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# Growth Regressions and Economic Theory

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# Growth Regressions and Economic Theory

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## 1 Introduction

What is the theoretical basis for growth regressions? The answer would seem obvious: most textbooks start from the Solow model and show that the canonical growth regression can be derived from it by log-linearization around the steady state. That answer is unsatisfactory, for at least two reasons. First, in the Solow model capital accumulation is driven by an exogenous savings rate and hence has no choice-theoretic basis. Secondly, investment is unaffected by risk since the model is deterministic. Uncertainty enters only at the final stage of the derivation, when a stochastic error term is added - almost as an afterthought - to the estimating equation.

Both objections can, of course, be overcome. A Cass-Koopmans-Ramsey model in which intertemporal optimizaton determines the savings rate endogenously provides micro foundations for growth theory. Amending this model by introducing risk defines a class of stochastic Ramsey models, models with forward looking behavior in which investment decisions are taken under uncertainty.

In this paper we consider the relation between such models and the canonical growth regression. It is known that there exists at least one model in the class of stochastic Ramsey models which is consistent with the canonical (Barro-type) growth regression. However, we show that the canonical growth regression is consistent with more than one stochastic Ramsey model: the

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Ramsey representation is not identified. This impossibility result has disturbing implications. For example, the canonical growth regression is consistent with a model in which risk affects investment decisions only *ex post*, when the agent experiences a shock. In that model exposure to risk does not affect the agent's behavior *ex ante:* behavior is completely independent of the distribution of shocks. In particular, an increase in risk has no effect on today's savings rate. But the same regression is also consistent with models in which a change in risk does affect behavior *ex ante*. Clearly, if the model is not identified then major questions in growth theory cannot be unanswered.

We also show, however, that the model *is* identified if, in addition to income data, observations on consumption can be used. Hence structural form estimation of stochastic Ramsey models is in principle possible. This re-establishes a link between growth regressions and economic theory.

In the next section we present the model and state and prove a theorem on the impossibility of identification if only income observations are used. We also state and prove that identifiability is achieved if both income and consumption data are available. Section 3 concludes.

## 2 Identification of Stochastic Ramsey Models

Consider the following intertemporal optimization problem:

$$\begin{aligned} V(k_0, z_0) &= \max_{\{c_t, k_{t+1}\}} E \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad (1) \\ \text{subject to } \; k_{t+1} &= z_t f(k_t) + (1-\delta) x_t k_t - c_t \quad (t = 0, 1, 2, ...) \\ & \quad \text{given } k_0, z_0, x_0 \end{aligned}$$

where $c$ denotes consumption, $k$ the capital stock, $u$ the instantaneous utility function, $\beta$ a discount factor ($0 < \beta < 1$), $z$ an income shock, $x$ an asset shock, $f(k)$ the production function and $\delta$ the rate of depreciation ($0 < \delta \leqslant 1$). Time periods are identified by the subscript $t$. Shocks are serially independent and are continuously and identically distributed. The agent maximizes expected discounted utility taking the probability distribution $F(z, x)$ of the shocks $z, x$ as given.

At the time the agent decides on $c_t$ and $k_{t+1}$ the realizations $z_t, x_t$ are known.[1] We assume that $u(c)$ is increasing, strictly concave and continuously differentiable, and that it satisfies the Uzawa conditions. Finally, $f(k)$ is increasing, continuously differentiable and strictly concave.[2] Since $z$ and $f(k)$ appear only multiplicatively, we can scale $f(k)$ in such a way that $Ez_t = 1$. Similarly, we can choose $1 - \delta$ so as to ensure that $Ex_t = 1$.

