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COMPUTATION OF AN INDUSTRIAL EQUILIBRIUM*)

by

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Abstract.

In this paper we use and develop the concept of a semi-public good. A semi-public good is defined as an ordered pair of commodities, the first one being a private commodity and the second one a public good, which are related to each other by an individual inequality constraint for each individual agent. This approach allows us to design economic institutions which carry out price discrimination among users of a semi-public good. People who are seriously hampered by too small a provision of a public good, because it constrains their use of the private commodity, are willing to pay a mark-up on the price for the latter one if this mark-up is spent for expanding the provision of the public good. In the model the availability of a public good is planned and organized by a central planner. The consumer's willingness to pay an individual mark-up on the price of a private commodity reflects his preferences for the availibility of the public good. These mark-ups are collected by the private goods industry and transferred to the central planner in order to cover the costs of the public good infrastructure. This framework of a private industry and a central planner providing semi-public goods is called an industrial economy. The model will be illustrated by some numerical examples.

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1. Introduction.

General equilibrium models in economic theory are isomorphic to fixed point theorems. This insight is due to **Von Neumann** [18], who applied Brouwer's fixed point theorem to prove the existence of a process of proportional growth in a competitive economy. **McKenzie** [11], **Arrow** and **Debreu** [1] and other authors used this tool in the fifties to prove existence of an equilibrium for the model designed by **Walras** [19]. They thus have put the general equilibrium model for an economy with private goods only and with private ownership, on a solid axiomatic foundation.

The mathematical tools were strong enough to extend the economy with public goods, a concept introduced by **Samuelson** [15]. The concept of public goods has been studied intensively, see e.g. **Cornes** and **Sandler** [3]. The problems raised since Samuelson in public good models are more related to economic behaviour and institutions than to mathematical limitations. One of the fundamental issues in the theory of public goods is the individual's revelation of preferences about the provision of public goods. It is individually rational to behave as a free rider, but it is socially harmful. Many solutions for this problem have been proposed and rejected. It is still an unresolved issue in economic theory. For a recent survey we refer to **Blümel, Pethig** and **Von dem Hagen** [2].

In this paper we develop and use the concept of a semi-public good, introduced by **Ruys** [14]. A semi-public good is defined as an ordered pair of commodities, the first one being a private good and the second one a public good. The amount y^i of consumption of agent i of the private commodity and the amount z of availability of the public good are related to each other by an individual inequality constraint $y^i \leq \mathit{y}^i(z)$ for each agent i. This constraint might be implicitly expressed in the consumer's utility function or the producer's production function. But the explicit formulation makes it possible to distinguish between whether an individual constraint is binding or not. If for some agent, say consumer i, the constraint is binding, then an increase of z has a direct effect on his demand because of the fact that z appears in the consumer's utility function, but also it has an indirect effect through the weakening of the constraint. The price for raising z offered by a truth-telling consumer will reflect the impact of both effects on his utility. The part reflecting the constraint will show up as a mark-up on the market price the consumer is willing to pay for the private commodity. If no agent in the economy feels himself constrained, the semi-public good reduces to a private good having a uniform market price, and a pure public good with, if desired, Lindahl prices. In general, the definition of a semi-public good is relevant only if the constraints are binding for a considerable number of agents.

The main advantage of this approach is that economic institutions can be designed which make price discrimination possible among users of a semi-public good. People who are seriously hampered by too small a provision of a public good, because it constrains their use of the private commodity, are thought of forming (political) pressure groups to expand its provision, or are informing the industry otherwise. They are also willing to pay a mark-up on the price of the private commodity if this mark-up is spent on expanding the provision of the public good. In the context of an industrial economy the enterprises in an industry discriminate between consumers by setting different prices, and not the public authority or planner. These differentiated prices inform the planner and partially finance the public good.

We will explore a model in which there is just one industry producing private goods, which form semi-public goods with a public good. The infrastructure of this public good is planned and organized by a central planner. The consumer's willingness to pay an individual mark-up on each of the prices of the private commodities reflects his preferences for the infrastructure of the public good. These mark-ups are collected by the private goods industry and transferred to the central planner in order to cover the costs of the public good infrastructure. As an alternative the private goods industry may levy a uniform mark-up on the prices of the private commodities to provide an infrastructure necessary for using their products. We call this framework of a central planner and private firms providing together semi-public goods an industrial economy.

It is evident that there are many spill-over effects resulting from any decision about the provision of a semi-public good. This calls for a general equilibrium approach, with an associated fixed point or zero point formulation. In order to calculate a fixed point, simplicial algorithms first have been designed by **Scarf** [16,17] and **Kuhn** [6,7] for fixed point problems on the unit price simplex. **Van der Laan** and **Talman** [9] developed a variable dimension algorithm for problems on the unit simplex. Similar algorithms for fixed or zero point problems on R^n have been introduced by **van der Laan** and **Talman** [10], **Wright** [20], **Kojima** and **Yamamoto** [5], and others. These algorithms allow for fast movements in lower dimensional spaces and are therefore very efficient. A code for these algorithms has been implemented on the computer by Seelen, see [9]. We will use this code for solving some numerical examples to illustrate the framework of an industrial economy.

This paper is organized as follows. In the next section we discuss the framework of an industrial economy by giving some examples. The mathematical model is given in section 3. In this section we also state the first order conditions for a Pareto efficient allocation. The institutional framework to reach a Pareto efficient allocation is given in section 4. In section 5 we give some numerical examples to illustrate the concept of an industrial economy. Finally, in section 6 we make some concluding remarks and we discuss the possibilities for further research.

