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ON UNIFORMIZATION FOR NONHOMOGENEOUS MARKOV CHAINS

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Abstract The standard method of uniformization for continuous-time Markov chains is shown to be generalizable to time-inhomogeneous Markov chains. A finite grid approximation is also provided.

Key-words Continuous-time Markov chain * uniformization * approximation.

1. Introduction

The technique of uniformization, as per the pioneering paper by Jensen [6] and also known as randomization, for transforming continuous-time Markov chains in discrete time Markov chains, is widely known as a powerful tool for evaluating continuous-time Markov chain applications such as naturally arising in computer performance evaluation, telecommunication and reliability. Particularly, over the last decade it has been extensively employed for computational and sensitivity analysis of both steady-state and transient measures (e.g. [3], [4], [5], [8], [10], [11], [14]).

However, beyond the essential requirement of uniformly bounded transition rates, the method, so far, seems to be limited to time-homogeneous chains. This latter restriction does seem justified for steady-state analysis but is much less natural when transient analysis is in order. For example, the availability or reliability of a system which is subject to breakdowns will generally deteriorate by age. A time-inhomogeneous version of uniformization, though, does not seem to be available.

This note aims to show that such a version can be concluded by relying upon a uniqueness result that can be obtained from the literature. Roughly speaking, the result is similar in spirit to the homogeneous case in that one can randomly sample Poissonian event-epochs at which transitions can take place according to conditional jump probabilities. The result seems of practical interest as:

(i) It suggests a computational tool by combining iterative numerical computations with random sampling or Monte-Carlo simulation.
(ii) Truncation error bounds are more easily concluded.

To enable practical computation or rather to avoid storage of a continuum of transition matrices, a finite grid approximation is also provided along with an error bound.

2. Model

Consider a continuous-time inhomogeneous Markov chain with state space $S = \{1,2,...,\}$ and at time t a transition rate $q_t(i,j)$ for a transition from state i into state $j\neq i$ where we allow $q_t(i,i)\geq 0$. Assume that these transition rates are continuous in t for all i,j as well as that for some constant β and all i,t:

$$(1)\qquad q_t(i) = \sum_j q_t(i,j) \leq \beta < \infty.$$

Definition. A family of transition probabilities $\{P_{s,t} | 0\leq s\leq t<\infty\}$ is said to be Markov, hereafter called transition semigroup, if for all i, j, s, t and v:

$$(2)\qquad P_{s,s+t+v}(i,j) = \sum_k P_{s,s+t}(i,k)\, P_{s+t,s+t+v}(k,j).$$

Based on this Markov (or semigroup) property the following uniqueness result can be adopted from the literature.

Lemma 1. There exists a unique transition semigroup $\{P_{s,t} | 0 \le s \le t < \infty\}$ such that for all t and i,j:

$$(3) \qquad [P_{t,t+h}(i,j) - 1_{\{j=i\}}]h^{-1} \to q_t(i,j) - 1_{\{j=i\}} \textstyle\sum_j q_t(i,j)$$

as $h \to 0$, in weak convergent sense, that is with the left hand side of (3) uniformly bounded for all t, h, i and j, and where $1_{\{A\}}$ denotes an indicator of an event A.

Proof In analogy with theorem 4 at p.364 and the analysis at pp.347-353 and 364-366 of [2] the existence and uniqueness is shown by the construction:

$$(4) \qquad \begin{cases} P_{s,t}(i,j) = \sum_{k=0}^{\infty} P^k_{s,t}(i,j), \text{ where} \\ \\ P^0_{s,t}(i,j) = 1_{\{j=i\}} \exp\left[-{}_s\int^t q_u(i)du\right], \quad \text{and for} \quad n \ge 0: \\ \\ P^{k+1}_{s,t}\ (i,j) = {}_s\int^t \exp\left[-{}_s\int^u q_v\ (i)d_v\right]\left[\sum_m q_u(i,m)\ P^k_{u,t}(m,j)\right]du \end{cases}$$

As only difference with this reference we need to apply dominated rather than uniform convergence arguments in view of the weak rather than uniform convergence in (3). □

Remark 1. Theorem 4 at p. 364 of [2] also yields the construction of a Markov process $\{X_t | t \ge 0\}$ with transition probabilities $P_{s,t}$ given by (4) and right-continuous sample paths. As the finite-dimensional distributions are hereby uniquely determined, by virtue of theorem 14.5 of Billingsley [1], this Markov pocess is unique at the space of right-continuous sample paths, more precisely $D[0,\infty)$ (for the precise definition see [1] or [15]).

