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BATCH-ARRIVAL $M^X/G/1$ QUEUE

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Simple approximations for the batch-arrival $M^X/G/1$ queue.

by

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Abstract. In this paper we consider the $M^X/G/1$ queueing system with batch arrivals. We give simple approximations for the waiting-time probabilities of individual customers. These approximations are numerically checked and found to perform very well for a wide variety of batch-size distributions and service-time distributions.

0. Introduction.

Batch-arrival queueing models arise in many practical situations. In general it is difficult, if not impossible, to find tractable expressions for the waiting-time probabilities of individual customers. It is therefore useful to have easily computable approximations for these probabilities. The present paper gives such approximations for the single server $M^X/G/1$ model.

Exact methods for the computation of the waiting-time distribution in the $M^X/G/1$ queue are discussed in Eikeboom and Tijms [1987], cf. also Chaudry and Templeton[1983], Neuts[1981] and Tijms[1986]. However these methods apply only for special service-time distributions and are in general not suited for routine calculations in practice. Also a simple approximation for the tail probabilities of the waiting time was given in Eikeboom and Tijms [1987] by using interpolation of the asymptotic expansions for the particular cases of deterministic and exponential services. This approximative approach uses only the first two moment of the service time since the interpolation is based on the squared coefficient of variation of the service time. This paper presents an alternative approach that uses the actual service-time distribution rather then only its first two moments. In Van Ommeren [1988] it is shown that the complementary waiting-time probability has an exponentially fast decreasing tail under some mild assumptions. By calculating the decay parameter and the amplitude factor we get the asymptotic expansion of the waiting-time distribution. For nonlight traffic this asymptotic expansion can be used as a first-order approximation for the waiting-time probabilities. Next, by incorporating exact results for other quantities such as the delay probability and the first two moments of the waiting time we are able to give an improved second-order approximation. This approximative method performs very well for a wide range of values of the traffic intensity and the coefficients of variation of the service-time distribution and the batch-size distribution.

The organization of this paper is as follows. In section 1 the model is defined and some preliminaries, including the asymptotic expansion of the waiting-time distribution, are given. The second-order approximation is given in section 2. In section 3 we give numerical results and discuss the performance of the approximations. The appendix deals with the motivation of the second-order approximation.

1. The model and preliminaries.

In the $M^X/G/1$ queue customers arrive in batches and are served individually by a single server. The batches arrive according to a Poisson process with rate λ. The number of customers in the batches are independent and identically distributed positive random variables. Denote the number of customers in a batch by X and the probability distribution of X by $g_i := \Pr\{X=i\}$, $i=1,2,\ldots$. The generating function of $\{g_i\}$ is denoted by $G(z):=\sum_{j=1}^{\infty} g_j z^j$. The service times of individual customers are independent identically distributed random variables. Denote the service time of a customer by S and the distribution of S by $B(x):=\Pr\{S\leq x\}$. We assume that $B(0)=0$ and $B'(0):=\lim_{t\downarrow 0} B(t)/t$ exists. We denote the Laplace-Stieltjes transform of $B(.)$ by $\hat{B}(s):=\int_0^{\infty} e^{-st}dB(t)$. Let $\hat{Q}(s)$ denote the Laplace-Stieltjes transform of the total amount of service time required by the customers belonging to one batch. It follows that

$$\hat{Q}(s)=G(\hat{B}(s)).$$

The offered load is denoted by $\rho:=\lambda E(S)E(X)$ and it is assumed that $\rho<1$.

Customers belonging to different batches are served in order of arrival, while customers belonging to the same batch are served according to their random position in the batch. Let D_n denote the delay in queue of the n-th served customer . The limit distribution $\lim_{n\to\infty}\Pr\{D_n\leq x\}$ exists only when the batch-size distribution $\{g_i\}$ is aperiodic (i.e. when the g.c.d. $\{j\,|\,g_j>0\}=1$), cf. Cohen [1976]. As counter-example, consider a constant batch size of 2 in which case $\Pr\{D_{2k}=0\}=0$ for all $k\geq 1$ and $\lim_{k\to\infty}\Pr\{D_{2k+1}=0\}=(1-\rho)$. In Van Ommeren[1988] it is proved that the following limit

$$W_q(t) := \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \Pr\{\, D_j\leq t \,\}, \qquad t\geq 0,$$

always exists. Note that $W_q(t)$ represents with probability 1 the long-run fraction of customers having a waiting time of no more than t. Denote the Laplace-Stieltjes transform of $W_q(.)$ by $\hat{W}_q(s):=\int_0^{\infty} e^{-st}dW_q(t)$. From Cohen[1976] we can easily get the following theorem.

