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Chapter 10
On the Universal Combination Function and the Universal Difference Equation for Reified Temporal-Causal Network Models

Abstract The universal differential and difference equation form an important basis for reified temporal-causal networks and their implementation. In this chapter, a more in depth analysis is presented of the universal differential and difference equation. It is shown how these equations can be derived in a direct manner and they are illustrated by some examples. Due to the existence of these universal difference and differential equation, the class of temporal-causal networks is closed under reification: by them it can be guaranteed that any reification of a temporal-causal network is itself also a temporal-causal network. That means that dedicated modeling and analysis methods for temporal-causal networks can also be applied to reified temporal-causal networks. In particular, it guarantees that reification can be done iteratively in order to obtain multilevel reified network models that are very useful to model multiple orders of adaptation. Moreover, as shown in Chap. 9, the universal difference equation enables that software of a very compact form can be developed, as all reification levels are handled by one computational reified network engine in the same manner. Alternatively, it is shown how the universal difference or differential equation can be used for compilation by multiple substitution for all states, which leads to another form of implementation. The background of these issues is discussed in the current chapter.

10.1 Introduction

Modeling dynamic processes by simulating and analysing differential or difference equations has a long tradition in almost all scientific disciplines; e.g., Ashby (1960), Brauer and Nohel (1969), Lotka (1956), Port and van Gelder (1995). The Network-Oriented Modeling approach (Treur 2016, 2019a, b) based on temporal-causal networks also has an underlying differential equation format to model the dynamics. The recent extension of this approach to reified temporal-causal networks as addressed in the current book, has extended and generalised the format of the underlying difference and differential equations in order to enable modeling of networks for processes that are adaptive of any order; see Chaps. 3–8 or (Treur 2018a, b).

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In Chap. 3 it was shown that when network reification is applied to a temporal-causal network, it results in a reified network that itself is again a temporal-causal network. In more abstract mathematical terms this can be formulated by the class of temporal-causal networks being closed under the reification operator. This is a very convenient closure property for this class of networks, as because of that all methods developed for temporal-causal networks can also be applied to reified temporal-causal networks. One important example of this is that it enables that reification can also be applied to reified networks, so that the reification can be iterated easily. Thus multilevel reified networks are obtained that in Chaps. 4–8 turned out very useful to model higher order adaptive networks, for example to model plasticity and metaplasticity. Another important example is that mathematical analysis of emerging behaviour as known for temporal-causal networks can also be applied to reified temporal-causal networks, in particular both to the base states and the reification states in a reified network. Chapters 13 and 14 are examples of such mathematical analyses, where mathematical properties of the combination function of a reification state are related to emerging behaviour for the adaptation process.

To prove that the class of temporal-causal networks is closed under reification there is a central role for the universal combination function and the universal difference equation that can be used to describe the base states in the reified network. In Chap. 3, Sect. 3.5 this universal combination function was introduced more or less out of the blue, and the universal difference equation is just derived from this combination function; that function turned out correct, which provides a form of verification afterwards. However, this can be done in a better way. Strictly spoken, it would be possible to derive valid statements from invalid statements, so confirmative verification of a derived statement is not a strict proof in a logical sense. Therefore in this chapter the derivation of the universal combination function is addressed in some more depth and it is also illustrated for specific cases. However, first a short route is described in Sect. 10.2. After this, the longer route is described. In Sect. 10.3 each of the different roles is analysed separately and for each a combination function and difference equation are derived for the base states. Then, in Sect. 10.4 the universal combination function and universal difference equation are derived for all roles at the same time. Section 10.5 shows that the criterion for equilibria for temporal-causal networks also applies to the universal differential equation in reified temporal-causal networks. In Sect. 10.6 it is shown how it can be derived from the role matrices. In Sect. 10.7 it is shown how this universal difference equation can be used for a compilation process by for each state Y substituting the data from the role matrices in them before simulation time and not during simulation time. This may be a useful method to simulate very large reified networks in an efficient manner.

10.2 A Short Route to the Universal Difference and Differential Equation

Recall this expression (7) from Chap. 3, Sect. 3.5 for the combination function and (8) for the difference equation:

$$\begin{aligned}&\mathbf{c}_Y(t, \boldsymbol{\pi}_{1,1}(t), \boldsymbol{\pi}_{1,2}(t), \ldots, \boldsymbol{\pi}_{1,m}(t), \boldsymbol{\pi}_{1,m}(t), V_1, \ldots, V_k)\\ &= \frac{\boldsymbol{\gamma}_{1,Y}(t)\,\mathrm{bcf}_1\left(\boldsymbol{\pi}_{1,1,Y}(t), \boldsymbol{\pi}_{2,1,Y}(t), V_1, \ldots, V_k\right) + \cdots + \boldsymbol{\gamma}_{m,Y}(t)\mathrm{bcf}_m\left(\boldsymbol{\pi}_{1,m,Y}(t)\boldsymbol{\pi}_{2,m,Y}(t),\ V_1, \ldots, V_k\right)}{\boldsymbol{\gamma}_{1,Y}(t) + \cdots + \boldsymbol{\gamma}_{m,Y}(t)}\end{aligned} \tag{1}$$

$$Y(t + \Delta t) = Y(t) + \boldsymbol{\eta}_Y(t)[\mathbf{c}_Y(t, \boldsymbol{\omega}_{X_1,Y}(t)X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}(t)X_k(t)) - Y(t)]\Delta t \tag{2}$$

Substituting the former expression (1) in the latter (2), the difference equation becomes

$$\begin{aligned}&Y(t+\Delta t) = Y(t)\\ &+\boldsymbol{\eta}_Y(t)[\frac{\boldsymbol{\gamma}_{1,Y}(t)\,\mathrm{bcf}_1\left(\boldsymbol{\pi}_{1,1,Y}(t), \boldsymbol{\pi}_{2,1,Y}(t), \boldsymbol{\omega}_{X_1,Y}(t)X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}(t)X_k(t)\right) + \cdots + \boldsymbol{\gamma}_{m,Y}(t)\,\mathrm{bcf}_m\left(\boldsymbol{\pi}_{1,m,Y}(t), \boldsymbol{\pi}_{2,m,Y}(t), \boldsymbol{\gamma}_{X_1,Y}(t)X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}(t)X_k(t)\right)}{\boldsymbol{\gamma}_{1,Y}(t) + \cdots + \boldsymbol{\gamma}_{m,Y}(t)} - Y(t)]\Delta t\end{aligned} \tag{3}$$

Within the reified network the adaptive values of $\boldsymbol{\eta}$, $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\pi}$ are represented by their reification states **H**, **W**, **C** and **P**. By substituting these in (3), a difference equation for the reified network is obtained:

$$\begin{aligned}&Y(t+\Delta t) = Y(t)\\ &+\mathbf{H}_Y(t)[\frac{\mathbf{C}_{1,Y}(t)\,\mathrm{bcf}_1\left(\mathbf{P}_{1,1,Y}(t), \mathbf{P}_{2,1,Y}(t), \mathbf{W}_{X_1,Y}(t)X_1(t), \ldots, \mathbf{W}_{X_k,Y}(t)X_k(t)\right) + \cdots + \mathbf{C}_{m,Y}(t)\,\mathrm{bcf}_m\left(\mathbf{P}_{1,m,Y}(t), \mathbf{P}_{2,m,Y}(t), \mathbf{W}_{X_1,Y}(t)X_1(t), \ldots, \mathbf{W}_{X_k,Y}(t)X_k(t)\right)}{\mathbf{C}_{1,Y}(t) + \cdots + \mathbf{C}_{m,Y}(t)} - Y(t)]\Delta t\end{aligned} \tag{4}$$

In differential equation format (leaving out references to t), this is

$$\mathbf{d}Y/\mathbf{d}t = \mathbf{H}_Y[\frac{\mathbf{C}_{1,Y}\,\mathrm{bcf}_1\left(\mathbf{P}_{1,1,Y}, \mathbf{P}_{2,1,Y}, \mathbf{W}_{X_1,Y}X_1, \ldots, \mathbf{W}_{X_k,Y}X_k\right) + \cdots + \mathbf{C}_{m,Y}\mathrm{bcf}_m\left(\mathbf{P}_{1,m,Y}, \mathbf{P}_{2,m,Y}, \mathbf{W}_{X_1,Y}X_1, \ldots, \mathbf{W}_{X_k,Y}X_k\right)}{\mathbf{C}_{1,Y} + \cdots + \mathbf{C}_{m,Y}} - Y] \tag{5}$$

Note that this difference and differential equation is not yet in the standard format of a temporal-causal network, as $\mathbf{H}_Y$ is not a constant speed factor. However, it can be rewritten into the temporal-causal network format when a suitable *universal combination function* $\mathbf{c*}_Y$**(..)** is defined. It can be verified by rewriting that when using the combination function defined by (6) below, this universal difference equation (4) (a) becomes in the standard temporal-causal format, and (b) indeed is equivalent to the above difference equation in (3). So, define

$$\mathbf{c}^*_Y(H, C_1, \ldots, C_m, P_{1,1}, P_{2,1}, \ldots, P_{1,m}, P_{2,m}, W_1, \ldots, W_k, V_1, \ldots, V_k, V)$$
$$= H\frac{C_1\mathrm{bcf}_1\left(P_{1,1}, P_{2,1}, W_1V_1, \ldots, W_kV_k\right) + \cdots + C_m\mathrm{bcf}_m\left(P_{1,m}, P_{2,m}, W_1V_1, \ldots, W_kV_k\right)}{C_1 + \cdots + C_m} + (1-H)V \tag{6}$$

Then this goes as follows. For more explanation and background on this, see Sect. 10.3 and further. Consider the following *universal differential equation* variant, which is (leaving out the reference to *t*), and assuming speed factor 1:

$$\mathbf{d}Y/\mathbf{d}t = \mathbf{c}^*_Y(\mathbf{H}_Y, \mathbf{C}_{1,Y}, \ldots, \mathbf{C}_{m,Y}, \mathbf{P}_{1,1,Y}, \mathbf{P}_{2,1,Y}, \ldots, \mathbf{P}_{1,m,Y}, \mathbf{P}_{2,m,Y,}\mathbf{W}_{X_1,Y}, \ldots, \mathbf{W}_{X_k,Y}, X_1, \ldots, X_k, Y) - Y \tag{7}$$