If this problem has a solution the model can be written in recursive form as the Bellman equation:

$$V(k, z, x) = \max_{\tilde{k}} u(zf(k) + (1 - \delta)xk - \tilde{k}) + \beta EV(\tilde{k}, \tilde{z}, \tilde{x}) \tag{2}$$

with the associated policy function

$$\tilde{\varphi}(z, x, k) = \arg\max_{\tilde{k}} u(zf(k) + (1 - \delta)xk - \tilde{k}) + \beta EV(\tilde{k}, \tilde{z}, \tilde{x})$$

where $k$ and $\tilde{k}$ denote the capital stock at the beginning and the end of each period. In this form the model applies to every period so that the time subscripts can be suppressed. The policy function $\tilde{\varphi}$ maps the current $(z, x, k)$ into $\tilde{k}$, next period's $k$. A value function $V$ which satisfies the Bellman equation (2) for all $(k, z, x)$ is a solution to the original maximization problem (1); see e.g. Stokey and Lucas (1989, Theorem 9.2). Define wealth, $w$ as

$$w_t = z_t f(k_t) + (1 - \delta)x_t k_t \tag{3}$$

Note that $\tilde{\varphi}(z, k, x)$ can be written as $\varphi(w)$. Both this function $\varphi(w)$ and $h(w) = w - \varphi(w)$ are increasing.[3] Note that $k_t = \varphi(w_{t-1})$ and that $w_t$ satisfies the following stochastic difference equation:

$$w_{t+1} = z_{t+1} f(\varphi(w_t)) + (1 - \delta)x_{t+1}\varphi(w_t). \tag{4}$$

To avoid technicalities we will add the requirement that this Markov chain is irreducible for all initial points $w_0$.and has invariant distribution $\pi$ which does not have a mass point in 0.[4]

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[1] Obviously, this is somewhat restrictive: in a more general model $k_{t+1}$ would have to be chosen before the shocks were fully known.

[2] Our conditions ensure non-negativity of $c_t$ and $k_{t+1}$.

[3] Exercise 10.1 in Stokey and Lucas (1989) applies with minor modifications (notably to allow for the distinction between asset and income shocks).

[4] The irreducibility and invariance assumptions are necessary conditions for Theorem 4.1 in Tierney (1996) which we use below.

We call the class of models satisfying these conditions *stochastic Ramsey models.*

$V$ and $\varphi$ satisfy the first order condition[5]

$$u'(zf(k) + (1-\delta)xk - \varphi(\cdot)) = \beta E V_k(\varphi(\cdot), \tilde{z}, \tilde{x}) \tag{5}$$

and the envelope condition

$$V_k(k, z, x) = u'(zf(k) + (1-\delta)xk - \varphi(\cdot))(zf'(k) + (1-\delta)x). \tag{6}$$

The first condition equates the current marginal utility of consumption to its opportunity cost, the expected value of a future extra unit of capital. The second condition states that the marginal value of capital can be obtained by allocating an extra unit of capital to current uses only. The two conditions imply the Euler equation

$$u'(c_t) = \beta E_t u'(c_{t+1})(z_{t+1} f'(k_{t+1}) + (1-\delta)x_{t+1}) \tag{7}$$

where $E_t$ takes the expectation conditional on information up to time $t$.

An interesting special case (Stokey and Lucas, 1989, section 2.2; Obstfeld and Rogoff, 1996, section 7.4) is the one where $u(c) = a_0 \ln c$ - the case of unitary relative risk aversion - where capital depreciates fully within the period ($\delta = 1$) and where $f(k) = k^\alpha$, a Cobb-Douglas production function ($0 < \alpha < 1$). In this case, which we will denote *the logarithmic model*, the policy function is

$$\tilde{k} = \varphi(zf(k)) = \alpha\beta z k^a. \tag{8}$$

With one exception - the full depreciation assumption - this model is a stochastic version of the Solow growth model with income $y = zf(k)$ and the savings rate equal to $\alpha\beta$. Substituting $k_{t+1}$ for $\tilde{k}$ and $k_t$ for $k$, and taking logs gives

$$\ln k_{t+1} = \ln(\alpha\beta) + E \ln z + \alpha \ln(k_t) + \varepsilon_t,$$

where $\varepsilon_t = \ln z_t - E \ln z$. Multiplying by $\alpha$, we obtain the canonical growth regression[6]

$$\ln y_{t+1} - \ln y_t = [\alpha \ln(\alpha\beta) + E \ln z] + (\alpha - 1) \ln y_t + \varepsilon_{t+1}. \tag{9}$$

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[5] Partial derivatives are denoted by subscripts, e.g. $V_k = \partial V / \partial k$.

[6] See, for example, Barro and Sala-i-Martin (1995), chapter 11.