Theorem 1.1. The Laplace-Stieltjes transform of the stationary waiting-time distribution $W_q(.)$ is given by $\hat{W}_q(s)=\hat{W}_1(s)\hat{W}_2(s)$ with

$$\hat{W}_1(s) = \frac{(1-\rho)s}{s-\lambda(1-\hat{Q}(s))} ,$$

and

$$\hat{W}_2(s) = \frac{1-\hat{Q}(s)}{E(X)(1-\hat{B}(s)} .$$

From the Laplace-Stieltjes transform of the stationary waiting-time distribution we can easily derive exact results for the delay probability, the derivative of $W_q(x)$ at $x=0$ and the first two moments of the waiting time. These results that will be needed for a second-order approximation to $W_q(.)$ are given by

$$(1.1)\quad W_q(0) = \lim_{s\to\infty} \hat{W}_q(s) = \frac{1-\rho}{E(X)} ,$$

$$(1.2)\quad W_q'(0) = \lim_{s\to\infty} s(\hat{W}_q(s)-W_q(0)) = \frac{(1-\rho)[(1-g_1)B'(0)+\lambda]}{E(X)} ,$$

$$(1.3)\quad \int_0^\infty (1-W_q(t))dt = -\hat{W}_q'(0) = -\hat{W}_1'(0)-\hat{W}_2'(0) ,$$

and

$$(1.4)\quad \int_0^\infty t(1-W_q(t))dt = \tfrac{1}{2}\hat{W}_1''(0) + \hat{W}_1'(0)\hat{W}_2'(0) + \tfrac{1}{2}\hat{W}_2''(0);$$

where the derivatives should be interpreted as the right derivatives in $t=0$. In order to evaluate the right hand sides of (1.3) and (1.4) we have to use l'Hôpital's rule repeatedly to obtain

$$\hat{W}_1'(0) = \frac{-\lambda\hat{Q}''(0)}{2(1-\rho)\hat{B}'(0)} , \qquad \hat{W}_1''(0) = \frac{-\lambda\hat{Q}'''(0)}{3(1-\rho)} + 2[\hat{W}_1'(0)]^2 ,$$

$$\hat{W}_2'(0) = \frac{\hat{Q}''(0)-E(X)\hat{B}''(0)}{2E(X)\hat{B}'(0)} ,$$

and

$$\hat{W}_2''(0)= \frac{2[\hat{Q}'''(0)\hat{B}'(0)-\hat{Q}'(0)\hat{B}'''(0)]\hat{B}'(0)-3[\hat{Q}''(0)\hat{B}'(0)-\hat{Q}'(0)\hat{B}''(0)]\hat{B}''(0)}{6E(X)(\hat{B}'(0))^3} ,$$

where

$$\hat{B}'(0)=-E(S),\qquad \hat{B}''(0)=E(S^2),\qquad \hat{B}'''(0)=-E(S^3),$$

$$\hat{Q}'(0)=-E(S)E(X),\qquad \hat{Q}''(0)=E(X(X-1))E(S)^2+E(X)E(S^2),$$

and

$$\hat{Q}'''(0)=-E(X(X-1)(X-2))E(S)^3-3E(X(X-1))E(S)E(S^2)-E(X)E(S^3).$$

Here it is assumed that the service time and the batch size have finite third moments.

To give the asymptotically exponential expansion of the complementary waiting-time distribution $1-W_q(t)$ for $t\to\infty$, we need a mild assumption on the service-time and batch-size distributions. Roughly speaking the assumption requires that these distributions should have no extremely long tails. More precisely we make the following assumption.

Assumption. The power series $G(z)=\Sigma_{j=1}^{\infty} g_j z^j$ has a convergence radius $R>1$, the integral $\hat{B}(s)=\int_0^\infty e^{-st}dB(t)$ has an abscis of convergence $A<0$, and there exists a $T\geq A$ with $\lim_{s\downarrow T}\hat{B}(s)\leq R$ and $\lim_{s\downarrow T}[G(\hat{B}(s))]^{-1}=0$.

This assumption is satisfied in most cases of practical interest, e.g. when the service-time distribution is of the phase type and the batch-size distribution has a finite support. Under this assumption the following theorem is proved in Van Ommeren[1988].

Theorem 1.2. The stationary waiting-time distribution $W_q(.)$ satisfies

$$(1.5)\qquad \lim_{t\to\infty} e^{\beta t}(1-W_q(t)) = \alpha \ ,$$

where β is the smallest positive solution to

$$\lambda(\hat{Q}(-\beta)-1)=\beta,$$

and α is defined by

$$\alpha := \frac{(1-\rho)\beta}{\lambda E(X)(\lambda\hat{Q}'(-\beta)+1)(1-\hat{B}(-\beta))} .$$

2. Approximations.

The asymptotic expansion stated in theorem 1.2. suggests the following first-order approximation to $W_q(t)$:

$$1-W_q(t) \approx \alpha e^{-\beta t} \qquad \text{for t large enough.}$$

This approximation gives practically useful results already for moderate values of x when the traffic load is not low. The performance of this first-order approximation improves as ρ gets larger. As a rule of thumb, in terms of the p-th percentile of the waiting-time distribution function $W_q(t)$, the first-order approximation can be used also for practical purposes when $p \geq 1-\rho(1-W_q(0))$, cf. also the numerical results in section 3.