This is indeed in temporal-causal format. Using (6) it can be rewritten as

$$\mathbf{d}Y/\mathbf{d}t = \mathbf{H}_Y\left[\frac{\mathbf{C}_{1,Y}\mathrm{bcf}_1\left(\mathbf{P}_{1,1,Y}, \mathbf{P}_{2,1,Y}, \mathbf{W}_{X_1,Y}X_1, \ldots, \mathbf{W}_{X_k,Y}X_k\right) + \cdots + \mathbf{C}_{m,Y}\mathrm{bcf}_m\left(\mathbf{P}_{1,m,Y}, \mathbf{P}_{2,m,Y}, , \mathbf{W}_{X_1,Y}X_1, \ldots, \mathbf{W}_{X_k,Y}X_k\right)}{\mathbf{C}_{1,Y} + \cdots + \mathbf{C}_{m,Y}}\right]$$
$$+ (1 - \mathbf{H}_Y)Y - Y$$

Now note that this last part $(1 - \mathbf{H}_Y)\ Y - Y$ is just $-\mathbf{H}_Y\ Y$. Then this easily can be rewritten into:

$$\mathbf{d}Y/\mathbf{d}t = \mathbf{H}_Y\left[\frac{\mathbf{C}_{1,Y}\mathrm{bcf}_1\left(\mathbf{P}_{1,1,Y}, \mathbf{P}_{2,1,Y}, \mathbf{W}_{X_1,Y}X_1, \ldots, \mathbf{W}_{X_k,Y}X_k\right) + \cdots + \mathbf{C}_{m,Y}\mathrm{bcf}_m\left(\mathbf{P}_{1,m,Y}, \mathbf{P}_{2,m,Y}, \mathbf{W}_{X_1,Y}X_1, \ldots, \mathbf{W}_{X_k,Y}X_k\right)}{\mathbf{C}_{1,Y} + \cdots + \mathbf{C}_{m,Y}} - Y\right] \tag{8}$$

This (8) is exactly differential equation (5) earlier above; this confirms that the chosen universal combination function $\mathbf{c}^*_Y(..)$ in (6) to get the reified network in temporal-causal network format is right. So far this short route. Here the definition of the combination function (6) may seem to come out of the blue. In the next sections it is shown (via the longer route) how that can be motivated and derived.

10.3 Downward Causal Connections Defining the Special Effect of Reification States

The added reification states have to be integrated to obtain a well-connected overall network. In the first place outward connections from the reification states to the states in the base network are needed, in order to model how they have their special effect on the dynamics in the network. More specifically, it has to be defined how the reification states contribute causally to an aggregated impact on the base network state. In addition to a downward connection, also the combination function for the base state has to be defined for the aggregated impact. Both these downward causal relations and the combination functions will be defined in a generic manner, related to how a specific network characteristic functions in the overall dynamics as part of the intended semantics of a temporal-causal network. That will be discussed in the current section.

In addition, other connections of the reification states are added in order to model specific network adaptation principles. These may concern upward connections from the states of the base network to the reification states, or horizontal mutual connections between reification states within the upper plain, or both, depending on the specific network adaptation principles addressed. These connections are not generic as they are an essential part of the specification of a particular adaptation principle; they have been illustrated for a number of well-known adaptation principles in Chap. 3; see Figs. 3.4–3.10.

10.3.1 The Overall Picture

For the downward connections the general pattern is that each of the reification states $\mathbf{W}_{X_i,Y}$, $\mathbf{H}_Y$ and $\mathbf{C}_Y$ for the reified network characteristics, connection weights, speed factors and combination functions, has a specific causal connection to state Y in the base network, as they all affect Y. These are the (pink) downward arrows from the reification plane to the base plane in Chap. 3, Fig. 3.3; see also Fig. 10.1. Actually $\mathbf{C}_Y$ is a vector of states ($\mathbf{C}_{1,Y}$, $\mathbf{C}_{2,Y}$, ...) with a (small) number of different components $\mathbf{C}_{1,Y}$, $\mathbf{C}_{2,Y}$, ... for different basic combination functions that will be explained below. Note that combination functions may contain some parameters, for example, for the scaled sum combination function the scaling factor $\boldsymbol{\lambda}$, and for the advanced logistic sum combination function the steepness $\boldsymbol{\sigma}$ and the threshold $\boldsymbol{\tau}$. For these parameters also reification states $\mathbf{P}_{i,j,Y}$ can be added, with the possibility to make them adaptive as well. More specifically, for each basic combination function represented by $\mathbf{C}_{j,Y}$ there are two parameters $\boldsymbol{\pi}_{1,i}$ and $\boldsymbol{\pi}_{2,i}$ that are reified by parameter reification states $\mathbf{P}_{1,j,Y}$ and $\mathbf{P}_{2,j,Y}$. All depicted (downward and horizontal)

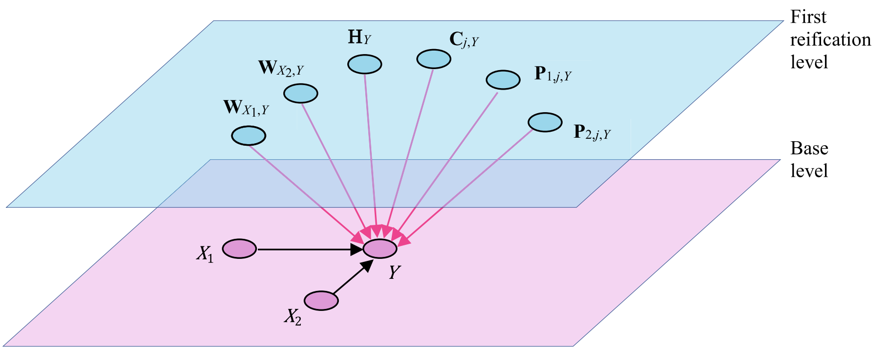

Fig. 10.1 Network reification for temporal-causal networks: downward causal connections from reification states to base network states

connections in Fig. 10.1 get weight 1. Note that this is also a way in which a weighted network can be transformed into an equivalent non-weighted network. In the extended network the speed factors of the base states are set at 1 too.

Note that the 3D layout of these figures and the depicted planes are just for understanding; in a mathematical or computational sense they are not part of the network specification. However, for each of the reification states it is crucial to know what it is that they are reifying and for what base state. Therefore the names of the reification states are chosen in such a way that this information is visible. For example, in the name $\mathbf{H}_Y$ the **H** indicates that it concerns speed factor (indicated by $\boldsymbol{\eta}$) reification and the subscript Y that it is for base state Y. So, in general the bold capital letter **R** in $\mathbf{R}_{subscript}$ indicates the type of reification and the subscript the concerning base state Y, or (for **W**) the pair of states X, Y. This **R** defines the *role* that is played by this reification state. This role corresponds one to one to the characteristics of the base network structure that is reified: connection weight $\boldsymbol{\omega}$, speed factor $\boldsymbol{\eta}$, basic combination function **c(..)**, parameter $\boldsymbol{\pi}$. In other words, there are four roles for reification states:

- the role of connection weight reification states $\mathbf{W}_{X_i,Y}$ reifying connection weights $\boldsymbol{\omega}_{X_i,Y}$
- the role of speed factor reification state $\mathbf{H}_Y$ reifying speed factor $\boldsymbol{\eta}_Y$
- the role of combination function reification states $\mathbf{C}_{j,Y}$ reifying combination function $\mathbf{c}_Y(..)$
- the role of parameter reification state $\mathbf{P}_{i,j,Y}$ reifying combination function parameter $\boldsymbol{\pi}_{i,j,Y}$

In accordance with this encoded role information, in principle each reification state has exactly one downward causal connection, which goes to the specified base state Y. In the reified network this downward connection is incorporated according to its role **R** in the aggregation of the causal impacts on Y by a new, dedicated universal combination function for that role. How this is done is explained in more detail in this section.

The general picture is that the base states have more incoming connections now, some of which have specific roles, with special effects according to their role. Therefore in the reified network new combination functions for the base states are needed. These new combination functions can be expressed in a universal manner based on the original combination functions, and the different reification states, but to define them some work is needed. As the overall approach is a bit complex, to get the idea, first the four roles **W**, **H**, **C** and **P** relating to the different types of characteristics are considered separately in Sects. 10.3.2–10.3.4; they are illustrated in Box 10.1–10.3. For the overall process, combining all three roles **W**, **H**, **C** and **P** for all base network structure characteristics, see Sect. 10.3 and Box 10.4.

10.3.2 Downward Causal Connections for Role W for Connection Weight Reification

First, consider only connection weight reification indicated by role **W**. The original difference equation for base state Y based on the original combination function $\mathbf{c}_Y(..)$ is

$$Y(t+\Delta t) = Y(t) + \boldsymbol{\eta}_Y[\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}(t)X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}(t)X_k(t)) - Y(t)]\Delta t \tag{9}$$

The new combination function $\mathbf{c}^*_Y(..)$ has to aggregate two types of values:

- the base state values $X_1(t), \ldots, X_k(t)$ for the base states from which state Y gets its incoming connections
- the reification state values $\mathbf{W}_{X_1,Y}(t)\ldots, \mathbf{W}_{X_k,Y}(t)$ for connection weights

Therefore it has to have arguments for all of these values:

$$\mathbf{c}^*_Y(\mathbf{W}_{X_1,Y}(t),\ldots,\mathbf{W}_{X_k,Y}(t), X_1(t),\ldots,X_k(t))$$

so $\mathbf{c}^*_Y(..)$ has to have this format:

$$\mathbf{c}^*_Y(W_1,\ldots,W_k, V_1,\ldots,V_k)$$

A requirement for this new combination function $\mathbf{c}^*_Y(..)$ in the reified network is

$$Y(t+\Delta t) = Y(t) + \boldsymbol{\eta}_Y\left[\mathbf{c}^*_Y(\mathbf{W}_{X_1,Y}(t),\ldots, \mathbf{W}_{X_k,Y}(t), X_1(t),\ldots,X_k(t)) - Y(t)\right]\Delta t \tag{10}$$