Remark 2. We prefer to use a weak convergent version in lemma 1 in order to avoid uniform continuity requirements for $q_t(i,j)$ in t.

Remark 3. Note that $q_t(i,i) > 0$ is allowed. This will be utilized below.

3. Uniformization

For any $t \ge 0$ define the matrix P_t by

$$(5) \qquad P_t(i,j) = \begin{cases} q_t(i,j)\beta^{-1} & , (j \ne i) \\ 1 - \sum_{j \ne i} q_t(i,j)\beta^{-1}, & (j=i) \end{cases}$$

which represents the uniformized transition probabilities at time t. Now, consider a continuous-time Markov chain with transition rates $\bar{q}_t(i,j) = \beta P_t(i,j)$ for all t and i,j (j=i included). Then by virtue of lemma 1 with $q_t(i,j)$ replaced by $\bar{q}_t(i,j)$ and

(6) $\quad \bar{q}_t(i) = \sum_j \bar{q}_t(i,j) = \beta \sum_j P_t(i,j) = \beta$

there exists a unique transition semigroup $(U_{s,t}|0 \le s \le t < \infty)$ satisfying (3) with $P_{s,t}$ and $q_t(i,j)$ replaced by $U_{s,t}$ and $\bar{q}_t(i,j)$ and, by (4),

$$(7) \quad \begin{cases} U_{s,t}(i,j) = \sum_{k=0}^{\infty} U^k_{s,t}(i,j), \text{ where} \\ U^0_{s,t}(i,j) = 1_{\{j=i\}} \exp[-(t-s)\beta], \text{ and for } n\geq 0 \\ U^{k+1}_{s,t}(i,j) = {}_s\int^t \exp[-(v-s)\beta]\beta \; [\sum_m P_v(i,k) \; U^k_{v,t}(m,j)]dv \end{cases}$$

Theorem 1 For all $0 \le s \le t < \infty$

$$(8) \quad P_{s,t} = U_{s,t} =$$

$$\sum_{k=0}^{\infty} \frac{[(t-s)\beta]^k}{k!} e^{-(t-s)\beta} \underset{(t_1\le \ldots \le t_k)}{{}_s\int^t \ldots {}_s\int^t} P_{t_1} P_{t_2} \ldots P_{t_k} d\, \bar{H}^k(t_1,\ldots,t_k)$$

where $\bar{H}^k(\ldots)$ is the density of the order statistics $X_{(1)} \le X_{(2)} \le \ldots \le X_{(k)}$ of a homogeneous distribution at $[s,t]\times\ldots\times[s,t]\subset R^k$ and where $P_{t_1}P_{t_2}\ldots P_{t_k}$ is the standard product matrix of transition probability matrices[1] (5) at times $t_1 \le t_2 \le \ldots \le t_k$.

Proof First note that for $t \le s+h$

$$\sum_{n=2}^{\infty} U^n_{s,t} \le [\beta(t-s)]^2/2! \le h^2 C$$

while

$$[U^1_{s,s+h}(i,j) + U^0_{s,s+h}(i,j)]h^{-1} - 1_{\{j=i\}} =$$

$$[1 + O(h)]h\beta \; [P_s(i,j) + o(1)] + [1 - h\beta + O(h^2)] \; 1_{\{j=i\}} - 1_{\{j=i\}} =$$

$$h[1+o(1)]q_s(i,j)\beta^{-1}1_{\{j\ne i\}} + 1_{\{j=i\}}h\beta[1+o(1)][1-\textstyle\sum_{j\ne i} q_s(i,j)\beta^{-1}]-1_{\{j=i\}}$$

where $O(h)h^{-1}\to 0$ and $o(1)\to 0$ as $h\to 0$. As a consequence, the weak convergence relation (3) of lemma 1 is satisfied if we replace $P_{s,t}$ by $U_{s,t}$. Lemma 1 thus proves the first equality of (8). To prove the second, by (7)and repetition we obtain for any k:

