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published in
Operator Theory, Analysis and the State Space Approach
2018

DOI (link to publisher)
10.1007/978-3-030-04269-1_8

document version
Publisher's PDF, also known as Version of record

citation for published version (APA)
Frazho, A. E., & Ran, A. C. M. (2018). A note on inner-outer factorization of wide matrix-valued functions. In H. Bart, S. ter Horst, A. C. M. Ran, & H. J. Woerdeman (Eds.), *Operator Theory, Analysis and the State Space Approach: In Honor of Rien Kaashoek* (pp. 201-214). (Operator Theory: Advances and Applications; Vol. 271). Springer International Publishing Switzerland. https://doi.org/10.1007/978-3-030-04269-1_8

A note on inner-outer factorization of wide matrix-valued functions

A.E. Frazho and A.C.M. Ran

Dedicated to our friend and mentor Rien Kaashoek on the occasion of his eightieth birthday, with gratitude for inspiring and motivating us to work on many interesting problems.

Abstract. In this paper we expand some of the results of [8, 9, 10]. In fact, using the techniques of [8, 9, 10], we provide formulas for the full rank inner-outer factorization of a wide matrix-valued rational function G with H^∞ entries, that is, functions G with more columns than rows. State space formulas are derived for the inner and outer factor of G.

Mathematics Subject Classification (2010). Primary 47B35, 47A68; Secondary 30J99 .

Keywords. Inner-outer factorization, matrix-valued function, Toeplitz operators, state space representation.

1. Introduction

In this note, $\mathcal{E}$, $\mathcal{U}$ and $\mathcal{Y}$ are finite-dimensional complex vector spaces and $\dim\mathcal{Y} \leq \dim\mathcal{U}$. We will present a method to compute the inner-outer factorization for certain matrix-valued rational functions G in $H^\infty(\mathcal{U}, \mathcal{Y})$, defined on the closure of the unit disc. Computing inner-outer factorizations for the case when $\dim\mathcal{U} \leq \dim\mathcal{Y}$ is well developed and presented in [4, 5, 13] and elsewhere.

Recall that a function G_i is *inner* if G_i is a function in $H^\infty(\mathcal{E}, \mathcal{Y})$ and $G_i(e^{i\omega})$ is almost everywhere an isometry. (In particular, $\dim\mathcal{E} \leq \dim\mathcal{Y}$.) Equivalently (see, e.g., [5, 13]), G_i in $H^\infty(\mathcal{E}, \mathcal{Y})$ is an inner function if and only if the Toeplitz operator T_{G_i} mapping $\ell^2_+(\mathcal{E})$ into $\ell^2_+(\mathcal{Y})$ is an isometry. A function G_o is *outer* if G_o is a function in $H^\infty(\mathcal{U}, \mathcal{E})$ and the range of the Toeplitz operator T_{G_o} is dense in $\ell^2_+(\mathcal{E})$.

Let G be a function in $H^\infty(\mathcal{U}, \mathcal{Y})$. Then G admits a unique inner-outer factorization of the form $G(\lambda) = G_i(\lambda)G_o(\lambda)$ where G_i is an inner function in $H^\infty(\mathcal{E}, \mathcal{Y})$ and G_o is an outer function in $H^\infty(\mathcal{U}, \mathcal{E})$ for some intermediate space $\mathcal{E}$. Because $G_i(e^{i\omega})$ is almost everywhere an isometry, $\dim\mathcal{E} \leq \dim\mathcal{Y}$.

H. Bart et al. (eds.), *Operator Theory, Analysis and the State Space Approach*, Operator Theory: Advances and Applications 271, https://doi.org/10.1007/978-3-030-04269-1_8

Since G_o is outer, $G_o(e^{i\omega})$ is almost everywhere onto $\mathcal{E}$, and thus, $\dim \mathcal{E} \leq \dim \mathcal{U}$. By unique we mean that if $G(\lambda) = F_i(\lambda)F_o(\lambda)$ is another inner-outer factorization of G where F_i is an inner function in $H^\infty(\mathcal{L}, \mathcal{Y})$ and F_o is an outer function in $H^\infty(\mathcal{U}, \mathcal{L})$, then there exists an constant unitary operator Ω mapping $\mathcal{E}$ onto $\mathcal{L}$ such that $G_i = F_i\Omega$ and $\Omega G_o = F_o$; see [1, 5, 6, 13, 14, 15] for further details.

Throughout we assume that $\mathcal{U}$, $\mathcal{E}$ and $\mathcal{Y}$ are all finite dimensional. We say that G_i in $H^\infty(\mathcal{E}, \mathcal{Y})$ is a *square inner function* if G_i is an inner function and $\mathcal{E}$ and $\mathcal{Y}$ have the same dimension, that is, $G_i(e^{i\omega})$ is almost everywhere a unitary operator, or equivalently, G_i is a two-sided inner function. So if G_iG_o is an inner-outer factorization of G where G_i is square, then without loss of generality we can assume that $\mathcal{E} = \mathcal{Y}$.

We say that the inner-outer factorization $G = G_iG_o$ is *full rank* if G_i is a square inner function in $H^\infty(\mathcal{Y}, \mathcal{Y})$ and the range of T_{G_o} equals $\ell^2_+(\mathcal{Y})$. An inner-outer factorization $G = G_iG_o$ is full rank if and only if G_i is a square inner function and the range of T_G is closed. If G is a rational function, then G admits a full rank inner-outer factorization if and only if

$$G(e^{i\omega})G(e^{i\omega})^* \geq \epsilon I \quad (\text{for all } \omega \in [0, 2\pi] \text{ and some } \epsilon > 0); \tag{1.1}$$

see Lemma 3.1 below. Finally, if G in $H^\infty(\mathcal{U}, \mathcal{Y})$ admits a full rank inner-outer factorization, then $\dim \mathcal{Y} \leq \dim \mathcal{U}$.