2. An industrial economy.

An industrial economy consists of a number of (small) enterprises which produce private commodities that are close substitutes or complements and which have a common interest in maintaining the availability of a public good, called the infrastructure. The presence of an infrastructure increases the utility of the private goods or may even be a necessary complement to them. Examples are:

a. airline transportation: several carrier companies provide substitutable transport services; they have a common interest in for example airports, a reservation network, safety measures.

b. tourist industry: there are many enterprises providing services that are close substitutes and complements (hotels, restaurants, entertainment, travel agencies); these enterprises have a common interest in for example a clean and attractive environment, promotion activities and a reputation for good quality of services.

c. surface transport: there are several modes of transportation which are close substitutes and complements (bicycle, car, taxi, tramway, bus, railroad); for each mode there are one or more enterprises providing transportation services; producers of a mode have a common interest, such as a road or a railroad system, and time and working schedules.

There are much more examples, of course, but the three given here are specific in some aspects. In example a) the private goods are close substitutes. The enterprises compete and they are comparable. Moreover, there is only one public good for all, called the infrastructure. In example b) the private goods are both substitutes and complements. The enterprises can be clustered in various branches each having a completely different production technology (hotels, attractions, souvenir shops), and most of the branches are competitive. The common infrastructure is induced rather than planned and organized. In example c) the private goods are again close substitutes and complements (trains have connections with buses). Some modes of transportation are competitive (taxis), but other are monopolistic and regulated. Again there is a common infrastructure from which some of the modes may benefit and some others may not. This infrastructure is planned.

The central problem in all examples is the way on which the infrastructure is provided and financed. In the air industry example it seems to be obvious that the enterprises organize and finance the infrastructure and pass on the costs in the prices the consumers have to pay. However, the consumers also benefit directly from the infrastructure. It enlarges their possibilities to travel and therefore they should also show a willingness to pay for having an airport. On the other hand, people living close to the airport may suffer from its noise. We have similar characteristics in the other examples. For organizing and financing the infrastructure we distinguish the following cases:

i) the infrastructure is not planned or decided upon, but it results from unorganized individual actions of the agents (e.g. it is attractive to do shopping in a city with a wide variety of supplies)

ii) there is an agent (a government or a private enterprise), who provides the infrastructure and who determines tariffs or prices for making use of it (e.g. a shopping center or airport). An agent can decide to take or leave the offer

iii) the infrastructure is planned and organized by an agent who has been established by the enterprises and by others who have interest in the production of the industry.

Case i) is not relevant for our problem. Case ii) gives a way out of our problem if the agent providing the infrastructure is economically self-supporting or can make profits. It remains interesting to analyse the rules of price setting with the theory developed here. Our approach is mainly relevant for case iii). In this case either the infrastructure is not apt for private (or profitable) exploitation, or there are political, juridicial and other non-economic elements involved that influence the productivity of an industry and its chances of survival. In the next section we present a model for this case. From this model we derive conditions for an efficient allocation. These conditions show that the prices the agents are willing to pay for the private commodities reveal the preferences for the infrastructure.

3. The mathematical model.

We consider a model of an industrial economy with two semi-public goods, composed from private goods a and b and a public good. For example, the public good is a road system that is used both by private cars, a, and by public buses, b. There are two other commodities, private goods 1 and 2. There is a (possibly private) producer who plans and organizes the level of the infrastructure, z, taking into account the wishes of the (transportation) industry. This industry has two branches, Y^a and Y^b, each consisting of a representative private firm producing commodity a and b respectively. For instance, the first firm leases private cars to consumers and the second firm exploits the public bus system.

There are h consumers, indexed by i=1,...,h. Each consumer i has a utility function $u^i(x_1^i,x_2^i,y_a^i,y_b^i,z)$ on $X^i=R^5_+$. Furthermore, each consumer i faces individual semi-public (quantity) constraints on the consumption y_a^i and y_b^i of the private goods a and b. That means, each consumer is constrained in his' or her's car driving and public transportation because of the limitations of the road system. So, we assume that there are constraints $\mathcal{y}_a^i(z)$ and $\mathcal{y}_b^i(z)$ for i=1,...,h, such that the consumption of consumer i is restricted by

$$y_a^i \leq \mathcal{y}_a^i(z) \tag{3.1}$$
$$y_b^i \leq \mathcal{y}_b^i(z). \tag{3.2}$$

The industry is aware of these (subjective) constraints because it observes rationing in the demand functions. Separate from these subjective feasibility constraints, the respective technical production constraints of the firms Y^a and Y^b are given by

$$F^a(y_a;x_1^a,x_2^a) \leq 0 \tag{3.3}$$
$$F^b(y_b;x_1^b,x_2^b) \leq 0, \tag{3.4}$$

where x_1^a, x_2^a and x_1^b, x_2^b respectively the amounts of inputs in the production of a and b respectively, and y_a and y_b are the amounts of output of commodity a and b respectively. Moreover, we assume that firm Y^b faces a constraint

$$y_b \leq \mathcal{y}_b(z), \tag{3.5}$$

This constraint reflects the fact that the system of public transportation is restricted by the limitations of the road system.

The enterprise producing the (public) infrastructure is given by the technical constraint

$$F^{z}(z;x_1^{z},x_2^{z}) \leq 0, \qquad (3.6)$$

with x_1^{z} and x_2^{z} the amounts of inputs. Finally, there is a firm which produces the commodities 1 and 2 from a production factor. Initially there is a total amount w of this production factor available. The technical constraint of this firm is given by

$$F^{o}(x_1^{o},x_2^{o};w) \leq 0, \qquad (3.7)$$

where x_1^{o} and x_2^{o} are the output amounts of the commodities 1 and 2 respectively.

