On the existence and computation of an equilibrium in an economy with constant returns to scale production

Kremers, H.; van der Laan, G.; Talman, A.J.J.

1991

Faculteit der Economische Wetenschappen en Econometrie

Serie Research Memoranda

On the Existence and Computation of an Equilibrium in an Economy with Constant Returns to Scale Production

Hans Kremers
Gerard van der Laan
Dolf Talman

Research Memorandum 1991-82
December 1991

vrije Universiteit *amsterdam*

ON THE EXISTENCE AND COMPUTATION OF AN EQUILIBRIUM IN AN ECONOMY WITH CONSTANT RETURNS TO SCALE PRODUCTION

by

Hans Kremers [1)]

Gerard van der Laan [2)]

and

Dolf Talman [1)]

Abstract

In this paper we consider the problem of finding an equilibrium in an economy with nonlinear constant returns to scale production activities. To find an equilibrium we propose an adjustment process in which the prices of the commodities and the activity levels of production adjust simultaneously. The process starts at a price vector at which each production activity has non positive profit. We show that the process follows a path which connects the starting point with an equilibrium of the economy. From this it follows that the existence of a price vector at which each production activity has non positive profit implies the existence of an equilibrium. The equilibrium can be computed by using a simplicial algorithm or by solving a sequence of Linear Stationary Point Problems.

This research is part of the VF-program "Competition and Cooperation".

[1)] Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.

[2)] Department of Econometrics, Free University, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands.

ON THE EXISTENCE AND COMPUTATION OF AN EQUILIBRIUM IN AN ECONOMY WITH CONSTANT RETURNS TO SCALE PRODUCTION

1. Introduction.

In this paper we consider the equilibrium problem in an economy with nonlinear constant returns to scale production activities. In Van den Elzen, Van der Laan and Talman [4] an adjustment process has been introduced to find an equilibrium in an economy with linear production activities. This adjustment process starts at an arbitrarily chosen price vector in the unit price simplex, such that all firms have negative profits per unit production. Therefore the activity levels of all firms are set initially equal to zero. Along the path followed by the process prices and activity levels are adjusted according to the sign vector of the net excess demand and keeping the levels of the activities equal to zero as long as the profits are negative. It is shown that the adjustment process converges to a pair of equilibrium prices and activity levels. In Kremers, Van der Laan and Talman [6] a simplicial algorithm has been described to follow the path of the adjustment process. In Kremers and Talman [7] an iterative algorithm to solve the equilibrium problem has been given. This algorithm solves the problem through solving a sequence of Linear Stationary Point Problems on the set of feasible prices.

In this paper we generalize the adjustment process of Van den Elzen, Van der Laan and Talman [4] to find an equilibrium in an economy with nonlinear constant returns to scale production. The production side of such an economy can be described by an activity matrix whose entries depend continuously on the prices. A sufficient condition for the existence of an equilibrium is that at any price vector there can be no production without input. Under this condition there exists a price vector at which all production technologies have negative profits per unit output. Starting at such a price vector we propose an adjustment process which generically converges to an equilibrium of the economy. This process increases initially the prices of the commodities wit hpositive excess demand and decreases the prices of the commodities with negative excess demand. Firms only produce when they mke zero profits.

In Section 2 we describe the model and we state conditions under which there exists a price vector for which all production technologies have negative profits. In Section 3 we describe the adjustment process and prove that the process converges to an equilibrium. The Sections 4 and 5 are concerned with the computation of an equilibrium.

2. The model.

We consider an economy with a finite number of consumers, m firms having constant returns to scale production technologies, indexed by $i = 1,\dots,m$, and n+1 commodities, indexed by $j = 1,\dots,n+1$.

The consumers are assumed to maximize their utilities under their budget constraints. Each consumer is endowed with an (n+1)-dimensional strictly positive vector of commodities. At some price vector $p \in R^{n+1}_{+}\backslash\{0\}$, the budget of a consumer is the value of his endowment. Let d(p) be the aggregate demand of the consumers at price vector p and let $z(p)$ be the excess demand, i.e. $z(p)$ is the aggregate demand $d(p)$ minus the total initial endowments. Under standard assumptions on the utility functions of the consumers, utility maximization yields an excess demand function z being homogeneous of degree zero in the prices, i.e., $z(\lambda p) = z(p)$ for all $\lambda > 0$, and which satisfies Walras' law, i.e., for all p, $p^T z(p) = 0$. Moreover, z is a continuous function of p on R^{n+1}_{++}.

Concerning the production technologies we assume that each firm produces just one output. Let $\pi: \{1,...,m\} \to \{1,...,n+1\}$ be a function which assigns the output commodity $\pi(i) \in \{1,...,n+1\}$ to the production technology $i \in \{1,...,m\}$. For $a \in R^{n+1}$, let a^{-h} be the n-vector $(a_1,...,a_{h-1},a_{h+1},...,a_{n+1})^T$. Then firm i produces commodity $\pi(i)$ according to a continuous production function $f^i: R^n_+ \to R_+$, i.e., if $y_{\pi(i)} = f^i(y^{-\pi(i)})$, then $y_{\pi(i)}$ is the amount of commodity $\pi(i)$ produced by firm i when $y^{-\pi(i)}$ is the n-vector of inputs of the other commodities $j \neq \pi(i)$. Observe that for some given commodity j, the set $\{i \mid \pi(i) = j\}$ might be empty or contain more than one element. We assume that there is constant returns to scale production.

<u>Assumption P.</u>
For all $i = 1,....,m$, $f^i(\lambda y^{-\pi(i)}) = \lambda f^i(y^{-\pi(i)})$ for all $\lambda > 0$.

From production theory (e.g. see Varian [13]) we know that cost-minimizing behaviour of producer i yields a cost function $c^i: R^n_+ \to R_+$ per unit of output of commodity $\pi(i)$ satisfying

(i) c^i is non decreasing in p_j, $j \neq \pi(i)$,
(ii) c^i is homogeneous of degree 1 in $p^{-\pi(i)}$,
(iii) c^i is concave in p_j, $j \neq \pi(i)$,
(iv) c^i is continuous in $p^{-\pi(i)}$ for $p^{-\pi(i)} \in R^n_{++}$.

Furthermore, we assume differentiability of the cost functions

<u>Assumption C.</u>
For all $i = 1,....,m$, c^i is twice-differentiable in $p^{-\pi(i)} \in R^n_{++}$.

With Euler's law it follows from (ii) that for every $p^{-\pi(i)} \in R^n_{++}$

(1) $c^i(p^{-\pi(i)}) = \Sigma_{j\neq\pi(i)} p_j \delta c^i/\delta p_j$.