Hence within the class of stochastic Ramsey models there exists a model, the logarithmic model, which is consistent with the canonical growth regression specification. We define the latter more formally as the log-linear growth path

$$\ln y_{t+1} - \ln y_t = b + (a-1)\ln y_t + \varepsilon_{t+1}, \tag{10}$$

where $(0 < a < 1)$ and $b$ includes a country effect.[7]

In practice we observe time series for income $\{\tilde{y}_t\}$, and sometimes also on consumption $\{\tilde{c}_t\}$.[8] If these are taken to be generated by a model in the class of stochastic Ramsey models the question arises whether that model can be identified if estimates of the coefficients $a$ and $b$ were available. The following theorem states that this is impossible.

**Theorem 1** *If an income path $\{\tilde{y}_t\}$ satifies (10) and is generated by a model in the class of stochastic Ramsey models then the depreciation rate $\delta$ is identified ($\delta = 1$) but neither the production function $f$ nor the utility function $u$ is identified.*

**Proof.** see Appendix. ■

In textbooks the deterministic Ramsey model is often presented as a theoretical basis for the canonical growth regression. An immediate corollary of Theorem 1 is that this model is not identified.

The Theorem limits the usefulness of growth regressions. Since it is impossible to use the regression results to recover an underlying structural optimisation model it is, for example, not feasible to estimate the effects on growth of changes in technology. This may come as a surprise: the loglinearization used in textbooks suggests that the production function is identified. In fact it is not, let alone that the results can be used to calculate the effects of a change in technology (e.g. a change in the parameter $\alpha$ in the Cobb-Douglas case) on growth.

Why identification is desirable may be illustrated with a second example. Suppose the question at hand is whether an increase in risk (in the sense of a mean-preserving spread) affects growth. It is easily seen that the answer is negative for the logarithmic model: the policy function (8) involves $z$ but not its distribution. It is also known (e.g. Elbers *et al.*, 2002) that for other stochastic Ramsey models (e.g. with CES rather than Cobb-Douglas production functions) the effect of risk on growth can be substantial. Hence it is important to be able to identify the production function. Suppose all one could estimate is a loglinear approximation of the income dynamics, *i.e.* (10), then one would have to conclude that the question cannot be answered: the loglinear approximation destroys any distinction between models in which risk matters for growth and those in which there is no such effect.[9]

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[7] When the canonical growth regression is derived from a loglinearization of the Solow model the coefficient $a$ is not equal to the exponent $\alpha$, but it does satisfy $0 < a < 1$.

[8] In microeconomic applications there may be observations on the capital stock (e.g. Elbers *et al.*, 2002). This greatly simplifies identification of the structural model.

In some growth regressions the initial investment share is included as one of the regressors so that information on consumption is used at least implicitly.

Of course, some models can be identified from income observations alone; an obvious example is the class of Solow models. However, as Theorem 1 states, the class of stochastic Ramsey models cannot be so identified.

However, in many empirical applications observations are available not only on $y$ but also on $c$. The following theorem shows that this additional information is sufficient to ensure identification of stochastic Ramsey models.

**Theorem 2** *If income and consumption paths* $\{y_t, c_t\}$ *generated by a model in the class of stochastic Ramsey models are available then the model can be identified.*

**Proof.** see Appendix. ■

Clearly, this Theorem offers no comfort to those who consider identification of the structural model desirable but want to maintain the growth regressions tradition of using only income dynamics $\{\tilde{y}_t\}$. However, if that limitation is lifted the model is identified. Also, estimating a stochastic Ramsey model (while much more complicated than Barro-regressions) is feasible, using estimation by simulation (Elbers *et al.*, 2002, provide an example using micro data).

# 3 Conclusion

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[9] It might seem that there is a simple way out of this dilemma: the effect of risk on growth could be assessed directly by using some measure of risk as an additional regressor in a Barro-type regression. However, this introduces a specification error by imposing that the effect of risk on growth is independent of the level of income. Such separablity is in general not satisfied by stochastic Ramsey models.

If growth regressions are to be based in economic theory it seems reasonable to require that they must be consistent with a model in which investment is endogenous and investors are exposed to risk. This rules out both the Solow model (in which the savings rate is exogenous) and deterministic intertemporal optimisation models. We have suggested that the model should belong to the class of stochastic Ramsey models.

