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Off-shell length for the two-nucleon T matrix in the $^3S_1$-$^3D_1$ state

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It is shown that the off-shell behavior of the low energy two-nucleon T matrix, in the coupled $^3S_1$-$^3D_1$ state, can be characterized by a single parameter, which is the generalization from central to noncentral forces of the equivalent or off-shell length. An approximate linear relation between the square of the off-shell length and the intrinsic range, which holds for central forces, is shown to be also valid for noncentral forces. It is found that the approximate relation is exact for a separable potential of the Yamaguchi type, and hence for the unitary pole approximation.

I. INTRODUCTION

Several authors[1-5] have shown that the low-energy off-shell two-nucleon scattering matrix is described by a single parameter in each spin state, in addition to the usual on-shell effective range parameters. This parameter is referred to as the equivalent length[1,6] or off-shell length[4,5] and is the coefficient of the first order term in an expansion of the T matrix or K matrix about zero energy.

Following a suggestion by Sprung,[7] Fiedeldey and McGurk[8] have shown that for a certain class of phase-shift-equivalent separable potentials, there is an approximate linear relation between the triton binding energy and the square of the off-shell length. The calculations of Bruinsma *et al.*[9] indicate a strong correlation between the various nucleon-deuteron reaction quantities and the off-shell lengths. It appears, therefore, that this parameter provides a useful way of characterizing the off-shell T matrix, at least insofar as the three-nucleon system is concerned. One of the purposes of the present note is to show that, even when the coupling between the $^3S_1$ and $^3D_1$ two-nucleon states is taken into account, there is still only one parameter needed to describe the low-energy T matrix in each spin state.

It has been shown by Kok[6] that for central forces there is an approximate linear relation between the intrinsic range of Blatt and Jackson[10] and the square of the off-shell length. This relation is exact for a rank-one central separable potential.[6,9] It will be shown here that this approximate relation is also valid for the coupled $^3S_1$-$^3D_1$ state and, furthermore, is exact for a noncentral separable potential of the Yamaguchi and Yamaguchi[11] type. Since the unitary pole approximation[12] (UPA) gives rise to a T matrix with the same structure as the Yamaguchi[11] T matrix, it follows that the linear relation between the intrinsic range and the square of the off-shell length is exact for the UPA.

II. OFF-SHELL LENGTH AND NONCENTRAL FORCES

It is most convenient, here, to write the two-nucleon T matrix in the coupled $^3S_1$-$^3D_1$ state in the form of a 2×2 matrix as shown below:

$$T(p,q;s)=\begin{bmatrix}\langle\alpha(p)|T(s)|\alpha(q)\rangle & \langle\alpha(p)|T(s)|\beta(q)\rangle\\ \langle\beta(p)|T(s)|\alpha(q)\rangle & \langle\beta(p)|T(s)|\beta(q)\rangle\end{bmatrix}, \tag{1}$$

where the matrix elements are taken with respect to the states

$$|\alpha(k)\rangle=|k011M\rangle\cos\epsilon(k)+|k211M\rangle\sin\epsilon(k), \tag{2}$$

$$|\beta(k)\rangle=-|k011M\rangle\sin\epsilon(k)+|k211M\rangle\cos\epsilon(k). \tag{3}$$

Here $\epsilon(k)$ is the mixture parameter of Blatt and Biedenharn,[13] and the states on the right-hand sides of (2) and (3) are given by

$$\langle\vec{r}\,|kLSJM\rangle=\sqrt{2/\pi}\,j_L(kr)\mathcal{Y}^M_{LSJ}(\hat{r}). \tag{4}$$

j_L is the usual spherical Bessel function, and $\mathcal{Y}^M_{LSJ}$ is a vector spherical harmonic.[14] Throughout, the complex energy parameter s will be taken to be

$$s=k^2+i\eta, \tag{5}$$

where k^2 is the on-shell energy in inverse fm^2, and η is a small positive real quantity. On the energy shell the matrix (1) is diagonal, and is given in terms of the Blatt-Biedenharn[13] phases δ_α and δ_β by

$$T(k,k;s)=-\frac{2}{\pi k}\begin{bmatrix}e^{i\delta_\alpha(k)}\sin[\delta_\alpha(k)] & 0\\ 0 & e^{i\delta_\beta(k)}\sin[\delta_\beta(k)]\end{bmatrix}. \tag{6}$$

We now consider expanding the various matrix elements in (1) to first order in p^2, q^2, and k^2. We have, from Refs. 13 and 14 and Eqs. (2)–(4),

$$\epsilon(k) = ck^2 + \cdots , \tag{7}$$

$$\langle \vec{r} \mid \alpha(k)\rangle = \sqrt{2/\pi}\,(1 - \tfrac{1}{6}k^2r^2)\mathcal{Y}^M_{011} + \cdots , \tag{8}$$

$$\langle \vec{r} \mid \beta(k)\rangle = \sqrt{2/\pi}\,k^2(-c\mathcal{Y}^M_{011} + \tfrac{1}{15}r^2\mathcal{Y}^M_{211}) + \cdots . \tag{9}$$