Since production goes to infinity if profits per unit of output are positive, we have that in equilibrium prices have to satisfy the nonpositive profit conditions, i.e.,

(2) $p_{\pi(i)} \leq c^i(p^{-\pi(i)})$, $i = 1,...,m$.

Together with (1) condition (2) can be written as

$$p^T a^i(p) \leq 0, \ i = 1,...,m,$$

where $a^i_{\pi(i)}(p) = 1$ and $a^i_j(p) = -\delta c^i/\delta p_j$, $j \neq \pi(i)$. From Hotelling's Lemma we know that $y^i_j(p) = \delta c^i/\delta p_j$, $j \neq \pi(i)$, is the cost-minimizing amount of input of good j needed to produce one unit of output of commodity $\pi(i)$. So, $a^i_j(p)$, $j \neq \pi(i)$, is the negative supply of commodity j by producer i per unit of output of commodity $\pi(i)$ at prices p. Following others (e.g. see Morishima [11]) we have now written the nonpositive profit condition as a generalization of the nonpositive profit condition $p^\top a^i \leq 0$ for a linear production technology with activity vector a^i. The only difference is that the the amount of inputs of goods $j \neq \pi(i)$ per unit of output of good $\pi(i)$ depend on p. Observe that according to property (ii) of the cost functions, the input $a^i_j(p)$ is homogeneous of degree zero in p. This formulation is well-known in AGE (Applied General Equilibrium) models, e.g. see Mathiesen [10] or Fischer, Frohberg, Keyzer and Parikh [5].

Summarizing we have that for positive prices the economy is characterized by the excess demand function z and the input-output functions a^i, $i = 1,...,m$, with for all $p \in R^{n+1}_+$, $p^\top z(p) = 0$ and $p^\top a^i(p)$ the profit of firm i per unit production of commodity $\pi(i)$, $i = 1,....,m$. Moreover, z and a^i, $i = 1,....,m$, are homogeneous in p of degree zero. To avoid technical discussions, in the following we also assume that both z and a^i, $i = 1,...,m$, are continuous and differentiable on R^{n+1}_+.

Let x be a nonnegative m-vector $(x_1,...,x_m)^\top$ of production levels, i.e., at production level x_i, producer i produces x_i units of output of commodity $\pi(i)$. Hence at price vector p and production level x_i the input-output vector or supply vector of producer i is $x_i a^i(p)$. With $A(p)$ the $(n+1) \times m$ matrix $[a^1(p),...,a^m(p)]$, the net supply of the production side at price vector p and production level vector x is $A(p)x$. Now an equilibrium for this economy is a price vector p^* and a production level vector x^*, such that the excess demand of the consumption sector is less than or equal to the net supply of the production sector and that each production technology satisfies the nonpositive profit condition. Let the net excess demand $f\colon R^{n+1}_+\backslash\{0\} \times R^m_+ \to R^{n+1}$ be defined as the excess demand of the consumption sector at p minus the the net supply of the production sector at (p,x), i.e., $f(p,x) = z(p) - A(p)x$.

<u>Definition 2.1.</u> A pair $(p^*, x^*) \in R^{n+1}_+\backslash\{0\} \times R^m_+$ is an equilibrium if

1) $f(p^*,x^*) \leq 0$,

2) $p^{*\top}A(p^*) \leq 0$.

From 1) it follows that at an equilibrium $p^{*\top}f(p^*,x^*) = p^{*\top}z(p^*) - p^{*\top}A(p^*)x^* = -p^{*\top}A(p^*)x^* \leq 0$, whereas 2) implies that $p^{*\top}A(p^*)x^* \leq 0$. Hence $p^{*\top}A(p^*)x^* = 0$, so that we have complementarity between x^* and $p^{*\top}A(p^*)$, i.e., $x^*_i = 0$ if $p^{*\top}a^i(p^*) < 0$ and $p^{*\top}a^i(p^*) = 0$ if $x^*_i > 0$. Furthermore, it follows that $p^{*\top}f(p^*,x^*) = 0$, so that we also have complementarity between p^* and $f(p^*,x^*)$. In equilibrium commodities can only be in excess demand if the corresponding prices are zero.

Since both z and a^i, $i = 1,...,m$, are homogeneous of degree zero in the prices we have that if (p^*,x^*) is an equilibrium pair, then also $(\lambda p^*,x^*)$ for any $\lambda > 0$ is an equilibrium pair. Hence we can normalize the price vectors to the n-dimensional unit simplex

$$S^n = \{ p \in R_+^{n+1} \mid \Sigma_j p_j = 1 \}.$$

From condition 2) of Definition 2.1 it follows that the set of feasible prices in S^n is the set

$$T^n = \{ p \in S^n \mid p^T A(p) \leq 0 \}.$$

The concavity of the cost functions implies that T^n is a convex subset of S^n. Since T^n is the set of prices satisfying the nonpositive profit condition, a necessary condition for the existence of an equilibrium is that the set T^n is not empty. Therefore we make the following no free production assumption.

Assumption F.
For any $p \in S^n$, $A(p)x \geq 0$ and $x \geq 0$ implies that $x = 0$.

Assumption F says that for any price vector there can not be production without input.

Theorem 2.2. The set T^n is not empty.

Proof. Let μ: $S^n \to S^n$ be a correspondence defined by

$$\mu(p) = \{ q \in S^n \mid q^T A(p) \leq 0 \},\ p \in S^n.$$

Assumption F says that $A(p)x = 0$ if $A(p)x \geq 0$ and $x \geq 0$. According to Farkas' lemma, this implies that $\mu(p)$ is not empty. For given p, $A(p)$ is a matrix of fixed coefficients. Hence, $\mu(p)$ is a subset of S^n defined by linear constraints and so $\mu(p)$ is convex. Furthermore, $\mu(p)$ is closed and hence also compact. Finally, since the the functions a^i, $i = 1,...,m$, are continuous in p, it follows that μ is upper semi-continuous in p. Hence μ satisfies the conditions of Kakutani's fixed point theorem. From this theorem we have that there exists a fixed point $p^* \in \mu(p^*) = \{q \in S^n | q^T A(p^*) \leq 0\}$. Hence $p^* \in T^n$. This proves that T^n is not empty.

The next lemma shows that assumption F is not only sufficient to prove that T^n is not empty, but also that the relative interior of T^n, defined by

$$\mathrm{int}(T^n) = \{ p \in S^n \mid p_j > 0,\ j = 1,...,n+1 \text{ and } p^T a^i(p) < 0,\ i = 1,...,m \}$$

has dimension n and hence $\mathrm{int}(T^n)$ is not empty. We will use this lemma to prove the existence of an equilibrium by defining a path which connects an arbitrarily chosen price vector in $\mathrm{int}(T^n)$ with an equilibrium price vector. Let $\mathrm{int}(\mu(q)) = \{p \in S^n | p_j > 0,\ j = 1,...,n+1$ and $p^T a^i(q) < 0,\ i = 1,...,m\}$ denote the relative interior of $\mu(q)$.