As these two difference Eqs. (9) and (10) must have the same result for $Y(t)$ and $Y(t + \Delta t)$, the requirement for $\mathbf{c}^*_Y(..)$ is that (when $\mathbf{W}_{X_i,Y}(t) = \boldsymbol{\omega}_{X_i,Y}(t)$) it holds

$$\mathbf{c}^*_Y(\mathbf{W}_{X_1,Y}(t),\ldots,\mathbf{W}_{X_k,Y}(t), X_1(t),\ldots,X_k(t)) = \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}(t)X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}(t)X_k(t))$$

So the new combination function $\mathbf{c}^*_Y(..)$ for this role has to be defined by

$$\mathbf{c}^*_Y(W_1,\ldots,W_k, V_1,\ldots,V_k) = \mathbf{c}_Y(W_1V_1,\ldots,W_kV_k) \tag{11}$$

where

- W_i stands for $\mathbf{W}_{X_i,Y}(t)$
- V_i stands for $X_i(t)$

In Box 10.1 an example of this combination function relating to Fig. 10.1 is shown. Indeed the requirement is fulfilled when $\mathbf{W}_{X_i,Y}(t) = \boldsymbol{\omega}_{X_i,Y}(t)$:

$$\mathbf{c}^*_Y(\mathbf{W}_{X_1,Y}(t), \ldots, \mathbf{W}_{X_k,Y}(t), X_1(t), \ldots, X_k(t)) = \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}(t)X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}(t)X_k(t))$$

Box 10.1 Example of the derived combination function for connection weight reification role $\mathbf{W}_{X_1,Y}$ and $\mathbf{W}_{X_2,Y}$ in the reified network for base state Y from Fig. 10.1.

In this box an example relating to Fig. 10.1 where $m = 2$, $\mathrm{bcf}_1(..) = \mathbf{eucl}_{n,\lambda}(..)$ for $n = 1$, $\mathrm{bcf}_2(..) = \mathbf{alogistic}_{\sigma,\tau}(..)$, where $\mathbf{eucl}_{1,\lambda}(..)$ is assumed for Y.

For connection weight reification the new combination function $\mathbf{c}^*_Y(..)$ for Y is

$$\begin{aligned}\mathbf{c}^*_Y(W_1, W_2, V_1, V_2) &= \mathbf{c}_Y(W_1V_1, W_2V_2) = \mathbf{eucl}(1, \boldsymbol{\lambda}, W_1V_1, W_2V_2)\\ &= (W_1V_1 + W_2V_2)/\boldsymbol{\lambda}\end{aligned}$$

where

$$\begin{aligned}W_1 &= \mathbf{W}_{X_1,Y}(t)\\ W_2 &= \mathbf{W}_{X_2,Y}(t)\\ V_1 &= X_1(t)\\ V_2 &= X_2(t)\end{aligned}$$

10.3.3 Downward Causal Connections for Role H for Speed Factor Reification

Second, reification of speed factors in terms of role **H** is addressed separately; in the new situation in the reified network the combination function needs an extra argument for $\mathbf{H}_Y(t)$. It turns out that to make it work also an extra argument for the current value $Y(t)$ is needed, for the timing modeled by the speed factor:

$$\mathbf{c}^*_Y(\mathbf{H}_Y(t), \boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t), Y(t))$$

So, the format for $\mathbf{c}^*_Y(.)$ becomes:

$$\mathbf{c}^*_Y(H, V_1, \ldots, V_k, V)$$

The requirement for this new function is that (when $\mathbf{H}_Y(t) = \boldsymbol{\eta}_Y(t)$) it holds

$$Y(t+\Delta t) = Y(t) + \boldsymbol{\eta}^*_Y[\mathbf{c}^*_Y(\mathbf{H}_Y(t), \boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t),\ Y(t)) - Y(t)]\Delta t \tag{12}$$

It is assumed that the new speed factor $\boldsymbol{\eta}^*_Y$ is 1; then since (9) and (12) should describe the same values for Y the requirement becomes:

$$\begin{aligned}&\mathbf{c}^*_Y(\mathbf{H}_Y(t), \boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t),\ Y(t)) - Y(t)\\ &\quad = \boldsymbol{\eta}_Y(t)[\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) - Y(t)]\end{aligned}$$

This can be rewritten into

$$\begin{aligned}&\mathbf{c}^*_Y(\mathbf{H}_Y(t), \boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t),\ Y(t))\\ &\quad = \boldsymbol{\eta}_Y(t)\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) + (\mathbf{1} - \boldsymbol{\eta}_Y(t))Y(t)\end{aligned}$$

Now define the combination function $\mathbf{c}^*_Y(..)$ by

$$\mathbf{c}^*_Y(H, V_1, \ldots, V_k,\ V) = H\mathbf{c}_Y(V_1,\ \ldots, V_k) + (1 - H)V \tag{13}$$

where

- H stands for $\mathbf{H}_Y(t)$
- V_i stands for $\boldsymbol{\omega}_{X_i,Y}\ X_i(t)$
- V stands for $Y(t)$

This is a weighted average (with weights speed factor H and $1 - H$) of $\mathbf{c}_Y(V_1, \ldots, V_k)$ and V. Again, in Box 10.2 an example of this combination function relating to Fig. 10.1 is shown. Also here the requirement is fulfilled for $\mathbf{H}_Y(t) = \boldsymbol{\eta}_Y(t)$.

Box 10.2 Example of the derived combination function for speed factor reification role **H** in the reified network for base state Y from Fig. 10.1.
In this box an example relating to Fig. 10.1 where $m = 2$, $\text{bcf}_1(..) = \mathbf{eucl}_{n,\lambda}(..)$ for $n = 1$, $\text{bcf}_2(..) = \mathbf{alogistic}_{\sigma,\tau}(..)$, where $\mathbf{eucl}_{1,\lambda}(..)$ is assumed for Y.
For speed factor reification the new combination function $\mathbf{c}^*_Y(..)$ for Y is

$$\begin{aligned}\mathbf{c}^*_Y(H, V_1, V_2,\ V) &= H\mathbf{c}_Y(V_1, V_2) + (1 - H)V\\ &= H\,\mathbf{eucl}(1, \boldsymbol{\lambda}, V_1, V_2) + (1 - H)V\\ &= H(V_1 + V_2)/\boldsymbol{\lambda} + (1 - H)V\end{aligned}$$

where

- H stands for $\mathbf{H}_Y(t)$
- V_i stands for $\boldsymbol{\omega}_{X_i,Y}X_i(t)$
- V stands for $Y(t)$

10.3.4 Downward Causal Connections for Roles C and P for Combination Function Weight and Parameter Reification

To make reification of combination functions more practical, for the base network a countable number of basic combination functions bcf(..) is assumed. From this sequence of basic combination functions for any arbitrary m a finite subsequence $\text{bcf}_1(..)$,.., $\text{bcf}_m(..)$ of m basic combination functions can be chosen to be used in a specific application. For example with $m = 3$:

$$\text{bcf}_1(..) = \mathbf{id}(..), \quad \text{bcf}_2(..) = \mathbf{ssum}_\lambda(..), \quad \text{bcf}_3(..) = \mathbf{alogistic}_{\sigma,\tau}(..)$$

Note that when more than one argument is used in **id(..)**, the outcome is the sum of these arguments (only one of them will be nonzero when Y has only one incoming connection). For each state Y in the base network *combination function weights* $\gamma_{j,Y}$ are assumed: numbers $\gamma_{1,Y}, \gamma_{2,Y}, \ldots \geq 0$ that change over time. Moreover, combination function parameters $\boldsymbol{\pi}_{1,i,Y}, \boldsymbol{\pi}_{2,i,Y}$ are assumed for each basic combination function $\text{bcf}_i(..)$ for Y. The actual combination function $\mathbf{c}_Y(.)$ at time t is expressed as a weighted average by:

$$\begin{aligned}&\mathbf{c}_Y(t, \boldsymbol{\pi}_{1,1,Y}, \boldsymbol{\pi}_{2,1,Y,\ldots,}\boldsymbol{\pi}_{1,m,Y}, \boldsymbol{\pi}_{2,m,Y,,} V_1, \ldots, V_k)\\ &= \frac{\gamma_{1,Y}(t)\,\text{bcf}_1\left(\boldsymbol{\pi}_{1,1,Y}, \boldsymbol{\pi}_{2,1,Y}, V_1, \ldots, V_k\right) + \cdots + \gamma_{m,Y}(t)\text{bcf}_m\left(\boldsymbol{\pi}_{1,m,Y}, \boldsymbol{\pi}_{2,m,Y}, V_1, \ldots, V_k\right)}{\gamma_{1,Y}(t) + \cdots + \gamma_{m,Y}(t)}\end{aligned} \tag{14}$$

In this way it can be expressed that for Y at each time point t a weighted average of the indicated basic combination functions is applied. This involves multiple basic combination functions if more than one of $\gamma_{j,Y}(t)$ has a nonzero value; just one basic combination function is selected for $\mathbf{c}_Y(.)$, if exactly one of the $\gamma_{j,Y}(t)$ is nonzero. This approach makes it possible, for example, to smoothly switch to another combination function over time by decreasing the value of $\gamma_{j,Y}(t)$ for the earlier chosen basic combination function and increasing the value of $\gamma_{j,Y}(t)$ for the new choice of combination function; see Chap. 3, Sect. 3.7 for an example.