A refinement of the first-order approximation for the complementary waiting-time distribution $1-W_q(t)$ that can be used for all values of t, is given by

$$(2.1)\quad \overline{W}_{app}(t) = \alpha e^{-\beta t} + \gamma e^{-\delta t} + \eta e^{-\varphi t} , \qquad t \geq 0.$$

Here α and β are the coefficients of the asymptotic expansion (1.5). A motivation of this approximation with three (rather than two) exponential terms is as follows. A close look at the derivation of the asymptotic expansion reveals that the (complex) poles and the residues at these points of the Laplace-Stieltjes transform of the waiting-time distribution determine the behaviour of this distribution. The pole with the largest negative real part is simple and real and gives a first-order approximation. The poles which have the second largest negative real parts lead to a second-order approximation. However in general it is difficult to find these poles since they no longer have to be real. We therefore try to determine γ,δ,η and φ by matching the exact explicit results for the delay probability, the derivative of $W_q(x)$ at $x=0$ and the first two moments of the stationary waiting time. This yields the relations

(2.2) $\overline{W}_{app}(0) = 1-W_q(0)$,

(2.3) $\overline{W}'_{app}(0) = -W'_q(0)$,

(2.4) $\int_0^\infty \overline{W}_{app}(t)dt = \int_0^\infty (1-W_q(t))dt$,

and

(2.5) $\int_0^\infty t\overline{W}_{app}(t)dt = \int_0^\infty t(1-W_q(t))dt$.

Here it is in principle allowed that the numbers γ,δ,η and φ are complex. In case they are not real, they should be complex conjugates (i.e. $\delta=\overline{\varphi}$ and $\gamma=\overline{\eta}$) and we get a cosinus term in (2.1). Furthermore we require that both $\mathrm{Re}(\delta)>\beta$ and $\mathrm{Re}(\varphi)>\beta$ hold, since otherwise the asymptotically exponential expansion would be violated for t large. In view of the fact that the poles may be complex it is preferable to approximate $1-W_q(t)$ by three exponential terms rather than by two exponential terms. Indeed the numerical investigations indicate that for smaller values of t the three-term approximation to $1-W_q(t)$ performs usually much better than the two-term approximation.

In support to the approximation (2.1) we make the following observations. It follows from results in Van Ommeren [1988] that the approximation is exact when the service time has a K_2-distribution and the batch size is geometrically distributed. If the service time has a K_3-distribution and the batch size is geometrically distributed the approximation is also exact when the function $\lambda(\hat{Q}(-s)-1)-s$ has no double zero as is usually the case.

In order to give closed forms expressions for γ,δ,η and φ, we define the constants

(2.6) $C_1 := (1-W_q(0))-\alpha, \quad C_2 := -W'_q(0)-\alpha\beta,$

$$C_3 := \int_0^\infty (1-W_q(t))dt-\alpha/\beta, \quad C_4 := \int_0^\infty t(1-W_q(t))dt-\alpha/\beta^2,$$

and

$$\Delta := (C_1C_3-C_2C_4)^2-4(C_3^2-C_4C_1)(C_1^2-C_2C_3).$$

Note that the C_i's represents the deviations of the "moments" of $1-W_q(x)$ and the "moments" of the first-order approximation $\alpha e^{-\beta t}$. We can give a simple scheme for computation of the numbers γ,δ,η and φ. For clarity of presentation we give here only the approximation and refer to the appendix for a motivation of these results. Let β_0 denote some constant with $\beta_0>\beta$, e.g. $\beta_0=2\beta$. We have to distinguish between the following cases:

case i) $C_1=C_2=C_3=C_4=0$. In this case the first-order approximation matches already the four proposed condition (2.2) to (2.5), and we set $\gamma=\eta=0$ and $\delta=\varphi=\beta_0$.

case ii) $C_3\neq 0$, $C_1/C_3>\beta$, $C_1^2=C_2C_3$ and $C_3^2=C_1C_4$. In this case we only need two exponential terms in the approximation (2.1) and we set $\gamma=C_1$, $\delta=C_1/C_3$, $\eta=0$ and $\varphi=\beta_0$.