Lemma 2.3. $\mathrm{Int}(T^n)$ has dimension n.

Proof. Let p^* be a fixed point of μ. According to Farkas' lemma the relative interior of $\mu(p^*)$ has dimension n. Now, let p^k, $k = 1,2...$, be any sequence of price vectors in the relative interior of $\mu(p^*)$ converging to p^*. So, for all k, $p^k{}_j > 0$, $j = 1,\dots,n+1$, and $p^{k\top}a^i(p^*) < 0$, $i = 1,\dots,m$. Since a^i is continuous in p, there exists an M such that $p^{k\top}a^i(p^k) < 0$ for $k > M$. Hence p^k, $k > M$, is in the relative interior of T^n. Therefore the dimension of $int(T^n)$ is also n.

Also the next lemma will be used in the the following sessions.

Lemma 2.4. Let v be a point in the relative interior of T^n. Then $v \in int(\mu(q))$ for any $q \in S^n$.

Proof. Since v is in $int(T^n)$, we have that $v_j > 0$ for all j. So, it remains to prove that for any $q \in S^n$, $v^\top a^i(q) < 0$ for $i = 1,\dots,m$. Define the function g^i: $S^n \rightarrow R$ by

$$g^i(p) = p_{\pi(i)} - c^i(p^{-\pi(i)}).$$

Since c^i is concave, it follows that g^i is a convex function of p. Hence, for any q we have

$$g^i(v) \geq g^i(q) + (v-q)^\top Dg^i(q),$$

where $Dg^i(p) = (\delta g^i(p)/\delta p_1,\dots,\delta g^i(p)/\delta p_{n+1})^\top$. By definition we have that $Dg^i(p) = a^i(p)$. Hence

$$g^i(v) \geq g^i(q) + v^\top a^i(q) - q^\top a^i(q)$$

Since $g^i(q) = q^\top a^i(q)$ it follows that $v^\top a^i(q) \leq g^i(v) = v^\top a^i(v) < 0$.

Observe that by the same reasoning we have that $v \in T^n$ implies that $v^\top a^i(q) \leq 0$ for all $q \in S^n$. So, if we linearize the production side of the economy in a point $q \in S^n$ by taking the input-output coefficients $a^i(q)$ at point q as fixed coefficients, then we have that $T^n \subset \{p \in S^n | p^\top A(q) \leq 0\}$. Thus we have the following corollary.

Corollary 2.5. For any $q \in S^n$, the set of feasible prices T^n is a subset of the set of feasible prices of the economy with linear technologies $a^i(q)$, $i = 1,\dots,m$.

Example. Let $n = 2$ and suppose there is one technology given by $a(p) = (1, -\sqrt{(p_3/p_2)}, -\sqrt{(p_2/p_3)})^\top$. Then T^n is given by

$$T^n = \{ p \in S^2 \mid p_1 - 2\sqrt{p_2 p_3} \leq 0 \}.$$

Take $q = (1-2b,b,b)^\top$ for $0 \leq b \leq 1/2$. Then for any b, $a(q) = (1, -1, -1)^\top$ and

$$T^n \subset \{ p \in S^2 \mid p^\top a(q) \leq 0 \} = \{ p \in S^2 \mid p_1 - p_2 - p_3 \leq 0\}.$$

Observe that for any $q = (1-2b,b,b)^\top$, the set $\{p \in S^2 | p_1 - p_2 - p_3 = 0\}$ is tangent on T^n in the point $p = (1/2, 1/4,/1/4)^\top$.

3. An adjustment process.

In Van den Elzen, Van der Laan and Talman [4] an adjustment process is given for an economy with linear production activities represented by (n+1)-vectors a^i, $i = 1,...,m$. In this section we generalize this process for an economy with nonlinear production technologies represented by the functions a^i, $i = 1,...,m$. We assume that Assumption F is satisfied. Then, let v be a point in $\text{int}(T^n)$. Such a point can be found by applying a simplicial fixed point algorithm to find a fixed point of the correspondence μ. Without loss of generality we assume that z(v) does not contain zero components.

For a subset $U \subset \{1,...,m\}$, let A(U) be defined by

$$A(U) = \{ (p,x) \in T^n \times R^m_+ \mid p^T a^i(p) = 0 \text{ for } i \in U,\ x_i = 0 \text{ for } i \notin U\}.$$

Observe that for $(p,x) \in A(U)$, $p^T f(p,x) = p^T[z(p) - A(p)x] = p^T z(p) - \Sigma_i x_i p^T a^i(p) = 0$, i.e., the value of the excess demand z(p) minus the value of the net production A(p)x is equal to zero. So, for positive prices, either $f(p,x) = 0$ and we have an equilibrium, or f(p,x) has both a negative and a positive component. Let the set S of feasible sign vectors be defined as the set of sign vectors in R^{n+1} with at least one component positive and one component negative, i.e.

$$S = \{s \in R^{n+1} \mid s_j \in \{-1,0,1\},\ j = 1,...,n+1, \text{ with } s_j = -1 \text{ for at least one } j \text{ and } s_k = +1 \text{ for at least one } k\}.$$

Given the starting price vector in the relative interior of T^n, we define for $s \in S$ and $U \subset \{1,...,m\}$ the sets A(s,U) by

$$A(s,U) = \{ (p,x) \in A(U) \mid p_j/v_j = \min_k p_k/v_k \text{ when } s_j = -1,\ p_j/v_j = \max_k p_k/v_k \text{ when } s_j = +1\}.$$

This definition says that for a pair $(p,x) \in A(s,U)$, $p_j = \alpha v_j$ if $s_j = -1$, $p_j = \lambda v_j$ if $s_j = +1$ and $\alpha v_j \le p_j \le \lambda v_j$ if $s_j = 0$, with $\alpha = \min_k p_k/v_k$ and $\lambda = \max_k p_k/v_k$.

Now, let |U| be the number of elements of U and let |s| be the number of components of s equal to zero. Then the restrictions $p_j = \alpha v_j$ if $s_j = -1$, $p_j = \lambda v_j$ if $s_j = +1$ put $n-1-|s|$ restrictions on the prices. Moreover, $p^T a^i(p) = 0$ for $i \in U$ puts |U| restrictions on the prices. So, for $(p,x) \in A(s,U)$, the number of restrictions on the prices is $n-1-|s|+|U|$. Since the dimension of S^n is equal to n, $n - (n-1-|s|+|U|) = 1+|s|-|U|$ degrees of freedom are left for the price variables. So, in case $|U| > 1 + |s|$, no degrees of freedom are left and in general there is no price vector satisfying all the conditions, i.e., A(s,U) is empty. On the other hand, for $(p,x) \in A(s,U)$ and hence $(p,x) \in A(U)$, there are |U| degrees of freedom for the x-variables. Hence, if $|U| \le 1 + |s|$ we have that A(s,U) is generically a well-defined manifold of dimension $1+|s|-|U| + |U| = 1+|s|$.