For each basic combination function weight $\gamma_{j,Y}$ a different reification state $\mathbf{C}_{j,Y}$ is added. The value of that state represents the extent to which that basic combination function $\text{bcf}_j(..)$ is applied for state Y. Moreover, the combination function parameters are reified by $\mathbf{P}_{1,1,Y}(t)$, $\mathbf{P}_{2,1,Y}(t)$,…, $\mathbf{P}_{1,m,Y}(t)$, $\mathbf{P}_{2,m,Y}(t)$. The new combination function $\mathbf{c}^*_Y(..)$ needs additional arguments for them, so it gets this format:

$$\mathbf{c}^*_Y(\mathbf{C}_{1,Y}(t),\ldots\mathbf{C}_{m,Y}(t),\mathbf{P}_{1,1,Y}(t),\mathbf{P}_{2,1,Y}(t),\ldots,\mathbf{P}_{1,m,Y}(t),\mathbf{P}_{2,m,Y}(t),\boldsymbol{\omega}_{X_1,Y}X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}X_k(t))$$

By using variables C_j, and $P_{i,j}$ for the reified weights $\mathbf{C}_{j,Y}(t)$ and reified parameter values $\mathbf{P}_{i,j,Y}(t)$, the combination function format for $\mathbf{c}^*_Y(..)$ becomes

$$\mathbf{c}^*_Y(C_1,\ldots,C_m,P_{1,1},P_{2,1,\ldots,}P_{1,m},P_{2,m,}V_1,\ldots,V_k)$$

Now the following two difference equations should make the same values for Y:

$$Y(t+\Delta t) = Y(t) + \boldsymbol{\eta}_Y[\mathbf{c}_Y(t,\boldsymbol{\pi}_{1,1,Y},\boldsymbol{\pi}_{2,1,Y,\ldots,}\boldsymbol{\pi}_{1,m,Y},\boldsymbol{\pi}_{2,m,Y,,}\boldsymbol{\omega}_{X_1,Y}X_1(t),\ \ldots,\boldsymbol{\omega}_{X_k,Y}X_k(t)) - Y(t)]\Delta t$$
$$Y(t+\Delta t) = Y(t) + \boldsymbol{\eta}_Y[\mathbf{c}^*_Y(\mathbf{C}_{1,Y}(t),\ldots\mathbf{C}_{m,Y}(t),\mathbf{P}_{1,1,Y},\mathbf{P}_{2,1,Y,\ldots,}\mathbf{P}_{1,m,Y},\mathbf{P}_{2,m,Y,,}\boldsymbol{\omega}_{X_1,Y}X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}X_k(t),\ Y(t)) - Y(t)]\Delta t$$

Therefore the following requirement for the combination function $\mathbf{c}^*_Y(C_1, \ldots, C_m, P_{1,1}, P_{2,1}, \ldots, P_{1,m}, P_{2,m}, V_1, \ldots, V_k)$ is obtained:

$$\begin{aligned}&\mathbf{c}^*_Y(\mathbf{C}_{1,Y}(t),\ldots\mathbf{C}_{m,Y}(t),\mathbf{P}_{1,1,Y}(t),\mathbf{P}_{2,1,Y}(t),\ldots,\mathbf{P}_{1,m,Y}(t),\mathbf{P}_{2,m,Y}(t),\boldsymbol{\omega}_{X_1,Y}X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}X_k(t))\\&= \mathbf{c}_Y(t,\boldsymbol{\pi}_{1,1,Y}(t),\boldsymbol{\pi}_{2,1,Y}(t),\ldots,\boldsymbol{\pi}_{1,m,Y}(t),\boldsymbol{\pi}_{2,m,Y}(t),\boldsymbol{\omega}_{X_1,Y}X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}X_k(t))\end{aligned}$$

which is

$$\begin{aligned}&\mathbf{c}^*_Y(\mathbf{C}_{1,Y}(t),\ldots\mathbf{C}_{m,Y}(t),\mathbf{P}_{1,1,Y}(t),\mathbf{P}_{2,1,Y}(t),\ldots,\mathbf{P}_{1,m,Y}(t),\mathbf{P}_{2,m,Y}(t),\boldsymbol{\omega}_{X_1,Y}X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}X_k(t))\\&= \frac{\boldsymbol{\gamma}_{1,Y}(t)\text{bcf}_1\left(\boldsymbol{\pi}_{1,1,Y}(t),\boldsymbol{\pi}_{2,1,Y}(t),\boldsymbol{\omega}_{X_1,Y}X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}X_k(t)\right)+\ldots+\boldsymbol{\gamma}_{m,Y}(t)\,\text{bcf}_m\left(\boldsymbol{\pi}_{1,1,Y(t)},\boldsymbol{\pi}_{2,1,Y}(t),\boldsymbol{\omega}_{X_1,Y}X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}X_k(t)\right)}{\boldsymbol{\gamma}_{1,Y}(t)+\ldots+\boldsymbol{\gamma}_{m,Y}(t)}\end{aligned}$$

To fullfill this requirement the combination function $\mathbf{c}^*_Y(C_1, \ldots, C_m, P_{1,1}, P_{2,1}, \ldots, P_{1,m}, P_{2,m}, V_1, \ldots, V_k)$ has to be defined by

$$\begin{aligned}&\mathbf{c}^*_Y(C_1,\ldots,C_m,P_{1,1},P_{2,1},\ldots,P_{1,m},P_{2,m},V_1,\ldots,V_k)\\&= \frac{C_1\text{bcf}_1\left(P_{1,1,Y},P_{2,1,Y},V_1,\cdots,V_k\right)+\cdots+C_m\,\text{bcf}_m\left(P_{1,m,Y},\ P_{2,m,Y},\ V_1,\ldots,V_k\right)}{C_1+\ldots+C_m}\end{aligned} \quad (15)$$

where

- C_j stands for the combination function weight reification $\mathbf{C}_{j,Y}(t)$
- $P_{i,j}$ for the combination function parameter reification $\mathbf{P}_{i,j,Y}(t)$
- V_i for the value $\boldsymbol{\omega}_{X_i,Y}\ X_i(t)$ for base state X_i.

Box 10.3 Example of a derived combination function in the reified network for base states Y from Fig. 10.1 for combination function reification roles **C** and **P**

In this box an example relating to Fig. 10.1 where $m = 2$, $\text{bcf}_1(..) = \mathbf{eucl}_{n,\lambda}(..)$ for $\boldsymbol{n} = 1$, $\text{bcf}_2(..) = \mathbf{alogistic}_{\sigma,\tau}(..)$, where first $\mathbf{eucl}_{1,\lambda}(..)$ is assumed for Y

For combination function reification, assuming $C_{1,Y}(0) = 1$, $C_{2,Y}(0) = 0$, the new combination function $\mathbf{c}^*_Y(..)$ for Y is

$$\begin{aligned}
&\mathbf{c}^*_Y(C_1, C_2, P_{1,1}, P_{2,1}, P_{1,2}, P_{2,2}, V_1, V_2) \\
&= \frac{C_1\,\text{bcf}_1\left(P_{1,1,Y},\ P_{2,1,Y}, V_1, V_2\right) + C_2\,\text{bcf}_2\left(P_{1,2,Y},\ P_{2,2,Y}, V_1, V_2\right)}{C_1 + C_2} \\
&= \frac{C_1\,\mathbf{eucl}(1, \boldsymbol{\lambda}, V_1, V_2) + C_2\mathbf{alogistic}(\boldsymbol{\sigma}, \boldsymbol{\tau}, V_1, V_2)}{C_1 + C_2} \\
&= \frac{C_1\,\frac{V_1 + V_2}{\boldsymbol{\lambda}} + C_2\mathbf{alogistic}(\boldsymbol{\sigma}, \boldsymbol{\tau}, V_1, V_2)}{C_1 + C_2}
\end{aligned}$$

where

- C_j stands for the combination function weight reification $\mathbf{C}_{j,Y}(t)$
- $P_{i,j}$ for the combination function parameter reification $\mathbf{P}_{i,j,Y}(t)$
- V_i for the state value $X_i(t)$ of base state X_i.

This enables over time change from combination function $\mathbf{eucl}_{n,\lambda}(..)$ to combination function $\mathbf{alogistic}_{\sigma,\tau}(..)$ where first $C_1 = 1$ and $C_2 = 0$, and later C_2 becomes 1 and C_1 becomes 0.

Using this combination function, by substitution for the variables it can easily be verified that the requirement is indeed fulfilled. Note that it has to be guaranteed that the case that all C_j become 0 does not occur. For a given combination function adaptation principle, this easily can be achieved by normalising the C_j for each adaptation step so that their sum always stays 1. In Box 10.3 an example of this combination function relating to Fig. 10.1 is shown.

10.4 Deriving the Universal Combination Function and Difference Equation for Reified Networks

Based on the preparation in the previous section, in the current section the universal combination function and universal difference equation for reified networks which apply to all roles at once are presented.

10.4.1 Deriving the Universal Combination Function for Reified Networks

It has been discussed above how in the reified network the causal relations for the base network states can be defined separately for each of the three types of network characteristics. By combining these three in one it can be found that this *universal combination function* for base states Y does all at once:

$$\begin{aligned}
&\mathbf{c}^*_Y(H, C_1, \ldots, C_m, P_{1,1}, P_{2,1}, \ldots, P_{1,m}, P_{2,m}, W_1, \ldots, W_k, V_1, \ldots, V_k, V) \\
&= H\frac{C_1 \mathrm{bcf}_1\left(P_{1,1,Y},\, P_{2,1,Y},\, W_1V_1, \ldots, W_kV_k\right) + \cdots + C_m \mathrm{bcf}_m\left(P_{1,m,Y},\, P_{2,m,Y},\, W_1V_1, \ldots, W_kV_k\right)}{C_1 + \cdots + C_m} + (1-H)V \\
&= H[\frac{C_1 \mathrm{bcf}_1\left(P_{1,1,Y}, P_{2,1,Y}, W_1V_1, \ldots, W_kV_k\right) + \cdots + C_m \mathrm{bcf}_m\left(P_{1,m,Y}, P_{2,m,Y}, W_1V_1, \ldots, W_kV_k\right)}{C_1 + \cdots + C_m} - V] + V
\end{aligned} \tag{16}$$

where

- H stands for the speed factor reification $\mathbf{H}_Y(t)$
- C_j for the combination function weight reification $\mathbf{C}_{j,Y}(t)$
- $P_{i,j}$ for the combination function parameter reification $\mathbf{P}_{i,j,Y}(t)$
- W_i for the connection weight reification $\mathbf{W}_{X_i,Y}(t)$
- V_i for the state value $X_i(t)$ of base state X_i
- V for the state value $Y(t)$ of base state Y

See Box 10.4 for a general derivation of this universal combination function and Box 10.5 for an example of its use.

Box 10.4 Deriving the universal combination function and universal difference equation in the reified network for base states.