We have shown that that type of model cannot be identified from the canonical, Barro-type growth regression. This makes it impossible to use such regressions for answering key questions, e.g. regarding the effect of changes in technology or in risk on growth. Such questions require identification of the structural model. However, we have also shown that the model is indeed identified if observations on income and consumption are available. If the link with theory is considered important then the present practice of growth regressions should be abandoned in favor of estimating structural models.

# Appendix: Proofs of Theorems 1 and 2

**Theorem 1** *If an income path $\{\tilde{y}_t\}$ satifies (10) and is generated by a model in the class of stochastic Ramsey models then the depreciation rate $\delta$ is identified ($\delta = 1$) but neither the production function $f$ nor the utility function $u$ is identified.*

**Proof.** Using $y_{t+1} = z_{t+1} f(k_{t+1})$ and $k_{t+1} = \varphi(y_t + (1-\delta)x_t k_t)$ we find

$$E(y_{t+1}|y_t, k_t) = f(k_{t+1}) = f(\varphi(y_t + (1-\delta)x_t k_t)).$$

On the other hand from (10)

$$y_{t+1} = z_{t+1} B y_t^a$$

where $B = e^{b - E\ln z}$. Hence

$$E(y_{t+1}|y_t, k_t) = E(y_{t+1}|y_t) = B y_t^a.$$

This can only be true if $\delta = 1$, proving the first part of the theorem.

We prove the second part of the theorem by constructing a counterexample. Note that the policy function (8) of the logarithmic model is consistent with the loglinear income dynamics (10). Define $T(c) = c + (e^{-c/2} - 1)$. Then it is easily verified that the process described by (10) is also consistent with a stochastic Ramsey model with utility function $v(\underline{c}) \equiv u(c) = u(T^{-1}(\underline{c}))$, the same $F(z)$, the same depreciation rate ($\delta = 1$) and the production function parametrically defined by $\underline{k}= s - T((1-\alpha\beta)s)$, $f(\underline{k}) = (\alpha\beta s)^{\alpha}$. (Since $\delta = 1$ the distribution of $x$ is irrelevant.) ■

**Theorem 2** *If income and consumption paths $\{y_t, c_t\}$ generated by a model in the class of stochastic Ramsey models are available then the model can be identified.*

**Proof.** We proceed in two steps. First we show - for a class of models of which the stochastic Ramsey models are a subset - that from the distribution of observed income and consumption paths $\{\tilde{y}_t, \tilde{c}_t\}$ we can identify the production function $f(k)$, the accumulation function $\psi$, the depreciation rate $\delta$, and the distribution of the shocks $\{z_t, x_t\}$. We then show - for the class of stochastic Ramsey models - that the utility function $u$ and the discount factor $\beta$ are also identified.

Consider the class of dynamic models (DM) defined by:

$$\begin{aligned}
y_t &= z_t f(k_t) \\
k_{t+1} &= \psi(z_t f(k_t) + (1-\delta)x_t k_t) \\
c_t &= z_t f(k_t) + (1-\delta)x_t k_t - k_{t+1} \\
& \quad k_0, z_0, x_0 \text{ given} \\
& \quad \{z_t, x_t\} \text{ are continuous and i.i.d with mean 1} \\
& \quad \psi \text{ is differentiable and increasing} \\
h(w) &= w - \psi(w) \text{ is increasing}
\end{aligned}$$

where $f$ satisfies the same assumptions made for the class of stochastic Ramsey models and units of $k$ are chosen so that

$$\text{whenever } E(y_1|y_0, c_0) = \widehat{y}_1 \text{ then } k_1 = 1.$$

The accumulation function $\psi$ does not necessarily reflect an optimisation, unlike the function $\varphi$.