We can write

$$\begin{aligned}\langle \alpha(p) \mid T(s) \mid \alpha(q)\rangle &= \langle \alpha(k) \mid T(s) \mid \alpha(k)\rangle \\ &+ [\langle \alpha(p) \mid - \langle \alpha(k) \mid]\, T(s) \mid \alpha(k)\rangle \\ &+ \langle \alpha(k) \mid T(s)[\mid \alpha(q)\rangle - \mid \alpha(k)\rangle] \\ &+ [\langle \alpha(p) \mid - \langle \alpha(k) \mid]\, T(s) \\ &\times [\mid \alpha(q)\rangle - \mid \alpha(k)\rangle] .\end{aligned} \tag{10}$$

From (8) it follows that the last term on the right-hand side of (10) is second order in energy, and can therefore be ignored. As is well known,[14] we can write

$$T(s) \mid \alpha(k)\rangle = V \mid \Psi_\alpha(k)\rangle \tag{11}$$

with

$$\mid \Psi_\alpha(k)\rangle = [1 + (s - H_0)^{-1}T(s)] \mid \alpha(k)\rangle , \tag{12}$$

where H_0, V, and $\mid \Psi_\alpha(k)\rangle$ are the two-particle kinetic energy operator, potential energy operator, and wave function, respectively. Using Ref. 15, it is straightforward to show that

$$\langle \vec{r} \mid \Psi_\alpha(k)\rangle = \sqrt{2/\pi}\,(kr)^{-1}e^{i\delta_\alpha(k)}\sin[\delta_\alpha(k)] \times [u_\alpha(k,r)\mathcal{Y}^M_{011}(\hat{r}) + w_\alpha(k,r)\mathcal{Y}^M_{211}(\hat{r})], \tag{13}$$

where the S and D radial wave functions u_α and w_α become outside the range of forces:

$$\begin{aligned} u_\alpha(k,r) &= kr[e^{-i\delta_\alpha(k)}\sin^{-1}[\delta_\alpha(k)]\,j_0(kr) + h_0^{(+)}(kr)]\cos[\epsilon(k)], \\ &\xrightarrow[k=0]{} 1 - r/a , \end{aligned} \tag{14}$$

$$\begin{aligned} w_\alpha(k,r) &= kr[e^{-i\delta_\alpha(k)}\sin^{-1}[\delta_\alpha(k)]\,j_2(kr) + h_2^{(+)}(kr)]\sin[\epsilon(k)], \\ &\xrightarrow[k=0]{} 3c/r^2 . \end{aligned} \tag{15}$$

Here $h_L^{(+)}(kr)$ is a spherical Hankel function,[14] and a is the triplet scattering length. From (11), (13), and the Schrödinger equation, it follows that

$$\langle \vec{r} \mid T(0) \mid \alpha(0)\rangle = \sqrt{2/\pi}\,ar^{-1}\left\{\frac{d^2}{dr^2}\left[1 - \frac{r}{a} - u_\alpha(0,r)\right]\mathcal{Y}^M_{011} + \left(\frac{d^2}{dr^2} - \frac{6}{r^2}\right)\left[\frac{3c}{r^2} - w_\alpha(0,r)\right]\mathcal{Y}^M_{211}\right\}. \tag{16}$$

By using (6), (8), and (16), and carrying out an integration by parts, we arrive at

$$\begin{aligned}&[\langle \alpha(p) \mid - \langle \alpha(k) \mid]\, T(s) \mid \alpha(k)\rangle \\ &\quad = \langle \alpha(0) \mid T(0) \mid \alpha(0)\rangle\, \tfrac{1}{2}\Lambda^2(k^2 - p^2) + \cdots ,\end{aligned} \tag{17}$$

where

$$\Lambda^2 = 2\int_0^\infty dr\, r[1 - r/a - u_\alpha(0,r)] . \tag{18}$$

The terms that are being neglected in Eq. (17), and in the rest of this section, are third order in the momenta p, q, and k.

From (10) and the well known symmetry relation

$$T(p,q;s) = \tilde{T}(q,p;s) \tag{19}$$

it follows that

$$\begin{aligned}\langle \alpha(p) \mid T(s) \mid \alpha(q)\rangle &= \langle \alpha(k) \mid T(s) \mid \alpha(k)\rangle \\ &\times [1 + \tfrac{1}{2}\Lambda^2(2k^2 - p^2 - q^2) + \cdots] .\end{aligned} \tag{20}$$

Clearly, Λ is the off-shell length for noncentral forces, and is given by a relation just like the one for central forces.[1-5]

