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Production, Manufacturing, Transportation and Logistics

## The Dynamic-Demand Joint Replenishment Problem with Approximated Transportation Costs

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## ABSTRACT

In a vendor-managed inventory setting, a supplier determines the timing and size of replenishments for its customers. In the Dynamic-Demand Joint Replenishment Problem (DJRP), one assumes that the supplier pays a fixed fee for replenishing a customer which often occurs if the supplier outsources transportation. Hence, there is no incentive for the supplier to schedule replenishments for nearby customers in the same period. This results in higher transportation costs for the carrier, decreased vehicle utilization and increased future fees for the supplier. To lower costs for both parties, this paper extends the traditional DJRP to the DJRP with Approximated Transportation Costs (DJRP-AT) by taking transportation considerations into account. Since routing problems are difficult to solve and it is not necessary to know the sequence of the deliveries to the customers as these are outsourced, the transportation costs for a given set of customers are approximated using classical schemes. A solution approach for the DJRP-AT based on Branch-and-Cut-and-Price is validated using test instances from the literature. Results show improvements of 4% on average and up to 14.4% for individual instances compared with the DJRP. Moreover, when the DJRP-AT is compared with the DJRP on instances derived from a real-life case, similar savings are obtained. Comparing the DJRP-AT to an equivalent problem with actual routing costs, the solution values of the DJRP-AT are on average only 0.77% higher showing the value of the approximation.

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## 1. Introduction

During the last decades, Vendor-Managed Inventory (VMI) systems have received a lot of attention in the literature (Andersson, Hoff, Christiansen, Hasle, & Løkketangen, 2010). In such a system, a supplier manages the inventory of its customers and arranges the transportation of the replenishments. The supplier bears both the inventory holding and transportation costs and therefore strives to minimize these costs by optimizing inventory and shipping decisions. If the supplier would decide on the replenishments and the routes to deliver the replenishments, the supplier faces a problem known as the Inventory Routing Problem (IRP) (Coelho, Cordeau, & Laporte, 2014). However, the transport of the replenishments is often outsourced to a Logistics Service Provider (LSP). As a consequence, the supplier pays a fixed transportation fee for a delivery that is specified in a long-term contract. The supplier therefore faces an optimization problem known as the Joint Replenishment Problem (JRP) (Khouja & Goyal, 2008). When customer demand

varies over time, this problem is known as the Dynamic-Demand Joint Replenishment Problem (DJRP). The DJRP decides which products to order or customers to serve in which periods of the planning horizon such that demand is satisfied at minimal inventory holding and servicing costs. More specifically, the cost of servicing a group of customers in a given period consists of two components, the first of which is a common set-up cost per period if at least one customer is served in that given period. The second component is a cost for each replenished customer. Because of the common set-up cost per period it can be beneficial for the supplier to have some customers replenished together with other customers even before stock runs low. In that case the inventory holding cost is higher because more inventory is kept, but it allows the supplier to save the fixed fee for a period.

The DJRP encompasses a number of key problem features occurring in real-life applications, but also suffers from a number of drawbacks. First, because the actual transportation costs are not directly considered, the DJRP cannot identify closely situated customers that would be well-suited for joint replenishment. Hence, customers served in one period may be randomly located in a region, resulting in high actual transportation costs for the LSP. Also, the number of replenishments that the LSP is able to perform on

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behalf of the supplier can be lower due to large distances between deliveries. Therefore, consideration of proximity of customers could result in better utilization of transportation resources, decrease actual transportation costs for the LSP and eventually decrease transportation fees for the supplier. In short, based on the DJRP, the supplier generates requests that are expensive or hard to fulfill. Second, the DJRP ignores duration constraints. Vehicle capacity constraints have been considered in the JRP literature (see e.g., Anily & Tzur (2005)) and in many routing problems, including most studies on the IRP. However, tour duration constraints have proven to be more binding in several practical applications, such as online ordered package delivery, blood product distribution (Hemmelmayr, Doerner, Hartl, & Savelsbergh, 2009) and replenishment of ATMs. Tour duration constraints are rarely found in the IRP literature. Finally, the DJRP does not take limited customer storage capacity into account, but in practice, storage capacity is often restricted. To address these shortcomings of the DJRP we propose an extension of the DJRP, the DJRP with Approximated Transportation Costs (DJRP-AT), that explicitly considers transportation costs. Furthermore, the DJRP-AT contains tour duration constraints and limits customer storage capacity. Because determining the optimal delivery tour is computationally expensive, we will approximate the transportation costs in the DJRP-AT by approximating the shortest traveling salesman tour. The only work that we are aware of that includes approximated transportation costs in a VMI setting is Larsen and Turkensteen (2014). The authors consider a VMI setting with stochastic demand and order-up-to-levels at the customers which they solve with a Markov Chain simulation model.

Our research is motivated by ATM replenishment in the Netherlands. A single supplier (vendor) decides on the timing of ATM cash replenishment and on the delivery quantity. The actual ATM replenishment orders per day are outsourced to an LSP, in this application often referred to as Cash-in-Transit company (CIT), that schedules and performs the daily delivery routes. Currently, the supplier pays a fixed fee to the LSP for each ATM replenishment. Therefore, the current replenishment policy ignores the impact that ordering decisions have on distance traveled and vehicle utilization. The supplier is reconsidering the replenishment cost structure to better align decisions. To provide insight for future negotiations between the supplier and the LSP, we examine the benefit of adopting a DJRP and a DJRP-AT perspective. How a new ordering policy is to be incorporated in the contract between the supplier and the LSP, is beyond the scope of this paper. With this work we contribute to the recent stream of publications on ATM replenishment. For example, Van Anholt, Coelho, Laporte, and Vis (2016) develop a heuristic for a pickup and delivery IRP for an advanced type of ATMs. Larrain, Coelho, and Cataldo (2017) focus on a local search based heuristic for an IRP in which stock-outs are allowed and cash is replenished by swapping cassettes.

To solve the DJRP-AT, this paper proposes a compact formulation in which transportation costs and inventory holding costs are minimized. Note that inventory holding costs in the application relate to the value of money or lost interest. The compact formulation is split by applying Dantzig–Wolfe decomposition (Desrosiers & Lübbecke, 2005). The resulting Master Problem and Pricing Problem are solved in a Branch-and-Cut-and-Price framework (Lübbecke & Desrosiers, 2005; Nemhauser & Park, 1991). The Master Problem selects customer subsets to be delivered and determines the corresponding delivery quantities. The Pricing Problem generates these customer subsets using a labeling algorithm with tailored dominance criteria to speed up the process. The solution method is tested on benchmark instances from the literature and on instances derived from a real-life case in ATM replenishment.