We distinguish between the observed process $\{\tilde{y}_t, \tilde{c}_t\}$ and the modelled process $\{y_t, c_t\}$. Since we assume that the observations are generated by a DM:

$$(y_0, y_1, .., y_t; c_0, c_1, .., c_t) \bumpeq (\tilde{y}_0, \tilde{y}_1, .., \tilde{y}_t; \tilde{c}_0, \tilde{c}_1, .., \tilde{c}_t)$$

it must be the case that (for $t \geqslant 1$):

$$f(k_t) = E(y_t|y_0, y_1, .., y_{t-1}; c_0, c_1, .., c_{t-1}) \bumpeq E(\tilde{y}_t|O_{t-1}).$$

where $O_{t-1}$ denotes the history of the observed process $\tilde{y}_0, \tilde{y}_1, .., \tilde{y}_{t-1}; \tilde{c}_0, \tilde{c}_1, .., \tilde{c}_{t-1}$. It follows that

$$z_t \bumpeq \tilde{y}_t / E(\tilde{y}_t|O_{t-1}).$$

Note that $c_t = h(y_t + (1-\delta)x_t k_t)$. It follows that

$$x_t \bumpeq \frac{h^{-1}(\tilde{c}_t) - \tilde{y}_t}{E([h^{-1}(\tilde{c}_t) - \tilde{y}_t]|O_{t-1})}.$$

It is convenient to construct the modelled process on the same probability space as the observed process. If we take $z_t = \tilde{y}_t/E(\tilde{y}_t|O_{t-1})$ (almost surely or a.s.) and $x_t = [h^{-1}(\tilde{c}_t) - \tilde{y}_t]/E([h^{-1}(\tilde{c}_t) - \tilde{y}_t]|O_{t-1})$ then $y_t = \tilde{y}_t$ (a.s.) and $c_t = \tilde{c}_t$ (a.s.) so we can drop the tilde without danger of confusion.

Now suppose there exists a second DM consistent with the observed process, denoted by underscored symbols. From the above we can construct the second process in such a way as to ensure that $f(k_t) = \underline{f}(\underline{k}_t)$ (a.s.) and $z_t = \underline{z}_t$ (a.s.) so that we can drop the underscoring of $z$. Since the production functions $f$ and $\underline{f}$ are increasing we can write $k_t$ as a function of $\underline{k}_t$ (and vice versa). From $c_t = \underline{c}_t = \tilde{c}_t$ (a.s.):

$$h(y_t + (1-\delta)x_t k_t) = \underline{h}(y_t + (1-\underline{\delta})\underline{x}_t\underline{k}_t) \text{ (a.s.)}$$

hence

$$y_t = I(y_t + (1-\underline{\delta})\underline{x}_t\underline{k}_t) - (1-\delta)x_t k_t \text{ (a.s.)} \tag{11}$$

where $I = h^{-1} \circ \underline{h}$.

Differentiating this last expression with respect to $z_t$ we obtain:

$$f(k_t) = I'(y_t + (1-\underline{\delta})\underline{x}_t\underline{k}_t) f(k_t) \text{ (a.s.)}$$

and hence

$$I'(y_t + (1-\underline{\delta})\underline{x}_t\underline{k}_t) = 1 \text{ (a.s.)}$$

so that the function $I$ satisfies

$$I(w) = a + w$$

where $a$ is a constant. We now show that $a = 0$ so that $h$ and $\underline{h}$ (and hence $\psi$ and $\underline{\psi}$) are identical.

From (11) it follows that

$$(1-\delta)x_t k_t - (1-\underline{\delta})\underline{x}_t \underline{k}_t = a \text{ (a.s.).} \tag{12}$$

Taking expectations (conditional on $O_{t-1}$):

$$(1-\delta)k_t - (1-\underline{\delta})\underline{k}_t = a \text{ (a.s.)}$$

and hence $k_t$ is a linear function of $\underline{k}_t$:

$$k_t = \frac{a}{1-\delta} + \frac{1-\underline{\delta}}{1-\delta}\underline{k}_t \text{ (a.s.).} \tag{13}$$

Similarly, from (12) it follows that $x_t$ is a linear function of $\underline{x}_t$:

$$x_t = \frac{a + (1-\underline{\delta})\underline{x}_t \underline{k}_t}{(1-\delta)k_t}.$$

Since $x_t$ and $\underline{x}_t$ are independently distributed from $k_t$ and $\underline{k}_t$, the ratio $k_t/\underline{k}_t$ must be constant. Using this in (13) gives $a = 0$ and since $\widehat{y}_1 = f(1) = \underline{f}(1)$ it follows that $\delta = \underline{\delta}$.

