(t)$ with $0 < \mathbf{W}_{X,Y}(t) < 1$ if $|X(t) - Y(t)| < \tau$.

Proof

(i) The proof is by contraposition. Suppose $0 < \mathbf{W}_{X,Y}(t) < 1$ holds and $|X(t) - Y(t)| > \tau$ does not hold. Then $|X(t) - Y(t)| \leq \tau$, and by Definition 1(a)(i) and (ii) it follows that c(V_1, V_2, W) $\geq$ W, and therefore from the difference equation it follows that $\mathbf{W}_{X,Y}(t + \Delta t) \geq \mathbf{W}_{X,Y}(t)$ will not become lower and in particular will not reach 0.
(ii) is similar using Definition 1(a)(ii) and (iii).

Theorem 1 (Relations between equilibrium values for states and for connection weights) Suppose the function c(V_1, V_2, W) has tipping point τ for V_1 and V_2 and an attracting equilibrium state is given with values $\underline{\mathbf{X}}$ for the states X and $\underline{\mathbf{W}}_{X,Y}$ for the connection weight reification states $\mathbf{W}_{X,Y}$. Then the following hold:

(a) If $|\underline{\mathbf{X}} - \underline{\mathbf{Y}}| < \tau$, then the equilibrium value $\underline{\mathbf{W}}_{X,Y}$ is 1; in particular this holds when $\underline{\mathbf{X}} = \underline{\mathbf{Y}}$. Therefore, if $\underline{\mathbf{W}}_{X,Y} < 1$, then $|\underline{\mathbf{X}} - \underline{\mathbf{Y}}| \geq \tau$, and, in particular, $\underline{\mathbf{X}} \neq \underline{\mathbf{Y}}$.
(b) If $|\underline{\mathbf{X}} - \underline{\mathbf{Y}}| > \tau$, then the equilibrium value $\underline{\mathbf{W}}_{X,Y}$ is 0. Therefore, if $\underline{\mathbf{W}}_{X,Y} > 0$, then $|\underline{\mathbf{X}} - \underline{\mathbf{Y}}| \leq \tau$.
(c) $0 < \underline{\mathbf{W}}_{X,Y} < 1$ implies $|\underline{\mathbf{X}} - \underline{\mathbf{Y}}| = \tau$.

Proof

(a) Suppose two states are given with equilibrium values $\underline{\mathbf{X}}$ and $\underline{\mathbf{Y}}$ with distance less than τ: $|\underline{\mathbf{X}} - \underline{\mathbf{Y}}| < \tau$. Given this, from the equilibrium equation **c(**$\underline{\mathbf{X}}$, $\underline{\mathbf{Y}}$, $\underline{\mathbf{W}}_{X,Y}$**)** = $\underline{\mathbf{W}}_{X,Y}$, by Definition 1(ii) it follows that $0 < \underline{\mathbf{W}}_{X,Y} < 1$ cannot be true, and therefore $\underline{\mathbf{W}}_{X,Y} = 0$ or $\underline{\mathbf{W}}_{X,Y} = 1$. By Lemma 3(i) and the equilibrium being attracting it follows that $\underline{\mathbf{W}}_{X,Y} = 0$ can be excluded, so $\underline{\mathbf{W}}_{X,Y} = 1$.

The other statement, that if $\underline{\mathbf{W}}_{X,Y} < 1$, then $|\underline{\mathbf{X}} - \underline{\mathbf{Y}}| \geq \tau$, follows by logical contraposition.

(b) For $|\underline{\mathbf{X}} - \underline{\mathbf{Y}}| > \tau$ this is similar, using Definition 1(iii). The last statement follows from the contraposition of the previous one.
(c) This immediately follows from (a) and (b).

Lemma 4 Let a normalised network with nonnegative connections be given with combination functions that are monotonically increasing and scalar-free; then the following hold:

(a) (i) If for some state Y at time t for all nodes X with $\omega_{X,Y} > 0$ it holds $X(t) \leq Y(t)$, then $Y(t)$ is decreasing at t: $\mathbf{d}Y(t)/\mathbf{d}t \leq 0$.
(ii) If, moreover, the combination function is strictly increasing and a state X exists with $X(t) < Y(t)$ and $\omega_{X,Y} > 0$, then $Y(t)$ is strictly decreasing at t: $\mathbf{d}Y(t)(t)/\mathbf{d}t < 0$.