The contributions of this paper are threefold. First, the DJRP is extended to incorporate transportation costs, limitations on storage capacity at the customers and restricted tour duration. The results

show that the proposed model leads to lower total costs compared with the DJRP. Second, we introduce novel dominance conditions for the labeling algorithm that is used to solve the Pricing Problem. Finally, existing valid inequalities originating from the inventory routing literature are tested, their impact on the integrality gap is demonstrated and it is shown that their effectiveness is different than for other models in which they have been applied.

The remainder of this paper is organized as follows: Section 2 discusses literature on the JRP and DJRP, together with their relation to the IRP. The DJRP-AT is described and modeled in Section 3. Section 4 proposes a decomposition of the model and specifies the Master and Pricing Problems. The algorithm to solve the Pricing Problem, including novel sufficient dominance conditions, the valid inequalities and the branching strategy are presented in Section 5. Section 6 presents the results of the experiments on benchmark instances from the literature, introduces the real-life case and reports on the results of instances derived from the real-life case. Finally, the conclusions and directions for further research are discussed in Section 7.

## 2. Literature review

The traditional JRP is the problem of minimizing holding and ordering costs, while ensuring that no customer runs out of stock in any period of the planning horizon. The ordering costs consist of a common set-up cost per period and a fixed fee per replenishment. An overview of the literature on the JRP from 1989 to 2005 distinguishes three types of models (Khouja & Goyal, 2008): first, the traditional JRP, considers deterministic and static demand. This means that demand is known beforehand and remains the same for every period of the planning horizon. For this problem, analytical expressions have been derived for the minimal total costs and heuristics have been designed to determine the corresponding cyclic replenishment policy. Second, the extension to stationary stochastic demand in which the objective is to minimize the expected total cost. Solution methods mainly consist of using a periodic review policy or a can-order policy. Finally, the JRP with deterministic and dynamic demand (DJRP) in which the demand is known but can vary across periods is discussed. The solution for this type of problem is not necessarily a cyclic replenishment policy as for the traditional JRP. For the DJRP different formulations and heuristic solution methods have been proposed and studied (Boctor, Laporte, & Renaud, 2004; Narayanan & Robinson, 2006; Robinson, Narayanan, & Gao, 2007; Webb, Buzby, & Campbell, 1997) and Robinson, Narayanan, and Sahin (2009) have provided an overview of available solution methods. Webb et al. (1997) studied fixed replenishment cycle models for the problem and compared these to optimal solutions that do not constrain the replenishment cycle. Boctor et al. (2004) proposed several linear programming formulations, tested several heuristic solution methods and proposed an improvement procedure that can be used in combination with a heuristic method.

To increase practical relevance of the DJRP, several extensions have been proposed such as capacitated aggregate order size (Anily & Tzur, 2005; Federgruen, Meissner, & Tzur, 2007; Narayanan & Robinson, 2010), supplier selection (Ventura, Valdebenito, & Golany, 2013), supplier selection with discounts (Kang, Lee, Wu, & Lee, 2017), inventory decisions at the supplier (Cunha & Melo, 2016; Solyali & Süral, 2012) as well as inventory decisions and capacitated production at the supplier (Senoussi, Mouss, Penz, Brahimi, & Dauzère-Pérés, 2016). A commonly occurring practical constraint is an inventory capacity limit at the customers. However, to our knowledge, this constraint has only been included in one paper on the DJRP (Senoussi et al., 2016); two papers on the traditional JRP also include this constraint (Hariga, Gumus, Daghfous, & Goyal, 2013; Hoque, 2006).

The IRP combines an inventory problem and a routing problem: it minimizes inventory holding and routings costs by optimizing replenishments for a set of customers and explicitly determining the delivery routes. The IRP is therefore related to the JRP, yet the IRP is structurally different from the JRP because the routing problem is explicitly solved. Various solution methods for the IRP have been proposed in the literature such as exact methods, matheuristics and metaheuristics (see Coelho et al. (2014) for an overview). Some of the exact solution methods for the IRP rely on the vehicle capacity constraint, for example in the Pricing Problem algorithms and valid inequalities. In our application a tour duration constraint is more appropriate. In some of the heuristic solution methods for the IRP, inventory and routing optimization are considered separately. In a first phase, decisions are made on the inventory policies, often incorporating a fixed replenishment cost per delivery, thus solving a variant of the JRP. In a second phase, routing is optimized given the replenishment decisions of the first phase. Iterative solution schemes have for example been proposed by Cordeau, Laganà, Musmanno, and Vocaturo (2015) and Absi, Archetti, Dauzère-Pérès, and Feillet (2015).

Some attention has also been paid to the fact that charging a fixed fee for servicing a customer in the DJRP is not always representative for the actual costs involved. A fixed fee per customer replenishment assumes that the costs for replenishing customers are independent, but in practice, this is not always true. Olsen (2008) used the example of using a refrigerated truck for canned food delivery, which increases the marginal replenishment costs of the canned food. Olsen (2008) and Wang, He, Wu, and Zeng (2012) proposed to model the marginal costs with additional fixed fees depending on the combination of items delivered. Senoussi et al. (2016) recognized that actual transportation cost are relevant, however, they assumed that the depot is located far away from a cluster of customers and that the transportation costs between the clustered customers are negligible, therefore the authors assumed that the costs of a tour are fixed. Rahmouni and Hen-net (2015) took actual routing costs into account by combining the deterministic and static JRP with the Traveling Salesman Problem (TSP). For each possible subset of customers the actual tour length was computed beforehand by solving a TSP, then a linear programming model was used to select the optimal subsets and to determine the delivery quantities. However, this solution method can only be applied to instances of very limited size since for all combinations of customers the traveling salesman tour has to be computed.

### 3. Problem description

In the DJRP-AT, a single supplier supplies  $N$  customers. The customers face a certain demand per period and have a limited storage capacity, and therefore require replenishments to prevent them from running out of stock. The supplier arranges the customer replenishments in a VMI setting with the objective of minimizing transportation and customer inventory holding costs. The transportation costs in a period are represented by the approximated tour length visiting the replenished customers and a fixed set-up fee for a period if at least one customer is replenished in that period. In the DJRP-AT, constraints are incorporated on the composition of the set of customers served in one period, e.g., the number of customers served or the tour duration. Note that the inventory holding costs at the supplier are not considered since in our practical application there is an infinite supply (similar to Larrain et al., 2017), but these costs could easily be added.