Here the overall situation is addressed in which all base network structure characteristics $\boldsymbol{\omega}$, $\boldsymbol{\eta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\pi}$ are reified together by reification states **W**, **H**, **C**, and **P**, respectively. The format for the new combination function $\mathbf{c}^*_Y(..)$ needs arguments for all states in the following manner:

$$\begin{aligned}
&\mathbf{c}^*_Y(\mathbf{H}_Y(t), \mathbf{C}_{1,Y}(t), \ldots, \mathbf{C}_{m,Y}(t), \mathbf{P}_{1,1,Y}(t), \mathbf{P}_{2,1,Y}(t), \ldots, \mathbf{P}_{1,m,Y}(t), \\
&\mathbf{P}_{2,m,Y}(t), \mathbf{W}_{X_1,Y}(t), \ldots, \mathbf{W}_{X_k,Y}(t),\, X_1(t), \ldots, X_k(t),\, Y(t))
\end{aligned}$$

Assuming speed factor $\boldsymbol{\eta}^*_Y = 1$, and connection weights are 1 for the reified network, a new combination function $\mathbf{c}^*_Y(..)$ is needed such that

$$\begin{aligned}
&Y(t+\Delta t) = Y(t) + \boldsymbol{\eta}_Y(t)[\mathbf{c}_Y(t, \boldsymbol{\omega}_{X_1,Y}(t)X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}(t)X_k(t)) - Y(t)]\Delta t \\
&Y(t+\Delta t) = Y(t) + [\mathbf{c}^*_Y(\mathbf{H}_Y(t), \mathbf{C}_{1,Y}(t), \ldots, \mathbf{C}_{m,Y}(t), \mathbf{P}_{1,1,Y}(t), \mathbf{P}_{2,1,Y}(t), \ldots, \mathbf{P}_{1,m,Y}(t), \mathbf{P}_{2,m,Y}(t), \mathbf{W}_{X_1,Y}(t), \ldots, \mathbf{W}_{X_k,Y}(t),\, X_1(t), \ldots, \\
&X_k(t),\, Y(t)) - Y(t)]\Delta t
\end{aligned}$$

So, the requirement for $\mathbf{c}^*_Y(..)$ is:

$$\begin{aligned}&\mathbf{c}^*_Y(\mathbf{H}_Y(t),\ \mathbf{C}_{1,Y}(t),\ldots,\mathbf{C}_{m,Y}(t),\mathbf{P}_{1,1,Y}(t),\mathbf{P}_{2,1,Y}(t),\ldots,\mathbf{P}_{1,m,Y}(t),\mathbf{P}_{2,m,Y}(t),\mathbf{W}_{X_1,Y}(t),\ldots,\mathbf{W}_{X_k,Y}(t),\ X_1(t),\ldots,X_k(t),\ Y(t))\\&= Y(t)+\boldsymbol{\eta}_Y(t)[\mathbf{c}_Y(t,\boldsymbol{\omega}_{X_1,Y}(t)X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}(t)X_k(t))-Y(t)]\end{aligned}$$

Assume

$$\begin{aligned}&\mathbf{c}_Y(t,\boldsymbol{\gamma}_{1,Y}(t),\ldots,\boldsymbol{\gamma}_{m,Y}(t),\boldsymbol{\pi}_{1,1,Y}(t),\boldsymbol{\pi}_{2,1,Y}(t),\ldots,\boldsymbol{\pi}_{1,m,Y}(t),\boldsymbol{\pi}_{2,m,Y}(t),V_1,\ldots,V_k)\\&=\frac{\boldsymbol{\gamma}_{1,Y}(t)\mathrm{bcf}_1\left(\boldsymbol{\pi}_{1,1,Y}(t),\boldsymbol{\pi}_{2,1,Y}(t),V_1,\ldots,V_k\right)+\cdots+\boldsymbol{\gamma}_{m,Y}(t)\mathrm{bcf}_m\left(\boldsymbol{\pi}_{1,m,Y}(t),\boldsymbol{\pi}_{2,m,Y}(t),V_1,\ldots,V_k\right)}{\boldsymbol{\gamma}_{1,Y}(t)+\cdots+\boldsymbol{\gamma}_{m,Y}(t)}\end{aligned}$$

and $\mathbf{C}_{j,Y}(t) = \boldsymbol{\gamma}_{j,Y}(t)$, $\mathbf{H}_Y(t) = \boldsymbol{\eta}_Y(t)$, $\mathbf{W}_{X_i,Y}(t) = \boldsymbol{\omega}_{X_i,Y}(t)$, and $\mathbf{P}_{i,jY}(t) = \boldsymbol{\pi}_{i,j,Y}(t)$ for all i and j.
Now given the above expression the new universal combination function $\mathbf{c}^*_Y(...)$ has to be defined by:

$$\begin{aligned}&\mathbf{c}^*_Y(H,C_1,\ldots,C_m,P_{1,1},P_{2,1},\ldots,P_{1,m},P_{2,m},W_1,\ldots,W_k,V_1,\ldots,V_k,V)\\&=H\frac{C_1\mathrm{bcf}_1\left(P_{1,1},P_{2,1},W_1V_1,..,W_kV_k\right)+\cdots+C_m\mathrm{bcf}_m\left(P_{1,m},P_{2,m},W_1V_1,..,W_kV_k\right)}{C_1+\cdots+C_m}+(1-H)V\end{aligned}$$

where

- H stands for the speed factor reification $\mathbf{H}_Y(t)$
- C_j for the combination function weight reification $\mathbf{C}_{j,Y}(t)$
- $P_{i,j}$ for the combination function weight reification $\mathbf{P}_{i,j,Y}(t)$
- W_i for the connection weight reification $\mathbf{W}_{X_i,Y}(t)$
- V_i for the state value of base state X_i
- V for the state value $Y(t)$

Then

$$\begin{aligned}&\mathbf{c}^*_Y(\mathbf{H}_Y(t),\mathbf{C}_{1,Y}(t),\ldots,\mathbf{C}_{m,Y}(t),\mathbf{P}_{1,1,Y}(t),\mathbf{P}_{2,1,Y}(t),\ldots,\mathbf{P}_{1,m,Y}(t),\mathbf{P}_{2,m,Y}(t),\mathbf{W}_{X_1,Y}(t),\ldots,\mathbf{W}_{X_k,Y}(t),\ X_1(t),\ldots,X_k(t),\ Y(t))\\&=\boldsymbol{\eta}_Y(t)\frac{\boldsymbol{\gamma}_{1,Y}(t)\mathrm{bcf}_1\left(\boldsymbol{\omega}_{X_1,Y}(t)X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}(t)X_k(t)\right)+\cdots+\boldsymbol{\gamma}_{m,Y}(t)\mathrm{bcf}_m\left(\boldsymbol{\omega}_{X_1,Y}(t)X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}(t)X_k(t)\right)}{\boldsymbol{\gamma}_{1,Y}(t)+\cdots+\boldsymbol{\gamma}_{m,Y}(t)}\\&\quad+(1-\boldsymbol{\eta}_Y(t))Y(t))\\&=Y(t)+\boldsymbol{\eta}_Y(t)[\\&\frac{\boldsymbol{\gamma}_{1,Y}(t)\mathrm{bcf}_1\left(\boldsymbol{\pi}_{1,1,Y},\boldsymbol{\pi}_{2,1,Y},\boldsymbol{\omega}_{X_1,Y}(t)X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}(t)X_k(t)\right)+\cdots+\boldsymbol{\gamma}_{m,Y}(t)\mathrm{bcf}_m\left(\boldsymbol{\pi}_{1,m,Y},\boldsymbol{\pi}_{2,m,Y},\boldsymbol{\omega}_{X_1,Y}(t)X_1(t),\ldots,\boldsymbol{\pi}_{X_k,Y}(t)X_k(t)\right)}{\boldsymbol{\gamma}_{1,Y}(t)+\cdots+\boldsymbol{\gamma}_{m,Y}(t)}-Y(t)]\\&=Y(t)+\boldsymbol{\eta}_Y(t)[\mathbf{c}_Y\left(t,\boldsymbol{\omega}_{X_1,Y}(t)X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}(t)X_k(t)\right)-Y(t)]\end{aligned}$$

So, this universal combination function $\mathbf{c}^*_Y(..)$ indeed fulfills the requirement.

Box 10.5 An example of the use of the universal combination function for all roles **H**, **C**, **P** and **W**.
Example for Fig. 10.1. For reification of connection weights, speed factors and combination functions and their parameters together, and $\text{bcf}_1(..)$ is the euclidean function **eucl(..)** with order n = 1 and $\text{bcf}_2(..)$ the logistic function **alogistic(..)**, and $C_{1,Y}(0) = 1$, $C_{2,Y}(0) = 0$ (so first $\mathbf{eucl_{1,\lambda}}$**(..)** is assumed for Y), the new combination function $\mathbf{c}^*_Y$**(..)** for Y is (where $P_{1,1}$ for the order $\boldsymbol{n}$ of $\mathbf{eucl}_{n,\lambda}$**(..)** is assumed 1):

$$\begin{aligned}
&\mathbf{c}^*_Y(H, C_1, C_2, P_{1,1}, P_{2,1}, P_{1,2}, P_{2,2}, W_1, W_2, V_1, V_2, V)\\
&= H\frac{C_1\text{bcf}_1(P_{1,1}, P_{2,1}, W_1V_1, W_2V_2) + C_2\text{bcf}_2(P_{1,2}, P_{2,2}, W_1V_1, W_2V_2)}{C_1 + C_2} + (1-H)V\\
&= H\frac{C_1\,\mathbf{eucl}(P_{1,1}, P_{2,1}, W_1V_1, W_2V_2) + C_2\,\mathbf{alogistic}(P_{1,2}, P_{2,2}, W_1V_1, W_2V_2)}{C_1 + C_2} + (1-H)V\\
&= H\frac{C_1\frac{W_1V_1 + W_2V_2}{P_{2,1}} + C_2\,\mathbf{alogistic}(P_{1,2}, P_{2,2}, W_1V_1, W_2V_2)}{C_1 + C_2} + (1-H)V
\end{aligned}$$