case iii) $C_3^2\neq C_1C_4$, $(C_1C_3-C_2C_4)/2(C_3^2-C_4C_1)>\beta$ and $0<\Delta<[(C_1C_3-C_2C_4)-2(C_3^2-C_4C_1)\beta]^2$. We then get three exponential terms in the approximation (2.1) where the numbers γ,δ,η and φ are all real and are given by

$$\delta=\frac{C_1C_3-C_2C_4+\sqrt{\Delta}}{2(C_3^2-C_4C_1)}, \qquad \varphi=\frac{C_1C_3-C_2C_4-\sqrt{\Delta}}{2(C_3^2-C_4C_1)},$$

$$\gamma=\frac{C_2-\varphi C_1}{\delta-\varphi}, \text{ and } \qquad \eta=\frac{C_2-\delta C_1}{\varphi-\delta}.$$

case iv) $C_3^2\neq C_1C_4$, $(C_1C_3-C_2C_4)/2(C_3^2-C_4C_1)>\beta$ and $\Delta<0$. In this case the numbers γ,δ,η and φ are complex and $\gamma=\overline{\eta}$ and $\delta=\overline{\varphi}$. This gives the following representation for the approximation (2.1):

$$\overline{W}_{app}(t) = \alpha e^{-\beta t} + \gamma^* \cos(\varphi^* t+\psi^*)e^{-\delta^* t},$$

where $\delta^*=\mathrm{Re}(\delta)$, $\varphi^*=\mathrm{Im}(\delta)$ and γ^* are given by

$$\delta^*=\frac{C_1C_3-C_2C_4}{2(C_3^2-C_4C_1)}, \qquad \varphi^*=\frac{\sqrt{-\Delta}}{2(C_3^2-C_4C_1)},$$

$$\gamma^*=\sqrt{C_1^2+((C_2-\delta^*C_1)/\varphi^*)^2},$$

and ψ^* is defined by $\cos(\psi^*)=C_1/\gamma^*$ and $\sin(\psi^*)=(\delta^*C_1-C_2)/\gamma^*\varphi^*$.

Remark 2.1. In most applications the constants C_1, C_2, C_3 and C_4 match one of these four cases. In case they do not, we propose to try the approximation (2.1) with two exponential terms (i.e. $\eta=0$), where the approximation matches the exact results for the delay probability and the first moment of the waiting time. This approximation is given by

$$\overline{W}_{app}(t)=\alpha e^{-\beta t} + C_1 e^{-(C_1/C_3)t}$$

and should of course satisfy the requirement that $C_3\neq 0$ and $\beta<C_1/C_3$. If this requirement is also not satisfied we propose to use the asymptotically exponential expansion. However, in our numerical investigations we never found that the approximation with two exponential terms did not work when the approximation with three exponential terms could not be used.

Remark 2.2. For the $M^X/D/1$ queue with deteministic services it may be hazardous to use the above approximations, in particular when the traffic load is low. Due to batches that arrive in an empty system the waiting-time distribution has a positive mass at nD when $\Sigma_{k=n+1}^{\infty} g_k\neq 0$. The effect of this phenomenon is considerable when the traffic load is low but becomes less important for high traffic when most batches arrive in a non-empty system. It was shown in Eikeboom and Tijms[1987] that the total mass of the waiting time distribution at the discrete points nD with $n\geq 1$ equals $(1-\rho)(1-1/E(X))$. Since the approximations given in this paper are all continuous, it cannot be expected that they perform well in the case of deterministic services and low traffic. In this particular case we therefore suggest to use the approximative method for system with constant service times which is given in Eikeboom and Tijms[1987]. Next assume that the service-time distribution has a similar shape as the constant distribution, i.e. there is a large probability that the service time is within a relatively narrow interval, for instance a distribution with a small coefficient of variation. For these systems we have the same effect as in the case that the service times are constant: when ρ is small the waiting-time distribution shall have "most" of its mass in narrow intervals. This explains the fact that our approximation performs slightly less for the E_{10}-distribution and $\rho=0.2$ (see section 3).

3. Numerical results.

In this section we present numerical results for various models. We consider four different batch-size distributions: i) the constant batch size ($C_X^2=0$), ii) the uniformly distributed batch size ($C_X^2=E(X-1)/3E(X)$), iii) the geometrically distributed batch size ($C_X^2=E(X-1)/E(X)$) and iv) a batch size with a mixed-geometric distribution with balanced means, where C_X^2 is taken equal to 2. A batch-size distribution $\{b_n, n\geq1\}$ is said to be a mixed-geometric distribution with balanced means when $b_n=qp_1(1-p_1)^n+(1-q)p_2(1-p_2)^n$, $n\geq1$, with $q/p_1=(1-q)/p_2$. Here C_X^2 denotes the squared coefficient of variation of the batch size X (i.e. the ratio of the variance to the squared mean). For the service time S of a customer we consider the Erlang-10 distribution ($C_S^2=1/10$), the Erlang-2 distribution ($C_S^2=1/2$) and the hyper-exponential distribution of order 2 with balanced means where $C_S^2=2$ is taken for the latter distribution. In all cases we have taken $E(S)=1$.