Finally, for $s \in S$ and $U \subset \{1,...,m\}$ we define the set B(s,U) by

$$B(s,U) = \{ (p,x) \in A(s,U) \mid f_j(p,x) \ge 0 \text{ when } s_j = +1,\ f_j(p,x) \le 0 \text{ when } s_j = -1,\ f_j(p,x) = 0 \text{ when } s_j = 0\}.$$

So, $B(s,U)$ is the closure of the set of pairs $(p,x) \in A(s,U)$, such that the componentswise sign of the net excess demand is equal to the sign vector s, i.e., s_j is the sign of $f_j(p,x)$, $j = 1,...,n+1$. Observe that $B(s,U)$ puts $|s|$ restrictions on the $(1+|s|)$-manifold $A(s,U)$. Assuming standard nondegeneracy, regularity and transversality conditions, this implies that the set $B(s,U)$ is either empty or a 1-manifold, i.e., a collection of smooth paths and loops. A path has either two end points, or one end point and goes to infinity, or comes from and goes to infinity. So, a bounded path has two end points. Let B be the union over all $s \in S$ and all $U \subset \{1,...,m\}$ of the sets $B(s,U)$. Using the same arguments as in Van den Elzen, Van der Laan and Talman for the case of linear production, we have that en end point (p,x) of a path in $B(s,U)$ either lies in a different set $B(s',U')$ or satisfies one of the following cases:

a) $(p,x) = (v,0)$

b) $\min_k p_k/v_k = 0$

c) $f_j(p,x) = 0$ and $s_k \in \{-1,0\}$ for all $k \neq j$

d) $f_j(p,x) = 0$ and $s_k \in \{+1,0\}$ for all $k \neq j$.

In case an end point (p,x) of a path in $B(s,U)$ lies in some other $B(s',U')$, then (p,x) is also an end point of a path in $B(s',U')$. We have that either for just one component j, $s'_j \in \{s_j+1,s_j-1\}$, while $s'_k = s_k$ for all $k \neq j$ and $U' = U$, or $s' = s$ and for some $i \in U$, $U' = U\backslash\{i\}$ or for some $i \notin U$, $U' = U \cup \{i\}$. Linking together all paths we get that the set B is generically a collection of piecewise smooth paths and loops. Each end point (p,x) of a path in B satisfies one of the cases a)-d).

Suppose that case a) occurs. Since $(v,0)$ is an end point of a path in B, there exist an s and U, such that $(v,0)$ is an end point of $B(s,U)$. We have assumed that $z(v)$ does not contain zero components and that $v^Ta^i(v) < 0$ for all i. Hence, s must be equal to s^0 with $s^0_j = \text{sign } z_j(v) = \text{sign } f_j(v,0)$, $j = 1,...,n+1$ and U must be the empty set. It follows that $(v,0)$ is an end point of a path in $B(s,U)$ if and only if $s = s^0$ and $U = \emptyset$.

In case b), by definition of the set $A(s,U)$, we must have that $p_j = 0$ for all j with $s_j = -1$. By definition of $B(s,U)$ we have that $f_j(p,x) \leq 0$ for all j with $s_j = -1$. Hence $f_j(p,x) \leq 0$ for all j with $p_j = 0$. Moreover, $p_j > 0$ for all j with $f_j(p,x) > 0$. Since $p^Tf(p,x) = 0$ for all $(p,x) \in A(U)$, it follows that $f_j(p,x) = 0$ for all j with $p_j > 0$. Hence $f(p,x)$ satisfies the conditions of Definition 2.1 and therefore (p,x) is an equilibrium pair.

In case c) we have by definition of the set $B(s,U)$ that $f_k(p,x) \leq 0$ for all $k \neq j$. Hence (p,x) is an equilibrium pair.

In case d) we have that $f_k(p,x) \geq 0$ for all $k \neq j$ and $f_j(p,x) = 0$. Suppose that for some k, $f_k(p,x) > 0$. Then by definition of the set $B(s,U)$ we have that $s_k = +1$. By definition of the set $A(s,U)$ we must have that $p_k/v_k = \max_h p_h/v_h > 0$. This contradicts the fact that $p^Tf(p,x) = 0$. Hence we must have that $f_k(p,x) = 0$ and (p,x) is an equilibrium pair.

Consequently we can conclude that B contains a path with $(v,0)$ in $B(s^0,0)$ as one of its end points. All other end points of a path in B satisfy one of the cases b), c) or d) and therefore is an equilibrium pair. So the path in B starting in $(v,0)$ leads to an equilibrium if this path is bounded.

Lemma 3.1. Each path in B is bounded.

Proof. Suppose that some path in B is unbounded. Then there exists a sequence (p^k,x^k), $k = 1,2,\ldots$, in B with some of the components of (p^k,x^k) going to infinity. Without loss of generality we may assume that for some $s \in S$ and some set U it holds that $(p^k,x^k) \in A(s,U)$ for all k. Since $p^k \in T^n$ and therefore the components of p^k are bounded, we must have that some components of the sequence x^k go to infinity. Moreover, T^n is compact and hence the sequence p^k has a subsequence converging to a point q in T^n. Since $(p^k,x^k) \in A(s,U)$ for all k we have that $q^Ta^i(q) = 0$ for all $i \in U$. Because $(p^k,x^k) \in B(s,U)$ we also have that $f_j(p^k,x^k) \geq 0$ when $s_j = +1$, $f_j(p^k,x^k) \leq 0$ when $s_j = -1$, and $f_j(p,x) = 0$ when $s_j = 0$. Hence there exist numbers $\mu^k_j \geq 0$, for all j with $s_j \neq 0$, such that

$$f(p^k,x^k) - \Sigma_j \mu^k_j s_j e(j) = z(p^k) - \Sigma_{i \in U} x^k_i a^i(p^k) - \Sigma_{\{j|s_j \neq 0\}} \mu^k_j s_j e(j) = 0$$

with $e(j)$ the j-th unit vector in R^{n+1}. Since p^k has a subsequence converging to q, this system of equations can only have a solution for all k if the homogeneous system of linear equations

$$(3) \qquad \Sigma_{i \in U} x_i a^i(q) + \Sigma_{\{j|s_j \neq 0\}} \mu_j s_j e(j) = 0$$

has a nonzero solution $x^*_i \geq 0$ for $i \in U$ and $\mu^*_j \geq 0$ for all j with $s_j \neq 0$. Since $q \in A(s,U)$, there is a number b, $0 \leq b \leq 1$, and a vector $w \in R^{n+1}_+$ with $w_j > 0$ if $s_j = +1$, $w_j \geq 0$ if $s_j = 0$ and $w_j = 0$ if $s_j = -1$, such that $q = av + w$. Since $q^Ta^i(q) = (bv + w)^Ta^i(q) = 0$ for $i \in U$, and according to Lemma 2.4 $v^Ta^i(q) < 0$, it follows that $w^Ta^i(q) > 0$ for $i \in U$. Premultiplying the system (3) with w^T yields

$$(4) \qquad \Sigma_{i \in U} x_i w^T a^i(q) + \Sigma_{\{j|s_j = 1\}} \mu_j w_j = 0.$$