(b) (i) If for some state Y at time t for all nodes X with $\omega_{X,Y} > 0$ it holds $X(t) \geq Y(t)$, then $Y(t)$ is increasing at t: $\mathbf{d}Y(t)/\mathbf{d}t \geq 0$.
(ii) If, moreover, the combination function is strictly increasing and a state X exists with $X(t) > Y(t)$ and $\omega_{X,Y} > 0$, then $Y(t)$ is strictly increasing at t: $\mathbf{d}Y(t)(t)/\mathbf{d}t > 0$.

Proof The proofs for (a) and (b) are similar. Therefore only the proof for (a) is given.

(a) (i) This proof shows that lower values of the states with incoming connections can never increase the state value of state Y. More specifically, assume for all states X_i with (positive) outgoing connections to Y it holds $X_i(t) \leq Y(t)$. Therefore

$$\begin{aligned} c_Y(\boldsymbol{\omega}_{X_1,Y} X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y} X_k(t)) &\leq \mathrm{c}_Y(\boldsymbol{\omega}_{X_1,Y} Y(t), \ldots, w_{X_k,Y} Y(t)) \\ &= \mathrm{c}_Y(\boldsymbol{\omega}_{X_1,Y} \ldots, \boldsymbol{\omega}_{X_k,Y}) Y(t) \\ &= Y(t) \end{aligned}$$

and by Lemma 2(i) and (iii) this implies $\mathbf{d}Y(t)/\mathbf{d}t \leq 0$.
(ii) This proof shows that a strictly lower values of one of the states with incoming connections will actually decrease the state value of state Y. More specifically, from being strictly monotonous it follows

$$\mathrm{c}_Y(\boldsymbol{\omega}_{X_1,Y} X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y} X_k(t)) < \mathrm{c}_Y(w_{X_1,Y} Y(t), \ldots, \boldsymbol{\omega}_{X_k,Y} Y(t)) = Y(t)$$

and by Lemma 2(iii) this implies $\mathbf{d}Y(t)/\mathbf{d}t < 0$.

(b) This can be proven in a similar manner.

Theorem 2 (Equilibrium values $\underline{\mathbf{W}}_{X,Y}$ all 0 or 1) Suppose the network is weakly symmetric and normalised, and the combination functions for the social contagion for the base states are strictly monotonically increasing and scalar-free. Suppose that the combination functions $\mathrm{c}(V_1, V_2, W)$ for the reification states for the connection weights have a tipping point $\boldsymbol{\tau}$. Then

(a) In an attracting equilibrium state for any states X, Y from $\underline{\mathbf{X}} \neq \underline{\mathbf{Y}}$ it follows $\underline{\mathbf{W}}_{X,Y} = 0$.
(b) In an attracting equilibrium state for any states X, Y with $\underline{\mathbf{X}} = \underline{\mathbf{Y}}$ it holds $\underline{\mathbf{W}}_{X,Y} = 0$ or $\underline{\mathbf{W}}_{X,Y} = 1$.
(c) If $c(V_1, V_2, W)$ has a strict tipping point τ, then in an equilibrium state for any X, Y with $\underline{\mathbf{X}} = \underline{\mathbf{Y}}$ it holds $\underline{\mathbf{W}}_{X,Y} = 1$.

