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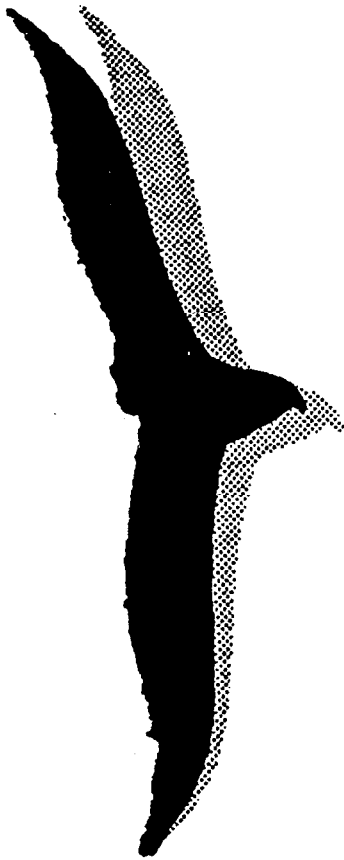
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integrated Model

Marjan W. Hofkes

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# SUSTAINABLE DEVELOPMENT IN AN ECONOMY – ECOLOGY INTEGRATED MODEL\* '1

by

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## **Abstract**

In this paper an endogenous growth model is analyzed which takes account of environmental deterioration and abatement. The environment plays a role both in production and welfare. It is **common** practice to solve growth models by looking at a balanced growth solution, which is often associated with sustainable development. We derive the conditions under which sustainable development is feasible. Furthermore, the conditions for optimal balanced growth are derived. Finally, some comparative statics of the balanced growth solutions are discussed and a numerical example is given to illustrate the model and to provide further insight.

## **JEL code**

## **Keywords**

Endogenous growth, abatement, sustainable development, renewable resources, environmental quality

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# 1. Introduction

In recent years, economists have been showing a growing interest in environmental issues. Especially the link between environmental policies and economic growth has the attention of both economists and politicians. One of the main questions is whether economic growth and environmental care can go together or are two competitive goals. With respect to environmental care the concept of sustainable development plays a central role in the discussion. With sustainable development one refers to ecologically sustainable development, which means maintaining the natural (i.e. ecological) basis of economic development. On the one hand the environment influences production possibilities and welfare, while on the other hand production diminishes the quality and quantity of environmental resources, by the use of resources and through pollution. A continuously decreasing quality and quantity of natural resources cannot support growing or even constant levels of physical economic output in the distant future. In order to be able to **analyse** the conditions under which sustainable development is possible, both the interactions between the environment and the economy have to be **modelled** and a dynamic specification of ecological relations, encompassing the absorption and regenerative capacities of the natural environment, has to be assessed. Furthermore, in studying the long term relation between economy and ecology, (the development of new) technology plays an important role. So, any model which aims at modelling the interactions between economy and ecology should account for technological development.

Once all the interactions between economy and ecology are integrated in one model, a so-called economy-ecology integrated model, there is no longer a contrast between sustainable development and economic development. In a long-term analysis, maximizing (social) welfare, ecological sustainability of economic development is taken automatically into account as welfare is longer determined only by economic development but also by environmental quality, both directly through the amenity value of the environment and indirectly through the impact of environmental quality on production possibilities. In other words in such a model sustainable development is lead by the concept of welfare. In this paper the relation between economic growth and environmental quality is analyzed in a two-sector (long term) growth model, in which the growth rate is endogenously determined and may depend upon preferences and technology. The one-sector model as described in Den Butter and Hofkes (1993) is expanded with a sector in which new technologies are developed. This extension builds on the model of Bovenberg and Smulders (1993), who develop a growth model that incorporates pollution saving. technological change. Above that, our model also allows for abatement activities. The optimal **growth rate is determined** by maximizing a social welfare function in which both economic and environmental **values** are represented. In the production process the environment enters in two ways. On the one hand, production depends upon the extractive use of the environment, which is **modelled** as a renewable resource. The environment deteriorates due to the extractive use in production. This deterioration of the natural environment can,

however, in our model, (partially) be offset by abatement. Moreover, the environment has some **self-regenerating** capacities. **On** the other hand, the quality of the environment itself plays a productive role by providing (non-extractive) services to the production process (think for example of the influence of air quality on **labour** productivity).

It is common practice to solve growth models by looking at a balanced growth solution, which is often associated with sustainable development. We find that under certain conditions with respect to production and substitution elasticities, there exists a feasible and optimal sustainable balanced growth path. In other words, under these conditions, there is an optimal growth path on which the economy grows at a constant positive growth rate, keeping environmental quality at a constant level. Growth in technology and abatement activities now compensate for the growing use of natural resources in production, such that net pollution is constant on the the optimal growth path and hence environmental quality remains constant. It must be noted that on such a growth path, with net use of environmental resources constant, input of materials in production relatively decreases. So, according to the laws of thermodynamics, it must be the case that the share of physical, c.q. material production in total production also decreases, as it is not possible to produce an ever increasing amount of physical output out of a constant amount of physical input. In other words the resulting growth path of the economy must be interpreted as one where non-physical products (such as information) take a growing share in total production.

This paper is organized as follows. In section 2 the model is described, while in section 3 we analyze conditions for feasibility and optimality of balanced growth. In section 4 some comparative static results for the optimal balanced growth solution are given, while section 5 provides a numerical example. Finally, section 6 concludes.