Since $w^Ta^i(q) > 0$ for all $i \in U$ and $w_j > 0$ for all j with $s_j = 1$, (4) can only hold when $x_i = 0$ for all $i \in U$ and $\mu_j = 0$ for all j with $s_j = 1$. Then (3) becomes

$$(5) \qquad \Sigma_{\{j|s_j = -1\}} \mu_j s_j e(j) = 0,$$

which can only hold when $\mu_j = 0$ for all j with $s_j = -1$. Hence, system (3) does not have a nonzero nonnegative solution. This proves that each path in B is bounded.

From Lemma 3.1. we obtain that following the path in B starting at $(v,0)$ we will reach an equilibrium pair (p^*,x^*) being the other end point of the path. This gives a constructive proof of the existence of an equilibrium under condition F.

Theorem 3.2. Under condition F the nonlinear constant returns to scale production economy has an equilibrium.

The path in B starting in (v,0) can be interpreted as an adjustment process. Prices and production levels are adjusted until an equilibrium has been reached. As long as all production technologies have negative profits, the prices are adjusted in the relative interior of T^n according to the sign vector of the excess demand, keeping the production levels equal to zero. If some production technology i has zero profit $p^Ta^i(p)$, then also the level of this activity is adjusted, while prices are adjusted according to the sign vector of the net demand and keeping the profit of activity i equal to zero. For the case of a linear production technology with fixed coefficients a detailed description of this adjustment process can be found in [4].

4. The simplicial approximation of an equilibrium .

In this section we propose a simplicial algorithm to compute an equilibrium of an economy with nonlinear constant returns to scale production. A simplicial algorithm to follow the path in case of linear production has been given in Kremers, Van der Laan and Talman [6]. In this algorithm each piece of a piecewise linear approximation of the path is followed by making a linear programming pivot step in a system of n+m+2 linear equations. The algorithm starts with a price vector $v \in S^n$, such that $v^Ta^i < 0$ for all linear technologies a^i, $i = 1,...m$. As soon as for some i, p^Ta^i becomes equal to zero, a column representing the linear activity a^i is pivoted into the system. This column remains in the system of equations as long as prices are generated satisfying $p^Ta^i = 0$. This makes that the algorithm can not be applied in the case of nonlinear technologies, because the vector of coefficients $a^i(p)$ depends on the prices. The linear programming technique does not allow to adapt all colums representing the technologies $a^i(p)$ with $p^Ta^i(p) = 0$ at each new price vector p generated by the algorithm. Therefore we propose an alternative simplicial algorithm to follow piecewise linearly the path described in the previous section. This algorithm can be seen as a modification of the simplicial algorithm introduced by Doup, Van der Laan and Talman [1] on S^n to find an equilibrium of an economy without production. The algorithm operates with a system of only n+2 linear equations, which is an advantage above the algorithm described in [6].

The algorithm of Doup, Van der Laan and Talman is one of the most efficient variable dimension simplicial algorithms on S^n, originally initiated by Van der Laan and Talman [9]. To solve the zero point problem of an excess demand function z on S^n, the algorithm can start in an arbitrarily chosen interior point of S^n, say v. Given the starting point v, the unit simplex S^n is subdivided in subsets A(s), $s \in S$, defined by

$$A(s) = \{\, p \in S^n \mid p_j/v_j = \min_k p_k/v_k \text{ when } s_j = -1$$
$$p_j/v_j = \max_k p_k/v_k \text{ when } s_j = +1\}.$$

Observe that $A(s) \cap T^n$ is equal to the set of prices in the projection of $A(S,\emptyset)$ on S^n. The subset A(s) has dimension $|s|+1$. The algorithm follows by simplicially approximating a path of prices in A(s) for various sign vectors $s \in S$, such that for any $p \in A(s)$ on the path it holds that $z_j(p) \geq 0$ when $s_j = +1$, $z_j(p) \leq 0$ when $s_j = -1$ and $z_j(p) = 0$ when $s_j =$

0. So the algorithm follows in S^n the path of prices described in the previous section in case $m = 0$ (no production technologies). To do so, the algorithm traces a piecewise linear path in a simplicial subdivision of S^n, which has the property that each $(|s|+1)$-dimensional set $A(s)$ is triangulated into a finite number of $(|s|+1)$-dimensional simplices. An appropiate subdivision having this property is the V-triangulation of S^n, described in Doup and Talman [2].

To adapt the Doup, Van der Laan and Talman algorithm for the case of an economy with nonlinear production technologies, each price vector p in S^n is labelled according to a vector labelling function b: $S^n \to R^{n+2}$.

Labelling Rule. For $p \in S^n$, compute $p^T a^i(p)$, $i = 1,\ldots,m$. Now, let h be the technology which maximizes the profit per unit production. In case there are several maximizing technologies, let h be the technology with the smallest index. So, $p^T a^h(p) \geq p^T a^i(p)$ for all $i = 1,\ldots,m$ and $h \leq k$ for all k with $p^T a^k(p) = p^T a^h(p)$. Then

$$b(p) = [(-a^h(p))^T, 0]^T \text{ when } p^T a^h(p) \geq 0$$

$$b(p) = [z^T(p), 1]^T \text{ when } p^T a^h(p) < 0.$$

Except for the last additional component with values 0 or 1, this labelling has been introduced already by Scarf [12] in order to compute an equilibrium for an economy with linear production activities. Now, let t be equal to $|s|+1$. Then we have the following definition.

Definition 4.1. Let $\sigma(p^1,\ldots,p^{t+1})$ be a t-simplex in the simplicial subdivision of S^n. Then the simplex σ is s-complete if the (n+2)-system of linear equations

$$(6) \qquad \Sigma_k \lambda_k b(p^k) - \Sigma_{\{j|s_j \neq 0\}} \mu_j s_j e(j) = e(n+2)$$

with $e(j)$ the j-th unit vector in R^{n+2}, has a nonnegative solution λ^*_k, $k = 1,\ldots,t+1$, and μ^*_j for each $j = 1,\ldots,n+1$ with $s_j \neq 0$.