Proof

(a) The proof goes by reductio ad absurdum (reduce to absurdity); it shows that the opposite of the claimed statement cannot be true by deriving a contradiction from this opposite statement. So, suppose (oppositely) that in an attracting equilibrium state, states X and Y exist such that $\underline{\mathbf{X}} \neq \underline{\mathbf{Y}}$ and $\underline{\mathbf{W}}_{X,Y}$, $\underline{\mathbf{W}}_{Y,X} > 0$. Take a state X with this property with highest value $\underline{\mathbf{X}}$. Then for all states Z with $\underline{\mathbf{Z}} > \underline{\mathbf{X}}$ it holds $\underline{\mathbf{W}}_{X,Z} = \underline{\mathbf{W}}_{Z,X} = 0$. Therefore all states X_i with a nonzero (positive) outgoing connection weight to state X satisfy $\underline{\mathbf{X}}_i \leq \underline{\mathbf{X}}$. Moreover, one of these X_i is state Y with $\underline{\mathbf{X}} \neq \underline{\mathbf{Y}}$, so, as X has the highest value, it holds $\underline{\mathbf{Y}} < \underline{\mathbf{X}}$. Now apply Lemma 4(a)(ii) to this state X. It follows that $\mathbf{d}X(t)/\mathbf{d}t < 0$; therefore $X(t)$ cannot be not in equilibrium. This contradicts the premise that the network is in equilibrium. Therefore no nodes X and Y exist such that $\underline{\mathbf{X}} \neq \underline{\mathbf{Y}}$ and $\underline{\mathbf{W}}_{X,Y}$, $\underline{\mathbf{W}}_{Y,X} > 0$. This implies that $\underline{\mathbf{W}}_{X,Y} = 0$ and $\underline{\mathbf{W}}_{\omega Y,X} = 0$ for all nodes X and Y with $\underline{\mathbf{X}} \neq \underline{\mathbf{Y}}$.
(b) Also this proof goes by reductio ad absurdum (reduce to absurdity); also here it is shown that the opposite of the claimed statement cannot be true by deriving a contradiction from this opposite statement. So, suppose (oppositely) $\underline{\mathbf{X}} = \underline{\mathbf{Y}}$ and $0 < \mathbf{W}_{X,Y}(t) < 1$. Then by Definition 1(a)(i) from $X(t) = Y(t)$ it follows that $c(X(t), Y(t), \mathbf{W}_{X,Y}(t)) > \mathbf{W}_{X,Y}(t)$. From this by Lemma 2(ii) it follows that $\mathbf{d}\mathbf{W}_{X,Y}(t)/\mathbf{d}t > 0$: $\mathbf{W}_{X,Y}(t)$ is strictly increasing and is not in equilibrium. This contradicts the premise that the network is in equilibrium. Therefore in the equilibrium state when $\underline{\mathbf{X}} = \underline{\mathbf{Y}}$ it holds $\underline{\mathbf{W}}_{X,Y} = 0$ or $\underline{\mathbf{W}}_{X,Y} = 1$.
(c) From (b) it is already known that $\underline{\mathbf{W}}_{X,Y} = 0$ or $\underline{\mathbf{W}}_{X,Y} = 1$. The former option $\underline{\mathbf{W}}_{X,Y} = 0$ has to be excluded now. Also this goes by reductio ad absurdum (reduce to absurdity); also here it is shown that the option $\underline{\mathbf{W}}_{X,Y} = 0$ cannot be true by deriving a contradiction when this option is assumed. So, suppose $\underline{\mathbf{W}}_{X,Y} = 0$. If $c(V_1, V_2, W)$ is strict, and $|V_1 - V_2| < \tau$ then by Definition 1(b)(i) it holds $c(V_1, V_2, 0) > 0$, so by Lemma 1(ii) it follows that when $\underline{\mathbf{X}} = \underline{\mathbf{Y}}$, the value $\underline{\mathbf{W}}_{X,Y} = 0$ cannot be an equilibrium value, which contradicts the premise that the network is in equilibrium. Therefore in an equilibrium for any X, Y with $\underline{\mathbf{X}} = \underline{\mathbf{Y}}$ it holds $\underline{\mathbf{W}}_{X,Y} = 1$.

Theorem 3 (Partition and equilibrium values of nodes) Suppose the network is weakly symmetric and normalised, the combination functions for the social contagion for the base states are strictly monotonically increasing and scalar-free, and the combination functions for the reification states for the connection weights use tipping point τ and is strict and symmetric. Then in any attracting equilibrium state a partition of the set of states into disjoint subsets $C_1, \ldots, C_p$ occurs such that:

(i) For each C_i the equilibrium values for all the states in C_i are equal: $\underline{\mathbf{X}} = \underline{\mathbf{Y}}$ for all X, $Y \in C_i$.
(ii) Every C_i forms a fully connected network with weights 1: $\underline{\mathbf{W}}_{X,Y} = 1$ for all X, $Y \in C_i$.
(iii) Every two nodes in different C_i have connection weight 0: when $i \neq j$, then $X \in C_i$ and $Y \in C_j$ implies $\underline{\mathbf{W}}_{X,Y} = 0$.
(iv) Any two distinct equilibrium values of states $\underline{\mathbf{X}} \neq \underline{\mathbf{Y}}$ have distance $\geq \tau$. Therefore there are at most $p \leq 1 + 1/\tau$ communities C_i and equilibrium values $\underline{\mathbf{X}}$.