We assume that this economy, denoted by $E = \{(u^i,y_a^{\,i},y_b^{\,i})$ i=1,...,h, F^a, (F^b,y_b), F^z, F^o, w} is *regular*, i.e., the utility and production functions and the constraint functions are continuously differentiable, the utility functions u^i are quasi-concave, the productions functions are concave, and w is positive. Furthermore we assume that in all technical constraints both the inputs and the outputs are measured positively. From this it follows that for r∈{a,b,z,o}, and for the variables v = x_1^{o}, x_2^{o}, x_1^{a}, x_2^{a}, x_1^{b}, x_2^{b}, x_1^{z}, x_2^{z}, y_a, y_b and z, holds

$\delta F^r/\delta v < 0$ if v is an input,

and

$\delta F^r/\delta v > 0$ if v is an output.

We are now ready to give some definitions.

<u>Definition 3.1.</u> An allocation e = $\{(x_1^{\,i},x_2^{\,i},y_a^{\,i},y_b^{\,i})$, i=1,...,h, (y_a,x_1^{a},x_2^{a}), (y_b,x_1^{b},x_2^{b}), (z,x_1^{z},x_2^{z}), $(x_1^{o},x_2^{o})\}$ is in the set *A* of *feasible* allocations if the constraints (3.1)-(3.7) hold, and if

$$\Sigma_i\, x_j^{\,i} + x_j^{\,a} + x_j^{\,b} + x_j^{\,z} \leq x_j^{\,o} \qquad j=1,2 \qquad (3.8)$$
$$\Sigma_i\, y_a^{\,i} \leq y_a \qquad (3.9)$$
$$\Sigma_i\, y_b^{\,i} \leq y_b. \qquad (3.10)$$

Observe that this definition includes the subjective constraints (3.1) and (3.2). The quantity constraints (3.8)-(3.10) say that total demand is less than or equal to total supply.

<u>Definition 3.2.</u> A feasible allocation e is *efficient* if there is a distribution of strictly positive individual weights ϑ_i, i=1,...,h, for which e maximizes the social welfare function

$$\Sigma_i\ \vartheta_i u^i(x_1{}^i, x_2{}^i, y_a{}^i, y_b{}^i, z)$$

over the set A of feasible allocations.

According to Definition 3.2 the necessary conditions for an allocation of a regular economy to be efficient follow from the maximization problem,

$$\max\ \Sigma_i\ \vartheta_i u^i(x_1{}^i, x_2{}^i, y_a{}^i, y_b{}^i, z), \qquad (3.11)$$

such that, with the shadow prices of the constraints between brackets,

(α^i)	$y_a{}^i - y_a{}^i(z) \le 0$	i=1,...,h
(β^i)	$y_b{}^i - y_b{}^i(z) \le 0$	i=1,...,h
(γ)	$y_b - y_b(z) \le 0$	
(λ^a)	$F^a(y_a; x_1{}^a, x_2{}^a) \le 0$	
(λ^b)	$F^b(y_b; x_1{}^b, x_2{}^b) \le 0$	
(λ^z)	$F^z(z; x_1{}^z, x_2{}^z) \le 0$	
(λ^o)	$F^o(x_1{}^o, x_2{}^o; w) \le 0$	
(μ^j)	$\Sigma_i\ x_j{}^i + x_j{}^a + x_j{}^b + x_j{}^z \le x_j{}^o$	j=1,2
(μ^a)	$\Sigma_i\ y_a{}^i \le y_a$	
(μ^b)	$\Sigma_i\ y_b{}^i \le y_b$.	

Differentiating the corresponding Lagrange function gives with respect to the variable between brackets, with j=1,2 and i=1,...,h:

$$
\begin{array}{ll}
(x_j^{\,i}) & \vartheta_i\, \delta u^i/\delta x_j^{\,i} - \mu^j \;\; = 0 \\
(y_a^{\,i}) & \vartheta_i\, \delta u^i/\delta y_a^{\,i} - \alpha^i - \mu^a = 0 \\
(y_b^{\,i}) & \vartheta_i\, \delta u^i/\delta y_b^{\,i} - \beta^i - \mu^b = 0 \\
(y_a) & -\lambda^a\, \delta F^a/\delta y_a + \mu^a \;\; = 0 \\
(y_b) & -\gamma - \lambda^b\, \delta F^b/\delta y_b + \mu^b = 0 \\
(z) & \Sigma_i\, \vartheta_i\, \delta u^i/\delta z + \Sigma_i\, \alpha^i\, \delta y_a^{\,i}/\delta z + \Sigma_i\, \beta^i\, \delta y_b^{\,i}/\delta z + \gamma\, \delta y_b/\delta z - \lambda^z\, \delta F^z/\delta z = 0 \\
(x_j^{\,a}) & \lambda^a\, \delta F^a/\delta x_j^{\,a} + \mu^j = 0 \\
(x_j^{\,b}) & \lambda^b\, \delta F^b/\delta x_j^{\,b} + \mu^j = 0 \\
(x_j^{\,z}) & \lambda^z\, \delta F^z/\delta x_j^{\,z} + \mu^j \;\; = 0 \\
(x_j^{\,o}) & \lambda^o\, \delta F^o/\delta x_j^{\,o} - \mu^j \;\; = 0,
\end{array}
$$

with all shadow prices nonnegative. With commodity 1 taken as the numeraire, we obtain from these equations the next first order conditions for an efficient allocation. For all $i=1,\dots,h$,

$$
\frac{\delta u^i/\delta x_2^{\,i}}{\delta u^i/\delta x_1^{\,i}} = \frac{\delta F^r/\delta x_2^{\,r}}{\delta F^r/\delta x_1^{\,r}} \qquad \text{for } r\in\{a,b,o,z\} \tag{3.12}
$$