10.4.2 The Universal Difference Equation for Reified Networks

In summary, the universal combination function found above in (8) is

$$\begin{aligned}
&\mathbf{c}^*_Y(H,\ C_1, \ldots, C_m, P_{1,1}, P_{2,1}, \ldots, P_{1,m}, P_{2,m}, W_1, \ldots, W_k, V_1, \ldots, V_k, V)\\
&= H\frac{C_1\text{bcf}_1(P_{1,1,Y}, P_{2,1,Y}, W_1V_1, .., W_kV_k) + \cdots + C_m\text{bcf}_m(P_{1,m,Y}, P_{2,m,Y}, W_1V_1, \ldots, W_kV_k)}{C_1 + \cdots + C_m} + (1-H)V\\
&= H[\frac{C_1\text{bcf}_1(P_{1,1,Y}, P_{2,1,Y}, W_1V_1, \ldots, W_kV_k) + \cdots + C_m\text{bcf}_m(P_{1,m,Y}, P_{2,m,Y}, W_1V_1, \ldots, W_kV_k)}{C_1 + \cdots + C_m} - V] + V
\end{aligned}$$

Based on this, the following *universal difference equation* describes the dynamics of each base state Y within the reified network; in cases of full reification it has no state-specific parameters for network structure characteristics, only variables; therefore it is the same for all states Y:

$$
\begin{aligned}
Y(t+\Delta t) &= Y(t) \\
&\quad + \left[\mathbf{c}^{*}_{Y}(\mathbf{H}_{Y}(t),\, \mathbf{C}_{1,Y}(t), \ldots, \mathbf{C}_{m,Y}(t), \mathbf{P}_{1,1,Y}(t), \mathbf{P}_{2,1,Y}(t), \ldots, \mathbf{P}_{1,m,Y}(t), \mathbf{P}_{2,m,Y}(t), \mathbf{W}_{X_1,Y}(t), \ldots, \mathbf{W}_{X_k,Y}(t),\, X_1(t), \ldots, X_k(t),\, Y(t)) - Y(t)\right]\Delta t \\
&= Y(t) \\
&\quad + [\mathbf{H}_{Y}(t)\frac{\mathbf{C}_{1,Y}(t)\mathrm{bcf}_1\left(\mathbf{P}_{1,1,Y}(t), \mathbf{P}_{2,1,Y}(t), \mathbf{W}_{X_1,Y}(t)X_1(t), \ldots, \mathbf{W}_{X_k,Y}(t)X_k(t)\right) + \cdots + \mathbf{C}_{m,Y}(t)\mathrm{bcf}_m\left(\mathbf{P}_{1,m,Y}(t), \mathbf{P}_{2,m,Y}(t), \mathbf{W}_{X_1,Y}(t)X_1(t), \ldots, \mathbf{W}_{X_k,Y}(t)X_k(t)\right)}{\mathbf{C}_{1,Y}(t) + \cdots + \mathbf{C}_{m,Y}(t)} \\
&\quad + (1 - \mathbf{H}_{Y}(t))Y(t) - Y(t)]\Delta t \\
&= Y(t) \\
&\quad + [\mathbf{H}_{Y}(t)\frac{\mathbf{C}_{1,Y}(t)\mathrm{bcf}_1\left(\mathbf{P}_{1,1,Y}(t), \mathbf{P}_{2,1,Y}(t), \mathbf{W}_{X_1,Y}(t)X_1(t), \ldots, \mathbf{W}_{X_k,Y}(t)X_k(t)\right) + \cdots + \mathbf{C}_{m,Y}(t)\mathrm{bcf}_m\left(\mathbf{P}_{1,m,Y}(t), \mathbf{P}_{2,m,Y}(t), \mathbf{W}_{X_1,Y}(t)X_1(t), \ldots, \mathbf{W}_{X_k,Y}(t)X_k(t)\right)}{\mathbf{C}_{1,Y}(t) + \cdots + \mathbf{C}_{m,Y}(t)} \\
&\quad - \mathbf{H}_{Y}(t)Y(t)]\Delta t \\
&= Y(t) \\
&\quad + \mathbf{H}_{Y}(t)[\frac{\mathbf{C}_{1,Y}(t)\mathrm{bcf}_1\left(\mathbf{P}_{1,1,Y}(t), \mathbf{P}_{2,1,Y}(t), \mathbf{W}_{X_1,Y}(t)X_1(t), \ldots, \mathbf{W}_{X_k,Y}(t)X_k(t)\right) + \cdots + \mathbf{C}_{m,Y}(t)\mathrm{bcf}_m\left(\mathbf{P}_{1,m,Y}(t), \mathbf{P}_{2,m,Y}(t), \mathbf{W}_{X_1,Y}(t)X_1(t), \ldots, \mathbf{W}_{X_k,Y}(t)X_k(t)\right)}{\mathbf{C}_{1,Y}(t) + \cdots + \mathbf{C}_{m,Y}(t)} - Y(t)]\Delta t
\end{aligned}
\tag{17}
$$

So, this universal difference equation is what defines the dynamics of the whole base network within the reified network. Its differential equation variant is

$$
\begin{aligned}
&\mathbf{d}Y(t)/\mathbf{d}t \\
&= \mathbf{H}_{Y}(t)[\frac{\mathbf{C}_{1,Y}(t)\mathrm{bcf}_1\left(\mathbf{P}_{1,1,Y}(t), \mathbf{P}_{2,1,Y}(t), \mathbf{W}_{X_1,Y}(t)X_1(t), \ldots, \mathbf{W}_{X_k,Y}(t)X_k(t)\right) + \cdots + \mathbf{C}_{m,Y}(t)\mathrm{bcf}_m\left(\mathbf{P}_{1,m,Y}(t), \mathbf{P}_{2,m,Y}(t), \mathbf{W}_{X_1,Y}(t)X_1(t), \ldots, \mathbf{W}_{X_k,Y}(t)X_k(t)\right)}{\mathbf{C}_{1,Y}(t) + \cdots + \mathbf{C}_{m,Y}(t)} - Y(t)]
\end{aligned}
$$

or by leaving out t:

$$
\begin{aligned}
&\mathbf{d}Y/\mathbf{d}t \\
&= \mathbf{H}_{Y}[\frac{\mathbf{C}_{1,Y}\mathrm{bcf}_1\left(\mathbf{P}_{1,1,Y}, \mathbf{P}_{2,1,Y}, \mathbf{W}_{X_1,Y}X_1, \ldots, \mathbf{W}_{X_k,Y}X_k\right) + \cdots + \mathbf{C}_{m,Y}\mathrm{bcf}_m\left(\mathbf{P}_{1,m,Y}, \mathbf{P}_{2,m,Y}, \mathbf{W}_{X_1,Y}X_1, \ldots, \mathbf{W}_{X_k,Y}X_k\right)}{\mathbf{C}_{1,Y} + \cdots + \mathbf{C}_{m,Y}} - Y]
\end{aligned}
\tag{18}
$$

By structure-preserving implementation based on the above universal difference equation, the software environment as described in Chap. 9 has been developed. As can be seen there in Boxes 9.3 and 9.4, the above universal difference equation just occurs in Matlab format in that software environment. However, starting from the above universal difference equation, a different path to implementation can be followed as well. This will be discussed in next section.

10.5 The Criterion for a Stationary Point for the Universal Difference Equation

Recall the criterion for a stationary point:

Criterion for stationary points and equilibria in temporal-causal network models

A state Y in an adaptive temporal-causal network model has a stationary point at t if and only if

$$\boldsymbol{\eta}_Y = 0 \quad \text{or} \quad \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}(t)X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}(t)X_k(t)) = Y(t)$$

where X_1, …, X_k are the states with outgoing connections to Y.

An adaptive temporal-causal network model is in an equilibrium state at t if and only if for all states the above criteria hold at t.

Now suppose that some or all of the characteristics $\boldsymbol{\eta}_Y$, $\boldsymbol{\omega}_{X_k,Y}$, $\mathbf{c}_Y(..)$ are reified. Then the above equation becomes the universal differential equation. What is the criterion then? The format shown in (17) above can be rewritten into the format of (18) above; when is the right hand side 0?

$$\mathbf{H}_Y[\frac{\mathbf{C}_{1,Y}\text{bcf}_1(\mathbf{P}_{1,1,Y}, \mathbf{P}_{2,1,Y}, \mathbf{W}_{X_1,Y}X_1, \ldots, \mathbf{W}_{X_k,Y}X_k) + \cdots + \mathbf{C}_{m,Y}\text{bcf}_m(\mathbf{P}_{1,m,Y}, \mathbf{P}_{2,m,Y}, \mathbf{W}_{X_1,Y}X_1, \ldots, \mathbf{W}_{X_k,Y}X_k)}{\mathbf{C}_{1,Y} + \cdots + \mathbf{C}_{m,Y}} - Y] = 0$$

This right hand side of this is 0 if and only if

$$\mathbf{H}_Y = 0 \quad \text{or}$$
$$\frac{\mathbf{C}_{1,Y}\text{bcf}_1(\mathbf{P}_{1,1,Y}, \mathbf{P}_{2,1,Y}, \mathbf{W}_{X_1,Y}X_1, \ldots, \mathbf{W}_{X_k,Y}X_k) + \cdots + \mathbf{C}_{m,Y}\text{bcf}_m(\mathbf{P}_{1,m,Y}, \mathbf{P}_{2,m,Y}, \mathbf{W}_{X_1,Y}X_1, \ldots, \mathbf{W}_{X_k,Y}X_k)}{\mathbf{C}_{1,Y} + \cdots + \mathbf{C}_{m,Y}} = Y$$

Now notice that the part

$$\frac{\mathbf{C}_{1,Y}\text{bcf}_1(\mathbf{P}_{1,1,Y}, \mathbf{P}_{2,1,Y}, \mathbf{W}_{X_1,Y}X_1, \ldots, \mathbf{W}_{X_k,Y}X_k) + \cdots + \mathbf{C}_{m,Y}\text{bcf}_m(\mathbf{P}_{1,m,Y}, \mathbf{P}_{2,m,Y}, \mathbf{W}_{X_1,Y}X_1, \ldots, \mathbf{W}_{X_k,Y}X_k)}{\mathbf{C}_{1,Y} + \cdots + \mathbf{C}_{m,Y}}$$

is precisely $\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}(t)\, X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}(t)\, X_k(t))$ in the old criterion above, so this has exactly the same form as the old criterion. Therefore the above criterion also can be used when some or all of the characteristics $\boldsymbol{\eta}_Y$, $\boldsymbol{\omega}_{X_k,Y}(t)$, $\mathbf{c}_Y(..)$ are adaptive.