$$U^k_{s,t} = {}_0\int^t \exp[-(\tau_1-s)] \; \beta P_{\tau_1} \Big[{}_{\tau_1}\int^t \exp[-(\tau_2-\tau_1)\beta] \; \beta P_{\tau_2} [\ldots.$$

$$[{}_{\tau_{k-1}}\int^t \exp[-(\tau_k-\tau_{k-1})\beta] \; \beta P_{\tau_k} [{}_{\tau_k}\int^t \exp[-(t-\tau_k)\beta]] \ldots \Big] d\tau_1 d\tau_2 \ldots d\tau_k$$

$$= e^{-(t-s)\beta} \quad \beta^k \, {}_0\int^t {}_{\tau_1}\int^t \cdots {}_{\tau_{k-1}}\int^t P_{\tau_1} \ldots P_{\tau_k} d\tau_1 d\tau_2 \ldots d\tau_k$$

$$= e^{-(t-s)\beta} \frac{[(t-s)\beta]^k}{k!} \underset{\{t_1 \leq \ldots \leq t_k\}}{{}_s\int^t \cdots {}_s\int^t} P_{t_1} \ldots P_{t_k} d\bar{H}^k \ (t_1, \ldots, t_k)$$

as the density of a k-dimensional distribution of the order statistics $\tilde{X}_1 \leq \tilde{X}_2 \leq \ldots \leq \tilde{X}_k$ of k independent random variables $X_1, X_2, \ldots, X_k$, each homogeneously distributed at [s,t], is standardly known (e.g., pp.100-103 of [7] or exercise 29, p.240 of [12]) to be $k!/(t-s)^k$. □

Remark 4 Note that the homogeneous case is included by $P_{t_1} = P_{t_2} = \ldots = P_{t_k} = P$, so that with the above density $k!/(t-s)^k$ for outcomes $t_1 \leq t_2 \leq \ldots \leq t_k$ we obtain

$$(9) \qquad \underset{\{t_1 \leq t_2 \leq \ldots \leq t_k\}}{{}_s\int^t \cdots {}_s\int^t} d\,\bar{H}^k (t_1, \ldots, t_k) = \frac{k!}{(t-s)^k} \quad [{}_s\int^t {}_{t_1}\int^t \cdots {}_{t_{k-1}}\int^t dt_1 \ldots dt_n] = 1$$

Remark 5 The Poisson expansion (8) directly provides one the error bound

$$(10) \qquad ||\bar{P}^L_{s,t} - P_{s,t}|| \leq \frac{\lambda^L (t-s)^L}{L!}$$

if $\bar{P}^L_{s,t}$ is a truncated version with the sum in (8) truncated at k=L. Here $||.||$ denotes the standard supremum norm.

Remark 6 An approximate though efficient computation of the integrals in (8) might be established by employing (Monte Carlo) simulation as follows.

(i) Independently generate k random numbers $x_i \in [s,t]$.
(ii) Determine their order statistics $t_1 = x_{(1)} \leq t_2 \leq x_{(2)} \leq \ldots \leq t_k = x_{(k)}$.
(iii) Compute $P_{t_1} \ldots P_{t_k}$.
(iv) Repeat these steps a reasonably large number of times and determine the average of the obtained individual results (matrices or weighted values).

4. Approximation

The major drawback of the above representation (as well as any other) for nonhomogeneous calculations is the fact that a continuum of transition matrices (and thus rates) is needed. To overcome this, below we investigate a finite-grid approximation under the simple Lipschitz assumption (for a relaxation see remark 7 below) that for some constant K and all t:

$$(11) \qquad ||Q_{t+\Delta t} - Q_t|| \leq \Delta t K$$

where Q_t denotes the matrix of transition rates $q_t(i,j)$ with $q_t(i,i)=-q_t(i)$ while $||A||=\max_i \Sigma_j |a_{ij}|$ for a matrix A.