## 2. The model

Consider an economy consisting of two sectors. Following Bovenberg and Smulders (1993) we distinguish a production-sector producing a **final** good,  $Y$ , and a knowledge or learning sector producing knowledge,  $h$ , about an efficient or pollution-saving use of environmental resources. The production technologies for the **final** good and for knowledge are given by:

$$Y(K_Y, Z_Y, E),$$

respectively

$$H(K_H, Z_H)$$

where  $K_Y$  is the use of physical capital,  $K$ , in the production of  $Y$ ,  $K_H$  is the use of physical capital in the production of  $h$ ,  $Z_Y$  is the effective input of environmental resources in the production of  $Y$ , given by  $h \cdot Q_Y$ , where  $Q_Y$  is the raw input of environmental resources,  $Q$ , in sector  $Y$ . Analogously,  $Z_H$  is the effective input of environmental resources in sector  $H$ , given by  $h \cdot Q_H$ , where  $Q_H$  is the raw input of environmental resources in sector  $H$ . Finally,  $E$  represents the natural environment, i.e., the aggregate stock of natural capital, which serves as an input in the production process. A better state of the natural environment involves for example healthier workers with higher marginal productivity. Both the production functions  $Y$  and  $H$  are assumed to be twice continuously differentiable and concave. Note that we assume that there is both extractive use ( $Q$ ) and non-extractive use ( $E$ ) of the natural environment, where  $Q$  is a flow and  $E$  is a stock.  $Q$  can be thought of as the use of natural resources like energy, or other kinds of (polluting) use of environmental resources. The **final** good can either be consumed ( $C$ ), used for abatement ( $A$ ) or invested ( $K$ ) in order to accumulate physical capital, which serves future consumption and abatement. So physical capital accumulation is given by the (usual) equation:

$$\dot{K} = Y(K_Y, Z_Y, E) \cdot c \cdot A.$$

Growth of the stock of knowledge,  $h$ , is given by:

$$\dot{h} = H(K_H, Z_H)$$

A dot represents a time derivative. The total stock of physical capital and the total effective use of environmental resources is allocated between the final good sector and the knowledge sector. Let  $u$  respectively  $v$  be the share of physical capital and the share of (effective) environmental resources used in the final goods sector, and let  $(1-u)$  respectively  $(1-v)$  be the share of physical capital and the share of (effective) environmental resources used in the knowledge sector:

$$K_Y = u \cdot K,$$

$$Z_Y = v \cdot h \cdot Q,$$

$$K_H = (1-u) \cdot K \text{ and}$$

$$Z_H = (1-v) \cdot h \cdot Q.$$

The use of environmental resources in production reduces the quality of the natural environment. On the other hand the quality of the environment can be improved by abatement activities (A). These abatement activities go at the expense of consumption and investment in physical capital, as final goods can either be consumed, or invested in order to accumulate physical capital, or used for abatement activities. We distinguish between gross and net pollution. Net pollution P is a function of the amount of environmental resources Q used in production (gross pollution) and the amount of abatement activities A. Furthermore, the natural environment has some self-regenerative capacities, described by the function N, depending upon the quality of the natural environment itself and upon the level of net pollution. So,

$$\dot{E} = N(E, P(Q, A))$$

N is assumed to be twice continuously differentiable and concave. Furthermore,  $N_P < 0$ ,  $N_{EP} > 0$ ,  $P_Q > 0$ ,  $P_A < 0$  and  $P_{QA} < 0$ . Subscripts attached to a function symbol denote partial derivatives. The regenerative capacity of the natural environment will decrease with an increasing level of net pollution ( $N_P < 0$ ), while the level of net pollution decreases with an increasing level of abatement ( $P_A < 0$ ). Furthermore, it is assumed that the higher the quality of the natural environment the smaller the (negative) influence of pollution on the regenerative capacity ( $N_{EP} > 0$ ). Finally, the higher the level of (gross) pollution, the larger the effect of an extra unit of abatement ( $P_{QA} < 0$ ). This means that reducing pollution gets increasingly difficult. For each level of net pollution there exists a stable level of the quality of the natural environment for which the regenerative capacities are such that the quality of the natural environment remains constant, which is ensured by assuming  $N_E(E, P(Q, A)) < 0$  in a neighbourhood of this stable level. These assumptions ensure that in order to have the quality of the environment growing at a constant rate, pollution should be reduced at an increasing rate (see appendix A).

Social welfare, W, is assumed to be dependent upon the utility of a representative consumer, who is supposed to be infinitely lived. Instantaneous individual utility, u, depends upon individual consumption c ( $c = C/L$ , where C is aggregate consumption and L is population which is assumed to be constant over time) and upon the quality of the natural environment E. The rate of time preference is given by  $\theta$ . So, we have:

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‘Although labour is not modelled explicitly in the production function, it can be assumed that the production function is dependent upon the size of the working force.’

$$W = \int_0^{\infty} e^{-\rho t} u(c,E) dt$$

### 3. Balanced Growth

We will first consider the conditions under which balanced growth is feasible in our model. Balanced growth is defined as a situation in which all variables grow at a constant (possibly zero) rate and in which the allocative variables, i.e.,  $C/Y$  and  $A/Y$ , are constant. It is easy to see that in a situation of balanced growth  $A$ ,  $C$ ,  $Y$  and  $\dot{K}$  grow at the same rate and consequently also  $K$  grows at this common rate. Let us denote this common growth rate by  $g$  and the growth rates of  $Y$ ,  $C$ ,  $K$ ,  $A$ ,  $Q$  and  $E$  respectively by  $g_Y$ ,  $g_C$ ,  $g_K$ ,  $g_A$ ,  $g_Q$  and  $g_E$ . So, in balanced growth we have:

$$g = g_Y = g_C = g_A = g_K \quad (3.1)$$

Furthermore, Bovenberg and Smulders (1993) show that balanced growth requires that the production elasticities are constant over time and the elasticity of substitution between the natural environment and the other inputs is equal to one. Let us denote the production elasticities of  $K_Y$ ,  $Z_Y$  and  $E$  and the knowledge elasticities of  $K_H$  and  $Z_H$  by respectively  $\lambda_K$ ,  $\lambda_Z$ ,  $\lambda_E$ ,  $\nu_K$  and  $\nu_Z$ . Loglinearizing the production function of the final good we find:

$$g_Y = \lambda_K g_K + \lambda_Z (g_Q + g_h) + \lambda_E g_E \quad (3.2)$$

Loglinearizing the production function of knowledge we find:

$$g_H = \nu_K g_K + \nu_Z (g_Q + g_h) \quad (3.3)$$

Furthermore, in a situation of balanced growth:

$$\dot{g}_h = g_h - g_h = g_H - g_h = 0 \quad (3.4)$$

Finally, **loglinearizing** the pollution function, given implicitly in the regeneration function, we find:

$$g_P = \lambda_Q g_Q + \lambda_A g_A \quad (3.5)$$



where  $\lambda_Q$  is the pollution-elasticity of Q and  $\lambda_A$  is the pollution-elasticity of A.