The numerical results are displayed by using the waiting-time percentiles. Here it is convenient to use the percentile $\nu(p)$ of the conditional waiting-time distribution of the delayed customer rather than the percentiles $\xi(p)$ of the unconditional waiting-time distribution $W_q(.)$, since the former percentiles are defined for all $0<p<1$. Note that $\nu(p)$ is determined by $(1-(W_q(\nu(p))/(1-W_q(0))=1-p$ and thus $\xi(p_0)=\nu(p_1)$ when $p_0=1-(1-p_1)(1-W_q(0))$. The numerical investigations reveal that for nonlight traffic the first-order approximation can be used already for relatively small values of t. In terms of the conditional waiting-time percentile $\nu(p)$, the first-order approximation $(1/\beta)\ln(\alpha/(1-p)\rho)$ to $\nu(p)$ can be used for practical purposes when $p\geq1-\rho$. This rule of thumb reflects the fact that the performance of the first-order approximation improves as ρ gets larger. The numerical results show the excellent performance of the second-order approximation to $W_q(t)$ for all values of t. Therefore this approximation is well-suited for practical purposes because it combines accuracy with ease of computation.

Table 3.1. Conditional waiting-time percentiles when E(X)=2.

			$E_{10}, C_S^2=0.1$				$E_2, C_S^2=0.5$				$H_2, C_S^2=2$			
		C_X^2	0.00	0.17	0.50	2.00	0.00	0.17	0.50	2.00	0.00	0.17	0.50	2.00
ρ=0.2	p=0.2	asy	0.56	0.80	0.85	0.52	0.41	0.68	0.75	0.38	0.00	0.00	0.00	0.00
		app	0.70	0.82	0.88	1.18	0.48	0.61	0.72	1.06	0.24	0.33	0.46	0.81
		exa	0.75	0.83	0.90	1.11	0.48	0.60	0.72	1.06	0.24	0.34	0.46	0.82
	p=0.5	asy	0.94	1.28	1.73	3.34	0.95	1.31	1.76	3.32	0.00	0.05	0.97	2.98
		app	1.07	1.35	1.73	3.35	1.04	1.34	1.76	3.41	0.84	1.12	1.57	3.38
		exa	1.07	1.35	1.74	3.38	1.04	1.34	1.76	3.41	0.84	1.12	1.57	3.37
	p=0.8	asy	1.71	2.22	3.46	8.85	2.01	2.54	3.73	9.03	2.15	2.85	4.25	9.64
		app	1.64	2.24	3.46	8.85	2.04	2.58	3.73	9.03	2.53	3.13	4.35	9.70
		exa	1.63	2.27	3.46	8.85	2.04	2.58	3.73	9.03	2.52	3.11	4.35	9.70
	p=0.9	asy	2.28	2.94	4.76	13.01	2.81	3.48	5.21	13.36	4.18	4.97	6.73	14.67
		app	2.23	2.92	4.76	13.01	2.82	3.50	5.21	13.36	4.31	5.04	6.75	14.68
		exa	2.27	2.88	4.76	13.01	2.81	3.49	5.21	13.36	4.30	5.05	6.75	14.69
ρ=0.5	p=0.2	asy	0.72	0.92	1.04	0.86	0.63	0.84	0.98	0.83	0.00	0.00	0.21	0.53
		app	0.83	0.97	1.05	1.41	0.67	0.83	0.98	1.39	0.47	0.61	0.79	1.32
		exa	0.85	0.95	1.05	1.33	0.67	0.83	0.98	1.39	0.47	0.62	0.79	1.32
	p=0.5	asy	1.52	1.88	2.47	4.68	1.65	2.02	2.62	4.82	1.50	1.95	2.74	5.21
		app	1.50	1.89	2.47	4.68	1.66	2.03	2.62	4.86	1.83	2.21	2.88	5.40
		exa	1.48	1.92	2.47	4.70	1.66	2.04	2.62	4.86	1.83	2.21	2.88	5.40
	p=0.8	asy	3.08	3.76	5.26	12.14	3.65	4.32	5.79	12.59	5.50	6.20	7.68	14.34
		app	3.09	3.76	5.26	12.14	3.65	4.32	5.79	12.59	5.52	6.21	7.68	14.36
		exa	3.09	3.76	5.26	12.14	3.65	4.32	5.79	12.59	5.52	6.21	7.68	14.36
	p=0.9	asy	4.27	5.18	7.37	17.78	5.16	6.06	8.19	18.48	8.52	9.42	11.41	21.25
		app	4.27	5.18	7.37	17.78	5.16	6.06	8.19	18.48	8.53	9.42	11.41	21.25
		exa	4.27	5.18	7.37	17.78	5.16	6.06	8.19	18.48	8.53	9.42	11.41	21.25
ρ=0.8	p=0.2	asy	1.40	1.68	2.02	2.95	1.47	1.74	2.12	3.12	1.32	1.65	2.18	3.58
		app	1.39	1.69	2.02	3.06	1.47	1.75	2.12	3.27	1.55	1.84	2.29	3.77
		exa	1.38	1.70	2.03	3.05	1.47	1.75	2.12	3.27	1.53	1.83	2.29	3.77
	p=0.5	asy	3.71	4.38	5.64	10.98	4.27	4.95	6.22	11.59	6.16	6.86	8.21	13.77
		app	3.71	4.38	5.64	10.98	4.27	4.95	6.22	11.59	6.16	6.86	8.21	13.78
		exa	3.71	4.38	5.64	10.98	4.27	4.95	6.22	11.59	6.16	6.86	8.21	13.78
	p=0.8	asy	8.21	9.66	12.69	26.65	9.75	11.20	14.20	28.09	15.58	17.02	19.96	33.63
		app	8.21	9.66	12.69	26.65	9.75	11.20	14.20	28.09	15.58	17.02	19.96	33.63
		exa	8.21	9.66	12.69	26.65	9.75	11.20	14.20	28.09	15.58	17.02	19.96	33.63
	p=0.9	asy	11.61	13.65	18.03	38.50	13.90	15.92	20.24	40.57	22.72	24.70	28.86	48.65
		app	11.61	13.65	18.03	38.50	13.90	15.92	20.24	40.57	22.72	24.70	28.86	48.65
		exa	11.61	13.65	18.03	38.50	13.90	15.92	20.24	40.57	22.72	24.70	28.86	48.65