10.6 Deriving the Difference and Differential Equation from the Role Matrices

In the role matrices all information is available to determine the difference or differential equations. In Box 10.6 it is shown how that can be done. Here it is assumed that the indicated matrix cell provides the static value from the matrix, or, if not a static value, the indicated state name X_k for the adaptive value.

Box 10.6 Derivation of the basic differential equation for the network's dynamics from the role matrices.
Substitute every characteristic by the reference to the cell in the role matrix where this is indicated. So, for the combination function of X_j, in Eq. (2) (use the parameters $\boldsymbol{\pi}_{i,1,X_j}$ and $\boldsymbol{\pi}_{i,2,X_j}$ as first two arguments of a basic combination function):

- for the combination function weight $\boldsymbol{\gamma}_{i,X_j}$ substitute $\mathbf{mcfw}(j, i)$
- for the parameter $\boldsymbol{\pi}_{i,1,X_j}$ or $\boldsymbol{\pi}_{i,2,X_j}$ substitute $\mathbf{mcfp}(j, 1, i)$ resp. $\mathbf{mcfp}(j, 2, i)$

Then from (2) the following expression in terms of the role matrices results:

$$\mathbf{c}_Y(V_1,\ ..,\ V_k) = \left[\frac{\begin{array}{c}\mathbf{mcfw}(j,1)\mathbf{bcf}_1(\mathbf{mcfp}(j,1,1),\mathbf{mcfp}(j,2,1),V_1,\ldots,V_k)\\ +\cdots+\mathbf{mcfw}(j,m)\mathbf{bcf}_m(\mathbf{mcfp}(j,1,m),\ \mathbf{mcfp}(j,2,m),V_1,\ldots,V_k)\end{array}}{\mathbf{mcfw}(j,1)+\cdots+\mathbf{mcfw}(j,m)}\right] \tag{19}$$

Suppose in the role base connectivity matrix **mb** the states specified in the row for X_j are the states X_{i_1}, …, X_{i_k}; these also can be denoted by **mb(***j*, 1**)**, …, **mb(***j*, *k***)**. To get the basic differential equation in terms of the role matrices, as a next step:

- in (3) substitute the single impact $\boldsymbol{\omega}_{\mathbf{mb}(j,i),X_j}$ **mb(***j*, *i***)** for V_i
- for connection weight $\boldsymbol{\omega}_{\mathbf{mb}(j,i),X_j}$ substitute $\mathbf{mcw}(j, i)$

Then the following is obtained:

$$\mathbf{c}_Y(\ldots) = \left[\frac{\begin{array}{c}\mathbf{mcfw}(j,1)\mathbf{bcf}_1(\mathbf{mcfp}(j,1,1),\mathbf{mcfp}(j,2,1),\mathbf{mcw}(j,1)\mathbf{mb}(j,1),\ldots,\mathbf{mcw}(j,k)\mathbf{mb}(j,k))\\ +\cdots+\mathbf{mcfw}(j,m)\mathbf{bcf}_m(\mathbf{mcfp}(j,1,m),\mathbf{mcfp}(j,2,m),\mathbf{mcw}(j,1)\mathbf{mb}(j.1),\ldots,\mathbf{mcw}(j,k)\mathbf{mb}(j,k))\end{array}}{\mathbf{mcfw}(j,1)+\cdots+\mathbf{mcfw}(j,m)}\right] \tag{20}$$

Now to get the differential equation, as a final step

- for the speed factor $\boldsymbol{\eta}_{X_j}$ substitute $\mathbf{ms}(j, 1)$

Then the differential equation expression becomes:

$$\mathbf{d}X_j/\mathbf{d}t = \mathbf{ms}(j,\ 1)\left[\frac{\begin{array}{c}\mathbf{mcfw}(j,1)\mathbf{bcf}_1(\mathbf{mcfp}(j,1,1),\mathbf{mcfp}(j,2,1),\mathbf{mcw}(j,1)\mathbf{mb}(j,1),\ldots,\mathbf{mcw}(j,k)\mathbf{mb}(j,k))\\ +\cdots+\mathbf{mcfw}(j,m)\mathbf{bcf}_m(\mathbf{mcfp}(j,1,m),\mathbf{mcfp}(j,2,m),\mathbf{mcw}(j,1)\mathbf{mb}(j.1),\ldots,\mathbf{mcw}(j,k)\mathbf{mb}(j,k))\end{array}}{\mathbf{mcfw}(j,1)+\cdots+\mathbf{mcfw}(j,m)} - X_j\right] \tag{21}$$

Note that for states often only one combination function is selected and has nonzero weight. Then expression (21) in Box 10.6 simplifies to (e.g., for $\mathbf{bcf}_i(..)$):

$$\begin{aligned}&\mathbf{d}X_j/\mathbf{d}t\\ &= \mathbf{ms}(j,1)[\mathbf{bcf}_i(\mathbf{mcfp}(j,1,i),\mathbf{mcfp}(j,2,i),\mathbf{mcw}(j,1)\mathbf{mb}(j,1),\ldots,\mathbf{mcw}(j,k)\mathbf{mb}(j,k)) - X_j]\end{aligned} \tag{22}$$

As a form of verification, this can be filled for state X_2 in the example Social Network from Chap. 2, Box 2.2, so $j = 2$, and $i = 1$ as can be seen in **mcfw**. It can be found in **mb** that $k = 9$ for X_2.

$$\begin{aligned}&\mathbf{d}X_2/\mathbf{d}t\\&= \mathbf{ms}(2,1)[\mathbf{bcf}_1(\mathbf{mcfp}(2,1,1), \mathbf{mcfp}(2,2,1), \mathbf{mcw}(2,1)\mathbf{mb}(2,1), \ldots, \mathbf{mcw}(2,9)\mathbf{mb}(2,9)) - X_2]\end{aligned} \tag{23}$$

To get the idea, from the role matrices in Chap. 2, Box 2.2, all values can be found, for example:

$$\begin{aligned}&\mathbf{ms}(2,1) = 0.5\\&\mathbf{mcfp}(2,1,1) = 1\\&\mathbf{mcfp}(2,2,1) = 1.55\\&\mathbf{mcw}(2,1) = 0.1\\&\mathbf{mb}(2,1) = X_1\\&\text{et cetera}\end{aligned}$$

This leads to:

$$\begin{aligned}&\mathbf{d}X_2/\mathbf{d}t\\&= 0.5[\mathbf{bcf}_1(1, 1.55, 0.1X_1, 0.25X_3, 0.15X_4, 0.2X_5, 0.1X_6, 0.1X_7, 0.25X_8, 0.15X_9, 0.25X_{10}) - X_2]\end{aligned}$$

Finally, after also incorporating the combination function weights represented in **mcfw** it provides:

$$\begin{aligned}&\mathbf{d}X_2/\mathbf{d}t\\&= 0.5[\mathbf{eucl}(1, 1.55, 0.1X_1, 0.25X_3, 0.15X_4, 0.2X_5, 0.1X_6, 0.1X_7, 0.25X_8, 0.15X_9, 0.25X_{10}) - X_2]\\&= 0.5[\frac{0.1X_1 + 0.25X_3 + 0.15X_4 + 0.2X_5 + 0.1X_6 + 0.1X_7 + 0.25X_8 + 0.15X_9 + 0.25X_{10}}{1.55} - X_2]\end{aligned} \tag{25}$$

10.7 Compilation of the Universal Differential Equation by Substitution

In the software as described in Chap. 9, the role matrices defining the model are inspected at every simulation step. There is a second option for implementation by separating this work from simulation time, in the form of compiling. Doing so, the one universal difference equation as shown above is instantiated for each of the states with the entries from the role matrices for that state, so it is replaced by n specific difference equations with n the number of states. The resulting set of specific difference (or differential) equations can be run by any general purpose software environment for differential equation simulation. As there are many quite efficient software environments for this, for large-scale reified networks of thousands or even millions of states such environments can be used for successful simulation. This compilation process will be illustrated for the plasticity and metaplasticity example network from Chap. 4 (see Fig. 4.3 and Box 4.1).

Suppose in role base connectivity matrix **mb** the states specified in the row for X_j are the states $X_{i_1}, \ldots, X_{i_k}$, which also can be denoted by **mb**(j, 1), …, **mb**(j, k). So consider again the universal differential equation

$$\mathbf{d}X_j/\mathbf{d}t = \mathbf{H}_Y \left[\frac{\mathbf{C}_{1,Y}\, \mathrm{bcf}_1(\mathbf{P}_{1,1,Y}, \mathbf{P}_{2,1,Y}, \mathbf{W}_{X_1,Y} X_{i_1}, \ldots, \mathbf{W}_{X_k,Y} X_{i_k}) + \ldots + \mathbf{C}_{m,Y}\, \mathrm{bcf}_m(\mathbf{P}_{1,m,Y}, \mathbf{P}_{2,m,Y}, \mathbf{W}_{X_1,Y} X_{i_1}, \ldots, \mathbf{W}_{X_k,Y} X_{i_k})}{\mathbf{C}_{1,Y} + \ldots + \mathbf{C}_{m,Y}} - X_j \right]$$

Here the parts that need substitution have been highlighted, and the role matrix where the entries to be substituted can be found are indicated as follows:

Yellow	$\mathbf{H}_Y$	from role matrix **ms** for speed factors
Green	$\mathbf{C}_{j,Y}$	from role matrix **mcfw** for combination function weights
Blue	$\mathbf{P}_{1,1,Y}$	from role matrix **mcfp** for combination function parameters
Purple	$\mathbf{W}_{X\,j\,Y}$	from role matrix **mcw** for connection weights

Here it is assumed that the indicated matrix cell provides the static value from the matrix, or, if not a static value, the indicated state name X_k for the adaptive value. For example, in the role matrices in Chap. 4, Box 4.1 it can be seen that X_1, X_3, X_6, X_7, X_8, X_9 have the standard difference equation with values for the characteristics. So for these states there are just constant values substituted in the universal difference equation. In other cases, such as X_2, there are entries in role matrices that are just names X_j of states; in these cases just that name X_j has to be substituted. Note first the number m of combination functions has to be read from the role matrices **mcfw**, and per state Y, the number k of incoming base connections from role matrix **mb**. For example, from Chap. 4, Box 4.1 it is seen in role matrix **mcfw** that $m = 3$ and for state X_4 from the fourth row in role matrix **mb** that $k = 2$, and the states with incoming connections (from **mb**) are X_2 and X_3.