Now, note that the specification of the regeneration function implies certain restrictions with respect to the growth rate of the quality of **the** natural environment on a balanced growth path. From the specification of the regeneration function it follows that the quality of the natural environment can only increase at a constant rate if net pollution decreases at an increasing rate (see appendix A), which would require either abatement to grow at an increasing rate or the use of natural resources to decrease at an increasing rate. Hence on a balanced growth path the quality of the environment and the level of net pollution have to be constant ( $g_E = g_P = 0$ ). Substituting  $g_P = 0$  into (3.5) gives:

$$g_Q = -\frac{\lambda_A}{\lambda_Q} g_A \quad (3.6)$$

Solving (3.2) and (3.3) for  $g$ , using (3.1), (3.4) and the fact that  $g_E = 0$ , we get:

$$g_Q = \frac{(1-\lambda_K)(1-\nu_Z) - \lambda_Z \nu_K}{\lambda_Z} g \quad (3.7)$$

Combining (3.6) and (3.7) yields:

$$\frac{\lambda_A}{\lambda_Q} = \nu_K - \frac{(1-\lambda_K)(1-\nu_Z)}{\lambda_Z} \quad (3.8)$$

This latter equality is rather restricting, since it implies that, given the assumptions with respect to the regeneration function, only for a very special relationship between on the one hand net pollution and on the other hand the use of environmental resources and abatement there is a feasible balanced growth path. The quality of the environment will only be constant if the pollution elasticities of Q and A are such that the growth rates of Q and A exactly outweigh each other given these pollution elasticities, such that net pollution remains constant<sup>2</sup>. Finally, substituting (3.7), (3.1) and (3.4) into (3.3) gives:

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<sup>2</sup> It must be noted that we do not find any restrictions with respect to the production elasticities in the two different sectors. Bovenberg and Smulders (1993) for example **find** that either both sectors should exhibit constant returns to scale, or if one of the sectors exhibits decreasing returns to scale this should be exactly compensated by increasing returns to scale in the other sector.

$$g_h = \left[ \nu_K + \nu_Z \frac{(1-\lambda_K)}{\lambda_Z} \right] g \quad (3.9)$$

Now, we will **analyse** the conditions under which the economy is on an optimal growth path. Society's optimisation problem is given by:

$$\begin{aligned} \max \int_0^{\infty} e^{-\theta t} U(c(t), E(t)) dt \\ \text{s.t. } \dot{K} &= Y(K_Y, h, Q_Y, E) - C - A \\ \dot{E} &= N(E, P(Q, A)) \\ \dot{h} &= H(K_H, h, Q_H) \end{aligned} \quad (3.10)$$

The social optimal plan implies the following conditions (see appendix B):

$$\frac{\dot{c}}{c} = \frac{1}{\rho} \left[ \frac{\partial Y}{\partial K_Y} + \frac{U_{cE} \dot{E}}{U_c} - \theta \right] \quad (3.11)$$

where  $\rho = -c \cdot U_{cc}/U_c$ ,

$$\frac{\partial Y}{\partial K_Y} = \frac{\partial N}{\partial P} \frac{\partial P}{\partial A} \left[ L \frac{U_E}{U_c} + \frac{\partial Y}{\partial E} \right] + \frac{\partial N}{\partial E} + \frac{\dot{\mu}}{\mu} - \frac{\dot{\lambda}}{\lambda} = \frac{\partial H}{\partial(h, Q_H)} \cdot Q + \frac{\dot{\kappa}}{\kappa} - \frac{\dot{\lambda}}{\lambda} \quad (3.12)$$

where  $\lambda$ ,  $\mu$  and  $\kappa$  denote respectively the shadow price of physical capital, of natural resources and of knowledge,

$$\frac{\partial Y / \partial K_Y}{\partial Y / \partial(h, Q_Y)} = \frac{\partial H / \partial K_H}{\partial H / \partial(h, Q_H)} \quad (3.13)$$

and

$$\frac{\partial Y}{\partial Z_{\neq}} h = - \frac{\partial P / \partial Q}{\partial P / \partial A} \quad (3.14)$$

Equation (3.11) gives the optimal allocation between current and future consumption. Consumption is postponed if the marginal contribution to future utility of consumption foregone exceeds the rate of time preference. The marginal contribution to future utility of consumption foregone, which can be called the social interest rate, is represented by the first two terms in long brackets in equation (3.11). Savings add to the physical capital stock and increase future output of the final good. Furthermore, future consumption is higher valued if the environment improves. So, a necessary condition for positive per capita growth is that the social interest rate exceeds the rate of time preference. Bovenberg and Smulders (1993) show that optimality of balanced growth requires that the intertemporal substitution elasticity,  $\rho$ , is constant. Without loss of generality we assume that  $\rho$  equals one. Furthermore, King, Rebelo and Plosser (1988) show that the elasticity of substitution in utility between consumption and the index of environmental services should be equal to one.

Equation (3.12) states that returns on physical capital, the natural environment and knowledge should be the same. Returns on the natural environment (right hand side of equation (3.12)) consist of increased marginal utility, increased marginal productivity, and increased regenerative capacity of the natural environment and furthermore it consists of changes in relative prices (capital gains).

On a balanced growth path (3.11) and (3.12), using (3.1) and the fact that  $g_E = 0$ , can be written as (see also appendix B):

$$g = (r - \theta) \quad (3.16)$$

respectively

$$r = \frac{\partial N}{\partial P} \frac{\partial P}{\partial A} \left[ L \frac{U_E}{U_c} + \frac{\partial Y}{\partial E} \right] + \frac{\partial N}{\partial E} + g \quad (3.17)$$

and

$$r = \frac{\partial H}{\partial (h \cdot Q_H)} \cdot Q + \left[ 1 - \nu_K - \nu_Z \frac{(1 - \lambda_K)}{\lambda_Z} \right] g \quad (3.18)$$

where

$$r = \frac{\partial Y}{\partial K_Y} \quad (3.19)$$

The optimal balanced growth rate is now given by the intersection of the two lines represented by (3.16) and (3.17). (3.16) gives the growth rate associated to any rate of return ( $r$ ) that is preferred given inter-temporal preferences, while (3.17) gives the growth rate that is sustainable for any  $r$  in the long run and that is consistent with optimal allocation. (3.18) determines together with (3.13) and (3.14) the optimal (static) allocation of capital and natural resources over the two sectors Y and H and the optimal level of abatement, given the optimal growth rate,  $g$ . In the next section we will derive some comparative static results with respect to the optimal balanced growth solution, which can be of interest.