$$
\frac{\delta u^i/\delta y_a^{\,i}}{\delta u^i/\delta x_1^{\,i}} = \frac{\delta F^a/\delta y_a}{-\delta F^a/\delta x_1^{\,a}} + \frac{\alpha^i}{-\lambda^z \delta F^z/\delta x_1^{\,z}} \tag{3.13}
$$

$$
\frac{\delta u^i/\delta y_b^{\,i}}{\delta u^i/\delta x_1^{\,i}} = \frac{\delta F^b/\delta y_b}{-\delta F^b/\delta x_1^{\,b}} + \frac{\gamma}{-\lambda^z \delta F^z/\delta x_1^{\,z}} + \frac{\beta^i}{-\lambda^z \delta F^z/\delta x_1^{\,z}} \tag{3.14}
$$

$$
\Sigma_k \frac{\delta u^k/\delta z}{\delta u^k/\delta x_1^{\,k}} + \frac{\Sigma_k\, \alpha^k \delta y_a^{\,k}/\delta z}{-\lambda^z \delta F^z/\delta x_1^{\,z}} + \frac{\Sigma_k\, \beta^k \delta y_b^{\,k}/\delta z}{-\lambda^z \delta F^z/\delta x_1^{\,z}} + \frac{\gamma \delta y_b/\delta z}{-\lambda^z \delta F^z/\delta x_1^{\,z}} = \frac{\delta F^z/\delta z}{-\delta F^z/\delta x_1^{\,z}} \tag{3.15}
$$

Condition (3.12) is the usual condition for pure private goods, saying that the marginal rate of substitution (MRS) equals the marginal rate of transformation (MRT). Notice that for each firm the two private commodities are either both an output with positive derivative, or both an input with negative derivative. The latter fact explains the minus signs in (3.13)-(3.15). If $\alpha^i=0$ for all i, then no consumer feels himself constrained in the use of commodity a. This commodity is then a private good having a uniform MRS equal to the MRT between a and the numeraire commodity. However, if $\alpha^i>0$ for some i, then consumer i is willing to pay a mark-up on the MRT of commodity a in order to subsidize an expansion of the infrastructure. In the condition (3.14) for commodity b an extra term appears in the equation. This term reflects the constraint of producer Y^b with respect to the availability z of the public good. If γ=0 then the producer is not constrained and we have the same situation as for commodity a. If γ>0therwise, the second term on the right hand side of equation (3.14) reflects the additional costs the producer is willing to make for getting an expansion of the infrastructure, in order to enlarge his production possibilities. If β^i>0, then consumer i is willing to pay a mark-up on the costs of commodity b, including the costs the producer has to pay for the expansion. All the mark-ups and the producer's costs for expanding the public good reappear in (3.15). Notice that the mark-ups in (3.13) and (3.14) reveal the willingness to pay for weakening of the constraints y_a^i, y_b^i and y_b, whereas the terms in (3.15) reveal the willingness to pay for an expansion of the infrastructure. We see that the sum of the MRS plus the sum of the mark-ups of the consumers plus the mark-up of the producer is equal to the MRT of the public good. If all mark-ups are equal to zero, then the public good behaves as a pure public good.

The main advantage of introducing semi-public goods in this way is that an industrial economy can discriminate between agents who are and who are not constrained by the infrastructure, because it can observe demand-behaviour. This information can solve partially (and sometimes completely) the difficult problem of determining the individual contributions to the provision of a public good.

4. The institutional framework.

In this section we describe the institutional framework under which an industrial equilibrium can be formulated satisfying the first order conditions for efficiency. This institutional framework is the private ownership industrial economy. In the economy *E* there are four private good markets in operation: one for each good 1, 2, a and b. The demands and supplies on these markets depend on the prices p_1, p_2, p_a and p_b respectively, with the price of the numeraire commodity, p_1, equal to one. In an efficient allocation these prices are equal to the respective MRT's. For the fifth commodity, the public good, the situation is much more complicated. Later on we will make some simplifying assumptions. For the moment we deal with the general model given in the previous section.

We assume that the industry is able to discriminate among consumers who are constrained and who are not. At some allocation e, let, for i=1,...,h,

$$t_a^i(e) = -\alpha^i(\lambda^z \delta F^z/\delta x_1^z)^{-1}$$

and

$$t_b^i(e) = -\beta^i(\lambda^z \delta F^z/\delta x_1^z)^{-1}$$

be the willingness of consumer i to pay for the weakening of the constraints $y_a^i(z)$ and $y_b^i(z)$ respectively. Then the sum of $T_a^i(e) = t_a^i(e)\delta y_a^i/\delta z$ and $T_b^i(e) = t_b^i(e)\delta y_b^i/\delta z$ is his willingness to pay for the expansion of the infrastructure. Suppose that the willingness to pay is known to the industry. Of course this is not an innocuous assumption, but it can be approached in reality under the simplifications we will make later on. Furthermore, let $t_b(e)=-\gamma/\lambda^z\delta F^z/\delta x_1^z$ and $T_b(e)=t_b(e)\delta y_b/\delta z$ be the willingness to pay of firm Y^b for weakening $y_p(z)$ and expanding z respectively. This information is of course known to the industry. Finally, at some allocation e, denote the marginal rate of substitution of consumer i between z and x_1 by $p_z^i(e)$, i=1,...,h. Now, the planner's task is to find the desired level of the infrastructure, i.e., to plan and to organize an amount z such that the sum of the MRS's plus the total willingness to pay is equal to the marginal rate of transformation, denoted by $p_z(e)$.