Let Z be the the piecewise linear approximation of the excess demand function z with respect to the underlying simplicial subdivision, i.e. for p in a t-simplex $\sigma(p^1,\ldots,p^{t+1})$, $Z(p) = \Sigma_k \alpha_k z(p^k)$ if $p = \Sigma_k \alpha_k p^k$ with $\alpha_k \geq 0$ for all k and $\Sigma_k \alpha_k = 1$. For an s-complete simplex $\sigma(p^1,\ldots,p^{t+1})$ with solution (λ,μ) to the system (6) of linear equations, let U be the subset of $\{1,\ldots,m\}$, such that for each $i \in U$, there exists a (unique) index $k(i) \in \{1,\ldots,t+1\}$ for which $b(p^{k(i)}) = [(-a^i(p))^T, 0]^T$. So for every index $i \in U$, there is a vertex in σ whose label corresponds to technology i. Furthermore, let K be the subset of $\{1,\ldots,t+1\}$, such that vertex p^k, $k \in K$ has label $b(p^k) = [z^T(p^k))^T, 1]^T$. System (6) can now be written as

$$(7) \qquad \Sigma_{k \in K} \lambda_k \begin{pmatrix} z(p^k) \\ 1 \end{pmatrix} - \Sigma_{i \in U} \lambda_{k(i)} \begin{pmatrix} a^i(p^{k(i)}) \\ 0 \end{pmatrix} - \Sigma_{\{j|s_j \neq 0\}} \mu_j s_j e(j) = e(n+2).$$

From the last equation it follows that $\Sigma_{k\in K}\ \lambda_k = 1$. Hence, K contains at least one element, so that the label of at least one vertex of σ, say p^h, corresponds to the excess demand $z(p^h)$ at p^h. This implies that for at least one vertex p^h of an s-complete simplex σ we have that $p^{hT}a^i(p^h) < 0$ for all $i = 1,\dots,m$ and hence p^h lies in the relative interior of T^n when also $p^h{}_j > 0$ for all j. The first n+1 equations of (7) can now be written as

$$(8) \qquad Z(p^*) - \Sigma_{i\in U}\ \lambda_{k(i)}\ a^i(p^{k(i)}) = \Sigma_{\{j|s_j\neq 0\}}\ \mu_j s_j e(j),$$

where $p^* = \Sigma_{k\in K}\ \lambda_k p^k$. Finally, set $x^*_i = \lambda_{k(i)}$ for $i \in U$ and $x^*_i = 0$ for $i \notin U$. Observe that for all $k \in K$, $p^{kT}a^i(p^k) < 0$ for all $i = 1,\dots,m$. So, if $p^*_j > 0$ for all j, $p^* \in \mathrm{int}(T^n)$ since T^n is convex.

<u>Theorem 4.2.</u> Let the starting point v be in the interior of S^n and let for some $s \in S$, $\sigma(p^1,\dots,p^{t+1})$ be an s-complete simplex in A(s) with solution (λ,μ) to the system (6) of linear equations. Then σ yields an approximating solution to the equilibrium problem if one of the following properties holds:

a) $\mu_j = 0$ for all j with $s_j = -1$
b) $\mu_j = 0$ for all j with $s_j = +1$
c) for all $k = 1,\dots,t+1$ with $\lambda_k > 0$, $p_j{}^k = 0$ for all j with $s_j = -1$.

<u>Proof.</u> Let U, K, p^* and x^* be defined as above. To prove That (p^*,x^*) is an approximating equilibrium, observe that $p^* \in T^n$, and hence p^* satisfies condition (2) of Definition 2.1. We now consider condition (1). Since p^* and $p^{k(i)}$ are in the same simplex σ, we have that p^* is close to $p^{k(i)}$. So $a^i(p^{k(i)})$ is close to $a^i(p^*)$ and from (8) we obtain that

$$(9) \qquad Z(p^*) - \Sigma_{i\in U}\ x^{*i}\ a^i(p^*) \simeq \Sigma_{\{j|s_j\neq 0\}}\ \mu_j s_j e(j),$$

where $\simeq$ stands for 'approximately equal to'. Since $(p^{k(i)})^T a^i(p^{k(i)}) > 0$, it follows that for some positive ϵ_1 close to zero, $p^{*T}a^i(p^*) > -\epsilon_1$ for $i \in U$. Hence, the profit per unit production $p^{*T}a^i(p^*) \simeq 0$ for the technologies $i \in U$ with positive production level x^*_i. Now, in case a) we have that $\Sigma_{\{j|s_j\neq 0\}}\ \mu_j s_j e(j) = \Sigma_{\{j|s_j=+1\}}\ \mu_j s_j e(j) \geq 0$. For ease of notation, let $F_j(p^*,x^*)$ be the j-th component of $Z(p^*) - \Sigma_{i\in U}\ x^{*i}\ a^i(p^*)$. Then it follows from (9) that $F_j(p^*,x^*) \simeq 0$ if $s_j \neq +1$ and $F_j(p^*,x^*) \simeq 0$ or $F_j(p^*,x^*) \geq 0$ if $s_j = +1$. However, v is in the interior of S^n and σ lies in A(s). By definition of A(s) this implies that $p^*_j > 0$ for all j with $s_j = +1$. Moreover Z is the piecewise linear approximation to z and hence $p^{*T}Z(p^*) \simeq 0$, while also $p^{*T}a^i(p^*) \simeq 0$ for all $i \in U$. From this it follows that also $F_j(p^*,x^*) \simeq 0$ in case $s_j = +1$. Hence (p^*,x^*) satisfies approximately condition (1) of Definition 2.1.

In case b), we have that $\Sigma_{\{j|s_j\neq 0\}}\ \mu_j s_j e(j) = \Sigma_{\{j|s_j=-1\}}\ \mu_j s_j e(j) \leq 0$. Then $F_j(p^*,x^*) \simeq 0$ if $s_j \neq -1$ and $F_j(p^*,x^*) \leq 0$ or $F_j(p^*,x^*) \simeq 0$ if $s_j = -1$. Hence (p^*,x^*) satisfies approximately condition (1) of Definition 2.1.

Finally, in case c) we have that $F_j(p^*,x^*) \simeq 0$ if $s_j = 0$, $F_j(p^*,x^*) \simeq 0$ or $F_j(p^*,x^*) \leq 0$ if $s_j = -1$ and $F_j(p^*,x^*) \simeq 0$ or $F_j(p^*,x^*) \geq 0$ if $s_j = +1$. However, $p^*{}_j = 0$ in case $s_j = -1$ and hence $p_j^* F_j(p^*,x^*) \simeq 0$ if $s_j = -1$. By the same reasoning as in case a) it follows that $F_j(p^*,x^*) \simeq 0$ in case $s_j = +1$. Hence for all j, $F_j(p^*,x^*) \simeq 0$ or $F_j(p^*,x^*) \leq 0$. This completes the proof.