## 4. Comparative Statics

On an optimal balanced growth path, the return on physical capital should be equal to the return on natural resources. From (3.17) we have:

$$r - g - N_E = \frac{\partial N}{\partial P} \frac{\partial P}{\partial A} \left[ L \frac{U_E}{U_c} + \frac{\partial Y}{\partial E} \right] \quad (4.1)$$

Equation (4.1) states that it is optimal to invest in natural resources up to the point where marginal costs are equal to marginal benefits. Marginal costs (left hand side of (4.1)) consist of the return on alternative investment corrected for capital gains on natural resources (the relative price of natural resources increases as the quality of the natural environment remains constant on a balanced growth path, while physical capital and knowledge grow) and corrected for the reducing regenerative capacities of the natural environment as the quality of the natural environment increases. Marginal benefits (right hand side of (4.1)) consist of higher utility and higher productivity due to a higher quality of the natural environment, both measured in terms of the marginal contribution of abatement to the natural environment.

From (4.1) we can derive the impact of a shift in preferences that raises the marginal utility of environmental quality. **Loglinearization** of (4.1) yields (see appendix C):

$$L \frac{U_E}{U_c} N_A \tilde{\phi} = (\theta - N_E - N_{EE}E) \tilde{E} - \left[ (\theta - N_E) \left| \frac{N_{PP}P + N_P}{N_P} \right| + N_{EP} \right] \tilde{P} \quad (4.2)$$

where  $\tilde{\phi}$  denotes an exogenous relative change in the marginal rate of substitution between E and C.

A tilde (-) denotes a relative change. From (4.2) we derive that a green preference shift ( $\tilde{\phi} > 0$ ) typically implies a reduction in net pollution and hence an increase in environmental quality. Increased marginal utility of environmental quality gives an incentive to invest more in the natural environment which requires a reduction in pollution which can be achieved by increased abatement expenditures and/or a decreased use of environmental resources in production. In the following we will analyze the long term effects of an exogenous reduction in pollution on **sectoral** factor intensities and on the growth **rate** of the economy.

Loglinearizing the conditions for optimal balanced growth we can derive expressions for changes in the **sectoral** factor intensities due to exogenous shocks in net pollution (see appendix C):

$$\begin{aligned} (\tilde{K}_Y - \tilde{Z}_Y) &= \frac{\sigma_Y}{\sigma_H} (\tilde{K}_H - \tilde{Z}_H) \\ &= \frac{(\lambda_K + \lambda_Z - 1)}{\lambda_Z} \frac{(1 - \nu)}{\nu} \left( \frac{r}{g} - \frac{xr}{xr + (1-x)\theta} \tilde{r} \right) - \frac{(\lambda_K + \lambda_Z)}{\lambda_Z} r^* + \\ &\quad + \frac{(\lambda_K + \lambda_Z - 1)}{\lambda_Z} \tilde{A} + \frac{\beta}{\lambda_Z} \tilde{E} + \frac{1}{\lambda_Z} \tilde{\Lambda}_1 \end{aligned} \quad (4.3)$$

where

$$x = \nu_K + \nu_Z \frac{(1 - \lambda_K)}{\lambda_Z}$$

Since the **coefficient** of  $\tilde{r}$  in (4.3) is negative, an increase in aggregate productivity ( $\tilde{r} > 0$ ) decreases the capital intensity of production. The impact of an exogenous reduction of net pollution depends upon the sizes of  $\beta$  and  $\lambda_K + \lambda_Z$ , as the direct impact of a positive shock in A is a lower capital intensity of production, while the indirect effect, through an improvement of environmental quality, increases capital intensity of production.

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<sup>3</sup> Note that  $N_E < 0$  in a neighbourhood of the stable equilibrium of N. Furthermore,  $N_{EE} < 0$ ,  $N_{pp} < 0$ ,  $N_p < 0$  and  $N_{EP} > 0$ .

Finally, we derive in appendix C the impact of an exogenous reduction in pollution on the rate of return. The resulting expression, however, does not provide much insight in the effects of such shocks. To provide some further insight we give a numerical example in the next section.

## 5. Numerical Example

In this section a numerical example of the model is provided. **This** example shows first of all that there are indeed specifications, satisfying the derived restrictions with respect to the pollution elasticities, for which an optimal balanced growth solution exists. Furthermore, it will illustrate the mechanisms at work in the model, by analysing the influence of changing parameter values on the long-term steady state growth rates for specified production, consumption and regeneration functions.

We assume that both the production of the final good  $Y$  and the production of knowledge are given by a Cobb Douglas function and moreover, for the sake of simplicity, that  $Y$  exhibits constant returns to scale with **respect** to physical capital and the effective use of environmental resources, i.e.,  $Y = \Lambda_1 \cdot K_Y^\alpha Z_Y^{1-\alpha} E^\beta$  and  $H = \Lambda_2 \cdot K_H^\epsilon Z_H^{\delta}$ , where  $\Lambda_1$  and  $\Lambda_2$  are technology parameters. The utility function is given by:  $u(c, E) = \ln(c \cdot E^\phi)$  ( $\ln$  denotes the natural logarithm), satisfying the conditions derived in section 3 of a constant intertemporal elasticity of substitution (taken to be equal to one) and an elasticity between consumption and the index of environmental quality of one. Finally, the regeneration function of the natural environment is given by:

$$N(E, P(Q, A)) = -\gamma_1 P^2/E \cdot \gamma_2 (E - \bar{E})^2 + \Gamma,$$

with  $P = Q/A^{(1-\alpha-\delta)}$  and where  $\bar{E}$  and  $\Gamma$  are constants. For each level of net pollution there is a stable level of the quality of the natural environment, i.e. a level such that  $N=0$  and the quality of the natural environment remains the same over time.

