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A note on
Monotonicity Results in Multicasting

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Abstract

In multicast communications responses (acknowledgements) from all recipients are to be returned to the sender. The sender buffers these responses in a finite storage buffer for a one by one processing. When this buffer is saturated responses are lost.

The mean number of successful responses per unit time is analytically proven to be monotone in the buffersize. This result is of practical assistance such as

(i) to support (e.g. reduce) numerical computations,

(ii) to determine a critical buffer size,

(iii) to apply optimal design.

Extensions to other performance measures are direct. A rough error bound on the marginal value of one buffer place will also be derived.

Keywords

Multicasting, Monotonicity result, Successful responses

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1 Introduction

Background

One of the many new technological developments in present-day telecommunications is the ability of sending a message (signal or packet) to a specific group of recipients (or destinations), called multicasting, rather than to just one receiver (unicasting) or to a total neighbouring environment (broadcasting). Substantial improvements of throughput and service quality are hereby achieved. However, as transmitting sources or responding recipients may interfere, design and control problems are also significantly increased.

A generic component of multicasting design is the determination of buffer capacity of a sender in order to process the responses (or acknowledgments) from recipients one by one. Recently, in [1] an elegant study on this topic was performed by using recursive Markov reward equations to numerically calculate the mean number of lost responses per multicast (i.e. the lost responses to one message). This study has been continued in [6] by showing that the underlying Markov chain can be explicitly solved recursively. Also other performance measures such as the mean number of successful (non-lost) responses per unit time are then obtained easily such as to trade off the buffersize against some cost structure.

Motivation

In executing such calculations and trading off different parameter values, a priori known structural results, that is monotonicity properties, of the performance measure of interest are helpful such as:

(i) to check the calculations,

(ii) to stop the numerical computations upon excess of a threshold value,

(iii) to determine a minimal buffer size to achieve a service level,

(iv) to obtain qualitative rather than only quantitative insights,

(v) to assist the optimization using monotonicity arguments.

Results

This note will illustrate how such monotonicity structures can be established analytically. More precisely, the mean number of successful responses per unit time will be shown to be monotone in the buffersize. This may seem obvious, but intuitively, an increased buffersize will not only increase the number of successful responses per multicast but also the total duration of the multicast, so that the resultant per unit time is no longer obvious. In fact a counterintuitive sample path realization example will be given.

In addition, the monotonicity prooftechnique will also be used to provide a rough upper bound on the marginal effect on one extra place in the buffer.

2 Model and Example

Consider a single sender which simultaneously transmits a message to N recipients. Without restriction of generality the actual transmission time is assumed to be 0. All of the N recipients have to respond by returning a response of receipt (acknowledgement) to the sender. The response times of these recipients are independent and exponentially distributed with parameter λ. The responses in turn are to be processed by the sender, one by one with an exponential processing time with parameter μ. To this end, the sender has a finite storage buffer for no more than b responses, the one in process included. When the buffer is saturated an incoming response is rejected and lost. Also referring to remark 3.3 below, for simplicity we further assume that the sender immediately multicasts a new message as soon as all recipients have responded and all responses are handled (either successfully processed or rejected).

We will be interested in qualitative or monotone behaviour of specific performance measures such as the mean number of successful responses per unit time when the buffersize b is increased. To motivate this interest, let us first consider a possible stochastic realization of the response and processing times.

Counterintuitive Example

With $N = 6$ assume that the following random generations for the response and processing times have been obtained. (Note that this would be possible, for example when, one runs a simulation under the exponential assumptions).

Recipient	1	2	3	4	5	6
Response time	5	7	12	18	24	30
Processing Time	4	26	4	3	5	18

The following realizations, when $b = 1$ and $b = 2$ would then be obtained. To clarify the notation, for example under $b = 2$ at time 18, recipient 4 responds but its response is rejected which leads to 1 lost response, as responses of recipients 2 and 3 are still present requiring 15 and 4 units time of processing respectively. The symbol ϕ stands for no responses being present.