Planner's problem: Find z such that

$$\Sigma_i\, [p_z^i(e) + T_a^i(e) + T_b^i(e)] + T_b(e) = p_z(e). \qquad (4.1)$$

The price p_z is the price to be paid by the planner for each unit of the public good and equals the MRT. On the other, hand the revenues of the planner consist of the consumers' contributions p_z^i per unit, and the mark-ups t_a^i, t_b^i and t_b, per unit of consumption y_a^i, y_b^i and per unit of production y_b, respectively. Since $y_b=\Sigma_i y_b^i$ if $t_b>0$, the planner's profit $\pi^q(p,z)$, where $p=(p_1,p_2,p_a,p_b)^T$, equals

$$\pi^q(p,z) = \Sigma_i\, p_z^i z + \Sigma_i\, [t_a^i y_a^i + (t_b^i + t_b)y_b^i] - p_z z =$$

$$= \Sigma_i\, (t_a^i y_a^i + t_b^i y_b^i) + t_b y_b - \Sigma_i\, (T_a^i + T_b^i)z - T_b z.$$

To complete the description of the economy, we assume that the private firms are profit maximizing producers. We denote the respective profits by $\pi^o(p,z)$, $\pi^a(p,z)$, $\pi^b(p,z)$ and $\pi^z(p,z)$. Since we assume that only firm producing the private commodities 1 and 2 is endowed with a production factor, all individual labour and wealth in the economy is put in the production function F^o. Wages are paid as profits. All profits are distributed among the consumers, with, for i=1,...,h and r∈{o,a,b,z,q}, ϕ^{ir} the share of consumer i in the profit of firm (or planner) r. All shares are nonnegative and $\Sigma_i \phi^{ir}=1$ for all r. The income of consumer i at (p,z) is given by $w^i(p,z) = \Sigma_r \phi^{ir}\pi^r(p,z)$.

We are now able to define an industrial equilibrium for the economy E. Recall that a feasible allocation satisfies (3.1)-(3.10).

Definition 4.1. An industrial equilibrium for the economy E is a feasible allocation $e = \{(x_1^i,x_2^i,y_a^i,y_b^i),\ i=1,...,h,\ (y_a,x_1^a,x_2^a),\ (y_b,x_1^b,x_2^b),\ (z,x_1^z,x_2^z),\ (x_1^o,x_2^o)\}$, a set of prices p_1, p_2, p_a, p_b for the private commodities and a price p_z for the public good, a set of individual prices p_z^i, i=1,...,h, and a set of mark-ups t_a^i, t_b^i, i=1,...,h, and t_b, such that

1. for all i, $(x_1^i,x_2^i,y_a^i,y_b^i,z)$ maximizes $u^i(x_1^i,x_2^i,y_a^i,y_b^i,z)$ under the budget constraint

$$p_1x_1^i + p_2x_2^i + (p_a+t_a^i)y_a^i + (p_b+t_b+t_b^i)y_b^i + p_z^iz = w^i(p,z)$$

2. each producer maximizes profit subject to his technical constraint, i.e.,

$$\pi^o(p,z) = p_1x_1^o+p_2x_2^o = \max\{p_1x_1^o+p_2x_2^o|F^o(x_1^o,x_2^o;w)\le 0\}$$
$$\pi^z(p,z) = p_zz-p_1x_1^z-p_2x_2^z = \max\{p_zz-p_1x_1^z-p_2x_2^z|F^z(z;x_1^z,x_2^z\le 0)\}$$
$$\pi^a(p,z) = p_ay_a-p_1x_1^a-p_2x_2^a = \max\{p_ay_a-p_1x_1^a-p_2x_2^a|F^a(y^a;x_1^a,x_2^a\le 0)\}$$
$$\pi^b(p,z) = p_by_b-p_1x_1^b-p_2x_2^b = \max\{p_by_b-p_1x_1^b-p_2x_2^b|F^b(y^b;x_1^b,x_2^b)\le 0\}$$

3. for all i: $t_a^i > 0$, implies $y_a^i = \mathcal{y}_a^i(z)$ and $t_b^i > 0$, implies $y_b^i = \mathcal{y}_b^i(z)$
4. $t_b > 0$ implies $y_b = \mathcal{y}_b(z)$
5. (4.1) is satisfied, i.e., the planner equates marginal social costs with marginal social benefits of the public good
6. (3.8)-(3.10) are satisfied with equalities, i.e., the markets clear demand and supply.

Notice that the availability z of public good is completely determined by the planner. So, actually the consumers do not maximize their utility over z. Instead, the prices p_z^i are determined such that for all i, z is optimal under p_z^i. The same reasoning holds for the public good's producer, who determines p_z given the amount z. The third condition has analogies in fixed price theory, from which it is well-known that quantity-constrained allocations can be sustained by virtual prices (see e.g. Neary and **Roberts** [12], **Ruys** [13] and **Cornielje and van der Laan** [4]. Here condition 3) says that a consumer is not willing to pay a mark-up on the cost price of a commodity if he or she is not constrained in the use of that commodity. Analogously, condition 4) says that the producer is willing to levy a mark-up on his output price p_b if he is constrained by the infrastructure level z. In this paper we assume that an equilibrium exists. We will address the existence problem in a subsequent paper, see also section 6.