**In** the model 10 parameters **characterise** the relationship between economic activity and the environment. These are:

- $\alpha$ : the weight of capital in the production function; hence  $(1-\alpha)$  is the weight of environmental resources;
- $\beta$ : the weight of environmental quality in the production function;
- $\epsilon$ : the weight of capital in the knowledge function;

$\delta$ :	the weight of environmental quality in the knowledge function;
$\phi$ :	the weight of environmental quality in the social welfare function;
$\gamma_1, \gamma_2$ :	impact-parameters of the regeneration function;
$\theta$ :	the discount rate as indicator of time preference;
$\Lambda_1, \Lambda_2$ :	the efficiency constants of the production function representing the state of technology.

By way of sensitivity analysis Table 1 gives the long-run solutions for some specific parameter values. In the benchmark the parameter values are given by:  $\alpha=0.7$ ,  $\beta=0.001$ ,  $\epsilon=0.4$ ,  $\delta=0.4$ ,  $\phi=1$ ,  $\gamma_1=1$ ,  $\gamma_2=0.5$ ,  $\theta=0.05$ ,  $\Lambda_1=0.2$  and  $\Lambda_2=0.2$ . In the first row of Table 1 the long run solution of the model in the benchmark is given. In each of the other rows the long run solution of the model is given if the value of one of the parameters is changed with respect to the benchmark. So,  $\phi=2$  means that only the value of  $\phi$  is changed with respect to the benchmark, while the values of the other parameters are the same as in the benchmark.

We see from Table 1 that increased environmental concern (increasing  $\phi$ ), lowers the long term steady state growth rate: abatement activities are increased, while the capital intensity of production increases and the environmental quality stabilizes at a higher level than in the benchmark. It is, however, theoretically possible to have increasing growth rates in this model under increasing environmental concern. When the impact of the natural environment on production is very large, this can dominate the adverse effect of the negative impact on the absorption capacity of the environment and in this case growth rates can increase if environmental concern increases.

Table 1. Balanced growth solutions

	$g$	$Y/K$	$P$	$E$	$u$	$v$
benchmark	4.74%	0.13	1.37	1.68	0.92	0.83
$\phi=2$	4.24%	0.12	1.17	1.78	0.92	0.83
$\alpha=0.75$	5.16%	0.13	1.34	1.68	0.93	0.82
$\beta=0.005$	4.75%	0.13	1.36	1.66	0.92	0.83
$\epsilon=0.35$	4.52%	0.13	1.36	1.66	0.93	0.84
$\gamma_1=2$	4.67%	0.13	1.34	1.43	0.92	0.83
$\Lambda_1=0.25$	6.45%	0.15	1.40	1.63	0.90	0.80
$\theta=0.04$	5.77%	0.13	1.41	1.62	0.90	0.79
• $\phi=1, \alpha=0.7, \beta=0.001, \epsilon=0.4, \delta=0.4, \gamma_1=0.5, \gamma_2=1, \Lambda_1=0.2, \Lambda_2=0.2, \theta=0.05, \bar{E}=\Gamma=1$						

Increasing the impact of the natural environment on production (increasing  $\beta$ ), does not influence the results very much. An increasing negative impact of pollution on the regenerative capacities of the natural environment (increasing  $\gamma_1$ ), gives an increasing intensity of abatement activities and a slightly

lower long **run** growth rate. Nevertheless environmental quality worsens. A decreased rate of time preference increases long term growth, while more resources are allocated to the knowledge sector (decreasing  $u$  and  $v$ ). Furthermore, environmental quality slightly worsens. Similar effects result from an exogenous increase in  $A$ .

All in all, the sensitivity analysis illustrates that changes in the production technology have a major impact on the pace of balanced economic growth, but do not affect environmental quality very much. A change in the regenerative capacity, given the present specification of the regeneration function, on the other hand does not influence the long term growth rate very much, but does influence environmental quality. However, more knowledge is needed in particular on the proper specification and parameter values of the regeneration function. The highest level of environmental quality with balanced growth is obtained in case of green preferences. According to our numerical example economic growth is **0.5%-points** lower as compared to the benchmark growth path when the weight of environmental quality in the welfare function is duplicated.

## 6. Conclusions

In this paper we have analysed balanced growth paths which are ecological sustainable, in a two sector growth model which incorporates environmental issues. We have derived the conditions under which balanced growth is both feasible and optimal. It appears that these conditions not only relate to production and substitution elasticities, but also to the regeneration function. The restrictions with respect to the regeneration function are closely related to the inclusion of the non-extractive use of the environment in production and to the concept of sustainable development. Introducing sustainable development in the context of balanced growth models implies that not only all traditional economic variables should grow at a constant rate, but also that the quality of the natural environment should grow at a constant (possibly zero) rate, which is a strong condition to impose. The growth rate of the quality of the natural environment influences the growth rate of output, since the quality of the natural environment is also a factor of production. Consequently, when we would drop the assumption of a constant growth rate of the quality of the natural environment, we would also have to drop the assumption of a constant growth rate with respect to one or more other (economic) variables. So, given the concept of balanced growth the restrictions with respect to the regeneration function are unavoidable.



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## Appendix A. The Regeneration Function

The regeneration function is given by:

$$\dot{E} = N(E, P(Q, A)) \quad (\text{A.1})$$

$N$  is assumed to be twice continuously differentiable. Furthermore, it is assumed that  $N_P < 0$ ,  $N_{EP} > 0$ ,  $N_E < 0$  in the neighbourhood of a stable equilibrium  $E(P)$ , with  $N(\bar{E}(P), P) = 0$ ,  $P_Q > 0$ ,  $P_A < 0$  and  $P_{QA} < 0$ .

### Theorem

If  $g_E > 0$  and  $d(g_E) = 0$  then  $d(g_P) < 0$ .