Here we are interested in computing the inner-outer factorization for full rank rational functions G in $H^\infty(\mathcal{U}, \mathcal{Y})$. So throughout we assume that $\dim \mathcal{Y} \leq \dim \mathcal{U}$. Computing inner-outer factorizations when G does not admit a full rank factorization is numerically sensitive. (In this case, our algebraic Riccati equation may not have a stabilizing solution.) Moreover, if G does not admit a full rank inner-outer factorization, then a small H^∞ perturbation of G does admit such a factorization. (If G in $H^\infty(\mathcal{U}, \mathcal{Y})$, does not satisfy (1.1), then a "small random" rational H^∞ perturbation of G will satisfy (1.1).) First we will present necessary and sufficient conditions to determine when G admits a full rank inner-outer factorization. Then we will give a state space algorithm to compute G_i and then G_o. Finally, it is emphasized that this note is devoted to finding inner-outer factorizations for wide rational functions G in $H^\infty(\mathcal{U}, \mathcal{Y})$ when $\dim \mathcal{Y} \leq \dim \mathcal{U}$. Finding inner-outer factorizations when $\dim \mathcal{U} \leq \dim \mathcal{Y}$ is well developed and presented in [4, 5] and elsewhere.

2. Preliminaries

Let $R = \sum_{-\infty}^{\infty} e^{i\omega n} R_n$ be the Fourier series expansion for a function R in $L^\infty(\mathcal{Y}, \mathcal{Y})$. Then T_R is the Toeplitz operator on $\ell^2_+(\mathcal{Y})$ defined by

$$T_R = \begin{bmatrix} R_0 & R_{-1} & R_{-2} & \cdots \\ R_1 & R_0 & R_{-1} & \cdots \\ R_2 & R_1 & R_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ on } \ell^2_+(\mathcal{Y}). \tag{2.2}$$

The function R is called the *symbol* for T_R. Recall that the Toeplitz operator T_R is strictly positive if and only if there exists an $\epsilon > 0$ such that $R(e^{i\omega}) \geq \epsilon I$ almost everywhere. The Toeplitz operator T_G with symbol G in $H^\infty(\mathcal{U}, \mathcal{Y})$, is given by

$$T_G = \begin{bmatrix} G_0 & 0 & 0 & \cdots \\ G_1 & G_0 & 0 & \cdots \\ G_2 & G_1 & G_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} : \ell_+^2(\mathcal{U}) \to \ell_+^2(\mathcal{Y}), \tag{2.3}$$

where $G(\lambda) = \sum_0^\infty \lambda^n G_n$ is the Taylor series expansion for G about the origin. Moreover, if G is in $H^\infty(\mathcal{U}, \mathcal{Y})$, then the Hankel operator H_G mapping $\ell_+^2(\mathcal{U})$ into $\ell_+^2(\mathcal{Y})$ is defined by

$$H_G = \begin{bmatrix} G_1 & G_2 & G_3 & \cdots \\ G_2 & G_3 & G_4 & \cdots \\ G_3 & G_4 & G_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} : \ell_+^2(\mathcal{U}) \to \ell_+^2(\mathcal{Y}). \tag{2.4}$$

Finally, for G in $H^\infty(\mathcal{U}, \mathcal{Y})$ it is well know and easy to verify that

$$T_G T_G^* = T_{GG^*} - H_G H_G^*. \tag{2.5}$$

3. Inner-outer factorization

First a characterization of the existence of a full rank inner-outer factorization is presented.

Lemma 3.1. *Let G be a rational function in $H^\infty(\mathcal{U}, \mathcal{Y})$ where $\mathcal{U}$ and $\mathcal{Y}$ are finite-dimensional spaces satisfying $\dim \mathcal{Y} \leq \dim \mathcal{U}$. Then G admits a full rank inner-outer factorization if and only if*

$$G(e^{i\omega})G(e^{i\omega})^* \geq \epsilon I \qquad (\textit{for all } \omega \in [0, 2\pi] \textit{ and some } \epsilon > 0), \tag{3.6}$$

or equivalently, the Toeplitz operator T_{GG^*} is strictly positive.

Proof. Let $G = G_i G_o$ be the inner-outer factorization for G where G_i is an inner function in $H^\infty(\mathcal{E}, \mathcal{Y})$ and G_o is an outer function in $H^\infty(\mathcal{U}, \mathcal{E})$. Clearly,

$$G(e^{i\omega})G(e^{i\omega})^* = G_i(e^{i\omega})G_o(e^{i\omega})G_o(e^{i\omega})^* G_i(e^{i\omega})^*. \tag{3.7}$$

Because G_i is an inner function, $G(e^{i\omega})G(e^{i\omega})^*$ and $G_o(e^{i\omega})G_o(e^{i\omega})^*$ have the same nonzero spectrum and rank almost everywhere. The range of T_{G_o} equals $\ell_+^2(\mathcal{E})$ if and only if the operator $T_{G_o}T_{G_o}^*$ is strictly positive. If $T_{G_o}T_{G_o}^*$ is strictly positive, then $T_{G_o G_o^*} = T_{G_o}T_{G_o}^* + H_{G_o}H_{G_o}^*$ implies that $T_{G_o G_o^*}$ is also strictly positive. So if the range of T_{G_o} equals $\ell_+^2(\mathcal{E})$, then $G_o(e^{i\omega})G_o(e^{i\omega})^* \geq \epsilon I_{\mathcal{E}}$ for some $\epsilon > 0$.