Table 3.2. Conditional waiting-time percentiles when E(X)=5.

			$E_{10},C_S^2=0.1$				$E_2,C_S^2=0.5$				$H_2,C_S^2=2$			
		C_X^2	0.00	0.27	0.80	2.00	0.00	0.27	0.80	2.00	0.00	0.27	0.80	2.00
ρ=0.2	p=0.2	asy	1.64	2.29	1.71	0.00	1.53	2.18	1.58	0.00	0.00	0.93	0.73	0.00
		app	1.53	1.70	1.71	1.94	1.27	1.61	1.58	1.78	0.78	0.98	1.16	1.38
		exa	1.41	1.56	1.72	1.89	1.23	1.40	1.58	1.78	0.80	0.97	1.16	1.38
	p=0.5	asy	2.56	3.56	4.37	4.10	2.60	3.59	4.36	4.02	1.77	3.02	4.04	3.61
		app	2.69	3.45	4.37	5.78	2.62	3.37	4.36	5.82	2.29	3.05	4.11	5.79
		exa	2.84	3.51	4.37	5.78	2.70	3.44	4.36	5.82	2.28	3.06	4.11	5.79
	p=0.8	asy	4.36	6.03	9.57	16.97	4.68	6.34	9.79	17.10	5.28	7.10	10.49	17.54
		app	4.44	6.15	9.57	17.05	4.77	6.29	9.79	17.21	5.39	7.11	10.49	17.84
		exa	4.33	6.26	9.57	17.05	4.74	6.49	9.79	17.21	5.37	7.10	10.49	17.84
	p=0.9	asy	5.71	7.89	13.50	26.71	6.25	8.42	13.90	27.00	7.94	10.18	15.38	28.08
		app	5.72	8.02	13.50	26.71	6.31	8.40	13.90	27.01	7.96	10.19	15.38	28.13
		exa	5.54	7.86	13.50	26.71	6.23	8.47	13.90	27.01	7.97	10.19	15.38	28.13
ρ=0.5	p=0.2	asy	1.88	2.44	2.44	0.00	1.76	2.32	2.36	0.00	0.78	1.48	1.79	0.00
		app	1.94	2.25	2.44	2.82	1.73	2.12	2.36	2.74	1.29	1.62	1.98	2.44
		exa	1.93	2.17	2.44	2.78	1.74	2.04	2.36	2.74	1.30	1.63	1.98	2.44
	p=0.5	asy	3.81	4.98	6.71	8.98	3.91	5.08	6.81	9.08	3.87	5.14	7.03	9.37
		app	3.85	5.00	6.71	9.56	3.94	5.05	6.81	9.72	4.00	5.18	7.04	10.20
		exa	3.87	5.10	6.71	9.57	3.96	5.14	6.81	9.72	3.98	5.17	7.04	10.20
	p=0.8	asy	7.56	9.93	15.03	26.58	8.09	10.45	15.51	27.00	9.89	12.26	17.25	28.56
		app	7.56	9.95	15.03	26.59	8.09	10.45	15.51	27.01	9.89	12.27	17.25	28.61
		exa	7.59	9.90	15.03	26.59	8.09	10.43	15.51	27.01	9.89	12.27	17.25	28.61
	p=0.9	asy	10.40	13.67	21.33	39.90	11.25	14.51	22.08	40.55	14.44	17.65	24.98	43.08
		app	10.40	13.68	21.33	39.90	11.25	14.51	22.08	40.56	14.44	17.65	24.98	43.09
		exa	10.39	13.68	21.33	39.90	11.25	14.51	22.08	40.56	14.44	17.65	24.98	43.09
ρ=0.8	p=0.2	asy	3.49	4.39	5.47	6.22	3.50	4.41	5.52	6.34	3.30	4.28	5.57	6.67
		app	3.52	4.39	5.47	7.00	3.52	4.39	5.52	7.13	3.41	4.32	5.59	7.46
		exa	3.57	4.45	5.47	7.00	3.54	4.43	5.52	7.13	3.39	4.32	5.59	7.46
	p=0.5	asy	9.07	11.54	16.15	25.63	9.58	12.05	16.68	26.18	11.35	13.86	18.57	28.20
		app	9.07	11.54	16.15	25.64	9.58	12.05	16.68	26.20	11.35	13.86	18.57	28.23
		exa	9.06	11.54	16.15	25.64	9.58	12.05	16.68	26.20	11.35	13.86	18.57	28.23
	p=0.8	asy	19.95	25.48	36.98	63.47	21.44	26.96	38.43	64.87	27.03	32.53	43.90	70.17
		app	19.95	25.48	36.98	63.47	21.44	26.96	38.43	64.87	27.03	32.53	43.90	70.17
		exa	19.95	25.48	36.98	63.47	21.44	26.96	38.43	64.87	27.03	32.53	43.90	70.17
	p=0.9	asy	28.18	36.03	52.74	92.09	30.40	38.24	54.88	94.14	38.90	46.66	63.06	101.9
		app	28.18	36.03	52.74	92.09	30.40	38.24	54.88	94.14	38.90	46.66	63.06	101.9
		exa	28.18	36.03	52.74	92.09	30.40	38.24	54.88	94.14	38.90	46.66	63.06	101.9