For the calculation of the transportation costs, consider that these costs must be estimated for a large number of customer subsets, which requires careful balance of approximation accuracy and calculation effort. Also, two sets of customers with the same car-

dinality, but with customers at different locations should result in different transportation costs. Therefore, based on the short literature review in Appendix A, we adopt the tour length approximation model of Chien (1992):

$$D \approx 0.98\sqrt{RM'}, \tag{1}$$

in which  $R$  is the area of the smallest rectangle covering both the customers and the depot, and  $M'$  is the number of points in the tour (depot and customers). Note that this function underestimates the actual TSP tour length. The transportation costs also include a fixed cost  $B$  that is independent of the distance traveled, but is for example a setup cost related to vehicle use. Define binary vector  $\mathbf{Y}$  to indicate which customers are served and binary variable  $\hat{Y}$  to indicate whether any customer is served ( $\sum_i Y_i \geq 1$ ). The following transportation cost function will be used to approximate the transportation costs for the customers in  $\mathbf{Y}$  and the depot

$$f(\mathbf{Y}) = B\hat{Y} + 0.98\sqrt{R(\mathbf{Y})M'(\mathbf{Y})}. \tag{2}$$

To formulate the DJRP-AT, consider the following notation in Boctor et al. (2004). A single depot and a set of  $\mathcal{N} = \{1, 2, \dots, N\}$  customers are positioned in Euclidean space and there is a finite time horizon  $\mathcal{T} = \{1, 2, \dots, T\}$ . Let  $Y_{it}$  denote the binary decision variable that takes value 1 if and only if customer  $i \in \mathcal{N}$  is visited in period  $t \in \mathcal{T}$ . Let  $\mathbf{Y}_t$  denote the vector  $\{Y_{1t}, Y_{2t}, \dots, Y_{Nt}\}$ . Define  $X_{it}$  as the quantity delivered to customer  $i \in \mathcal{N}$  in period  $t \in \mathcal{T}$  and let  $I_{it}$  be the quantity in stock at customer  $i \in \mathcal{N}$  at the end of period  $t \in \mathcal{T}$ .  $I_{it}$  should be non-negative, because stock-outs are not allowed. The inventory level is measured at the end of the period assuming the following order of events: delivery of new stock, consumption, inventory calculation. This assumption coincides with JRP literature (Boctor et al., 2004) and with most literature on the IRP (Archetti, Bianchessi, Irnich, & Speranza, 2014).  $I_{i0}$  denotes the initial inventory level and  $d_{it}$  is the dynamic and deterministic demand in period  $t \in \mathcal{T}$  at customer  $i \in \mathcal{N}$ . For the items in stock at a customer  $i \in \mathcal{N}$  an inventory holding rate of  $h_{it}$  is charged per period  $t \in \mathcal{T}$ .

We introduce the following additional notation for the DJRP-AT. Each customer  $i \in \mathcal{N}$  has a storage capacity  $u_i$ . Define for each  $i \in \mathcal{N}$  and  $t \in \mathcal{T}$  big-M value  $M_{it} = \min\{u_i, \sum_{s=t}^T d_{is}\}$ . Furthermore, a single vehicle with unlimited load capacity performs at most one route in each time period, beginning and ending at the depot. Transportation costs in a period are represented by function  $f(\cdot)$  as defined in Eq. (2). Finally, let function  $g(\cdot)$  assess the composition of the tour in a given period. We consider two different functions for  $g(\cdot)$ . First, let  $g(\cdot)$  be the approximated tour duration and define  $k_D \in \mathbb{R}$  as the maximum tour duration. Second, we let  $g(\cdot)$  be the number of customers in a tour and impose that at most  $k_M \in \mathbb{N}$  customers can be served in a single tour. These constraints will be referred to as ‘subset composition constraints’ for the remainder of the paper. DJRP-AT models will only contain one of these two types of constraints to assess the tour composition.

The goal of the DJRP-AT is to minimize inventory holding and transportation costs by selecting, for each period, which customers to replenish, while avoiding stock-out at any customer, without violating the customer’s storage capacity restrictions and the additional restrictions on the tour composition. This problem can be formulated as follows:

$$z = \min \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}} h_{it} I_{it} + \sum_{t \in \mathcal{T}} f(\mathbf{Y}_t) \tag{3a}$$

$$\text{s.t. } I_{it} = I_{i,t-1} - d_{it} + X_{it} \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T} \tag{3b}$$

$$X_{it} \leq u_i - I_{i,t-1} \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T} \tag{3c}$$

$$X_{it} \leq M_{it} Y_{it} \quad \forall i \in \mathcal{N}, \quad \forall t \in \mathcal{T} \tag{3d}$$

$$g(\mathbf{Y}_t) \leq k \quad \forall t \in \mathcal{T} \quad (3e)$$

$$I_{it} \geq 0 \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \quad (3f)$$

$$X_{it} \geq 0 \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \quad (3g)$$

$$Y_{it} \in \{0, 1\} \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \quad (3h)$$

in which  $k = k_D$  or  $k = k_M$ , depending on the applied subset composition constraints. The objective function (3a) minimizes the costs for inventory holding and transportation  $f(\cdot)$  defined in (2). The inventory balance for each customer in each period is maintained by constraints (3b). Constraints (3c) ensure that the customer's capacity is not exceeded when a delivery is made and constraints (3d) force the amount delivered to zero if a customer is not visited. Furthermore, constraints (3e) represent the additional constraints on the composition of the subset of customers replenished in a period. Constraints (3f)–(3h) impose binary and non-negativity constraints on the decision variables.

#### 4. Column generation

When considering the complexity of the DJRP-AT with cost function (2), it is important to note that the traditional JRP is not a special case of the DJRP-AT, due to the different cost structure. Hence, although the traditional JRP is NP-complete (Arkin, Joneja, & Roundy, 1989), this conclusion cannot be directly made for the DJRP-AT. Furthermore, analysis of the literature on the complexity of related problems shows that the so-called Pinwheel Scheduling Problem is a special case of the DJRP-AT with cost function (2). The Pinwheel Scheduling Problem is likely to be an NP-complete problem (Jacobs & Longo, 2014), but this has not yet been proven despite several attempts. A mapping between the Pinwheel Scheduling Problem and the DJRP-AT, including details on the complexity of the Pinwheel Scheduling Problem, are presented in Appendix B. Because the existence of a polynomial-time algorithm is unlikely and column generation has proven to be efficient for similar problem structures, this solution method will be used to solve the DJRP-AT.