In addition, if $G = G_i G_o$ is a full rank inner-outer factorization, then $G_i(e^{i\omega})$ is a unitary operator. In this case, equation (3.7) shows that (3.6) holds.

On the other hand, assume that (3.6) holds, or equivalently, the Toeplitz operator T_{GG^*} is strictly positive. Because G is rational, the range of H_G is finite dimensional. Using $T_G T_G^* = T_{GG^*} - H_G H_G^*$, we see that $T_G T_G^*$ equals a strictly positive operator T_{GG^*} minus a finite rank positive operator $H_G H_G^*$. Clearly, T_{GG^*} is a Fredholm operator with index zero. Since $T_G T_G^*$ is a finite rank perturbation of T_{GG^*}, it follows that $T_G T_G^*$ is also a Fredholm operator with index zero. In particular, the range of T_G is closed. Hence the range of T_{G_o} is also closed. Because $G(e^{i\omega})G(e^{i\omega})^*$ and $G_o(e^{i\omega})G_o(e^{i\omega})^*$ have the same rank and $\dim \mathcal{E} \leq \dim \mathcal{Y}$, equation (3.7) with (3.6) shows that $\mathcal{E}$ and $\mathcal{Y}$ are of the same dimension. In particular, G_i is a square inner function. Therefore the inner-outer factorization $G = G_i G_o$ is of full rank. $\square$

Next, we recall some results on the inner-outer factorization in terms of a stable finite-dimensional realization for a rational function G. To this end, let $\{A \text{ on } \mathcal{X}, B, C, D\}$ be a stable realization for G in $H^\infty(\mathcal{U}, \mathcal{Y})$, that is,

$$G(\lambda) = D + \lambda C\,(I - \lambda A)^{-1}\,B. \tag{3.8}$$

Here A is a stable operator on a finite-dimensional space $\mathcal{X}$ and B maps $\mathcal{U}$ into $\mathcal{X}$ while C maps $\mathcal{X}$ into $\mathcal{Y}$ and D maps $\mathcal{U}$ into $\mathcal{Y}$. By *stable* we mean that all the eigenvalues for A are inside the open unit disc. Note that $\{A, B, C, D\}$ is a realization for G if and only if

$$G_0 = D \quad \text{and} \quad G_n = CA^{n-1}B \quad (\text{for } n \geq 1) \tag{3.9}$$

where $G(\lambda) = \sum_0^\infty \lambda^n G_n$ is the Taylor series expansion for G. Let W_o be the *observability operator* mapping $\mathcal{X}$ into $\ell_+^2(\mathcal{Y})$ and W_c the *controllability operator* mapping $\ell_+^2(\mathcal{U})$ into $\mathcal{X}$ defined by

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} : \mathcal{X} \to \ell_+^2(\mathcal{Y}),$$

$$W_c = \begin{bmatrix} B & AB & A^2B \cdots \end{bmatrix} : \ell_+^2(\mathcal{U}) \to \mathcal{X}. \tag{3.10}$$

Let $P = W_c W_c^* = \sum_0^\infty A^n BB^* A^{*n}$ be the *controllability Gramian* for the pair $\{A, B\}$. Then P is the solution to the following Stein equation

$$P = APA^* + BB^*. \tag{3.11}$$

Using (3.9), we see that the Hankel operator H_G is equal to

$$H_G = W_o W_c. \tag{3.12}$$

In particular, it follows that the Hankel operator H_G admits a factorization of the form $H_G = W_o W_c$ where W_o is an operator mapping $\mathcal{X}$ into $\ell_+^2(\mathcal{Y})$ and W_c is an operator mapping $\ell_+^2(\mathcal{U})$ into $\mathcal{X}$. Using $P = W_c W_c^*$ with (2.5), we obtain

$$H_G H_G^* = W_o P W_o^* \quad \text{and} \quad T_G T_G^* = T_{GG^*} - W_o P W_o^*. \tag{3.13}$$

Consider the *algebraic Riccati equation*

$$\begin{aligned} Q &= A^*QA + (C - \Gamma^*QA)^*(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA) \\ \Gamma &= BD^* + APC^* \quad \text{and} \quad R_0 = DD^* + CPC^*. \end{aligned} \tag{3.14}$$

We say that Q is a *stabilizing solution* to this algebraic Riccati equation if Q is positive, $R_0 - \Gamma^*Q\Gamma$ is strictly positive, and the following operator A_o on $\mathcal{X}$ is stable:

$$A_o = A - \Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA). \tag{3.15}$$

Moreover, if the algebraic Riccati equation (3.14) admits a stabilizing solution Q, then the stabilizing solution Q can be computed by

$$Q = \lim_{n\to\infty} Q_n \tag{3.16}$$

$$Q_{n+1} = A^*Q_nA + (C - \Gamma^*Q_nA)^*(R_0 - \Gamma^*Q_n\Gamma)^{-1}(C - \Gamma^*Q_nA)$$

subject to the initial condition $Q_0 = 0$. In particular, if the limit in (3.16) does not exist or A_o is not stable, then the algebraic Riccati equation (3.14) does not have a stabilizing solution; see [8, 9] for further details.