### Proof

$$d(g_E) = d\left(\frac{N}{E}\right) = \frac{EdN - NdE}{E^2} = 0$$

So,

$$EdN = EN_E dE + EN_P dP = NdE \quad (\text{A.2})$$

From (A.2) we derive:

$$g_P = \frac{dP}{P} = \frac{NdE - EN_E dE}{EN_P P} = g_E \left( \frac{N - N_E E}{N_P P} \right) \quad (\text{A.3})$$

Taking time derivative of (A.3) yields:

$$\begin{aligned} d(g_P) &= g_E \frac{N_P P (N_E dE + N_P dP - N_{EP} dP \cdot E - N_{EE} dE \cdot E - N_E dE)}{(N_P P)^2} \\ &= g_E \frac{(N - N_E E)(N_{PP} dP \cdot P + N_{EP} dE \cdot P + N_P dP)}{(N_P P)^2} \end{aligned} \quad (\text{A.4})$$

Substituting (A.3) in (A.4) gives:

$$d(g_P) = \frac{g_E}{N_P P} (N_P dP - N_{EP} dP \cdot E - N_{EE} dE \cdot E) - \frac{g_P}{N_P P} (N_{PP} dP \cdot P + N_{EP} dE \cdot P + N_P dP) \quad (\text{A.5})$$

Using the assumptions made with respect to the regeneration function and using that  $g_E > 0$  requires  $g_P < 0$ , it is straightforward from (AS) that  $d(g_P) < 0$ .

□

Appendix B. Derivation of optimum conditions

The present value Hamiltonian of the maximization problem is given by:

$$H = e^{-\rho t} [ U(c,E) + \lambda.(Y(K_Y,Z_Y,E)-C-A) + \mu.N(E,P(Q,A)) + \kappa.H(K_H,Z_H) ] \quad (\text{B.1})$$

From (B. 1), using the maximum principle, we derive the following optimum conditions:

$$U_c = \lambda \quad (\text{B.2})$$

$$\dot{\lambda} = \lambda \left( \mu \frac{\partial N}{\partial P} \frac{\partial P}{\partial A} \right) \quad (\text{B.3})$$

$$\lambda \left( \frac{\partial Y}{\partial Z_Y} v h + \kappa \frac{\partial H}{\partial Z_H} (1-v) h \right) = -\mu \frac{\partial N}{\partial P} \frac{\partial P}{\partial Q} \quad (\text{B.4})$$

$$\lambda \frac{\partial Y}{\partial K_Y} K = \kappa \frac{\partial H}{\partial K_H} K \quad (\text{B.5})$$

$$\lambda \frac{\partial Y}{\partial Z_Y} h Q = \kappa \frac{\partial H}{\partial Z_H} h Q \quad (\text{B.6})$$

$$\dot{\lambda} = \theta \lambda - \lambda \frac{\partial Y}{\partial K_Y} u - \kappa \frac{\partial H}{\partial K_H} (1-u) \quad (\text{B.7})$$

$$\dot{\mu} = \theta \mu - U_E - \lambda \frac{\partial Y}{\partial E} - \mu \frac{\partial N}{\partial E} \quad (\text{B.8})$$

$$\dot{\kappa} = \theta \kappa - \kappa \frac{\partial Y}{\partial Z_Y} v Q - \nu \frac{\partial H}{\partial Z_H} (1-v) Q \quad (\text{B.9})$$

where  $\lambda$ ,  $\mu$  and  $\kappa$  denote respectively the shadow price of physical capital, of natural resources and of knowledge.

Taking derivatives at both sides in (B.2), dividing by  $U_c$  and substituting (B.2) we get:

$$\rho \frac{\dot{c}}{c} = \frac{U_{cE} \dot{E}}{U_c} - \frac{\dot{\lambda}}{\lambda} \quad (\text{B.10})$$

where  $\rho = -c \cdot \frac{U_{cc}}{U_c}$

Furthermore, dividing both sides of (B.7) by  $\lambda$  and substituting (B.5) yields:

$$\frac{\dot{\lambda}}{\lambda} = \theta - \frac{\partial Y}{\partial K_Y} \quad (\text{B.11})$$

Substituting (B.11) in (B.10) gives (3.11).

Dividing both sides of (B.8) by  $\mu$ , substituting (B.2) and (B.3) and subtracting (B.11) gives the first equality of (3.12). Dividing both sides of (B.9) by  $\kappa$  substituting (B.6) and subtracting (B. 11) gives the second equality of (3.12).

Dividing (B.5) by (B.6) gives (3.13). Finally, dividing (B.4) by (B.3), substituting (B.6) yields (3.14).

Loglinearizing (B.3) yields:

$$\frac{\dot{\lambda}}{\lambda} - \frac{\dot{\mu}}{\mu} = \frac{dN_A}{N_A} = \frac{N_{AA}dA + N_{AQ}dQ}{N_A} \quad (\text{B. 12})$$

where

$$N_A = N_P P_A = N_P \lambda_A \frac{P}{A} \quad (\text{B. 13})$$

so.

$$\begin{aligned} N_{AA} &= \frac{\partial N_A}{\partial A} = N_{PP} P_A \lambda_A \frac{P}{A} + N_P \lambda_A \left( \frac{AP_A - P}{A^2} \right) \\ &= \lambda_A^2 \frac{P}{A^2} (N_{PP} P + N_P) = \frac{N_A}{A} \end{aligned} \quad (\text{B. 14})$$

Analogously,

$$N_{AQ} = \lambda_A \lambda_Q \frac{P}{AQ} (N_{PP} P + N_P) \quad (\text{B. 15})$$

Substituting (B. 13), (B. 14) and (B. 15) in (B. 12) yields:

$$\frac{\dot{\lambda}}{\lambda} - \frac{\dot{\mu}}{\mu} = \frac{dN_A}{N_A} = \lambda_A \left[ \frac{N_{PP} P + N_P}{N_P} \right] g_A - g_A + \lambda_Q \left[ \frac{N_{PP} P + N_P}{N_P} \right] g_Q \quad (\text{B. 16})$$

Substituting (3.6) into (B. 16) yields:

$$\frac{\dot{\lambda}}{\lambda} - \frac{\dot{\mu}}{\mu} = -g_A = -g \quad (\text{B. 17})$$

Substituting (B. 17) into the first part of (3.12) yields (3.17).