Proof Suppose in the equilibrium there are p distinct state values $V_1, \ldots, V_p$; then define the sets

$$C_i = \{X | \underline{\mathbf{X}} = V_i\}$$

It can easily be verified in a straightforward manner that these sets fulfill what is claimed:

(i) By definition all state values in one C_i are equal.
(ii) From Theorem 2(c) it follows that all states with equal values have connections 1, therefore any C_i is fully connected.
(iii) This follows from Theorem 2(a).
(iv) Suppose for some X, Y it holds $|\, \underline{\mathbf{X}} - \underline{\mathbf{Y}} \,| < \tau$. Then by Definition 1(i) it follows $c(V_1, V_2, 0) > 0$. Therefore 0 cannot be the equilibrium value $\underline{\mathbf{W}}_{X,Y}$; from Theorem 2(a) it follows that $\underline{\mathbf{X}} = \underline{\mathbf{Y}}$, and therefore X and Y are in one C_i. This implies that the state values in different C_i have distance $\geq \tau$.

■

Theorem 6 (Strongly connected components characterisation) Suppose the network is weakly symmetric, the combination functions for social contagion between the base nodes are strictly monotonically increasing, normalised and scalar-free, and the homophily combination functions for the connections weight reification states use tipping point τ and are strict and symmetric. Suppose at some time point t the following hold:

(i) Each strongly connected component C is fully connected and all states in C have a common state value.
(ii) All connections between states from different strongly connected components have weight 0 and the equilibrium values of these states have distance $> \tau$.

Then the network is in an equilibrium state.

Proof First in (a) it is proven that the state values are stationary; next, in b) it is proven that the connection weights are stationary. Having both stationary, the network is in equilibrium.

(a) Consider within any component C any state Y which has only nonzero incoming connections from states X_1, …, X_k. Due to (ii) these necessarily belong to the same component C. As within C the state values are equal to one value V and each connection has weight 1 the following holds

$$\begin{aligned}\mathbf{aggimpact}_Y(t) &= \mathbf{c}_Y(X_1(t), \ldots, X_k(t)) \\ &= \mathbf{c}_Y(V, \ldots, V) \\ &= V\,\mathbf{c}_Y(1, \ldots, 1) \\ &= V \\ &= Y(t)\end{aligned}$$

and therefore by Lemma 2(i) it holds d$Y(t)$/dt = 0, so $Y(t)$ is stationary.

(b) Next, it is proven that the connection weights are stationary. Consider the connection weight reification state $\mathbf{W}_{X,Y}$ for the connection from states X to Y in the same component C. Suppose as a perturbation from 1 it holds $\mathbf{W}_{X,Y}(t) < 1$. Given that $|X(t) - Y(t)| < \tau$, from Definition 1(i) it follows that $\mathbf{c}(X(t), Y(t), \mathbf{W}_{X,Y}(t)) > \mathbf{W}_{X,Y}(t)$, and therefore $\mathbf{dW}_{X,Y}(t)/\mathbf{d}t > 0$, so it would move upward to 1. Therefore $\mathbf{W}_{X,Y}$ it is stationary. A similar argument for states X and Y in different components shows that $\mathbf{W}_{X,Y}$ would move downward to 0, and therefore is stationary.

As both the states and the connection weights have been proven stationary, it has been found that the network is in equilibrium.

■

15.9 Analysis of Emerging Behaviour for Classes of Hebbian Learning Functions

This section presents a number of proofs that were left out from Chap. 14.

Proposition 1 (functional relation for W) Suppose that c(V_1, V_2, W) is a Hebbian learning function with persistence parameter $\boldsymbol{\mu}$.

(a) Suppose $\boldsymbol{\mu} < 1$. Then the following hold:

  (i) The function $W \rightarrow \mathrm{c}(V_1, V_2, W) - W$ on [0, 1] is strictly monotonically decreasing

  (ii) There is a unique function $\mathrm{f}_{\boldsymbol{\mu}} : [0,1]\mathrm{x}[0,1] \rightarrow [0,1]$ such for any V_1, V_2 it holds

$$c(V_1, V_2, f_{\boldsymbol{\mu}}(V_1, V_2)) = f_{\boldsymbol{\mu}}(V_1, V_2)$$

This function $f_{\boldsymbol{\mu}}$ is a monotonically increasing function of V_1, V_2, and is implicitly defined by the above equation. Its maximal value is $f_{\boldsymbol{\mu}}(1, 1)$ and minimum $f_{\boldsymbol{\mu}}(0, 0) = 0$.

(b) Suppose $\boldsymbol{\mu} = 1$. Then there is a unique function $f_1 : (0, 1]x(0, 1] \rightarrow [0, 1]$, such for any V_1, V_2 it holds

$$c(V_1, V_2, f_1(V_1, V_2)) = f_1(V_1, V_2)$$

This function f_1 is a constant function of V_1, V_2 with $f_1(V_1, V_2) = 1$ for all V_1, $V_2 > 0$ and is implicitly defined on (0, 1] x (0, 1] by the above equation.
If one of V_1, V_2 is 0, then any value of W satisfies the equation $c(V_1, V_2, W) = W$, so no unique function value for $f_1(V_1, V_2)$ can be defined then.