If Θ is an inner function in $H^\infty(\mathcal{E},\mathcal{Y})$, then $\mathfrak{H}(\Theta)$ is the subspace of $\ell_+^2(\mathcal{Y})$ defined by

$$\mathfrak{H}(\Theta) = \ell_+^2(\mathcal{Y}) \ominus T_\Theta \ell_+^2(\mathcal{E}) = \ker T_\Theta^*. \tag{3.17}$$

Because T_Θ is an isometry, $I - T_\Theta T_\Theta^*$ is the orthogonal projection onto $\mathfrak{H}(\Theta)$. It is noted that $\mathfrak{H}(\Theta)$ is an invariant subspace for the backward shift $S_\mathcal{Y}^*$ on $\ell_+^2(\mathcal{Y})$. According to the Beurling–Lax–Halmos Theorem if $\mathfrak{H}$ is any invariant subspace for the backward shift, then there exists a unique inner function Θ in $H^\infty(\mathcal{E},\mathcal{Y})$ such that $\mathfrak{H} = \mathfrak{H}(\Theta)$. By unique we mean that if $\mathfrak{H} = \mathfrak{H}(\Psi)$ where Ψ is an inner function in $H^\infty(\mathcal{L},\mathcal{Y})$, then there exists a constant unitary operator Ω from $\mathcal{E}$ onto $\mathcal{L}$ such that $\Theta = \Psi\Omega$; see [5, 11, 12, 13, 14, 15] for further details. By combining Lemma 3.1 with the results in [9], we obtain the following result. (For part (v) compare also Lemma 4.1 below.)

Theorem 3.2. *Let $\{A \text{ on } \mathcal{X}, B, C, D\}$ be a minimal realization for a rational function G in $H^\infty(\mathcal{U},\mathcal{Y})$ where $\dim\mathcal{Y} \le \dim\mathcal{U}$. Let R be the function in $L^\infty(\mathcal{Y},\mathcal{Y})$ defined by $R(e^{i\omega}) = G(e^{i\omega})G(e^{i\omega})^*$. Let P the unique solution to the Stein equation $P = APA^* + BB^*$. Then the following statements are equivalent.*

- (i) *The function G admits a full rank inner-outer factorization;*
- (ii) *the Toeplitz operator T_R is invertible;*
- (iii) *there exists a stabilizing solution Q to the algebraic Riccati equation* (3.14).

In this case, $Q = W_o^* T_R^{-1} W_o$ and the following holds.

- (iv) *The eigenvalues of QP are real numbers contained in the interval* $[0, 1]$.
- (v) *If G_i is the inner factor of G, then the dimension of $\mathfrak{H}(G_i)$ is given by*

$$\dim \mathfrak{H}(G_i) = \dim\ker T_{G_i}^* = \dim\ker T_G^* = \dim\ker(I - QP). \tag{3.18}$$

(vi) *The McMillan degree of G_i is given by*

$$\delta(G_i) = \dim \mathfrak{H}(G_i) = \dim \ker(I - QP). \tag{3.19}$$

In particular, the McMillan degree of G_i is less than or equal to the McMillan degree of G.

(vii) *The operator $T_R^{-1} W_o$ is given by*

$$T_R^{-1} W_o = \begin{bmatrix} C_o \\ C_o A_o \\ C_o A_o^2 \\ \vdots \end{bmatrix} : \mathcal{X} \to \ell_+^2(\mathcal{Y}),$$

$$C_o = (R_0 - \Gamma^* Q \Gamma)^{-1}(C - \Gamma^* Q A) : \mathcal{X} \to \mathcal{Y}. \tag{3.20}$$

Finally, because $\{C, A\}$ is observable, $T_R^{-1} W_o$ is one-to-one and $\{C_o, A_o\}$ is a stable observable pair.

Let us present the following classical result; see Theorem 7.1 in [7], Sections 4.2 and 4.3 in [5] and Section XXVIII.7 in [11].

Lemma 3.3. *Let Θ be an inner function in $H^\infty(\mathcal{Y}, \mathcal{Y})$ where $\mathcal{Y}$ is finite dimensional. Then the Hankel operator H_Θ is a partial isometry whose range equals $\mathfrak{H}(\Theta)$, that is,*

$$P_{\mathfrak{H}(\Theta)} = H_\Theta H_\Theta^* \tag{3.21}$$

where $P_{\mathfrak{H}(\Theta)}$ denotes the orthogonal projection onto $\mathfrak{H}(\Theta)$. Furthermore, the following holds.

(i) *The subspace $\mathfrak{H}(\Theta)$ is finite dimensional if and only if Θ is rational.*
(ii) *The dimension of $\mathfrak{H}(\Theta)$ equals the McMillan degree of Θ.*

Proof. For completeness a proof is given. By replacing G by Θ in (2.5), we see that

$$T_\Theta T_\Theta^* = T_{\Theta\Theta^*} - H_\Theta H_\Theta^*.$$

Because Θ is a square inner function, $\Theta(e^{i\omega})\Theta(e^{i\omega})^* = I$ almost everywhere on the unit circle. Hence $T_{\Theta\Theta^*} = I$. This readily implies that

$$H_\Theta H_\Theta^* = I - T_\Theta T_\Theta^* = P_{\mathfrak{H}(\Theta)}.$$

Therefore (3.21) holds and H_Θ is a partial isometry whose range equals $\mathfrak{H}(\Theta)$.