**Loglinearizing** (B.5) yields:

$$\frac{\dot{\kappa}}{\kappa} - \frac{\dot{\lambda}}{\lambda} = d\log\left(\frac{\partial Y}{\partial K_Y}\right) - d\log\left(\frac{\partial H}{\partial K_H}\right) \quad (\text{B.18})$$

Using the (necessary) condition that the production elasticities are constant we derive from (B. 18):

$$\frac{\dot{\kappa}}{\kappa} - \frac{\dot{\lambda}}{\lambda} = \frac{dY}{Y} - \frac{dK_Y}{K_Y} - \frac{dH}{H} + \frac{dK_H}{K_H} \quad (\text{B.19})$$

Substituting (3.1) and (3.9) in (B.19) gives

$$\frac{\dot{\kappa}}{\kappa} - \frac{\dot{\lambda}}{\lambda} = \left[ 1 - \nu_K - \nu_Z \cdot \frac{(1-\lambda_K)}{\lambda_Z} \right] g \quad (\text{B.20})$$

Substituting (B.20) into the second part of (3.12) yields (3.18).

Appendix C. Derivation of comparative static results

Loglinearizing (4.1) gives:

$$\frac{r\tilde{r} - g\tilde{g} - N_E\tilde{N}_E}{(r - g - N_E)} = \tilde{N}_A + \frac{L \cdot U_E/U_C \cdot d\log(L \cdot \frac{U_E}{U_C}) + Y_E \cdot d\log Y_E}{L \cdot U_E/U_C + Y_E} \quad (\text{C.1})$$

where

$$N_A = \frac{\partial N}{\partial P} \frac{\partial P}{\partial A} \quad (\text{C.2})$$

A tilde (-) denotes a relative change. Note, that  $g_x = dx/x = \tilde{x}$ .

Loglinearization of (3.13) gives:

$$\tilde{g} = \frac{r}{g} \tilde{r} \quad (\text{C.3})$$

hence

$$r\tilde{r} - g\tilde{g} = 0 \quad (\text{C.4})$$

Furthermore,

$$N_E\tilde{N}_E = dN_E = N_{EE}dE + N_{EP}dP = N_{EE}E\tilde{E} + N_{EP}P\tilde{P} \quad (\text{C.5})$$

and from (B. 16) we have

$$\tilde{N}_A = \frac{dN_A}{N_A} = \lambda_A \left( \frac{N_{PP}P}{N_P} \tilde{P} + \lambda_Q \left| \frac{N_{PP}P + N_P}{N_P} \right| \tilde{Q} \right) \quad (\text{C.6})$$

Finally,

$$\tilde{P} = \lambda_A \tilde{A} + \lambda_Q \tilde{Q} \quad (\text{C.7})$$

Substituting (3.13), (C.4), (C.5) and (C.6) in (C.1) and using the fact that the elasticity of substitution in utility is equal to one and the elasticity of Y with respect to E is constant, yields:

$$\begin{aligned}
-N_{EE}E\tilde{E} - N_{EP}P\tilde{P} &= (\theta - N_E) \left[ \lambda_A \tilde{A} \left( \frac{N_{PP}P + N_P}{N_P} \right) - \tilde{A} + \lambda_Q \tilde{Q} \left( \frac{N_{PP}P + N_P}{N_P} \right) \right] \\
&+ (\theta - N_E) \frac{L \frac{U_E}{U_C} (\tilde{\phi} + \tilde{C} - \tilde{E}) + Y_E (\tilde{Y} - \tilde{E})}{L \frac{U_E}{U_C} + Y_E}
\end{aligned} \tag{C.8}$$

Rewriting (C.8), using (4.1), (C.7) and the fact that  $\tilde{Y} = \tilde{C} = \tilde{A}$  yields:

$$L \frac{U_E}{U_C} N_A \tilde{\phi} = (\theta - N_E - N_{EE}E) \tilde{E} - \left[ (\theta - N_E) \left( \frac{N_{PP}P + N_P}{N_P} \right) + N_{EP}P \right] \tilde{P} \tag{C.9}$$

which is equal to (4.2).

Using the definition of the elasticities of substitution and noting that:

$$\begin{aligned}
\tilde{K}_Y &= \tilde{K} + \tilde{u} \\
\tilde{K}_H &= \tilde{K} - \frac{u}{1-u} \tilde{u} \\
\tilde{Z}_Y &= \tilde{Z} + \tilde{v} \\
\tilde{Z}_H &= \tilde{Z} - \frac{v}{1-v} \tilde{v}
\end{aligned} \tag{C.10}$$

we derive straightforward from (3.13) that:

$$\frac{1}{\sigma_Y} [\tilde{v} + \tilde{Z} - \tilde{u} - \tilde{K}] = \frac{1}{\sigma_H} \left[ \tilde{Z} - \frac{v}{(1-v)} \tilde{v} + \frac{u}{(1-u)} \tilde{u} - \tilde{K} \right] \tag{C.11}$$

Loglinearization of (3.14) yields:

$$\tilde{\lambda}_Z + \tilde{Y} - \tilde{Z} - \tilde{v} + \tilde{h} = \tilde{\lambda}_Q + \tilde{P} - \tilde{Q} - \tilde{\lambda}_A - \tilde{P} + \tilde{A}$$

So,

$$\tilde{v} = \tilde{\lambda}_Z + \tilde{A}_1 + \tilde{\lambda}_K (\tilde{K} + \tilde{u}) + \tilde{\lambda}_Z (\tilde{Z} + \tilde{v}) + \beta \tilde{E} - \tilde{A} + \tilde{\lambda}_A - \tilde{\lambda}_Q \tag{C.12}$$

Noting that, on a balanced growth path, the production elasticities should be constant over time and hence,

using (3.8), also the pollution elasticities should be constant over time, (C.12) gives:

$$\tilde{v} = \tilde{\Lambda}_1 + \lambda_K(\tilde{K} + \tilde{u}) + \lambda_Z(\tilde{Z} + \tilde{v}) + \beta\tilde{E} - \tilde{A} \quad (\text{C. 13})$$

Loglinearization of (3.16) gives:

$$\tilde{g} = \frac{r}{g} \tilde{r} \quad (\text{C. 14})$$

Loglinearization of (3.18) gives:

$$\begin{aligned} \tilde{r} = & \frac{r}{r} \left[ 1 - \nu_K - \nu_Z \frac{(1-\lambda_K)}{\lambda_Z} \right] g \left[ \tilde{v}_Z + \tilde{\Lambda}_2 + \nu_K \left( \tilde{K} - \frac{u}{1-u} \tilde{u} \right) + (\nu_Z - 1) \left( \tilde{Z} - \frac{v}{(1-v)} \tilde{v} \right) + \tilde{Q}_I \right. \\ & \left. + \frac{\left[ 1 - \nu_K - \nu_Z \frac{(1-\lambda_K)}{\lambda_Z} \right] g}{r} \tilde{g} \right] \end{aligned} \quad (\text{C. 15})$$