We conclude that the two DMs are identical: from observations $\tilde{y}$, $\tilde{c}$ we can identify the production function $f$, the accumulation function $\psi$, the parameter $\delta$, and the distributions of the shocks $z$ and $x$.[10]

It remains to identify the utility function $u$ and the discount factor $\beta$ for the subset of dynamic models in which capital accumulation is optimal, *i.e.* stochastic Ramsey models. For this second part of the proof we first show that the accumulation path can be characterized as a martingale. We then apply Doob's martingale convergence theorem to show that only one value of $\beta$ is compatible with optimality. The same theorem then implies uniqueness of the utility function $u$ (up to an affine transformation).

Note that $k_{t+1} = \varphi(w_t)$ and $c_t = h(w_t) = w_t - \varphi(w_t)$ are both increasing functions of $w_t$ so that we may write $c_t$ as a function of $k_{t+1}$ alone, say $c_t = a(k_{t+1})$. Define

$$q_t = u'(a(k_t))$$

$$B_t = \beta(z_t f'(k_t) + (1-\delta)x_t)$$

---

[10] Hence the class of DMs is identifiable irrespective of whether $\psi$ reflects a mechanistic process (as in the Solow model) or an optimisation.

then the Euler equation (7) may be written as

$$q_{t+1} = E_t B_{t+1} q_{t+2}.$$

We consider paths starting with positive $w_0$. By recursion, for all $T$,

$$\begin{aligned} q_{t+1} &= E_t B_{t+1} q_{t+2} \\ &= E_t B_{t+1}(E_{t+1} B_{t+2} q_{t+3}) \\ &= E_t E_{t+1}(B_{t+1} B_{t+2} q_{t+3}) \\ &= E_t(B_{t+1} B_{t+2} q_{t+3}) \\ &= .. \\ &= E_t B_{t+1} B_{t+2} .. B_{t+T} q_{t+T+1}. \end{aligned}$$

In particular,

$$\begin{aligned} q_1 &= E_0 B_1 B_2 .. B_T q_{T+1} \\ &= E B_1 B_2 .. B_T q_{T+1} \end{aligned}$$

since starting conditions are given. Define

$$S_t = B_1 B_2 .. B_t q_{t+1}.$$

Note that $E_t S_{t+T} = S_t$ and that $ES_t = E|S_t| = q_1$ (which is finite since $w_0 > 0$). Hence the sequence $\{S_t\}$ is a martingale and the expectation $E|S_t|$ is bounded. This allows us to apply Doob's martingale convergence theorem (Doob, 1953) and conclude that there exists a random variable $S$ such that

$$\lim_{t\to\infty} S_t = S \text{ a.s.}$$

Suppose there are two stochastic Ramsey models (characterized by $\beta$, $u$ and $\tilde{\beta}$, $\tilde{u}$ with $\beta > \tilde{\beta}$, without loss of generality) consistent with the observed process. Then, with probability 1:

$$\frac{\tilde{S}_t}{S_t} = (\frac{\tilde{\beta}}{\beta})^t \frac{\tilde{q}_t}{q_t} \to \frac{\tilde{S}}{S}. \tag{14}$$

Choose $0 < m < M < \infty$ for which $\lim_{t\to\infty} P\{m < w_t < M\} = \pi[(m, M)] > 0$.

From Theorem 4.1 in Tierney (1996) the Markov chain is positively recurrent so that $P\{m < w_t < M \text{ infinitely often}\} = 1$. Hence for $\omega \in A = \{m < w_t < M \text{ infinitely often}\}$ there exists a sequence $t_k$ such that

$$(\frac{\tilde{\beta}}{\beta})^{t_k} \frac{\tilde{q}_{t_k}}{q_{t_k}} \leq (\frac{\tilde{\beta}}{\beta})^{t_k} \frac{\tilde{u}'(h(m))}{u'(h(M))}. \tag{15}$$

Note that the RHS converges to 0 so that $\tilde{S}(\omega)/S(\omega) = 0$ and hence $\tilde{S}(\omega) = 0$.[11] This implies $\tilde{u}'(c_1) = 0$ and this would violate the Uzawa conditions. Hence $\beta = \tilde{\beta}$.