We now make some simplifying assumptions. First, we assume, without loss of the generality of our approach, that the public good does not appear in the utility function of the consumers, i.e., $p_z^i=0$ for all i. In this case the consumers are only interested in the infrastructure if they are constrained. Now, the planner's problem becomes: find z such that

$$\Sigma_i \, [T_a^i(e) + T_b^i(e)] + T_b(e) = p_z(e). \tag{4.2}$$

Secondly, assume that the consumers' constraint functions are linear with constant term equal to zero, i.e., for all i,

$$y_a^i(z) = a^i z$$
$$y_b^i(z) = b^i z.$$

In general, we are not able to say anything about the concavity or convexity of the constraint functions. Both cases may occur. Therefore the assumption of linear functions is very simplifying, but not too bad. For simplicity we also assume that $y_b(z)=b^p z$. Now (4.2) becomes

$$\Sigma_i \, [a^i t_a^i(e) + b^i t_b^i(e)] + b^p t_b(e) = p_z(e).$$

From this it follows that the planner's profit becomes equal to zero. Moreover, the coefficients a^i and b^i follow from the consumption level of the goods a and b of the constrained consumers. The willingness to pay can be approached in reality if the consumers are partitioned in classes with different needs to expand the infrastructure. These needs can be inferred from the unconstrained demands for the goods a and b.

Under these simplifying assumptions the planner can obtain enough information to decide upon the infrastructure level z, given the mark-ups on the cost prices p_a and p_b. In this linear case, the infrastructure is completely financed by the returns on the mark-ups on the prices of the private goods.

5. Examples.

For all firms we take constant returns to scale production functions, implying that the firm with production function F^a, F^b and F^z respectively, is cost minimizing with in equilibrium zero profit. The income of consumer i equals $\phi^i\pi^o$, with ϕ^i the share of i in the profit of the firm with production function F^o. For the consumers we take Cobb-Douglas utility functions. Recall that we assume that z does not appear in these functions. Furthermore we assume that the consumers are not constrained in the use of commodity b, i.e., $b^i = \infty$ for all i. This gives the next example. For i=1,...,h, the utility of consumer i is given by

$$u^i = \rho_1{}^i \ln x_1{}^i + \rho_2{}^i \ln x_2{}^i + \rho_a{}^i \ln y_a{}^i + \rho_b{}^i \ln y_b{}^i$$

under $y_a{}^i \leq a^i z$, where $\rho_1{}^i+\rho_2{}^i+\rho_a{}^i+\rho_b{}^i$ is normalized to one. The production constraints are given by, with all $\psi_j{}^r>0$,

$$F^o = \psi_1{}^o(x_1{}^o)^2 + \psi_2{}^o(x_2{}^o)^2 - w^2 \leq 0$$

$$F^z = \ln z - \psi_1{}^z \ln x_1{}^z - \psi_2{}^z \ln x_2{}^z \leq 0 \qquad \text{with } \psi_1{}^z+\psi_2{}^z=1,$$

and for r∈{a,b,}

$$F^r = \ln y_r - \psi_1{}^r \ln x_1{}^r - \psi_2{}^r \ln x_2{}^r \leq 0 \qquad \text{with } \psi_1{}^r+\psi_2{}^r=1.$$

For firm Y^b we have the quantity constraint $y_b \leq b^p z$.
Given prices p_1 and p_2 we obtain from cost minimizing that for r∈{a,b,z} the conditional factor demand per unit of output is given by

$$x_1{}^r = (Ap_2/Bp_1)^B \qquad (5.1)$$
$$x_2{}^r = (Bp_1/Ap_2)^A, \qquad (5.2)$$

where $A=\psi_1{}^r$, $B=\psi_2{}^r$. The zero profit condition gives

$$p_r = p_1 x_1{}^r + p_2 x_2{}^r. \qquad (5.3)$$

Maximizing profit under $F^o \leq 0$ gives the private goods supply functions

$$x_j^{\,o} = p_j w/\psi_j^{\,o} c \qquad j=1,2, \qquad (5.4)$$

while the profit is given by

$$\pi^0 = cw, \qquad (5.5)$$

with $c^2 = p_1^{\,2}/\psi_1^{\,o} + p_2^{\,2}/\psi_2^{\,o}$.

Utility maximizing of consumer i under the budget constraint

$$p_1 x_1^{\,i} + p_2 x_2^{\,i} + (p_a + t_a^{\,i}) y_a^{\,i} + (p_b + t_b) y_b^{\,i} = \phi^i cw$$

gives for the consumer's demand

$$x_j^{\,i} = \rho_j^{\,i} \phi^i cw/p_j \qquad j = 1,2 \qquad (5.6)$$

$$y_a^{\,i} = \rho_a^{\,i} \phi^i cw/(p_a + t_a^{\,i}) \qquad (5.7)$$

$$y_b^{\,i} = \rho_b^{\,i} \phi^i cw/(p_b + t_b). \qquad (5.8)$$

For given z and demand $y_a^{\,i}$, the mark-up $t_a^{\,i}$ is determined by firm Y^a by setting

$$t_a^{\,i} = \max\{0, (\rho_a^{\,i} \phi^i cw/a^i z) - p_a\}. \qquad (5.9)$$

So, the mark-ups are determined by the industry such that the individual demands do not exceed the individual constraints $a^i z$. From (5.7) and (5.9) we obtain that

$$y_a^{\,i} = \rho_a^{\,i} \phi^i cw/p_a \text{ and } t_a^{\,i} = 0 \qquad \text{if } \rho_a^{\,i} \phi^i cw/p_a \le a^i z \qquad (5.10)$$

and

$$y_a^{\,i} = a^i z \text{ and } t_a^{\,i} = \rho_a^{\,i} \phi^i cw/a^i z - p_a \text{ if } \rho_a^{\,i} \phi^i cw/p_a > a^i z. \qquad (5.11)$$