Application of the Dantzig–Wolfe decomposition to problem (3a)–(3h) results in a Master Problem that selects for every period a subset of customers to replenish out of a collection of subsets to minimize the inventory holding and transportation costs. Moreover, the Master Problem optimizes the delivery quantities corresponding to constraints (3b)–(3d) in the compact formulation. The Pricing Problem generates subsets of customers, taking constraints (3e) into account, and is solved for each period separately.

To formulate the Master Problem, let  $\mathcal{S}_t$  be the collection of subsets of customers that are generated by the Pricing Problem for period  $t \in \mathcal{T}$ . The binary decision variable  $Z_{st}$  equals 1 if subset  $s \in \mathcal{S}_t$  is selected for period  $t \in \mathcal{T}$ . For a specific subset  $s \in \mathcal{S}_t$  the transportation costs  $c_s$  for servicing its customers is given by the Pricing Problem. Furthermore, let  $a_{is}$  indicate whether customer  $i \in \mathcal{N}$  is present in subset  $s \in \mathcal{S}_t$ . The decomposition gives the following Master Problem:

$$z = \min \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{N}} h_{it} I_{it} + \sum_{t \in \mathcal{T}} \sum_{s \in \mathcal{S}_t} c_s Z_{st} \quad (4a)$$

$$\text{s.t. } I_{it} = I_{i,t-1} - d_{it} + X_{it} \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \quad (4b)$$

$$X_{it} \leq u_i - I_{i,t-1} \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \quad (4c)$$

$$X_{it} \leq u_i \sum_{s \in \mathcal{S}_t} a_{is} Z_{st} \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \quad (4d)$$

$$\sum_{s \in \mathcal{S}_t} Z_{st} \leq 1 \quad \forall t \in \mathcal{T} \quad (4e)$$

$$I_{it} \geq 0 \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \quad (4f)$$

$$X_{it} \geq 0 \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \quad (4g)$$

$$Z_{st} \in \{0, 1\} \quad \forall s \in \mathcal{S}_t, \forall t \in \mathcal{T} \quad (4h)$$

The objective function (4a) aims to minimize total costs. Constraints (4b)–(4d) are equivalent to constraints (3b)–(3d) of the compact formulation. Constraint (4e) ensures that at most one subset of customers is selected for each period. Finally, non-negativity and binary requirements on the decision variables are imposed by constraints (4f)–(4h).

We use column generation to solve the linear programming relaxation of (4a)–(4h) since the total number of variables  $Z_{st}$  is exponentially large. Starting with a small subset of all possible columns gives the Restricted Master Problem (RMP) and additional columns with negative reduced cost are generated by repeatedly solving the Pricing Problem. To formulate the Pricing Problem, let us associate the following dual variables with the Master Problem with respect to decision variables  $Z_{st}$ . Let  $\pi_{it}^1$  be a non-positive dual variable associated with constraints (4d) and let  $\pi_t^2$  be the non-positive dual variable of constraints (4e). Let us also reuse decision variables  $Y_{it}$  from the compact formulation: these variables indicate whether customer  $i \in \mathcal{N}$  is replenished in period  $t \in \mathcal{T}$  and remind that  $\mathbf{Y}_t = \{Y_{1t}, Y_{2t}, \dots, Y_{Nt}\}$ . For a given time period  $t \in \mathcal{T}$  the Pricing Problem can be formulated as follows:

$$\min \bar{c}_t(\mathbf{Y}_t) = f(\mathbf{Y}_t) + \sum_{i \in \mathcal{N}} u_i Y_{it} \pi_{it}^1 - \pi_t^2 \quad (5a)$$

$$\text{s.t. } g(\mathbf{Y}_t) \leq k \quad (5b)$$

$$\mathbf{Y}_t \in \{0, 1\}^N \quad (5c)$$

The objective (5a) is to minimize the reduced cost  $\bar{c}_t(\mathbf{Y}_t)$  while the subset composition constraints are satisfied (5b). The reduced cost consists of the transportation costs of the subset  $f(\mathbf{Y}_t)$  and dual terms corresponding to the current solution of the RMP. The subset composition constraints (5b) can, in general, concern any function of the combination of customers in the subset. However, these constraints cannot contain the delivery quantities, since these quantities are determined in the Master Problem. Hence, in our model, a load capacity constraint cannot be in the Pricing Problem, but, for example, a tour duration constraint is possible.

#### 5. Branch-and-Cut-and-Price

The Master Problem and Pricing Problem of Section 4 are solved in a Branch-and-Cut-and-Price framework. In Section 5.1, a tailored labeling algorithm to solve the Pricing Problem per period is described and novel sufficient conditions are presented that provide a dominance criterion to discard labels. Valid inequalities are presented in Sections 5.2 and 5.3 provides a description of the branching strategy.

##### 5.1. Labeling algorithm for the pricing problem

To solve the Pricing Problem, we propose a tailored labeling algorithm that identifies subsets of customers that will improve the current solution of the RMP. Note that during the process of generating subsets of customers, only the customer combination is relevant, there is no sequential relationship between the customers as opposed to routing problems (Feillet, Dejax, Gendreau, & Gueguen, 2004).

Define label  $L = \langle s(L), \bar{c}_t(L), g(L) \rangle$  in which  $s(L)$  is the subset of customers,  $\bar{c}_t(L)$  is the corresponding reduced cost and  $g(L)$

represents the value of the function  $g(\cdot)$  in the subset composition constraint for subset  $s(L)$ . Hence, each label corresponds to a subset of customers that is a candidate to be added to the RMP.

The labeling algorithm starts for each customer  $i \in \mathcal{N}$  separately and the labels are extended by adding the other customers one by one. The order of the customers in  $s(L)$  is not important, since the subset of customers is considered for replenishment, but the order in which they are served is not determined. Hence, each possible subset has to be considered at most once. Therefore, when starting with a label containing customer  $i$  and extending with customer  $j$ , the inverse order of these customers, starting with  $j$  and adding  $i$ , does not have to be considered. The labeling algorithm terminates when all possible subsets of customers are considered.