We now consider the piecewise linear path of prices followed by the algorithm. Firstly, observe that the left hand side of this system has $|s|+2 + n+1-|s| = n+3$ columns.

Nondegeneracy Assumption. For each solution to the system (6) of linear equations, at most one of the n+3 variables (λ,μ) is equal to zero.

Under this assumption the system has a line segment of solutions (λ^*,μ^*), if any. An end point of such a line segment is called a basic solution and has exactly one of the variables equal to zero.

Now, let $v \in \text{int}(T^n)$ be a price vector such that $z_j(v) \neq 0$ for all j and set $s^0 = \text{sgn } z(v)$. Then there is a unique 1-dimensional simplex $\sigma(p^1,p^2)$ in $A(s^0)$ with v as one of its vertices, say $v = p^1$. This simplex is s^0-complete with $\lambda_1 = 1$, $\lambda_2 = 0$ and $\mu_j = |z_j(v)|$ as one of its basic solutions. The algorithm starts by making a linear programming step with $b(p^2)$ in the corresponding system (6). In general, the algorithm traces a piecewise linear path of prices by making linear programming pivot steps in the system (6) with respect to a sequence of adjacent s-complete $(|s|+1)$-simplices in A(s) for various sign vectors s. At a solution (λ,μ), let $p = \Sigma_k\ \lambda_k p^k$. By making a linear programming step in the system (6) corresponding to a simplex $\sigma(p^1,\dots,p^{t+1})$ in some subset A(s), the following cases can occur.

Case 1: λ_g becomes zero for some $g \in \{1,\dots,t+1\}$. Then p lies in the facet τ of σ opposite the vertex p^g. This facet is either a facet of exactly one other t-simplex σ' in A(s), or τ lies in the boundary of A(s). In the first case, a pivot step is made in (6) with b(p') with p' the unique vertex of σ' not being a vertex of τ. In the latter case, either τ lies in the boundary of S^n, or τ is a simplex in a subset A(s'), where s' is a sign vector such that $s'_i \neq 0$ for some i with $s_i = 0$ and $s'_j = s_j$ for all $j \neq i$. If τ lies in the boundary of S^n, then by definition of A(s) and the fact that v lies in the interior of T^n, we have that $p^k{}_j = 0$ if $s_j = -1$ for all $k \neq g$. Hence by c) of Theorem 4.2, we have an approximate equilibrium. If τ lies in A(s') with s' as just defined, then τ is an s'-complete (t-1)-simplex in A(s') and the algorithm continues in A(s') by pivoting the (n+2)-column $-s_i e(i)$ into the system, thereby raising μ_i from zero.

Case 2: μ_k becomes zero for some k with $s_k \neq 0$. If $s_j \geq 0$ for all $j \neq k$, we have an approximate equilibrium according to a) of Theorem 4.2. If $s_j \leq 0$ for all $j \neq k$, we have an approximate equilibrium according to b) of Theorem 4.2. Otherwise, $\sigma(p^1,\dots,p^{t+1})$ is a facet of a unique (t+1)-simplex σ' in $A(s')$, where $s'_k = 0$ and $s'_j = s_j$ for all $j \neq k$. Let p' be the vertex of σ' not in σ. Then σ' is s'-complete and the algorithm continues in $A(s')$ by pivoting $b(p')$ into the system.

Since all steps are unique the algorithm either terminates with an approximating equilibrium within a finite number of steps, or one of the variables λ_k corresponding to a vertex p^k with a label $b(p^k) = [(-a^h(p^k))^T,0]^T$ for some h goes to infinity. Now, let $\sigma(p^1,...,p^{t+1})$ be an s-complete simplex in $A(s)$ and let U, K, p^* and x^* be as defined earlier. Then, analogously to the reasoning in Theorem 4.2, it can be shown that (p^*, x^*) is approximately close to a pair (p,x) in $B(s,U)$. So, the path of prices followed by the algorithm approximately follows the projection on T^n of the adjustment path in B of prices and quantities starting in $(v,0)$. Similarly to the proof that the path in B is bounded, it can be shown that all variables λ_k are bounded. This implies together with the finiteness of the number of simplices that the algorithm ends with an approximating equilibrium (p^*,x^*) as defined in Theorem 4.2. We have seen that p^* satisfies condition (2) of Definition 2.1. Therefore, we can measure the inaccuracy of the approximation by taking the smallest $\epsilon \geq 0$ such that $f_j(p^*,x^*) \leq \epsilon$, $j = 1,...,n+1$. If the accuracy is not sufficient, i.e. ϵ is too large, then the algorithm can be repeated with a finer simplicial subdivision of S^n with $p^* \in \text{int}(T^n)$ as the new starting point if $p^*_j > 0$ for all j, or a point in $\text{int}(T^n)$ close to p^* if $p^*_j = 0$ for some j. In this way a sequence of approximating equilibria can be found with inaccuracy going to zero.

5. An SLSPP-algorithm for the computation of an equilibrium.

Mathiesen [10] approximates the equilibrium problem by solving a sequence of Linear Complementarity Problems (LCP's). Each LCP in the sequence is solved by the Lemke-Howson algorithm. It can however not be assured that this algorithm solves each LCP in a finite number of steps. If not, the method breaks down. Otherwise, the sequence possibly converges to a solution of the equilibrium problem. For an economy with linear production technologies, Eaves [3] proposed an alternative formulation of the Linear Complementarity Problems. For this formulation it can be shown that the Lemke-Howson algorithm always finds a solution within a finite number of steps. So, the method always generates a sequence of approximating solutions which may converge to a solution of the equilibrium problem. In Kremers and Talman [7], the equilibrium problem for the economy with linear production is formulated as a stationary point problem on the set of feasible prices T^n. By linearizing the excess demand function, they obtain a Linear Stationary Point Problem (LSPP) on T^n. This LSPP is solved by an algorithm on T^n by using an algorithm of Kamiya and Talman [8] for solving a stationary point problem on a polytope. This algorithm always finds a solution within a finite number of steps. By linearizing the excess demand function at the solution of the previous problem, again a sequence of approximate solutions is generated, possibly converging to a solution of the equilibrium problem. In this section we generalize the algorithm of Kremers and Talman to deal with the case of nonlinear production.