It is well know that the range of a Hankel operator H_F is finite dimensional if and only if its symbol F is rational. Moreover, the dimension of the range of the Hankel operator H_F equals the McMillan degree of F. Therefore parts (i) and (ii) follow from the fact that $\mathfrak{H}(\Theta) = \operatorname{ran} H_\Theta$. □

Let $\{A_i \text{ on } \mathcal{X}_i, B_i, C_i, D_i\}$ be a minimal state space realization for a rational function Θ in $H^\infty(\mathcal{Y}, \mathcal{Y})$. It is well known (see, e.g., [7], Section III.7) that Θ is a square inner function if and only if

$$\begin{bmatrix} A_i^* & C_i^* \\ B_i^* & D_i^* \end{bmatrix} \begin{bmatrix} Q_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} = \begin{bmatrix} Q_i & 0 \\ 0 & I \end{bmatrix} \tag{3.22}$$

where $Q_i = A_i^* Q_i A_i + C_i^* C_i$. Moreover, in this case,

$$\mathfrak{H}(\Theta) = \operatorname{ran} H_\Theta = \operatorname{ran} W_i$$

where W_i is the observability operator for $\{C_i, A_i\}$ defined by

$$W_i = \begin{bmatrix} C_i \\ C_i A_i \\ C_i A_i^2 \\ \vdots \end{bmatrix} : \mathcal{X}_i \to \ell_+^2(\mathcal{Y}). \tag{3.23}$$

It is noted that $S_{\mathcal{Y}}^* W_i = W_i A_i$. So the range of W_i is a finite-dimensional invariant subspace for the backward shift $S_{\mathcal{Y}}^*$. Finally, $Q_i = W_i^* W_i$.

On the other hand, if $\{C_i, A_i \text{ on } \mathcal{X}_i\}$ is a stable observable pair where $\mathcal{X}_i$ is finite dimensional, then there exists operators B_i mapping $\mathcal{Y}$ into $\mathcal{X}_i$ and D_i on $\mathcal{Y}$ such that

$$\Theta(\lambda) = D_i + \lambda C_i (I - \lambda A_i)^{-1} B_i \tag{3.24}$$

is a square inner function in $H^\infty(\mathcal{Y}, \mathcal{Y})$. Moreover, $\mathfrak{H}(\Theta) = \operatorname{ran} W_i$ and (3.22) holds. The Beurling–Lax–Halmos Theorem guarantees that the inner function Θ is unique up to a unitary constant on the right. The operators B_i and D_i are called the *complementary operators* for the pair $\{C_i, A_i\}$. To compute the complementary operators B_i and D_i explicitly, let

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} : \mathcal{Y} \to \begin{bmatrix} \mathcal{X}_i \\ \mathcal{Y} \end{bmatrix} \tag{3.25}$$

be an isometry from $\mathcal{Y}$ onto the kernel of $\begin{bmatrix} A_i^* Q_i^{\frac{1}{2}} & C_i^* \end{bmatrix}$. Then set

$$B_i = Q_i^{-\frac{1}{2}} E_1 \quad \text{and} \quad D_i = E_2. \tag{3.26}$$

Because the pair $\{C_i, A_i\}$ is observable, the operator W_i defined in (3.23) is one to one, and the complementary operators B_i and D_i together with A_i and C_i form a minimal realization $\{A_i, B_i, C_i, D_i\}$ for a square inner function Θ such that $\operatorname{ran} W_i = \mathfrak{H}(\Theta)$. For further details see Lemma XXVIII7.7 in [11] and Sections 4.2 and 4.3 in [5].

We are now in a position to present our main result. The proof is given in Section 5.

Let $\{A, B, C, D\}$ be a minimal realization for a rational function G in $H^\infty(\mathcal{U}, \mathcal{Y})$ where $\dim \mathcal{Y} \le \dim \mathcal{U}$. To compute a full rank inner-outer factorization $G = G_i G_o$ for G, let P be the controllability Gramian for the pair $\{A, B\}$ (see (3.11)) and Q the stabilizing solution to the algebraic Riccati equation (3.14). If this algebraic Riccati equation does not admit a stabilizing solution, then G does not have a full rank inner-outer factorization.

Theorem 3.4. *Let $\{A, B, C, D\}$ be a minimal realization for a rational function G in $H^\infty(\mathcal{U}, \mathcal{Y})$ where* $\dim \mathcal{Y} \le \dim \mathcal{U}$. *Assume there exists a stabilizing solution Q to the algebraic Riccati equation* (3.14).

Let $\mathcal{X}_i$ be any space isomorphic to the kernel of $I - QP$. Let U be any isometry from $\mathcal{X}_i$ onto the kernel of $I - QP$. In particular, $U = QPU$. Let A_i on $\mathcal{X}_i$ and C_i mapping $\mathcal{X}_i$ into $\mathcal{Y}$ be the operators computed by

$$A_i = U^* Q A_o P U \quad \text{and} \quad C_i = C_o P U. \tag{3.27}$$

Then $\{C_i, A_i\}$ is a stable observable pair. Let B_i and D_i be the complementary operators for $\{C_i, A_i\}$ as constructed in (3.25) *and* (3.26)*. Then the square inner factor G_i for G is given by*

$$G_i(\lambda) = D_i + \lambda C_i (I - \lambda A_i)^{-1} B_i. \tag{3.28}$$

The outer factor G_o for G is given by

$$G_o(\lambda) = D_i^* D + B_i^* U^* B + \lambda \left(D_i^* C + B_i^* U^* A\right) (I - \lambda A)^{-1} B. \tag{3.29}$$

4. An auxiliary lemma

To prove that the inner-outer factorization of $G = G_i G_o$ is indeed given by (3.28) and (3.29), let us begin with the following auxiliary result.

Lemma 4.1. *Let T be a strictly positive operator on $\mathcal{H}$ and P a strictly positive operator on $\mathcal{X}$. Let W be an operator mapping $\mathcal{X}$ into $\mathcal{H}$ and set $Q = W^* T^{-1} W$. Then the following two assertions hold.*

(i) *Let $\mathfrak{X}$ and $\mathfrak{H}$ be the subspaces defined by*

$$\mathfrak{X} = \ker(I - QP) \quad \text{and} \quad \mathfrak{H} = \ker\left(T - WPW^*\right). \tag{4.1}$$

Then the operators

$$\Lambda_1 = W^*|\mathfrak{H} : \mathfrak{H} \to \mathfrak{X} \quad \text{and} \quad \Lambda_2 = T^{-1}WP|\mathfrak{X} : \mathfrak{X} \to \mathfrak{H} \tag{4.2}$$

are both well defined and invertible. Moreover, $\Lambda_1^{-1} = \Lambda_2$.