Proof

(a) Consider $\boldsymbol{\mu} < 1$. Then by Definition 2(b) the function $W \rightarrow c(V_1, V_2, W) - \boldsymbol{\mu} W$

is monotonically decreasing in W, and since $\boldsymbol{\mu} - 1 < 0$ the function $W \rightarrow (\boldsymbol{\mu} - 1)W$ is strictly monotonically decreasing in W. Therefore the sum of them is also strictly monotonically decreasing in W. Now this sum is

$$c(V_1, V_2, W) - \boldsymbol{\mu} W + (\boldsymbol{\mu} - 1)W = c(V_1, V_2, W) - W$$

So, the function $W \rightarrow c(V_1, V_2, W) - W$ is strictly monotonically decreasing in W; by Definition 2(d) it holds $c(V_1, V_2, 1) - 1 = \boldsymbol{\mu} - 1 < 0$, and by Definition 2(c) $c(V_1, V_2, 0) - 0 \geq 0$. Therefore $c(V_1, V_2, W) - W$ has exactly 1 point with $c(V_1, V_2, W) - W = 0$; so for each V_1, V_2 the equation $c(V_1, V_2, W) - W = 0$ has exactly one solution W, indicated by $f_{\boldsymbol{\mu}}(V_1, V_2)$; this provides a unique function $f_{\boldsymbol{\mu}}$: [0, 1] x [0, 1] $\rightarrow$ [0, 1] implicitly defined by $c(V_1, V_2, f_{\boldsymbol{\mu}}(V_1, V_2)) = f_{\boldsymbol{\mu}}(V_1, V_2)$. To prove that $f_{\boldsymbol{\mu}}$ is monotonically increasing, the following. Suppose $V_1 \leq V_1'$ and $V_2 \leq V_2'$, then by monotonicity of $V_1, V_2 \rightarrow c(V_1, V_2, W)$ in Definition 2(a) it holds

$$0 = c(V_1, V_2, f_{\boldsymbol{\mu}}(V_1, V_2)) - f_{\boldsymbol{\mu}}(V_1, V_2) \leq c(V_1', V_2', f_{\boldsymbol{\mu}}(V_1, V_2)) - f_{\boldsymbol{\mu}}(V_1, V_2)$$

So

$$c(V_1', V_2', f_{\boldsymbol{\mu}}(V_1, V_2)) - f_{\boldsymbol{\mu}}(V_1, V_2) \geq 0$$

whereas

$$c(V_1', V_2', f_{\boldsymbol{\mu}}(V_1', V_2')) - f_{\boldsymbol{\mu}}(V_1', V_2') = 0$$

and therefore

$$c(V_1', V_2', f_\mu(V_1', V_2')) - f_\mu(V_1', V_2') \leq c(V_1', V_2', f_\mu(V_1, V_2)) - f_\mu(V_1, V_2)$$

By strict decreasing monotonicity of $W \to c(V_1, V_2, W) - W$ it follows that $f_\mu(V_1, V_2) > f_\mu(V_1', V_2')$ cannot hold, so $f_\mu(V_1, V_2) \leq f_\mu(V_1', V_2')$. This proves that f_μ is monotonically increasing. From this monotonicity of $f_\mu(..)$ it follows that $f_\mu(1, 1)$ is the maximal value and $f_\mu(0, 0)$ the minimal value. Now by Definition 1(d) it follows that $f_\mu(0, 0) = c(0, 0, f_\mu(0, 0)) = \mu\, f_\mu(0, 0)$ so $f_\mu(0, 0) = \mu\, f_\mu(0, 0)$, and as $\mu < 1$ this implies $f_\mu(0, 0) = 0$.

(b) Consider $\mu = 1$. When both V_1, V_2 are > 0, and $c(V_1, V_2, W) = W$, then $W = 1$,

by Definition 1(d). This defines a function $f_1(V_1, V_2)$ of $V_1, V_2 \in (0, 1]$, this time $f_1(V_1, V_2) = 1$ for all $V_1, V_2 > 0$. When one of V_1, V_2 is 0 and $\mu = 1$, then also by Definition 1(d) always $c(V_1, V_2, W) = W$, so in this case multiple solutions for W are possible: every W is a solution, and therefore no unique function value for $f_1(V_1, V_2)$ can be defined then.