That makes the following format in particular for X_4

$$\begin{aligned}&\mathbf{d}X_4/\mathbf{d}t\\ &= \mathbf{H}_{X_4}\left[\frac{\mathbf{C}_{1,X_4}\mathrm{bcf}_1(\mathbf{P}_{1,1,X_4}, \mathbf{P}_{2,1,X_4}, \mathbf{W}_{X_2,X_4}X_2, \mathbf{W}_{X_3,X_4}X_3) + \mathbf{C}_{2,X_4}\mathrm{bcf}_2(\mathbf{P}_{1,2,X_4}, \mathbf{P}_{2,2,X_4}, \mathbf{W}_{X_2X_4}X_2, \mathbf{W}_{X_3,X_4}X_3) + \mathbf{C}_{3,X_4}\mathrm{bcf}_3(\mathbf{P}_{1,3,X_4}, \mathbf{P}_{2,3,Y}, \mathbf{W}_{X_2,X_4}X_2, \mathbf{W}_{X_3,X_4}X_3)}{C_{1,X_4} + C_{2,X_4} + \mathbf{C}_{3,X_4}} - X_4\right]\end{aligned}$$

So consider this state X_4 further. In role matrix **cfw** for connection function weights it can be seen that the weights $\mathbf{C}_{j,X_4}$ are values 0, 1 and 0, respectively. Substituting these values makes the equation much simpler:

$$\mathbf{d}X_4/\mathbf{d}t = \mathbf{H}_{X_4}\left[\mathrm{bcf}_2\left(\mathbf{P}_{1,2,X_4}, \mathbf{P}_{2,2,X_4}, \mathbf{W}_{X_2X_4}X_2, \mathbf{W}_{X_3,X_4}X_3\right) - X_4\right]$$

In role matrix **mcw** for connection weights it can be seen that state name X_5 is indicated for the connection from X_2. So that name has to be substituted for $\mathbf{W}_{X_2,X_4}$. The other weight has just constant value 1 in role matrix **mcw**, so then 1 can be substituted for $\mathbf{W}_{X_3,X_4}$. Then this is obtained with the remaining spots for further substitution highlighted:

$$\mathbf{d}X_4/\mathbf{d}t = \mathbf{H}_{X_4}[\mathrm{bcf}_2\left(\mathbf{P}_{1,2,X_4}, \mathbf{P}_{2,2,X_4}, X_5X_2, X_3\right) - X_4]$$

Also in role matrix **mcfp** for the parameters for X_4 there is an adaptive one, namely the second parameter or the second combination function indicates X_7 as an adaptive value, so this has to be substituted for $\mathbf{P}_{2,2,X_4}$; for $\mathbf{P}_{1,2,X_4}$ the value 5 is indicated. These substitutions make

$$\mathbf{d}X_4/\mathbf{d}t = \mathbf{H}_{X_4}[\mathrm{bcf}_2(5, X_7, X_5X_2, X_3) - X_4]$$

The speed factor 0.5 from role matrix **ms** can be substituted, and the function $\mathbf{alogistic}_{\sigma,\tau}(..)$ can be substituted for $\mathrm{bcf}_2(..)$:

$$\mathbf{d}X_4/\mathbf{d}t = 0.5[\mathbf{alogistic}_{5,X_7}(X_5X_2, X_3) - X_4] \tag{26}$$

Similarly the following instantiated difference equations can be found

$$\begin{aligned} \mathbf{d}X_2/\mathbf{d}t &= 0.5\left[\mathbf{alogistic}_{5,X_6}(X_1) - X_2\right] \\ \mathbf{d}X_5/\mathbf{d}t &= X_8[\mathbf{hebb}_{X_9}(X_2, X_4, X_5) - X_5] \end{aligned} \tag{27}$$

For these functions their detailed formulae can be substituted. For example, for Hebbian learning

$$\mathbf{hebb}_{X_9}(X_2, X_4, X_5) = X_2X_4(1 - X_5) + X_9X_5$$

this makes it

$$\mathbf{d}X_5/\mathbf{d}t = X_8[X_2X_4(1 - X_5) + X_9X_5 - X_5] \tag{28}$$

The equations for the other states do not involve adaptive characteristics, so then just values found in the role matrices are substituted. See Box 10.6 for the complete outcome of the compilation.

This illustrates how the universal differential equation can be compiled by replacing it by a set of specific differential equations for each of the states that can be entered in a general purpose differential equation solver. This may imply a gain in efficiency during simulation, which may be beneficial when large scale reified networks are simulated, for example, with thousands of states. The compilation process itself can be time consuming if done by hand, but in future that could also be automated.

Box 10.6 The result of complete compilation for the reified network for plasticity and metaplasticity

$\mathbf{d}X_1/\mathbf{d}t = 0$

$\mathbf{d}X_2/\mathbf{d}t = 0.5\ [\mathbf{alogistic}_{5,X_6}(X_1) - X_2]$

$\mathbf{d}X_3/\mathbf{d}t = 0.2\ [\mathbf{alogistic}_{5,0.2}(X_2) - X_3]$

$\mathbf{d}X_4/\mathbf{d}t = 0.5\ [\mathbf{alogistic}_{5,X_7}(X_5X_2, X_3) - X_4]$

$\mathbf{d}X_5/\mathbf{d}t = X_8\ [X_2X_4\ (1 - X_5) + X_9X_5 - X_5]$

$\mathbf{d}X_6/\mathbf{d}t = 0.3\ [\mathbf{alogistic}_{5,0.7}(-0.4X_2, -0.4X_4, X_6) - X_6]$

$\mathbf{d}X_7/\mathbf{d}t = 0.3\ [\mathbf{alogistic}_{5,0.7}(-0.4X_2, -0.4X_4, X_6) - X_7]$

$\mathbf{d}X_8/\mathbf{d}t = 0.5\ [\mathbf{alogistic}_{5,1}(X_2, X_4, -0.4\ X_5, X_8) - X_8]$

$\mathbf{d}X_9/\mathbf{d}t = 0.1\ [\mathbf{alogistic}_{5,1}(X_2, X_4, X_5, X_9) - X_9]$

There is a generic way to write the compiled differential equations down in a symbolic manner, in terms of the cell references in the role matrices as follows. Substitute every reification state by the reference to the cell in the role matrix where this is indicated. So, for the equation of X_j:

- for $\mathbf{H}_{X_j}$ substitute $\mathbf{ms}(j, 1)$
- for $\mathbf{C}_{i,X_j}$ substitute $\mathbf{mcfw}(j, i)$
- for $\mathbf{P}_{i,1,X_j}$ or $\mathbf{P}_{i,2,X_j}$ substitute $\mathbf{mcfp}(j, 1, i)$ or $\mathbf{mcfp}(j, 2, i)$
- for $\mathbf{W}_{X_i,X_j}$ substitute $\mathbf{mcw}(j, i)$

Then the following equation results

$$\mathbf{d}X_j/\mathbf{d}t = \mathbf{ms}(j,1)[\frac{\begin{array}{c}\mathbf{mcfw}(j,1)\mathbf{bcf}_1(\mathbf{mcfp}(j,1,1),\mathbf{mcfp}(j,2,1),\mathbf{mcw}(j,1)\mathbf{mb}(j,1),\ldots,\mathbf{mcw}(j,k)\mathbf{mb}(j,k))\\ +\cdots+\mathbf{mcfw}(j,m)\mathbf{bcf}_m(\mathbf{mcfp}(j,1,m),\mathbf{mcfp}(j,2,m),\mathbf{mcw}(j,1)\mathbf{mb}(j.1),\ldots,\mathbf{mcw}(j,k)\mathbf{mb}(j,k))\end{array}}{\mathbf{mcfw}(j,1)+\cdots+\mathbf{mcfw}(j,m)} - X_j] \tag{29}$$

When for states often only one combination function is selected and has nonzero weight, (5) simplifies to (e.g., for $\mathbf{bcf}_i$(..)):

$$\mathbf{d}X_j/\mathbf{d}t = \mathbf{ms}(j,1)[\mathbf{bcf}_i(\mathbf{mcfp}(j,1,i),\mathbf{mcfp}(j,2,i),\mathbf{mcw}(j,1)\mathbf{mb}(j,1),\ldots.,\mathbf{mcw}(j,k)\mathbf{mb}(j,k)) - X_j] \tag{30}$$

Here, to evaluate this expression, the references in the cells of the role matrices are interpreted as strings; so, for example, if in that cell it is written X_4, then that is substituted in the above expression to get the resulting differential equation.

10.8 Discussion

In this chapter a more in depth analysis was presented for the universal differential or difference equation that is an important basis for reified temporal-causal networks. It was shown how this equation can be derived and it was illustrated by some examples. Due to the existence of this specific universal difference or differential equation, it can be guaranteed that any reification of a temporal-causal network is itself also a temporal-causal network: the class of temporal-causal networks is closed under reification. That means that dedicated modeling and analysis methods for temporal-causal networks can also be applied to reified temporal-causal networks. In particular, reification can be done iteratively so that multilevel reified network models are obtained that are very useful to model multiple orders of adaptation. In addition, the fact that the universal difference or differential equation is the same for all states, and has not a number of instantiations for different states, makes that it indeed is universal. This supports structure preserving implementation where the core of the program code for the computational reified temporal-causal network engine has the same simple universal structure expressed in only a few lines of code, as can be seen in Chap. 9, Sect. 9.4.3, Box 9.4.

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