Denote  $L \oplus P$  as the resulting label from the extension of label  $L$  with the customers in set  $P \subseteq \mathcal{N} \setminus s(L)$ . The operation to extend a label  $L$  with the next customer  $j$  is to set  $s(L \oplus \{j\}) = s(L) \cup \{j\}$  and to compute  $\bar{c}_t(L \oplus \{j\})$  and  $g(L \oplus \{j\})$ :

$$\begin{aligned} \bar{c}_t(L \oplus \{j\}) &= \bar{c}_t(L) - f(\mathbf{Y}_t(s(L))) + f(\mathbf{Y}_t(s(L) \cup \{j\})) + u_j \pi_{jt}^1 \quad (6) \\ &= f(\mathbf{Y}_t(s(L) \cup \{j\})) + \sum_{i \in s(L) \cup \{j\}} u_i \pi_{it}^1 - \pi_t^2 \end{aligned}$$

in which  $\mathbf{Y}_t(s(L))$  is the vector in which the variables corresponding to  $s(L)$  equal 1. If we consider the model with subset composition constraints that set a maximum on the tour duration, we have

$$g(L \oplus \{j\}) = 0.98 \sqrt{R(s(L) \cup \{j\})(|s(L)| + 1)} \quad (7)$$

and if we consider the model with subset composition constraints that pose a maximum on the number of customers in the subset, we have

$$g(L \oplus \{j\}) = g(L) + 1 \quad (8)$$

The extended label is feasible if

$$s(L) \cap \{j\} = \emptyset \wedge g(L \oplus \{j\}) \leq k \quad (9)$$

When the number of customers  $N$  increases, the maximum number of labels becomes large ( $2^N - 1$ ). A dominance test will therefore be used to reduce the number of labels.

Denote the set of feasible extensions of label  $L$  by  $E(L)$  which consists of all combinations of the customers that have not already been considered and for which the extension will satisfy the subset composition constraints. The following definition for dominance holds

**Definition 1.** Label  $L$  dominates label  $L'$  if

- D.1  $E(L') \subseteq E(L)$
- D.2  $\bar{c}_t(L \oplus P) \leq \bar{c}_t(L' \oplus P), \forall P \in E(L')$

The first condition, D.1, states that a feasible extension of  $L'$  must also be a feasible extension of  $L$ . The second condition, D.2, requires that all feasible extensions of  $L$  do not result in worse solutions than the same extensions of  $L'$ . These conditions are difficult to check in practice, since all feasible extensions would have to be computed. Therefore, Proposition 1 introduces sufficient conditions for dominance of  $L$  over  $L'$ .

**Proposition 1.** Label  $L$  dominates label  $L'$  if the following conditions hold

- P.1  $s(L) \subseteq s(L')$
- P.2  $g(L) \leq g(L')$
- P.3  $\bar{c}_t(L) + \Delta(L, L') \leq \bar{c}_t(L')$

Conditions P.1 and P.2 combined imply condition D.1, such conditions are also used for shortest path problems (Feillet et al., 2004), and condition P.3 implies condition D.2. Before a formal proof for Proposition 1 is presented, an intuitive reasoning for condition P.3 is given and an expression for  $\Delta(L, L')$  is derived.

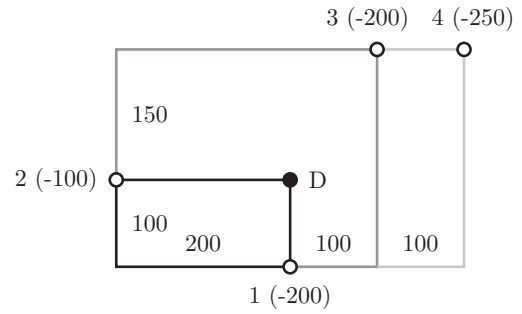


Fig. 1. Example of dominance in labeling algorithm.

The cost function, and therefore the reduced costs, are dependent on the number of customers and the area in which these customers are located. Consider a comparison of the two labels  $L$  and  $L'$  for which it holds that  $s(L) \subseteq s(L')$ ,  $\bar{c}_t(L) < \bar{c}_t(L')$  and  $g(L) \leq g(L')$ . One would like to conclude that  $L$  dominates  $L'$ . However, if a set of customers  $P$  is added to both labels, the area that is used in the cost function can increase more for  $L$  than for  $L'$ , i.e., the additional cost of the extension with  $P$  is not identical for both labels. This could result in  $\bar{c}_t(L \oplus P) > \bar{c}_t(L' \oplus P)$ , therefore, it cannot be concluded that  $L$  dominates  $L'$  since condition D.2 is violated. Hence, a sufficient dominance condition should be stricter than  $\bar{c}_t(L) \leq \bar{c}_t(L')$ , therefore sufficient condition P.3 is introduced. This will be illustrated in the following example. Consider an instance with four customers, indicated by white nodes and customer index in Fig. 1. The depot is indicated by the black node with label D. The current terms for the reduced cost corresponding to each customer are indicated between brackets (suppose  $\pi_t = 1000$ ).

Now, consider subset  $s_1 = \{1, 2\}$ , the corresponding cost is  $f(s_1) = 1000 + 0.98 \sqrt{100 \times 200 \times 3} \approx 1240$  and the reduced cost of this subset is  $\bar{c}_t(s_1) \approx 1240 - 200 - 100 - 1000 = -60$ . Similarly, subset  $s_2 = \{1, 2, 3\}$  has  $f(s_2) \approx 1537$  and  $\bar{c}_t(s_2) \approx 37$ . In this case  $s_1$  and  $s_2$  are comparable, since  $s_1 \subset s_2$ . Note that  $\bar{c}_t(s_1) < \bar{c}_t(s_2)$ , hence, one would like to conclude that the label with subset  $s_1$  dominates the label with  $s_2$ . However, suppose customer 4 is added to both subsets. This gives  $s_3 = \{1, 2, 4\}$  with  $f(s_3) \approx 1620$  and  $\bar{c}_t(s_3) \approx 70$  and  $s_4 = \{1, 2, 3, 4\}$  with  $f(s_4) \approx 1693$  and  $\bar{c}_t(s_4) \approx -57$ . Note that the cost increase from subset  $s_1$  to  $s_3$  is larger than from subset  $s_2$  to  $s_4$ . Because  $\bar{c}_t(s_3) > \bar{c}_t(s_4)$ , we cannot conclude from  $\bar{c}_t(s_1) < \bar{c}_t(s_2)$  that the label with  $s_1$  dominates the label with  $s_2$ .