$b = 1$

Time	Event	Present	Lost
5	$A(1)$	$(1,4)$	
7	$A(2)$	$(1,2)$	2
9	$D(1)$	ϕ	
12	$A(3)$	$(3,4)$	
16	$D(3)$	ϕ	
18	$A(4)$	$(4,3)$	
21	$D(4)$	ϕ	
24	$A(5)$	$(5,5)$	
29	$D(5)$	ϕ	
30	$A(6)$	$(6,18)$	
48	$D(6)$	ϕ, End	

$b = 2$

Time	Event	Present	Lost
5	$A(1)$	$(1,4)$	
7	$A(2)$	$(1,2)$	
		$(2,26)$	
9	$D(1)$	$(2,26)$	
12	$A(3)$	$(2,23)$	
		$(3,4)$	
18	$A(4)$	$(2,17)$	4
		$(3,4)$	
24	$A(5)$	$(2,11)$	5
		$(3,4)$	
30	$A(6)$	$(2,5)$	6
		$(3,4)$	
35	$D(2)$	$(3,4)$	
39	$D(3)$	ϕ, End	

$b=1$		$b=2$
Total Time : 48	$>$	Total Time : 39
Number Lost : 1	$<$	Number Lost : 3
Number Lost per unit time : 1/48	$<$	Number Lost per unit time : 3/39
Successful responses per unit time : 5/48	$>$	Successful responses per unit time : 3/39

In all respects, for this stochastic realization example a larger buffersize thus leads to reduction of performance. In the next section it will be proven that such counterintuitive results are impossible for expected values.

3 Comparison

In this section we aim to investigate whether the performance increases by increasing the buffersize. To this end a Markov reward technique will be employed as adopted from [5]. The actual verification of the essential conditions required for the present application is still to be studied which involves special technicalities. The presentation will therefore be kept self contained.

With (n,s) denoting the number of still to respond recipients for a specific multicast and s the number of occupied storage places (the one in process included), we first note that the model of section 2 constitutes an irreducible continuous-time Markov chain at

$S_b = \{(n,s)|n \le N, s \le b\}$ with transition rates

(3.1)
$$q([n,s],[n',s']) = \begin{cases} \mu & [n',s'] = [n,s-1] \quad (s \ge 1) \\ n\lambda & [n',s'] = [n-1,s+1] \quad (n \ge 1, s < b) \\ & [n',s'] = [n-1,b] \quad (n \ge 1, s = b) \end{cases}$$

To evaluate a specific performance measure L associated with a reward (cost) rate $r(n,s)$ per unit of time when the system is in state (n,s), we employ the uniformization technique (cf. [2], p 110) and define total reward functions $V_b^t, t = 0,1,2,\ldots$ by $V_b^0(n,s) = 0$ for all (n,s) and for $t \ge 0$:

(3.2)
$$\begin{aligned} V_b^{t+1}(n,s) &= \tfrac{r(n,s)}{Q} + \tfrac{\mu}{Q}1_{\{s>0\}}V_b^t(n,s-1) + \tfrac{n\lambda}{Q}1_{\{s<b\}}V_b^t(n-1,s+1) \\ &+ \tfrac{n\lambda}{Q}1_{\{s=b\}}V_b^t(n-1,b) + \left[1 - \tfrac{\mu}{Q} - \tfrac{n\lambda}{Q}1_{\{s<b\}} - \tfrac{n\lambda}{Q}1_{\{s=b\}}\right]V_b^t(n,s) \end{aligned}$$

where $Q = [\mu + N\lambda]$ and where $1_{\{A\}} = 1$ if event A is satisfied and 0 otherwise.

By virtue of this uniformization technique we then obtain

(3.3)
$$L = \lim_{t\to\infty} \frac{Q}{t} V_b^t(n,s)$$

for arbitrary initial state (n,s) at $t = 0$. For example, we obtain

(3.4)
L^1 : steady state probability $\pi(0,0)$by $r^1(n,s) = 1_{\{(n,s)=(0,0)\}}$
L^2 : mean number of lost responses per unit time by $r^2(n,s) = n\lambda 1_{\{s=b\}}$
L^3 : mean number of successful responses per unit time by $r^3(n,s) = n\lambda 1_{\{s<b\}}$

Also noting that the total time per multicast T is given by $T = 1/[n\lambda\pi(0,0)]$, performance measures per multicast are thus included.