Observe that the discrimination among consumers is determined by the parameters $\rho_a^{\,i}$, ϕ^i and a^i. In fact, the willingness to pay increases with $\rho_a^{\,i}$ and ϕ^i and decreases with a^i. Firm Y^b determines the mark-up t_b on his output price p_b such that the total demand y_b does not exceed the constraint $b^p z$. We obtain from (5.8) that

$$t_b = \max\{0, (\Sigma_i \, \rho_b^{\,i} \phi^i cw/b^p z) - p_b\}, \qquad (5.12)$$

Hence

$$y_b^{\,i} = \rho_b^{\,i}\phi^i cw/\;\; \text{and } t_b = 0 \qquad \text{if } \Sigma_i\, \rho_b^{\,i}\phi^i cw/p_b \le b^p z \qquad (5.13)$$

and

$$y_b^{\,i} = \rho_b^{\,i}\phi^i cw/\;\; +t_b) \text{ and } t_b = (\Sigma_i\, \rho_b^{\,i}\phi^i cw/b^p z) - p_b$$

$$\text{if } \Sigma_i\, \rho_b^{\,i}\phi^i cw/p_b > bz \qquad (5.14)$$

Finally the production vels y_a and y_b are set by the producers such that they are equal to the total consumpti i.e.,

$$y_a = \Sigma_i\, y_a^{\,i} \text{ anc } \;{}_b = \Sigma_i\, y_b^{\,i}. \qquad (5.15)$$

Notice that $y_b = b^p z$ if $\rho_b^{\,i}\phi^i cw/p_b > b^p z$. Consequently, given the prices p_1 and p_2 and the infrastructure vel z, the values of all other variables, prices, quantities and mark-ups, can be calc ted through (5.1)-(5.15). So, the equilibrium problem is to find market prices p_1 and p and a level z of the infrastructure such that the markets for the private commodities 1 d 2 clear and the mark-ups revenues are equal to the costs of the infrastructure, i.e.,

$$\Sigma_i\, x_j^{\,i} + x_j^{\,a} + x \;\; = x_j - x_j^{\,z}, \qquad j = 1,2 \qquad \text{(market-condition)}$$

$$\Sigma_i\, a^i t_a^{\,i} + b^p t_b \;\; {}_z. \qquad \text{(planner-condition)}$$

In the next sec n we discuss this problem both from a numerical and economic viewpoint. Here we cc entrate ourselves on the numerical results. Using the computer code described in **van** r **Laan** and **Seelen** [8] we have calculated the equilibrium with the following data.

<u>Example 1.</u> Number of nsumers: 4. Input: w = 100. Constraint coefficient producer Y^b: $b^p = 4$. The data of the ther coefficients are given in the Tables 1 and 2.

Table 1. Coefficients o he producers.

Producer r =	o	b	z
$\psi_1^{\,r}$	1	.5	.5
$\psi_2^{\,r}$	1	.5	.5

Table 2. Coefficients of the consumers.

Consumer i =	1	2	3	4
ρ_1^i	.4	.3	.2	.1
ρ_2^i	.1	.1	.1	.1
ρ_a^i	.5	.6	.7	.8
ρ_b^i	0	0	0	0
Profit shares ϕ^i	.1	.2	.3	.4
Constr. coef. a^i	1	1	1	1

Observe that the budget shares for commodity b are zero for all consumers. So, the demands for commodity b are zero, implying that firm Y^b is not active and t_b=0. The equilibrium values are given in Table 3 with the unconstrained demands (i.e., with t_a^i=0) between brackets.

Table 3. Equilibrium values Example 1.

	x_1	x_2	y_a	y_b	z	t_a^i	$\phi^i\pi^0$
price	1	0.905	1.902	1.902	1.902		
mark-up t_b				0			
Producers:							
output	74.2	67.1	37.1	0	12.5		
input a	35.3	39.0					
b	0	0					
z	11.9	13.2					
Consumers:							
1	5.4	1.5	3.5	0		0	13.5
2	8.1	3.0	8.5	0		0	27.0
3	8.1	4.5	12.5 (14.9)	0		0.36	40.5
4	5.4	6.0	12.5 (22.7)	0		1.54	53.9

Example 2. Same data as in Example 1, except that the budget shares for the commodities a and b are equal to ρ_a^i=0.1, 0.2, 0.3, 0.4, for i=1,...,4 respectively, and ρ_b^i=0.4 for all i. The equilibrium values are given in Table 4 with again the unconstained demands between brackets.

In the two examples the expenditures for the private commodities 1 and 2 are equal to each other. The only difference comes from the budget shares for a and b. So, the total budget share for the "public" sector is the same. This budget spent on the public sector finances the costs of the total output of the three firms. Since the three firms have identical cost structure and have constant returns to scale, the total output of the three firms is equal for the two examples. However, in Example 1 all income spent on the public sector is spent on commodity a. Because of the constraints on the use of this commodity it results in a higher need for the public good than in Example 2.

Table 4. Equilibrium values Example 2.

	x_1	x_2	y_a	y_b	z	t_a^i	$\phi^i_\pi 0$
price	1	0.905	1.902	1.902	1.902		
mark-up t_b				0.136			
Producers:							
output	74.2	67.1	16.5	26.5	6.617		
				(28.3)			
input a	15.8	17.3					
b	25.2	27.8					
z	6.3	7.0					
Consumers:							
1	5.4	1.5	0.71	2.65		0	13.5
				(2.84)			
2	8.1	3.0	2.84	5.30		0	27.0
				(5.67)			
3	8.1	4.5	6.38	7.94		0	40.5
				(8.50)			
4	5.4	6.0	6.62	10.59		1.36	53.9
			(11.3)	(11.3)			

The results show that in the first example the consumers 3 and 4, being the consumers with the highest profit shares and the highest budget shares for a, are constrained in the use of the private commodity a. Notice that the sum of the mark-ups these consumers are willing to pay for an expansion of the infrastructure equals to the price of one unit of the public good.