A more strict condition is required and therefore  $\Delta(L, L')$  is introduced, representing the maximum difference in costs between two labels. The value of  $\Delta(L, L')$  must be sufficient to guarantee  $\bar{c}_t(L \oplus P) \leq \bar{c}_t(L' \oplus P)$  ( $\forall P \in E(L')$ ) to conclude that  $L$  dominates  $L'$ . First, express  $\bar{c}_t(L \oplus P)$  and  $\bar{c}_t(L' \oplus P)$  in terms of  $\bar{c}_t(L)$  and  $\bar{c}_t(L')$ , respectively. Combined with  $\bar{c}_t(L) \leq \bar{c}_t(L')$ , an upper bound can be derived for the difference between  $\bar{c}_t(L \oplus P)$  and  $\bar{c}_t(L' \oplus P)$ . The derivation for the following sufficient value of  $\Delta(L, L')$  is given in Appendix C:

$$\Delta(L, L') = \phi \sqrt{R(s(L'))|s(L') \cup P| - R(s(L))|s(L) \cup P|} \quad (10)$$

with  $R(\cdot)$  as the area of the smallest rectangle and  $P = \mathcal{N} \setminus s(L')$ . A formal proof for Proposition 1 with  $\Delta(L, L')$  as defined in (10) can now be presented.

**Proof.** (Proof of Proposition 1) Assume two labels  $L$  and  $L'$  with corresponding subsets of customers  $s = s(L)$  and  $s' = s(L')$  satisfy the conditions in Proposition 1. Given  $s \subseteq s'$  and therefore  $s \cup P \subseteq s' \cup P$ , it holds that if  $g(L) \leq g(L')$ , then  $g(L \oplus P) \leq g(L' \oplus P)$  if  $g(\cdot)$  is a monotone function. Hence, if  $g(L \oplus P) \leq k$ , then  $g(L' \oplus P) \leq k$  and condition D.1 is satisfied. To show that condition D.2 holds, note that we have

already shown in [Appendix C](#) that

$$\bar{c}_t(L \oplus P) - \bar{c}_t(L' \oplus P) \leq \bar{c}_t(L) - \bar{c}_t(L') + \Delta(L, L') \quad (11)$$

Hence, if  $\bar{c}_t(L) + \Delta(L, L') \leq \bar{c}_t(L')$ , then condition D.2 holds which concludes the proof of [Proposition 1](#).  $\square$

An overview of the labeling algorithm is provided in [Algorithm 1](#). A label is not extended any further in the labeling

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#### Algorithm 1 Labeling algorithm