(ii) *The operator $T - WPW^*$ is positive if and only if $P^{-1} - Q$ is positive, or equivalently, $P^{\frac{1}{2}} Q P^{\frac{1}{2}}$ is a contraction. In this case, the spectrum of QP is contained in $[0,1]$. In particular, if $\mathcal{X}$ is finite dimensional, then the eigenvalues for QP are contained in $[0,1]$.*

Proof. The proof is based on some ideas involving Schur complements; see [2] and Section 2.2 in [3]. Consider the operator matrix

$$\begin{aligned} M = \begin{bmatrix} T & W \\ W^* & P^{-1} \end{bmatrix} &= \begin{bmatrix} I & WP \\ 0 & I \end{bmatrix} \begin{bmatrix} T - WPW^* & 0 \\ 0 & P^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ PW^* & I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ W^* T^{-1} & I \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & P^{-1} - Q \end{bmatrix} \begin{bmatrix} I & T^{-1} W \\ 0 & I \end{bmatrix}. \end{aligned}$$

From this we conclude several things: first, by the fact that both T and P are strictly positive, the congruences above imply that $T - W^* PW$ is positive if and only if $P^{-1} - Q$ is positive, or equivalently, $P^{\frac{1}{2}} Q P^{\frac{1}{2}}$ is a contraction. In particular, if $T - W^* PW$ is positive, then the spectrum of QP is contained in the interval $[0,1]$. This proves part (ii) of the lemma.

To prove part (i) observe that we can describe $\ker M$ in two different ways. Based on the first factorization we have

$$\ker M = \left\{ \begin{bmatrix} h \\ -PW^*h \end{bmatrix} \mid h \in \ker(T - WPW^*) = \mathfrak{H} \right\}.$$

The second factorization yields

$$\ker M = \left\{ \begin{bmatrix} -T^{-1}Wy \\ y \end{bmatrix} \mid y \in \ker(P^{-1} - Q) = P\mathfrak{X} \right\}$$

$$\ker M = \left\{ \begin{bmatrix} T^{-1}WPx \\ -Px \end{bmatrix} \mid x \in \ker(I - QP) = \mathfrak{X} \right\}.$$

Together these equalities prove the first assertion in Lemma 4.1. Indeed,

$$\Phi_1 = \begin{bmatrix} I \\ -PW^* \end{bmatrix} : \mathfrak{H} \to \ker M$$

is a one-to-one operator from $\mathfrak{H}$ onto $\ker M$. Likewise,

$$\Phi_2 = \begin{bmatrix} T^{-1}WP \\ -P \end{bmatrix} : \mathfrak{X} \to \ker M$$

is a one-to-one operator from $\mathfrak{X}$ onto $\ker M$. Because the first component of Φ_1 is the identity operator on $\mathfrak{H}$, we see that $T^{-1}WP$ maps $\mathfrak{X}$ onto $\mathfrak{H}$. Since the second component of Φ_2 is $-P$ and P is invertible, W^* maps $\mathfrak{H}$ onto $\mathfrak{X}$. Therefore the operators Λ_1 and Λ_2 in (4.2) are well defined.

If x is in $\mathfrak{X}$, then $\Phi_2 x = \Phi_1 h$ for some unique h in $\mathfrak{H}$, that is,

$$\begin{bmatrix} \Lambda_2 x \\ -Px \end{bmatrix} = \begin{bmatrix} T^{-1}WPx \\ -Px \end{bmatrix} = \Phi_2 x = \Phi_1 h = \begin{bmatrix} h \\ -PW^*h \end{bmatrix} = \begin{bmatrix} \Lambda_2 x \\ -PW^*\Lambda_2 x \end{bmatrix}.$$

The last equality follows from the fact that $h = \Lambda_2 x$. The second component of the previous equation shows that $x = W^*\Lambda_2 x$, and thus, $\Lambda_1 = W^*|\mathfrak{H}$ is the left inverse of Λ_2. On the other hand, if h is in $\mathfrak{H}$, then $\Phi_1 h = \Phi_2 x$ for some unique x in $\mathfrak{X}$, that is,

$$\begin{bmatrix} h \\ -PW^*h \end{bmatrix} = \Phi_1 h = \Phi_2 x = \begin{bmatrix} T^{-1}WPx \\ -Px \end{bmatrix} = \begin{bmatrix} \Lambda_2 x \\ -Px \end{bmatrix}.$$

By consulting the second component, we have $\Lambda_1 h = W^*h = x$. Substituting $x = \Lambda_1 h$ into the first component, yields $h = \Lambda_2\Lambda_1 h$. Therefore Λ_1 is the right inverse of Λ_2 and $\Lambda_1^{-1} = \Lambda_2$. □

5. Proof of the inner-outer factorization

Proof. Assume that the algebraic Riccati equation (3.14) admits a stabilizing solution Q. In other words, assume that T_R is strictly positive, or equivalently, G admits a full rank inner-outer factorization $G = G_i G_o$. Using $P = W_c W_c^*$ with $H_G = W_o W_c$, we have

$$T_G T_G^* = T_R - H_G H_G^* = T_R - W_o P W_o^*.$$

Recall that the subspace $\mathfrak{H}(G_i) = \ell_+^2(\mathcal{Y}) \ominus T_{G_i}\ell_+^2(\mathcal{Y})$. Then

$$\mathfrak{H}(G_i) = \ker T_{G_i}^* = \ker T_G^* = \ker\left(T_R - W_o P W_o^*\right).$$