To prove identification of $u(c)$ (up to an affine transformation) we must show that $\tilde{u}'(c) = \gamma u'(c)$ for some positive constant $\gamma$. Let $w^1 \neq w^2$ be in the support of $\pi$ and $c^i = h(w^i)$ the associated consumption values. We will use the fact that $h(w)$ is strictly monotonic and continuous. Assume that $\tilde{u}'(c^1)/u'(c^1) \neq \tilde{u}'(c^2)/u'(c^2)$. By continuity of $u'(c)$ and $\tilde{u}'(c)$ and $h(w)$ there exists $\delta_0 > 0$ such that for all $w \in B^1 = (w^1 - \delta_0, w^1 + \delta_0)$ and $\hat{w} \in B^2 = (w^2 - \delta_0, w^2 + \delta_0)$ we have

$$\left| \frac{\tilde{u}'(h(w))}{u'(h(w))} - \frac{\tilde{u}'(h(\hat{w}))}{u'(h(\hat{w}))} \right| > \varepsilon_0 = \frac{1}{2} \left| \frac{\tilde{u}'(c^1)}{u'(c^1)} - \frac{\tilde{u}'(c^2)}{u'(c^2)} \right| > 0. \tag{16}$$

Since $w^1$ and $w^2$ are in the support of $\pi$, it follows that $\pi(B^i) > 0$ and $P\{w_t \in B^i \text{ infinitely often}\} = 1$. Also, the set $A^1 = \{\omega | \tilde{q}_t(\omega)/q_t(\omega) \to \tilde{S}(\omega)/S(\omega)\}$ has probability 1. Hence, with $A = A^1 \cap \{w_t \in B^1 \cap B^2 \text{ infinitely often}\}$, $P(A) = 1$.

Take $\underline{\omega} \in A$. Then there exist subsequences $t_k$ and $\tau_k$ such that $w_{t_k}(\underline{\omega}) \in B^1$ and $w_{\tau_k}(\underline{\omega}) \in B^2$ for all $k$. Note that $\tilde{q}_{t_k}(\underline{\omega})/q_{t_k}(\underline{\omega}) - \tilde{q}_{\tau_k}(\underline{\omega})/q_{\tau_k}(\underline{\omega}) \to 0$ since $\underline{\omega} \in A^1$ while $|\tilde{q}_{t_k}(\underline{\omega})/q_{t_k}(\underline{\omega}) - \tilde{q}_{\tau_k}(\underline{\omega})/q_{\tau_k}(\underline{\omega})| > \varepsilon_0 > 0$ by equation (16). Hence we have a contradiction and it follows that $\tilde{u}'(c)/u'(c)$ is constant for all $c \in \{h(w) | w \in \text{support}(\pi)\}$. Denote the proportionality constant by $\gamma$. Note in particular from equation (14) that also $\tilde{S}/S = \gamma$ with probability 1. To prove that $\tilde{u}'(c)/u'(c) = \gamma$ for arbitrary $c > 0$, consider starting the DM process in $w_0 = h^{-1}(c)$. Now it immediately follows that

$$\tilde{u}'(c) = E\,\tilde{S} = \gamma E S = \gamma u'(c).$$

This completes the proof. ■

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[11] Since $ES$ is finite the case $S(\omega) = \infty$ cannot arise.

## References

Barro, Robert J. and Xavier Sala-i-Martin (1995), *Economic Growth,* New York: McGraw-Hill.

Doob, Joseph L. (1953), *Stochastic Processes*, New York: John Wiley.

Elbers, Chris, Jan Willem Gunning and Bill Kinsey (2002), ‘Convergence, Shocks and Poverty: Micro Evidence of Growth under Uncertainty’, Tinbergen Institute Discussion Paper 035/2.

Obstfeld, Maurice and Kenneth Rogoff (1996), *Foundations of International Macroeconomics*, Cambridge, Mass.: MIT Press.

Stokey, Nancy L. and Robert E. Lucas Jr. (1989), *Recursive Methods in Economic Dynamics*, Cambridge, Mass.: Harvard University Press.

Tierney, Luke (1996),‘Introduction to General State-Space Markov Chain Theory’, in W.R. Gilks, S. Richardson and D.J. Spiegelhalter (eds.), *Markov Chain Monte Carlo in Practice*, London: Chapman and Hall