Comparison Result

Without restriction of generality (see remark 3.1) consider $r = r^3$ in (3.2) and for convenience, we introduce the probability matrix notation P_b (not to be confused with a power matrix like P^b) to rewrite (3.2) as

(3.5)
$$V_b^{t+1}(n,s) = n\lambda 1_{\{s<b\}}Q^{-1} + P_b V_b^t(n,s)$$

Then by comparing the system with buffersizes b and $b+1$, for $(n,s)\epsilon S^b$ we can write

(3.6)
$$\begin{aligned} &\left(V_{b+1}^{t+1} - V_b^{t+1}\right)(n,s) \\ &= n\lambda 1_{\{s=b\}}Q^{-1} + \left(P_{b+1}V_{b+1}^t - P_bV_b^t\right)(n,s) \\ &= n\lambda 1_{\{s=b\}}Q^{-1} + (P_{b+1} - P_b)V_{b+1}^t(n,s) + P_b(V_{b+1}^t - V_b^t)(n,s) \end{aligned}$$

where the latter step is justified as V_{b+1}^t is well defined at S_b. Further, form (3.2) and (3.5), we conclude

(3.7)
$$(P_{b+1} - P_b)V_b^t(n,s) = n\lambda 1_{\{s=b\}}Q^{-1}\left[V_b^t(n-1,b+1) - V_b^t(n-1,b)\right]$$

Also note that $P_b g \geq 0$ for any vector $g \geq 0$ (both $\geq$ signs componentwise). As a consequence, by using Lemma 3.2 below and assuming that (induction hypothesis):

$$(V_{b+1}^t - V_b^t) \geq 0 \qquad \text{at } S_b, \tag{3.8}$$

(3.6) and (3.7) would show that (3.8) also holds for t replaced by $t+1$. With $V_{b+1}^0 = V_b^t = 0$ (componentwise), by induction we have thus proven (3.8) for all $t \geq 0$. Applying (3.3), for example with $(n,s) = (0,0)$ now yields:

Theorem 3.1

$$\boxed{L_{b+1}^3 \geq L_b^3} \tag{3.9}$$

Lemma 3.1 For any b, any $(n,s),(n,s+1)\epsilon S_b$ and all $t \geq 0$:

$$\boxed{0 \leq V_b^t(n,s) - V_b^t(n,s+1) \leq 1} \tag{3.10}$$

Proof For convenience we suppress the subscript b. The proof will follow by induction in t. Clearly (3.10) holds for $t = 0$ as $V^0 = 0$. Assume that (3.10) holds for $t \leq m$. Then, by (3.2):

$$\begin{aligned} V^{m+1}(n,s) - V^{m+1}(n,s-1) &= \Big\{n\lambda 1_{\{s<b\}}Q^{-1} + n\lambda 1_{\{s<b\}}Q^{-1}V^m(n-1,s+1) \\ &+ \mu Q^{-1}1_{\{s\geq 1\}}V^m(n,s-1) + [1 - n\lambda 1_{\{s<b\}}Q^{-1} - \mu Q^{-1}1_{\{s\geq b\}}] \\ &V^m(n,s)\Big\} - \Big\{n\lambda 1_{\{s+1<b\}}Q^{-1} + n\lambda 1_{\{s+1<b\}}Q^{-1}V^m(n-1,s+2) \\ &+ n\lambda 1_{\{s+1=b\}}Q^{-1}V^m(n-1,b) + \mu Q^{-1}V^m(n,s) \\ &+ [1 - n\lambda 1_{\{s+1=b\}}Q^{-1} - n\lambda 1_{\{s+1=b\}}Q^{-1} - \mu Q^{-1}]V^m(n,s+1)\Big\} \end{aligned} \tag{3.11}$$

By substituting $1_{\{s<b\}} = [1_{\{s+1<b\}} + 1_{\{s+1=b\}}]$ and $1_{\{s\geq 1\}} + 1_{\{s=0\}} = 1$, and comparing terms pairwise, this can be rewritten as:

$$\begin{aligned} V^{m+1}(n,s) - V^{m+1}(n,s+1) &= n\lambda 1_{\{s+1=b\}}Q^{-1} \\ &+ n\lambda 1_{\{s+1<b\}}Q^{-1}[V^m(n-1,s+1) - V^m(n-1,s+2)] \\ &+ n\lambda 1_{\{s+1=b\}}Q^{-1}[V^m(n-1,b) - V^m(n-1,b)] \\ &+ \mu Q^{-1}\lambda 1_{\{s\geq 1\}}[V^m(n,s-1) - V^m(n,s)] \\ &+ \mu Q^{-1}n\lambda 1_{\{s=0\}}[V^m(n,0) - V^m(n,0)] \\ &+ [1 - n\lambda 1_{\{s+1=b\}}Q^{-1} - n\lambda 1_{\{s+1=b\}}Q^{-1} - \mu Q] \\ &[V^m(n,s) - V^m(n,s+1)] \end{aligned} \tag{3.12}$$

Here indeed the third and fifth term in the right hand side are equal to 0 but kept in for clarity and an argument below. Substituting the lower estimate 0 from (3.10) for $t = m$ one directly verifies $V^{m+1}(n,s) - V^{m+1}(n,s+1) \geq 0$. By substituting the upper estimate 1 from (3.10) for $t = m$, noting that the first positive term:

$$n\lambda 1_{\{s+1=b\}}Q^{-1}$$

is exactly equal to the coefficient of the third term which is equal to 0 and recalling that all coefficients sum up to 1, we also verify: $V^{m+1}(n,s) - V^{m+1}(n,s+1) \leq 1$. This proves (3.10) for $t = m$. The induction completes the proof.