It is noted that $\mathfrak{H}(G_i)$ is an invariant subspace for the backward shift $S_{\mathcal{Y}}^*$ on $\ell_+^2(\mathcal{Y})$. Recall that $Q = W_o^* T_R^{-1} W_o$. Let $k = \dim\ker(I - QP)$, and put $\mathcal{X}_i = \mathbb{C}^k$. Let U be an isometry from $\mathcal{X}_i$ onto $\ker(I - QP)$. According to Lemma 4.1, the operator

$$\Lambda_2 = T_R^{-1} W_o P U = \begin{bmatrix} C_o \\ C_o A_o \\ C_o A_o^2 \\ C_o A_o^3 \\ \vdots \end{bmatrix} PU : \mathcal{X}_i \to \mathfrak{H}(G_i)$$

is invertible, where we also use (3.20). In particular, the dimension of $\mathfrak{H}(G_i)$ equals $\dim \mathcal{X}_i$. Since P is invertible and U is an isometry, the operator PU from $\mathcal{X}_i$ into $\mathcal{X}$ is one to one.

Because $\mathfrak{H}(G_i)$ is an invariant subspace for the backward shift $S_{\mathcal{Y}}^*$, there exists an operator A_i on $\mathcal{X}_i = \mathbb{C}^k$ such that

$$S_{\mathcal{Y}}^* T_R^{-1} W_o P U = T_R^{-1} W_o P U A_i. \tag{5.1}$$

Since $T_R^{-1} W_o P U$ is one to one and $S_{\mathcal{Y}}^{*n}$ converges to zero pointwise, A_i is stable.

Now observe that

$$\begin{bmatrix} C_o \\ C_o A_o \\ C_o A_o^2 \\ \vdots \end{bmatrix} A_o P U = S_{\mathcal{Y}}^* T_R^{-1} W_o P U = \begin{bmatrix} C_o \\ C_o A_o \\ C_o A_o^2 \\ \vdots \end{bmatrix} P U A_i.$$

Since the observability matrix for $\{C_o, A_o\}$ is one to one, $A_o P U = P U A_i$. Because PU is one to one, the spectrum of A_i is contained in the spectrum of A_o. Multiplying $A_o P U = P U A_i$ by $U^* Q$ on the left and using $QPU = U$ shows that

$$A_o P U = P U A_i \quad \text{and} \quad A_i = U^* Q A_o P U. \tag{5.2}$$

Setting $C_i = C_o P U$ and using $A_o^j P U = P U A_i^j$ for all positive integers j, we obtain

$$C_o A_o^j P U = C_o P U A_i^j = C_i A_i^j \quad \text{(for all integers } j \geq 0\text{)}. \tag{5.3}$$

In particular,

$$T_R^{-1} W_o P U = \begin{bmatrix} C_o \\ C_o A_o \\ C_o A_o^2 \\ \vdots \end{bmatrix} P U = \begin{bmatrix} C_i \\ C_i A_i \\ C_i A_i^2 \\ \vdots \end{bmatrix}.$$

Since $T_R^{-1} W_o P U$ is one to one, $\{C_i, A_i\}$ is a stable observable pair. Let B_i mapping $\mathcal{Y}$ into $\mathcal{X}_i = \mathbb{C}^k$ and D_i on $\mathcal{Y}$ be the complementary operators for the pair $\{C_i, A_i\}$. Since $\mathfrak{H}(G_i)$ equals the range of $T_R^{-1} W_o P U$, the inner function G_i (up to a unitary constant on the right) is given by

$$G_i(\lambda) = D_i + \lambda C_i (I - \lambda A_i)^{-1} B_i.$$

To find the outer factor G_o, first notice that

$$Q = W_o^* T_R^{-1} W_o = \begin{bmatrix} C_o^* & A_o^* C_o^* & A_o^{*2} C_o^* \cdots \end{bmatrix} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} = \sum_{j=0}^{\infty} A_o^{*j} C_o^* C A^j.$$

The second equality follows from (3.20). In other words, Q satisfies the Stein equation

$$Q = A_o^* Q A + C_o^* C. \tag{5.4}$$

Now note that $U^* P C_o^* C = C_i^* C$, so that $C_i^* C = U^* P (Q - A_o^* Q A)$. Moreover, $U^* P A_o^* = A_i^* U^* P$ and $U^* P Q = U^*$. Hence

$$C_i^* C = U^* - A_i^* U^* P Q A = U^* - A_i^* U^* A. \tag{5.5}$$

It follows that $U^* = \sum_{j=0}^{\infty} A_i^{*j} C_i^* C A^j$.