Using (3.16), (C.14) and the fact that the production elasticities are assumed to be constant over time, (C. 15) yields:

$$\tilde{r} = \frac{\left( \nu_K + \nu_Z \frac{(1-\lambda_K)}{\lambda_Z} \right) r + \left( 1 - \nu_K - \nu_Z \frac{(1-\lambda_K)}{\lambda_Z} \right) \beta}{\left( \nu_K + \nu_Z \frac{(1-\lambda_K)}{\lambda_Z} \right) r} \left[ \tilde{\Lambda}_2 + \nu_K \left( \tilde{K} - \frac{u}{1-u} \tilde{u} \right) + (\nu_Z - 1) \left( \tilde{Z} - \frac{v}{(1-v)} \tilde{v} \right) + \tilde{Q}_I \right] \quad (\text{C. 16})$$

Loglinearization of (3.19), using the fact that the production elasticities are constant over time yields:

$$\tilde{r} = \tilde{Y} - \tilde{K}_Y = \tilde{\Lambda}_1 + (\lambda_K - 1)(\tilde{K} + \tilde{u}) + \lambda_Z(\tilde{Z} + \tilde{v}) + \beta\tilde{E} \quad (\text{C. 17})$$

From (3.9) we have:

$$\tilde{g}_h = \tilde{g} \quad (\text{C. 18})$$

Furthermore,

$$\tilde{g}_h = \tilde{H} - \tilde{h} = \tilde{\Lambda}_2 + \nu_K \tilde{K}_H + \nu_Z \tilde{Z}_H - \tilde{h} \quad (\text{C. 19})$$

Substituting (C. 19) in (C. 18) and using (C. 10) we get:



$$\tilde{g} = \tilde{\Lambda}_2 + \nu_K \left( \tilde{K} - \frac{u}{(1-u)} \tilde{u} \right) + (\nu_Z - 1) \left( \tilde{Z} - \frac{v}{(1-v)} \tilde{v} \right) + \frac{v}{1-v} \tilde{v} + \tilde{Q} \quad (\text{C.20})$$

Equations (C.11), (C. 13), (C. 14), (C. 16), (C.17) and (C.20) provide 6 equations in 6 unknowns  $\tilde{g}$ ,  $\tilde{r}$ ,  $\tilde{u}$ ,  $\tilde{v}$ ,  $\tilde{K}$  and  $\tilde{Z}$ , which can be solved as functions of exogenous shocks in  $\tilde{A}$  and/or  $\tilde{Q}$ .

Substituting (C. 14) and (C. 16) in (C.20) yields:

$$\left[ \frac{r}{g} - \frac{xr}{xr+(1-x)\theta} \right] \tilde{r} = \frac{v}{(1-v)} \tilde{v} \quad (\text{C.21})$$

where

$$x = \nu_K + \nu_Z \frac{(1-\lambda_K)}{\lambda_Z}$$

Furthermore, substituting (C.21) and (C. 17) in (C. 13) yields:

$$\tilde{K} + \tilde{u} = \left[ \frac{(1-v)}{v} \left[ \frac{r}{g} - \frac{xr}{xr+(1-x)\theta} \right] \tilde{r} \right] + A \quad (\text{C.22})$$

Substituting (C.22) in (C. 17) gives:

$$\tilde{Z} + \tilde{v} = \frac{1}{\lambda_Z} \left[ (1-\lambda_K) \frac{(1-v)}{v} \left[ \frac{r}{g} - \frac{xr}{xr+(1-x)\theta} \right] \tilde{r} + \lambda_K \tilde{r} + (1-\lambda_K) \tilde{A} - \beta \tilde{E} - \tilde{\Lambda}_1 \right] \quad (\text{C.23})$$

Subtracting (C.23) from (C.22), using (C. 11) yields (4.3).

Substituting (4.3) in (C.11) and the result in (C.20) yields:

$$\begin{aligned} \tilde{Z} - \frac{v}{(1-v)} \tilde{v} = & \frac{1}{(\nu_K + \nu_Z - 1)} \left[ \left[ \frac{r}{g} + \nu_K \frac{\sigma_H (\lambda_K + \lambda_Z)}{\sigma_Y \lambda_Z} \right] \tilde{r} - \left[ \nu_K \frac{\sigma_H (\lambda_K + \lambda_Z - 1) (1-v)}{\sigma_Y \lambda_Z} - \frac{v}{(1-v)} \right] \left[ \frac{r}{g} - \frac{xr}{xr+(1-x)\theta} \right] \tilde{r} \right] \\ & \frac{1}{(\nu_K + \nu_Z - 1)} \left[ \nu_K \frac{\sigma_H}{\sigma_Y} \left[ \frac{(\lambda_K + \lambda_Z - 1)}{\lambda_Z} \tilde{A} + \frac{\beta}{\lambda_Z} \tilde{E} + \frac{1}{\lambda_Z} \tilde{\Lambda}_1 \right] + \tilde{Q} + \tilde{\Lambda}_2 \right] \end{aligned}$$

(C.24)

Subtracting (C.24) from (C.23) yields  $\tilde{v}$  as a function of  $\tilde{r}$ ,  $\tilde{\Lambda}_1$ ,  $\tilde{\Lambda}_2$ ,  $\tilde{A}$  and  $\tilde{Q}$ . Using (C.21)  $\tilde{v}$  can be eliminated and we end up with  $\tilde{r}$  as a function of exogenous shocks in  $\tilde{A}$  and  $\tilde{Q}$  and  $\tilde{\Lambda}_1$  and  $\tilde{\Lambda}_2$ . Finally, using (C.14) we can also derive  $\tilde{g}$  as a function of exogenous shocks in  $\tilde{\Lambda}_1$ ,  $\tilde{\Lambda}_2$ ,  $\tilde{A}$  and  $\tilde{Q}$ .