Appendix. The derivation of the second-order approximation.

In this appendix we will motivate the approximation given in section 2. The proposed conditions (2.2) to (2.6) for the determination of the numbers γ,δ,η and φ in the approximation (2.1) lead to the following equations for these numbers:

(A.1) $C_1=\gamma+\eta$, $C_2=\gamma\delta+\eta\varphi$, $C_3=\gamma/\delta+\eta/\varphi$, and $C_4=\gamma/\delta^2+\eta/\varphi^2$,

(see (2.6) for the definition of the C_i's). In these four nonlinear equations we restrict the feasible (complex) numbers as follows: for $\gamma\neq0$ we require that $Re(\delta)>\beta$ and for $\eta\neq0$ we require that $Re(\varphi)>\beta$. Note from (2.1) to (2.6) that when $\gamma=0$ the number δ is not determined by (A.1) and hence can be taken as any real (or complex) number with $Re(\delta)>\beta$. The same applies for φ when $\eta=0$. Also it will be used below that the roles of γ and δ in (A.1) are interchangeable with the roles of η and φ respectively. In the following let β_0 denote some real constant with $\beta_0>\beta$, e.g. $\beta_0=2\beta$, and let $\Delta:=(C_1C_3-C_2C_4)^2-4(C_3^2-C_4C_1)(C_1^2-C_2C_3)$.

Theorem A.1. The four nonlinear equations in (A.1) have a solution if and only if one of the following four (exclusive) cases applies:

i) $C_1=C_2=C_3=C_4=0$;

ii) $C_3\neq0$, $C_1/C_3>\beta$, $C_1^2=C_2C_3$ and $C_3^2=C_1C_4$;

iii) $C_3^2\neq C_1C_4$, $(C_1C_3-C_2C_4)/2(C_3^2-C_1C_4)>\beta$ and $0<\Delta<[(C_1C_3-C_2C_4)-2(C_3^2-C_1C_4)\beta]^2$;

iv) $C_3^2\neq C_1C_4$, $(C_1C_3-C_2C_4)/2(C_3^2-C_1C_4)>\beta$, $\Delta<0$.

For the respective cases we have as solutions:

i)

(A.2) $\gamma=\eta=0$ and $\delta=\varphi=\beta_0$;

ii)

(A.3) $\gamma=C_1$, $\delta=C_1/C_3$, $\eta=0$ and $\varphi=\beta_0$;

iii) and iv)

(A.4) $$\delta=\frac{C_1C_3-C_2C_4+\sqrt{\Delta}}{2(C_3^2-C_4C_1)}, \qquad \varphi=\frac{C_1C_3-C_2C_4+\sqrt{\Delta}}{2(C_3^2-C_4C_1)},$$

$$\gamma = \frac{C_2 - \varphi C_1}{\delta - \varphi}, \text{ and } \quad \eta = \frac{C_2 - \delta C_1}{\varphi - \delta}.$$

Proof.

(a) Suppose $(\gamma,\delta,\eta,\varphi)$ is a solution to (A.1) satisfying the restriction stated below (A.1). First consider the case of $\gamma\eta=0$. Since the roles of γ and η are interchangeable it is no restriction to assume $\eta=0$. Then the set of equations (A.1) reduces to

(A.1.1) $\quad C_1=\gamma,\ C_2=\gamma\delta,\ C_3=\gamma/\delta$, and $C_4=\gamma/\delta^2$.