Let $h>0$ be arbitrarily chosen such that $\beta=h^{-1}$ satisfies (1) and set $n_i=[t_i h^{-1}]$, $i=1,\ldots,n$ where $[x]$ denotes the entier of a number x. Then

$$(12)\qquad ||P_{t_1}\ldots P_{t_k} - P_{n_1 h}\ldots P_{n_k h}|| =$$

$$||P_{t_1}(P_{t_2}\ldots P_{t_k}) - P_{n_1 h}(P_{n_2 h}\ldots P_{n_k h})|| =$$

$$||[P_{t_1}-P_{n_1 h}]\ (P_{t_2}\ldots P_{t_k}) - P_{n_1 h}\ [(P_{n_2 h}\ldots P_{n_k h})-(P_{t_2}\ldots P_{t_k})]|| \le$$

$$||[I + hQ_{t_1}] - [I + hQ_{n_1 h}]|| + ||P_{n_2 h}\ldots P_{n_k h} - P_{t_2}\ldots P_{t_k}|| \le$$

$$||P_{n_2 h}\ldots P_{n_k h}-P_{t_2}\ldots P_{t_k}|| + h^2 K \le \ldots \le kh^2 K.$$

where the latter inequality follows by iterating these steps for $t_1, t_2, \ldots, t_k$.

Theorem 2 For $s=nh$ and $t=mh$ let:

$$(13)\qquad U^h_{s,t} = \sum_{k=0}^{\infty} \frac{[(t-s)\beta]^k}{k!} e^{-[t-s]\beta} \left\{ \sum_{\substack{n\\ \{n_1\le n_2\le\ldots\le n_k\}}}^{m-1} \ldots \sum_n^{m-1} P_{n_1 h}\ldots P_{n_2 h}\ \bar{H}^k_d(n_1,\ldots,n_2)\right\}$$

where $\bar{H}^k_d(\ldots)$ is the probability mass distribution of the order statistics $n_1\le n_2\le\ldots\le n_k$ of k independent discrete homogeneous random variables at $\{0,1,\ldots,m-1\}$. Then

$$(14)\qquad ||U^h_{s,t} - P_{s,t}|| \le h\ (t-s)K.$$

Proof Noting that

$$(15)\qquad \sum_{\substack{n\\ \{n_1\le\ldots\le n_k\}}}^{m-1}\ldots\sum_n^{m-1} P_{n_1 h}P_{n_2 h}\ldots P_{n_k h}\ \bar{H}^k_d(t_1,\ldots,t_k) =$$

$$\int_{\substack{s\\ \{t_1\le\ldots\le t_k\}\\ \{n_i = [t_i h^{-1}]\}}}^{t} \ldots \int_s^t \quad P_{n_1 h}P_{n_2 h}\ldots P_{n_k h}\, d\bar{H}^k(t_1,\ldots,t_k)$$

while by (12): (also (see (9)):

$$(16)\quad \left|\left| \underset{\substack{\{t_1\le\ldots\le t_k\}\\ \{n_i=[t_ih^{-1}]\}}}{\int_s^t \ldots \int_s^t} [P_{t_1}P_{t_2}\ldots P_{t_k} - P_{n_1h}P_{n_2h}\ldots P_{n_kh}]\,d\hat{H}^k(t_1,\ldots,t_k)\right|\right| \le hK(kh)$$

we conclude from (8), (13) and substitution of $h=\beta^{-1}$:

$$(17)\quad ||U^h_{s,t} - U_{s,t}|| \le hK\beta^{-1}\Sigma_{k=0}^{\infty} ke^{-(t-s)\beta}[(t-s)\beta]^k/k! = hK(t-s). \qquad \square$$

Remark 7 Clearly, the Lipschitz condition (11) is simple in form but can be too strong. In fact, following the proof we have only used that for the chosen h-grid and all $s\le rh<t$:

$$(18)\quad ||Q_v - Q_{rh}|| \le hK \qquad (v\in[rh,rh+1)).$$

For example, with Q_v piecewise constant at intervals $[rh,rh+1)$, the Lipschitz condition (11) fails, while (18) and thus (14) hold with $K=0$.

Remark 8 Result (14) can be seen as a time-inhomogeneous extension of the Euler approximation (cf. Meis and Marcowitz [9])

$$[I + hQ]^n \to T_t \qquad (n=[th^{-1}])$$

for homogeneous semigroups $\{T_t|t\ge 0\}$, satisfying $d/dt\ T_t=QT_t=T_tQ$. As such, the probabilistic and direct proof by means of uniformization is of interest in itself.

Remark 9 In [13] an elegant method is proposed for approximating homogeneous transition probabilities of continuous-time Markov chains by inspecting the process at exponential times. This method resembles uniformization but is not the same. It turned out to be amazingly efficient. Along the lines of this section extension of this method to nonhomogeneous Markov processes seems promising.

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