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1: Initialize list of improving labels  $\mathcal{I}$  and of labels to propagate  $\mathcal{P}$ 
2: for  $i = 0$  to  $N$  do
3:   Create label  $l$  containing  $i$  and add  $l$  to  $\mathcal{P}$ 
4:   if Reduced cost of  $l$  is negative then
5:     Add  $l$  to  $\mathcal{I}$ 
6:   while  $\mathcal{P} \neq \emptyset$  do
7:     Consider a waiting label  $p \in \mathcal{P}$ 
8:     for All customers  $j$  with higher index than the last added
customer to  $p$  do
9:       Extend  $p$  with  $j$  to  $q$ 
10:      if  $q$  is feasible and  $p$  does not dominate  $q$  then
11:        Add  $q$  to  $\mathcal{P}$ 
12:      if Reduced cost of  $u < 0$  then
13:        Add  $q$  to  $\mathcal{I}$ 
14:      Remove  $p$  from  $\mathcal{P}$ 

```

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algorithm if the subset composition constraint is violated, since no feasible subsets of customers can be found by adding more customers because of the subset composition constraints. Moreover, after extending a label, it is tested whether adding another customer violates a subset composition constraint, in which case an extension to this customer from the current label is not considered in a later stage of the labeling algorithm.

To accelerate the solution process, a heuristic variant of the labeling algorithm is applied. If the heuristic fails to find any improving subsets, the exact labeling algorithm is applied in which all combinations of customers are considered. In the heuristic pricing instrument, the process is identical to the exact pricing algorithm, but the number of customers to which a label can be extended is limited. Only the extensions to the  $b$  customers closest to the last added customer of a subset are evaluated. The initial value of  $b$  is small (2) and this value is doubled up to a certain limit (8) as long as no improving subsets of customers are found.

Preliminary experiments showed that adding all columns with negative reduced cost could be time consuming for the algorithm and moreover, many of these columns are not in the final solution. Therefore, at most 10,000 columns are added per call to the Pricing Problem for both the heuristic and the exact pricing instrument.

#### 5.2. Valid inequalities

The formulation of the problem can be strengthened with valid inequalities. Two valid inequalities that were introduced by [Archetti, Bertazzi, Laporte, and Speranza \(2007\)](#) for the IRP are also applicable to the DJRP-AT. The first inequality states that if a customer  $i \in \mathcal{N}$  is not replenished in periods  $t - r, t - r + 1, \dots, t$ , then the inventory in period  $t - r - 1$  should be sufficient to cover demand of all periods up to  $t$ :

$$(ILB) \quad I_{i,t-r-1} \geq \left( \sum_{j=0}^r d_{i,t-j} \right) \left( 1 - \sum_{j=0}^r \sum_{s \in \mathcal{S}_t} a_{is} Z_{s,t-j} \right) \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T}, \forall r = 0, 1, \dots, t - 1 \quad (12)$$

The second inequality gives a lower bound for the number of required visits to a specific customer  $i$  up to a period  $t$  taking the

customer's inventory capacity into account:

$$(NrVis) \quad \sum_{j=1}^t \sum_{s \in \mathcal{S}_t} a_{is} Z_{sj} \geq \left\lceil \frac{\sum_{j=1}^t d_{ij} - I_{i0}}{u_i} \right\rceil \quad \forall i \in \mathcal{N}, \forall t \in \mathcal{T} \quad (13)$$

Preliminary experiments showed that dynamic management of the valid inequalities, i.e., adding them whenever violated, slowed down the execution. Therefore, for all experiments, all valid inequalities are added to the model in the root node of the Branch-and-Bound tree.

#### 5.3. Branching

In the compact formulation (3a)–(3h), the variables indicating whether a customer is served in a certain time period, the assignment variables, are binary decision variables. The delivery quantity and inventory level decision variables are non-negative and continuous. To find binary results for the assignment variables a Branch-and-Bound tree is used, which is explored via a best bound strategy. First, the algorithm branches on the total number of replenishments of a customer  $i \in \mathcal{N}$  over all periods  $\sum_{t \in \mathcal{T}} Y_{it}$ . In the Master Problem variables this can be expressed as  $\sum_{t \in \mathcal{T}} \sum_{s \in \mathcal{S}_t} a_{is} Z_{st}$ . If all customers have an integer number of replenishments, the second branching method branches on whether any customer is replenished in a period  $t$  or no customer is replenished ( $\sum_{i \in \mathcal{N}} Y_{it}$ ). Expressed in the Master Problem variables, the corresponding constraints are  $\sum_{s \in \mathcal{S}_t} Z_{st} \leq 0$  and  $\sum_{s \in \mathcal{S}_t} Z_{st} \geq 1$ . If no new branches can be identified for the first two branching methods, the algorithm branches on whether a customer  $i \in \mathcal{N}$  is visited in a specific period  $t \in \mathcal{T}$  or not ( $Y_{it} = \sum_{s \in \mathcal{S}_t} a_{is} Z_{st}$ ). This branching method leads to binary solutions for the assignment variables in the compact formulation, hence, if no new branches are identified an integer solution is found. If there is more than one branch candidate, the following strategy to select a branch is followed for each type of branching. For each branch candidate the child nodes are quickly evaluated by solving the LP relaxation given the current set of columns. The branch that maximizes the lower bound is chosen. We consider at most 25 branch candidates in the first 20 nodes of the branch and bound tree and at most 15 candidates in the other nodes.

## 6. Computational results

The proposed model for the DJRP-AT is analyzed for two types of subset composition constraints: a tour duration constraint and a maximum number of customers served per period (i.e., in a subset). The effectiveness of the valid inequalities presented in [Section 5.2](#) is evaluated for both types of subset composition constraints in [Section 6.1](#). The model with a maximum tour duration most resembles practical cases; this model is tested with different values for the maximum tour duration. The model with a maximum number of customers per subset can be compared to the DJRP with fixed fees ([Boctor et al., 2004](#)) and to a variant of the IRP in which the actual routing costs are used. The first comparison provides insight in the potential improvements that can be achieved with the DJRP-AT compared to the DJRP. [Section 6.2](#) explains how the DJRP-AT and the DJRP are compared and provides the results. The second comparison shows how well the DJRP-AT performs compared to an equivalent IRP in which actual routing costs are used, on which [Section 6.3](#) reports the results. Note that results of the DJRP-AT are not comparable to the results of 'standard' IRP as often used in the literature ([Coelho et al., 2014](#)) because the constraints are different.

The instances for the IRP, created by [Archetti et al. \(2007\)](#), are used for the computational experiments. Although the IRP differs

from the DJRP-AT regarding the cost structure, decision variables and constraints, the IRP instances are used as a base, since they contain most of the data required for the experiments on the DJRP-AT. The time horizon is equal to either 3 or 6 periods and instances with 5, 10, 15 and 20 customers are considered; there are five instances for each combination of number of customers and periods. For each customer, a location is given by two coordinates, both randomly chosen within the interval  $[0, 500]$ ; demand is randomly selected between 10 and 100, and the customer holding rate is in the interval  $[0.1, 0.5]$ . The customer's inventory capacity is the demand of the customers multiplied by either 2 or 3, which is randomly selected. The initial inventory is the customer's capacity minus the demand of the first period. The instances are available online (Coelho, n.d.). For the tests on the model including maximal tour duration the maximum ( $k_D$ ) is set to 600, 800, 1000 and 1200 for all instances. If a maximum is set on the number of customers per subset ( $k_M$ ), this maximum depends on the number of customers in the instance. For 5 customers, maxima of 3 and 4 are considered, for 10 customers 5, 6, 7 and 8 are considered as maxima, for 15 customers 7, 8, 9, 10 and 11 are considered, and for instances with 20 customers 10, 11, 12 and 13 are the maxima. The fixed major cost is set to  $B = 1000$ .