A point $x^* \in R^k$ is a stationary point of a function $f: R^k \rightarrow R^k$ on a nonempty subset C of R^k if $x^{*T}f(x^*) \geq x^Tf(x^*)$ for all $x \in C$. For the excess demand function z of the consumption sector in our economy with nonlinear production technologies characterized by $a^i(p)$, $i = 1,...,m$, we have the following stationary point theorem (e.g. see Yamamoto [14]).

Theorem 5.1. A price vector $p^* \in T^n$ is a solution to the equilibrium problem of the economy with production technologies $a^i(p)$, $i = 1,...,m$, if and only if p^* is a stationary point of the excess demand function z on T^n.

The theorem implies that if p^* is a stationary point of z on T^n, there exists production levels x^*_i, $i = 1,...,m$, such that (p^*,x^*) satisfies the conditions of Definition 2.1. Given the stationary point $p^* \in T^n$, let U be the set of indices, such that $p^{*\top}a^i(p^*) = 0$ for all $i \in U$ and $p^{*\top}a^i(p^*) < 0$ for all $i \notin U$. Then $x^*_i = 0$ for $i \notin U$. For $i \in U$, x^*_i follows by solving the linear system $z(p^*) - \Sigma_{i \in U} x_i a^i(p^*) \leq 0$. From Theorem 5.1 it follows that the equilibrium problem can be reduced to the stationary point problem of the excess demand function z on T^n. This problem can possibly be solved by a sequence of Linear Stationary Point Problems as follows.

Let $p^k \in S^n$ be the solution to the k-th iterative in the sequence of Linear Stationary Point Problems, $k = 1,2....$, and take p^0 arbitrarily in S^n. Furthermore, let v be a point in the interior of T^n. We will see that p^k may lie outside T^n. Therefore, let λ^k be the solution to

(10) max λ, such that $p(\lambda)^\top a^i(p(\lambda)) \leq 0$ for all i, $0 \leq \lambda \leq 1$,

with $p(\lambda) = \lambda p^k + (1-\lambda)v$. Finally, let $v^k = \lambda^k p^k + (1-\lambda^k)v$. So, $\lambda^k = 1$ and $v^k = p^k$ if p^k is in the interior of the convex set T^n. If $p^k \notin T^n$, then $\lambda^k < 1$ and v^k lies on the boundary of T^n with $v^k a^i(v^k) = 0$ for some i. We now take $v^k \in T^n$ as the initial point in the (k+1)-th iterate and linearize the problem around v^k. So we define for $p \in S^n$,

$$z^{k+1}(p) = z(v^k) + [Jz(v^k)]p,$$

where $[Jz(v^k)]$ is the Jacobian matrix of first order derivatives of z to p in v^k. Furthermore, we set $a^{k+1,i}(p) = a^i(v^k)$. So, at the (k+1)-th iterate we consider the linear production technology given by $a^i(v^k)$, $i = 1,...,m$. Let $T^n(v^k)$ be defined by

$$T^n(v^k) = \{ p \in S^n \mid p^\top A(v^k) \leq 0 \}.$$

Then, by Corollary 2.5, $T^n \subset T^n(v^k)$. Since, $v^k \in T^n$ we have that $v^k \in T^n(v^k)$. Moreover, v^k lies on the boundary of $T^n(v^k)$ if v^k lies on the boundary of T^n. Applying the algorithm in [7], we can solve the stationary point problem of the linearized excess demand function $z^{k+1}(p)$ on the set of feasible prices $T^n(v^k)$ of the economy with linear technologies $a^i(v^k)$. The algorithm always finds a solution p^{k+1} within a finite number of steps. Since $T^n \subset T^n(v^k)$, p^{k+1} may lie outside T^n and we have to solve the one-dimensional maximization problem (10) for k+1 to find $v^{k+1} \in T^n$ as the approximate solution to the stationary point problem of z on T^n. This approximate solution v^{k+1} can be used as the new initial point in the next iterate. The sequence of approximate solutions v^k, $k = 1,2...$, may converge to a stationary point p^* of the excess demand function on T^n. If the sequences converges, the SLSPP is generally more efficient than the simplicial algorithm described in the previous section. If not, then the simplicial algorithm should be applied to solve the equilibrium problem because such an algorithm is globally convergent.

References

[1] T.M. Doup, G. van der Laan and A.J.J. Talman, "The (2^{n+1}-2)-ray algorithm: a new simplicial algorithm to compute economic equilibria", *Mathematical Programming* 39 (1987) 241-252.

[2] T.M. Doup and A.J.J. Talman, "A new simplicial variable dimension algorithm to find equilibria on the product space of simplices", *Mathematical Programming* 37 (1987) 319-355.

[3] B.C. Eaves, "Thoughts on computing market equilibrium with SLCP", in: A.J.J. Talman and G. van der Laan, eds., *The Computation and Modelling of Economic Equilibria* (North Holland, Amsterdam, 1987), pp. 1-17.

[4] A.H. van den Elzen, G. van der Laan and A.J.J. Talman, "An adjustment process for an exchange economy with linear production technologies", Research Memorandum 1989-82, Vrije Universiteit, Amsterdam, 1989, to appear in Mathematics of Operations Research.

[5] G. Fischer, K. Frohberg, M.A. Keyzer and K.S. Parikh, *Linked National Models: A Tool for International Food Policy Analysis* (Kluwer Academic Publishers, Dordrecht, 1989).

[6] J.A.W.M. Kremers, G. van der Laan and A.J.J. Talman, "A simplicial variable dimension restart algorithm for an economy with linear production technologies", 1991, to appear.

[7] J.A.W.M. Kremers and A.J.J. Talman, "An SLSPP-algorithm to compute an equilibrium in an economy with linear production technologies", Research Memorandum 496, Tilburg University, Tilburg, 1991.

[8] K. Kamiya and A.J.J. Talman, "Linear Stationary Point Problems", CentER Discussion Paper 9022 (Center for Economic Research, Tilburg University, Tilburg, 1990).

[9] G. van der Laan and A.J.J. Talman, "A restart algorithm for computing fixed points without an extra dimension", *Mathematical Programming* 17 (1979) 74-84.

[10] L. Mathiesen, "Computation of economic equilibria by a sequence of linear complementarity problems", *Mathematical Programming Study* 23 (1985) 144-162.

[11] M. Morishima, *Equilibrium, Stability and Growth, A Multi-Sectoral Analysis* (Oxford University Press, London, 1964).

[12] H. Scarf, *The Computation of Economic Equilibria* (Yale University Press, New Haven, 1973).

[13] H.R. Varian, *Microeconomic Analysis* (W.W. Norton & Company, New York, 1984, second edition).

[14] Y. Yamamoto, "Fixed point algorithms for stationary point problems" in: M.Iri and K. Tanabe, eds., *Mathematical Programming, Recent Developments and Applications* (KTK Scientific Publishers, Tokyo, 1989), pp. 283-307.