If $C_1=0$ it follows from (A.1.1) that all C_i's are zero and so $\gamma=\eta=0$ and $\delta=\varphi=\beta_0$ is a solution (case i). If $C_1\neq0$, then $\gamma\neq0$ and so by the convention made below (A.1) $\delta>\beta$ which implies $\delta\neq0$. Hence $C_1\neq0$ implies that all C_i's are unequal to zero and so $\delta=C_2/C_1=C_1/C_3=C_3/C_4$ implying the results for case ii). Secondly consider the case that $\gamma\eta\neq0$ and $\delta=\varphi$. By our convention $\delta\neq0$. The set of equations (A.1) reduces to

(A.1.2) $\quad C_1=\gamma+\eta,\ C_2=(\gamma+\eta)\delta,\ C_3=(\gamma+\eta)/\delta$, and $C_4=(\gamma+\eta)/\delta^2$.

This set of equations is identical to (A.1.1) with $\gamma'=\gamma+\eta$ and $\delta'=\delta$ and thus again one of the cases i) or ii) applies. Next we can replace the solution $(\gamma,\delta,\eta,\varphi)$ by a solution as in (A.2) or (A.3). Finally consider the left case of $\gamma\eta\neq0$ and $\delta\neq\varphi$. Since $\gamma\eta(\delta-\varphi)\neq0$ it is easily derived from (A.1) that (δC_1-C_2), (δC_3-C_1) and (δC_4-C_3) are all unequal to zero and that φ is equal to both $(\delta C_1-C_2)/(\delta C_3-C_1)$ and $(\delta C_3-C_1)/(\delta C_4-C_3)$. Thus δ must satisfy the relation

(A.6) $\quad (C_3^2-C_1C_4)\delta^2-(C_1C_3-C_2C_4)\delta+(C_1^2-C_2C_3)=0.$

By the interchangeability of the roles of δ and φ the same relation applies to φ, i.e.

(A.7) $\quad (C_3^2-C_1C_4)\varphi^2-(C_1C_3-C_2C_4)\varphi+(C_1^2-C_2C_3)=0.$

Hence, since $\delta\neq\varphi$ and $C_3^2-C_1C_4=\gamma\eta(1/\delta-1/\varphi)^2\neq0$, it follows from (A.6) and (A.7) that δ and φ can be taken as in (A.4). From the condition that

$\mathrm{Re}(\delta)>\beta$ and $\mathrm{Re}(\varphi)>\beta$ it easily follows that $(C_1C_3-C_2C_4)/2(C_3^2-C_1C_4)>\beta$ and $\Delta<[(C_1C_3-C_2C_4)-2(C_3^2-C_1C_4)\beta]^2$. We also must have $\Delta\neq0$ since otherwise $\delta=\varphi$. Next using (A.1) we find the equations $\gamma=(\varphi C_1-C_2)/(\varphi-\delta)$ and $\eta=(\delta C_1-C_2)/(\delta-\varphi)$. Note that η and γ are unequal to zero since (δC_1-C_2) and (φC_1-C_2) are unequal to zero. Hence $\gamma\eta\neq0$ and $\delta\neq\varphi$ imply the conditions of case iii) or case iv).

(b) By the construction of the solutions given in (a) it follows that under the conditions stated in case i) to iv) the corresponding solutions satisfy the nonlinear equations (A.1) with restrictions.

Remark A.1. Note that in case iv) of the previous theorem $\Delta<0$, which implies that the numbers γ,δ,η and φ are not real. In this case we have that $\gamma=\overline{\eta}$ and $\delta=\overline{\varphi}$ and therefore we also have that $\eta e^{-\varphi t}$ is the complex conjugate of $\gamma e^{-\delta t}$. In the remaining analysis we use the relations $e^{ix}=\cos(x)+i\sin(x)$ and $\theta\cos(x)+\omega\sin(x)=(\theta^2+\omega^2)^{\frac12}\cos(x+y)$ with $\theta^2+\omega^2>0$ and y such that $\cos(y)=\theta/(\theta^2+\omega^2)^{\frac12}$ and $\sin(y)=-\omega/(\theta^2+\omega^2)^{\frac12}$. After some algebra we find that $\gamma e^{-\delta t}+\eta e^{-\varphi t}=\gamma^*\cos(\varphi^* t+\psi^*)e^{-\delta^* t}$ where $\delta^*=\mathrm{Re}(\delta)$, $\varphi^*=\mathrm{Im}(\delta)$ and γ^* are given by

$$\delta^*=\frac{C_1C_3-C_2C_4}{2(C_3^2-C_4C_1)}\,,\qquad \varphi^*=\frac{\sqrt{-\Delta}}{2(C_3^2-C_4C_1)}\,,$$

$$\gamma^*=\sqrt{C_1^2+((C_2-\delta^*C_1)/\varphi^*)^2}\,,$$

and ψ^* is defined by $\cos(\psi^*)=C_1/\gamma^*$ and $\sin(\psi^*)=(\delta^*C_1-C_2)/\gamma^*\varphi^*$.

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