Remark 3.1 (Other measures) By choosing a different reward rate function similar results can be expected for other measures along the same lines of the proof. For example, with r^2 from (3.4) the monotonicity in (3.9) will be just opposite as the bounds 0 and 1 in (3.10) are to be replaced by -1 and 0 respectively.

Remark 3.2 (Nonexponential times) By using mixtures of Erlang distributions the proof should in principle be extendable to general nonexponential processing and response times. The technicalities however can become quite complicated (cf [3]). Another possible approach to deal with nonexponential times is to use sample path techniques with coupling arguments. The present Markov reward technique is preferred herein as it is rather direct and also leads to marginal value bounds as per the next section.

Remark 3.3 (Immediate multicasting) The assumption of an immediate new multicast upon completion of handling all responses may not be realistic in certain applications but nevertheless be justifiable for the following reasons:

- Random (say exponential) holding times for a next multicast are easily included without essentially affecting the proof.
- It is one natural form which takes into account both the effect of losses per multicast and the duration of multicasts. As such the result and proof can be seen as representative also for other forms.
- Even if only a single multicast is to be considered the assumption of immediate repetitive multicases allows computation per multicast by the transformation mentioned.

A rough bound on the marginal value.

As a further application of the prooftechnique employed, a rough upper bound will be derived on the marginal value of one extra buffer place. Though very rough it provides a first indication of order of magnitude or a stop criterion for investigation alternative buffersizes.

To this end, conclude from equation (3.6), (3.7), theorem 3.1 and lemma 3.2 for any $(n,s)\epsilon S_b$:

(3.13) $$0 \leq (V_{b+1}^t - V_b^t)(n,s) \leq n\lambda 1_{\{s=b\}} Q^{-1} + P_b(V_{b+1}^{t-1} - V_b^{t-1})(n,s)$$

Now note that the transition matrix P_b remains restricted to S_b. As a consequence, with P_b^k the k-th power of matrix P_b and $\Phi(n,s)$ the function defined by:

$$\Phi(n,s) = n\lambda 1_{\{s=b\}} Q^{-1}$$

we obtain by iterating (3.13) for $t = N, N-1, \ldots, 0$ and using $V_{b+1}^0 = V_b^0 = 0$, that for any $(n,s) \epsilon S_b$:

(3.14) $$0 \leq (V_{b+1}^N - V_b^N)(n,s) \leq \sum_{k=0}^{N-1} P_b^k \Phi(n,s)$$

Furthermore, similarly to steps as in [4] one can prove that

(3.15) $$P_b^k \Phi(0,0) \leq P_b^{k+1}\Phi(0,0) \leq \lim_{k\to\infty} P_b^k \Phi(o,o) = Q^{-1} L_b^2$$

with L_b^2 the mean number of Lost responses per unit time. By (3.14), (3.15) and (3.3) we have thus concluded:

Theorem 3.2

(3.16) $$\boxed{0 \leq L_{b+1}^3 - L_b^3 \leq L_b^2}$$

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References

[1] Danzig, P.B. (1989), "Finite buffers and fast multicast", Performance Evaluation Review, Vol 17, 108-117.

[2] Tijms, H.C. (1986), "Stochastic modelling and analysis", Wiley, New York.

[3] Van Dijk, N.M. (1990), "A formal proof for the insensitivity of simple bounds for finite tandem queues", Stochastic Proc. Appl.

[4] Van Dijk, N.M. (1989), "A simple throughput bound for queueing networks with finite capacities", Performance Evaluation.

[5] Van Dijk, N.M. (1990), "The importance of bias-terms for error bounds and comparison result", Proceedings of First International Workshop on Numerical Solutions of Markov Chains.

[6] Yunus, N. (1990), "A queueing model for buffer overflows in multicast communications", Research Report, Centre for Teletraffic Research, Bond University.

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