In example 2 for each consumer i the sum of the (unconstrained) demands for a and b is equal to the (unconstrained) demand for a in the example 1. Observe that both the individual unconstrained demands for a and the total unconstrained demand for b are less than the corresponding constraint function values given the level of the infrastructure found in Example 1. So, for this level neither an individual nor the firm Y^b is willing to pay. Consequently, the infrastructure has been cut down to the level at which the mark-ups are again high enough to cover the costs. In equilibrium, only consumer 4 is constrained in the use of a. Moreover the production of firm Y^b is constrained by the infrastructure, which results in a mark-up t_b on the price od commodity b, so that $t_a^4 + 4t_b = p_z$, (planner-condition).

It is not difficult to gain some more insight from these examples. Decreasing the coefficients a^i will result in a higher willingness of the consumer to pay (see formula 5.11)). To remain in equilibrium this induces a higher value of z, so that the producer would become unconstrained for low enough values of the consumers' constraint coefficients. In this case the infrastructure is financed by the consumers' mark-ups only. For example, taking a^i=0.75 for all i, the equilibrium values of $t_a{}^i$, t_b and z become

$$t_a{}^i = 0 \text{ for } i=1,2;\ t_a{}^3 = 0.381;\ t_a{}^4 = 2.155;\ t_b = 0; \text{ and } z = 7.09.$$

On the other hand, increasing the coefficients a^i and/or decreasing the producer's constraint coefficient b^p results in a lower willingness of the consumer to pay and/or a higher mark-up t_b on the producer's price p_b. For a^i=0.75i for i=1,...,4, and b^p=2 we obtain that in equilibrium

$$t_a{}^i=0 \text{ for all } i,\ t_b=0.95 \text{ and } z=9.452.$$

In this case the infrastructure is completely financed through the mark-up the producer is willing to levy on his price p_b. Because the consumers are willing to spend 40% of their income on commodity b, the low constraint coefficient b^p enforces a (relatively) high level of z. In the first case the infrastructure can be seen as a public good for which the willingness to pay expresses the marginal utility. In the latter case the infrastructure can be seen as an investment of producer Y^b, without which the producer is not able to produce anything. For both alternatives the prices are equal to those given in the examples.

6. Concluding remarks and further research.

This paper has been concerned with the problem of financing an infrastructure needed for operating and utilizing private services and commodities. The paper has to be seen as a first attempt to give a solid framework for the idea that the industry plays a central role in financing the infrastructure. In fact, the infrastructure is financed through mark-ups on the private services and commodities that make use of it. These mark-ups come from the constraints experienced by the agents. With respect to the consumers, the level of the infrastructure yields a (subjective) constraint on their private consumption. In case of producers the level of the infrastructure puts a constraint on their production possibilities. The mark-ups reveal these constraints and therefore the need for the infrastructure. Given the mark-ups the agents are willing to pay, the planner determines the optimal level of the infrastructure. In subsequent papers we will develop this idea.

A first question concerns the existence of an industrial equilibrium and the way in which the optimal level is determined. We want to make some remarks on this topic. Therefore we return to the previous section, in which we formulated the market-condition and the planner-condition. To solve these equilibrium conditions we used a computer code based on simplicial approximation. We remark that the computational procedure adjusts prices and quantities until an approximate equilibrium has been found. It should be observed that all quantities are homogeneous of degree zero in prices and mark-ups. So, by setting $p_1=1$, we can take commodity 1 as the numeraire commodity. Then, for the examples considered in the previous section, the problem reduces to finding a price p_2 and a quantity z such that the market-condition holds for j=2 and the planner-condition is satisfied. Then there is also equilibrium on the numeraire market (Walras' property), since all consumers spend all their income. The algorithm adjusts p_2 and z simultaneously until (approximate) equilibrium values have been reached. So, numerically the price p_2 and the quantity z are determined simultaneously. However, from an economic viewpoint we may consider the following procedure. Suppose that, given p_2, the industry (or planner) solves the planner-condition, i.e., given p_2 the planner searches for a quantity z for which the planner-condition holds. Let $z(p_2)$ be this quantity as function of p_2. On the other hand, let the market solve the market-condition for j=2 given a quantity z. So, the market determines a price $p_2(z)$ for which the market for commodity 2 is in equilibrium. Starting with either some p_2 or some z, the quantity z and price p_2 are adjusted subsequently and alternately until a price p_2^* and a quantity z^* are found such that

$$z^* = z(p_2^*) \text{ and } p_2^* = p_2(z^*).$$

Such a pair (p_2^*,z^*) solves the equilibrium problem. Using this "Nash formulation", in a subsequent paper we will investigate the conditions for the existence of an industrial equilibrium. One of the issues showing up is whether the constraint functions have to satisfy certain conditions.

A second question concerns the problem of determining the mark-ups. We want to elaborate the idea that the individual mark-ups are determined by the industry and are corporated in the prices the producers set for their products. We may think of a partitioning of the consumers into a number of groups. Then for each group the industry sets the mark-ups by considering a representative agent. So, in this way we get different prices for different types of agents.

A third topic concerns the characterization of public goods by classifying the agents who pay for it. The examples have shown that within the same model the equilibrium may result in a situation in which either the consumers, or the producers, or both types of agents finance the infrastructure. This result urges us to be careful in making recommendations for the way in which the costs of public goods should be shared. In the near future we plan to do "cost-sharing" analysis for some (Dutch) "public sector" industries.

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