Next observe that $T_G = T_{G_i G_o} = T_{G_i} T_{G_o}$. Multiplying by $T_{G_i}^*$ on the left, with the fact that T_{G_i} is an isometry, we have $T_{G_i}^* T_G = T_{G_o}$. Using this with $U^* = \sum_{j=0}^{\infty} A_i^{*j} C_i^* C A^j$, we see that the first column of T_{G_o} is given by

$$\begin{aligned} T_{G_i}^* \begin{bmatrix} D \\ CB \\ CAB \\ \vdots \end{bmatrix} &= \begin{bmatrix} D_i^* & B_i^* C_i^* & B_i^* A_i^* C_i^* & B_i^* A_i^{*2} C_i^* & \cdots \\ 0 & D_i^* & B_i^* C_i^* & B_i^* A_i^* C_i^* & \cdots \\ 0 & 0 & D_i^* & B_i^* C_i^* & \cdots \\ 0 & 0 & 0 & D_i^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} D \\ CB \\ CAB \\ CA^2B \\ \vdots \end{bmatrix} \\ &= \begin{bmatrix} D_i D^* + B_i^* U^* B \\ (D_i^* C + B_i^* U^* A) B \\ (D_i^* C + B_i^* U^* A) AB \\ (D_i^* C + B_i^* U^* A) A^2 B \\ \vdots \end{bmatrix}. \end{aligned}$$

By taking the Fourier transform of the first column of T_{G_o}, we obtain the following state space formula:

$$G_o(\lambda) = D_i^* D + B_i^* U^* B + \lambda \left(D_i^* C + B_i^* U^* A \right) (I - \lambda A)^{-1} B.$$

This completes the proof. $\square$

For completeness we shall also provide a slightly different derivation of the last part of the proof, that is, the formula for G_o. The idea is similar in nature but slightly different in execution: for $|\lambda| = 1$ we compute $G_o(\lambda) = G_i(\lambda)^* G(\lambda)$ using the realization formulas for G_i and G. This leads to

$$\begin{aligned} G_o(\lambda) &= G_i(\lambda)^* G(\lambda) \\ &= (D_i^* + \overline{\lambda} B_i^* (I - \overline{\lambda} A_i^*)^{-1} C_i^*)(D + \lambda C(I - \lambda A)^{-1} B) \\ &= D_i^* D + \overline{\lambda} B_i^* (I - \overline{\lambda} A_i^*)^{-1} C_i^* D + \lambda D_i^* C (I - \lambda A)^{-1} B \\ &\quad + |\lambda|^2 B_i^* (I - \overline{\lambda} A_i^*)^{-1} C_i^* C (I - \lambda A)^{-1} B. \end{aligned}$$

Since we consider $|\lambda| = 1$ this is equal to

$$\begin{aligned} G_o(\lambda) = D_i^* D &+ \frac{1}{\lambda} B_i^* (I - \frac{1}{\lambda} A_i^*)^{-1} C_i^* D + \lambda D_i^* C (I - \lambda A)^{-1} B \\ &+ B_i^* (I - \frac{1}{\lambda} A_i^*)^{-1} C_i^* C (I - \lambda A)^{-1} B. \end{aligned}$$

Consider the Stein equation $C_i^* C = U^* - A_i^* U^* A$; see (5.5). This may be used to compute

$$\begin{aligned} &(I - \frac{1}{\lambda} A_i^*)^{-1} C_i^* C (I - \lambda A)^{-1} \\ &= (I - \frac{1}{\lambda} A_i^*)^{-1} \Big(U^* - \frac{1}{\lambda} A_i^* U^* (\lambda A) \Big) (I - \lambda A)^{-1} \\ &= (I - \frac{1}{\lambda} A_i^*)^{-1} \Big(U^* (I - \lambda A) + (I - \frac{1}{\lambda} A_i^*) U^* (\lambda A) \Big) (I - \lambda A)^{-1} \\ &= (I - \frac{1}{\lambda} A_i^*)^{-1} U^* + \lambda U^* A (I - \lambda A)^{-1}. \end{aligned}$$

Inserting this in the formula for $G_o(\lambda)$ we obtain

$$\begin{aligned} G_o(\lambda) &= D_i^* D + \frac{1}{\lambda} B_i^* (I - \frac{1}{\lambda} A_i^*)^{-1} C_i^* D + \lambda D_i^* C (I - \lambda A)^{-1} B + \\ &\quad + B_i^* (I - \frac{1}{\lambda} A_i^*)^{-1} U^* B + \lambda B_i^* U^* A (I - \lambda A)^{-1} B \\ &= D_i^* D + B_i^* (\lambda I - A_i^*)^{-1} C_i^* D \\ &\quad + B_i^* \left(I + (I - \frac{1}{\lambda} A_i^*)^{-1} - I \right) U^* B \\ &\quad + \lambda (D_i^* C + B_i^* U^* A)(I - \lambda A)^{-1} B \\ &= D_i^* D + B_i^* U^* B + B_i^* (\lambda I - A_i^*)^{-1} (C_i^* D + A_i^* U^* B) \\ &\quad + \lambda (D_i^* C + B_i^* U^* A)(I - \lambda A)^{-1} B. \end{aligned}$$

Because G_o is analytic in the open unit disc, we know that the term $B_i^* (\lambda I - A_i^*)^{-1} (C_i^* D + A_i^* U^* B)$ must be zero. Let us give a direct proof of this fact. This turns out to be an easy consequence of formula (3.23) in [9]. Indeed, this formula states that

$$C_1^* C_1 = (Q - QPQ) - A_0^* (Q - QPQ) A_0,$$

where $C_1 = D^* C_0 + B^* Q A_0$. Multiplying the above formula with PU on the right and $U^* P$ on the left, we obtain

$$U^* P C_1^* C_1 P U = U^* P (Q - QPQ) P U - U^* P A_0^* (Q - QPQ) A_0 P U.$$

Since $QPU = U$ it follows that $(Q - QPQ)PU = 0$, so the first term on the right hand side is zero. Further, since $A_0PU = PUA_i$ it follows that also the second term on the right hand side is zero. Hence $C_1PU = 0$, which means

$$0 = (D^*C_0 + B^*QA_0)PU = D^*C_i + B^*QPUA_i = D^*C_i + B^*UA_i.$$

Thus the formula for G_o can also be established by a direct computation using the realizations of G_i and G.

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