■

Proposition 2 (functional relation for *W* based on variable separation) Assume the Hebbian function $c(V_1, V_2, W)$ with persistence parameter μ enables variable separation by the two functions $cs(V_1, V_2)$ monotonically increasing and $cc(W)$ monotonically decreasing:

$$c(V_1, V_2, W) = cs(V_1, V_2)\, cc(W) + \mu\, W$$

Let $h_\mu(W)$ be the function defined for $W \in [0, 1)$ *by*

$$h_\mu(W) = \frac{(1 - \mu)W}{cc(W)}$$

Then the following hold.

(a) When $\mu < 1$ the function $h_\mu(W)$ is strictly monotonically increasing, and has a strictly monotonically increasing inverse g_μ on the range $h_\mu([0, 1))$ of h_μ with $W = g_\mu(h_\mu(W))$ *for all* $W \in [0, 1)$.
(b) When $\mu < 1$ and $c(V_1, V_2, W) = W$, then $g_\mu(cs(V_1, V_2)) < 1$ and $W < 1$, and it holds

$$h_\mu(W) = cs(V_1, V_2)$$
$$W = g_\mu(cs(V_1, V_2))$$

So, in this case the function f_μ from Theorem 1 is the function composition g_μ o cs of cs followed by g_μ; it holds:

$$f_{\mu}(V_1, V_2) = g_{\mu}(cs(V_1, V_2))$$

(c) For $\mu = 1$ it holds $c(V_1, V_2, W) = W$ if and only if $V_1 = 0$ *or* $V_2 = 0$ or $W = 1$.
(d) For $\mu < 1$ the maximal value W with $c(V_1, V_2, W) = W$ is $g_{\mu}(cs(1, 1))$, and the minimal equilibrium value W is 0. For $\mu = 1$ the maximal value W *is* 1 (always when $V_1, V_2 > 0$ holds) and the minimal value is 0 (occurs when one of V_1, V_2 is 0).

Proof

(a) From $cc(W)$ monotonically decreasing in W it follows that $W \to 1/cc(W)$ is monotonically increasing on [0, 1). Moreover, the function W is strictly monotonically increasing; therefore for $\mu < 1$ the function $h_{\mu}(W) = (1 - \mu) W/cc(W)$ is strictly monotonically increasing. Therefore h_{μ} is injective and has an inverse function g_{μ} on the range of h_{μ}: a function g_{μ} with $g_{\mu}(h_{\mu}(W)) = W$ for all $W \in [0, 1)$.
(b) Suppose $\mu < 1$ and $c(V_1, V_2, W) = W$, then from Definition 2(d) it follows that $W = 1$ is excluded, since from both $c(V_1, V_2, W) = W$ and $c(V_1, V_2, W) = \mu W$ it would follow $\mu = 1$, which is not the case. Therefore $W < 1$, and the following hold

$$\begin{aligned} cs(V_1, V_2)cc(W) + \mu W &= W \\ cs(V_1, V_2)cc(W) &= (1 - \mu)W \\ cs(V_1, V_2) &= (1 - \mu)W/cc(W) = h_{\mu}(W) \end{aligned}$$

So, $h_{\mu}(W) = cs(V_1, V_2)$. Applying the inverse g_{μ} yields $W = g_{\mu}(h_{\mu}(W)) = g_{\mu}(cs(V_1, V_2))$.
Therefore in this case for the function f_{μ} from Theorem 1 it holds:

$$f_{\mu}(V_1, V_2) = W = g_{\mu}(cs(V_1, V_2)) < 1$$

so f_{μ} is the function composition g_{μ} o cs of cs(..) followed by g_{μ}.
(c) For $\mu = 1$ the equation $c(V_1, V_2, W) = W$ becomes $cs(V_1, V_2)\, cc(W) = 0$ and this is equivalent to $cs(V_1, V_2) = 0$ or $cc(W) = 0$. From the definition of separation of variables it follows that this is equivalent to $V_1 = 0$ or $V_2 = 0$ or $W = 1$.
(d) Suppose $\mu < 1$ and $c(V_1, V_2, W) = W$, then because cs(..) and g_{μ} are both monotonically increasing, the maximal W is $g_{\mu}(cs(1, 1))$, and the minimal W is $g_{\mu}(cs(0, 0))$. For $\mu = 1$ these values are 1 always when $V_1, V_2 > 0$, and any value in [0, 1] (including 0) when one of V_1, V_2 is 0. ■