We now consider one of the off-diagonal elements of (1). We can write

$$\begin{aligned}\langle \beta(p) \mid T(s) \mid \alpha(q)\rangle &= \langle \beta(p) \mid T(s) \mid \alpha(k)\rangle \\ &+ \langle \beta(p) \mid T(s)[\mid \alpha(q)\rangle - \mid \alpha(k)\rangle] ,\end{aligned} \tag{21}$$

where from (8) and (9) it follows that the second term on the right-hand side can be neglected. By using (9) and (16), it is easy to show that to first order in energy $\langle \beta(p) \mid T(s) \mid \alpha(k)\rangle$ is zero. To the same order $\langle \beta(p) \mid T(s) \mid \alpha(q)\rangle$ and, from (19), $\langle \alpha(p) \mid T(s) \mid \beta(q)\rangle$ are zero, as well as $\langle \beta(p) \mid T(s) \mid \beta(q)\rangle$. This means that in the low-energy region the off-shell length, given by (18), is the only parameter needed to characterize the off-shell behavior of the T matrix for the coupled $^3S_1$-$^3D_1$ state at low energies.

It is somewhat surprising that the off-diagonal matrix element $\langle \beta(p) | T(s) | \alpha(q)\rangle$ vanishes to first order in energy. It should be kept in mind, however, that if the T matrix given by Eq. (1) is transformed to the more conventional basis given by the states $| k011M\rangle$ and $| k211M\rangle$, there will be an off-diagonal term of this order. The point is that the mixture parameter [see Eq. (7)] is already first order in the energy and going off shell introduces no new energy dependence to this order.

III. INTRINSIC RANGE AND OFF-SHELL LENGTH

The effective range for the coupled ${}^3S_1$-${}^3D_1$ state is given by[16]

$$r_0 = 2\int_0^\infty dr[(1-r/a)^2 - u_\alpha{}^2(0,r) - w_\alpha{}^2(0,r)]. \tag{22}$$

The intrinsic range b[10] is the effective range for a potential whose strength has been adjusted to produce a bound state at zero energy or, what is equivalent, an infinite scattering length. From (18) and (22), it follows that

$$b = r_0 + \frac{2\Lambda^2}{a} + 2\int_0^\infty dr[2y_i(r) - 2y(r) - y_i{}^2(r) + y^2(r) - w_{\alpha i}{}^2(0,r) + w_\alpha{}^2(0,r)], \tag{23}$$

where

$$y(r) = 1 - r/a - u_\alpha(0,r) \tag{24}$$

and the subscript i refers to the potential which produces the intrinsic range. If we assume that y and w_α are approximately equal to y_i and $w_{\alpha i}$, respectively, then

$$b \simeq r_0 + 2\Lambda^2/a, \tag{25}$$

and we have a linear relation between b and Λ^2, for fixed a and r_0.

We will now show that this approximation is exact for a noncentral separable potential of the Yamaguchi[11] type. The partial wave matrix elements of such a potential have the form

$$\langle pL11M | V | qL'11M\rangle = -g_L(p)\lambda g_{L'}(q), \quad L, L' = 0, 2. \tag{26}$$

It is easy to show that the T matrix arising from (26), in the representation of (1) – (3), is given by

$$T(p,q;s) = \frac{1}{D(s)}\begin{bmatrix} g(p)g(q) & 0 \\ 0 & 0\end{bmatrix}, \tag{27}$$

$$g(k) = [g_0{}^2(k) + g_2{}^2(k)]^{1/2}, \tag{28}$$

$$\tan[\epsilon(k)] = g_2(k)/g_0(k), \tag{29}$$

$$D(s) = -\lambda^{-1} - \int_0^\infty [g_0{}^2(p) + g_2{}^2(p)]\frac{p^2dp}{s-p^2}. \tag{30}$$

From (1), (2), (4), (6), (12), and (27)–(30) it follows that

$$\langle \vec{r} | \Psi_\alpha(k)\rangle = \langle \vec{r} | \alpha(k)\rangle + \int_0^\infty \langle \vec{r} | \alpha(p)\rangle \frac{p^2dp}{s-p^2}\frac{g(p)g(k)}{D(s)} \tag{31}$$

$$= \sqrt{2/\pi}\,(kr)^{-1}e^{i\delta_\alpha(k)}\sin[\delta_\alpha(k)] \times\{kr[e^{-i\delta_\alpha(k)}\sin^{-1}[\delta_\alpha(k)]j_0(kr) + h_0^{(+)}(kr)]\cos[\epsilon(k)]\mathcal{Y}_{011}^M + kr[e^{-i\delta_\alpha(k)}\sin^{-1}[\delta_\alpha(k)]j_2(kr) + h_2^{(+)}(kr)]\sin[\epsilon(k)]\mathcal{Y}_{211}^M + \text{terms independent of } \lambda\}. \tag{32}$$

By comparing (13) and (32), we see that the difference between u_α and its asymptotic form (14), and the difference between w_α and its asymptotic form (15), are independent of the potential strength λ. This means that for the separable potential (26) the approximations leading to (25) are exact. As pointed out in the introduction the UPA[12] is of the same form as (27), and therefore for potentials whose T matrix is well approximated by the UPA, the relation (25) should be quite accurate.

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