The RMP is initialized with dummy columns with very high costs to guarantee a feasible solution for the initial linear program. To improve computation times, two heuristics are designed to attempt quickly identifying columns that provide a feasible integer solution. The first heuristic assigns customers to periods in a greedy way, ensuring customers do not run out of stock and respecting the subset composition constraints. If the subset composition constraint is a maximum on the tour duration, a second heuristic is applied if the first one did not yield a feasible integer solution. In the second heuristic, before applying the steps of the first heuristic, the customers are sorted by decreasing distance from the depot. Since the sorting process indirectly takes the subset composition constraint into account, this provides a greater chance of finding a feasible solution. Limited computational experiments showed that the first heuristic provides a better upper bound if a solution is found, therefore this heuristic is applied first.

The algorithm to solve the DJRP-AT as described in the previous sections is implemented using Java and Gurobi 6.5. All tests are performed on a desktop computer running Windows 10, equipped with an eight core Intel(R) Core(TM) i7-6700K, CPU 4.00 gigahertz processor with 24 gigabyte of RAM. A single core is used to generate the results and the maximum running time is two hours per instance.

### 6.1. Effectiveness of valid inequalities

Two valid inequalities are considered to strengthen the formulation for the DJRP-AT. These valid inequalities are tested for both types of subset composition constraints and the results are presented in Table 1. The results are aggregated per number of customers in the instance ( $N$ ) and the length of the time horizon ( $T$ ). The average solution time over all tested instances, the number of instances solved, and the average integrality gap for the solved instances are reported for the model without valid inequalities, the model with only the ILB inequalities (Eq. 12), the model with only NrVis inequalities (Eq. 13), and the model with both inequalities, respectively. Next to the number of instances solved, the total number of tested instances that have not been proved infeasible is indicated between brackets. The difference between the two numbers gives the number of instances that have not been solved in two hours. The integrality gap is the percentage difference between the optimal binary solution ( $LB_{best}$ ) and the solution of the

relaxation of the model in the root node of the Branch-and-Bound tree ( $LB_{root}$ ), which is calculated by  $(LB_{best} - LB_{root})/LB_{best}$ . To test the effectiveness of the valid inequalities, for instances with 15 customers and 3 periods, duration 600 is not considered; for 15 customers and 6 periods, both 600 and 800 are not considered, since most instances would be infeasible. For instances with 15 customers, a maximum number of customers per period ( $k_M$ ) of 8, 9, 10 and 11 are considered.

The observed integrality gaps, as reported in Table 1, are quite high and decrease if more valid inequalities are added. The high integrality gaps can be partially explained by the formulation of the Master Problem. The linear relaxation of the model allows for satisfying customer demand in a certain period using only a fractional value for the decision variable corresponding to a subset containing this customer. Therefore, the costs of the subset are only fractionally accounted for. Moreover, the linear relaxation allows infeasible combinations of customers to be served in one period in a fractional solution. Therefore, the costs of the fractional solution can be lower, even though the solution is certainly not feasible. Both of these effects were observed in our experiments. For example, consider instance *abs3n5* from Archetti et al. (2007) which contains 5 customers, 3 time periods and set  $k_D = 600$  as the maximum on the duration. The optimal solution selects customer subset  $\{3\}$  in period 1, subset  $\{1,2\}$  in period 2 and subset  $\{3,4,5\}$  in period 3; the objective value is 4310. In the solution of the relaxed model, subsets  $\{1,2\}$  and  $\{3,4,5\}$  are selected with value 0.5 in both period 1 and period 2. The objective value of this fractional solution is 3397, resulting in an integrality gap of 21%. Note that in the fractional solution all customers can be replenished in both period 1 and 2, while a subset consisting of all customers  $\{1,2,3,4,5\}$  is not a feasible subset given the tour duration constraint. This demonstrates that the fractional model provides the opportunity to select suboptimal subsets of customers, causing high integrality gaps. Also note that the instances with the highest integrality gaps do not necessarily have the highest computation times for both types of subset composition constraints.

For the model with a tour duration constraint, the computational results show that valid inequalities decrease computation times and integrality gaps, but that adding both types of valid inequalities does not always improve computation times, compared with adding one type of inequality. The integrality gaps are best if both types of inequalities are used, but only adding the NrVis inequalities yields almost the same average integrality gap. For this subset composition constraint, the best performance in terms of computation times are found if only the ILB inequalities are added to the model.

For the model with the constraint on the number of customers replenished per period, the results show that both types of valid inequalities improve the efficiency of the model. The average computation time decreases strongly for all instance sizes. For the largest instances (15 customers, 6 time periods) the number of solved instances increases from 12 to 17 out of 20 by adding the valid inequalities and the average integrality gap is only a third of the average gap without inequalities. Furthermore, the integrality gap of the model including only the NrVis inequalities is almost identical to the model with both types of inequalities; however, computation times show that the model including both types of inequalities performs better.

Archetti et al. (2007) concluded that the NrVis inequalities (Eq. 13) are ineffective for solving their IRP model. It is therefore important to note that, for the DJRP-AT, these valid inequalities lower the integrality gaps substantially and reduce the computation time. Hence, for the DJRP-AT, the effectiveness of these inequalities has been demonstrated for both types of subset composition constraints. The results per instance with all valid



**Table 1**  
Effectiveness of valid inequalities for DJRP-AT for two types of constraints.

N	T	None			Only ILB			Only NrVis			ILB and NrVis		
		Time (s)	# Solved	Gap (%)	Time (s)	# Solved	Gap (%)	Time (s)	# Solved	Gap (%)	Time (s)	# Solved	Gap (%)
<b>Constraint on duration</b>													
5	3	0	20 (20)	28	0	20 (20)	22	0	20 (20)	7	0	20 (20)	7
10	3	2	18 (18)	42	1	18 (18)	34	1	18 (18)	16	1	18 (18)	16
15	3	40	13 (13)	44	18	13 (13)	37	17	13 (13)	20	6	13 (13)	20
5	6	1	20 (20)	25	1	20 (20)	19	1	20 (20)	6	0	20 (20)	3
10	6	83	16 (16)	31	39	16 (16)	25	71	16 (16)	12	19	16 (16)	9
15	6	3801	6 (10)	31	3021	7 (10)	26	3316	7 (10)	13	3456	6 (10)	11
Sum			93 (97)			94 (97)			94 (97)			93 (97)	
<b>Constraint on number of customers</b>													
5	3	0	10 (10)	46	0	10 (10)	42	0	10 (10)	20	0	10 (10)	20
10	3	2	20 (20)	42	1	20 (20)	34	1	20 (20)	14	1	20 (20)	13
15	3	30	20 (20)	42	12	20 (20)	34	15	20 (20)	15	7	20 (20)	14
5	6	3	10 (10)	32	1	10 (10)	26	2	10 (10)	13	0	10 (10)	10
10	6	86	20 (20)	31	29	20 (20)	24	39	20 (20)	11	20	20 (20)	8
15	6	4269	12 (20)	30	3212	14 (20)	24	3264	15 (20)	11	2507	17 (20)	11
Sum			92 (100)			94 (100)			95 (100)			97 (100)	

inequalities in the model are available in [Appendix D](#) for both types of subset composition constraints.

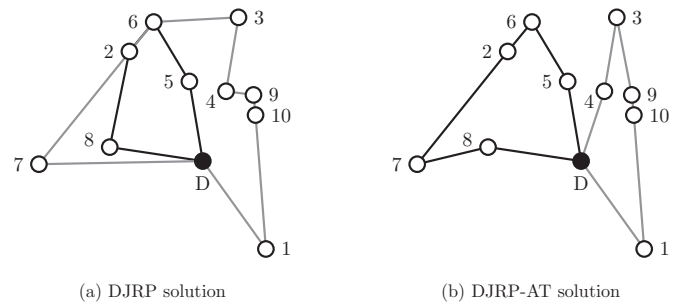
## 6.2. Comparison DJRP-AT and DJRP

The results of the DJRP-AT with a constraint on the number of customers per tour  $k_M$  can be compared with the existing DJRP with fixed fees ([Boctor et al., 2004](#)). In this model a common cost is paid for serving at least one customer in a period ( $B = 1000$ ) and an individual cost  $m_i$  is incurred for replenishing each customer. Note that the individual replenishment cost would in practice be given by the contract between the supplier and the LSP, and cannot be changed during the execution of the contract. To assess the impact of the individual replenishment costs, we test several values for  $m_i$ ,  $i \in \mathcal{N}$ . For the experiments the individual replenishment cost  $m_i$  is either set to the same value for all customers (25 and 100) or set according to one of the following schemes. First, we set  $m_i$  to a value proportional to the distance to the depot, which we denote by 'prop'. Second, we define a zone around the depot in which at least one-third of the customers is located ('zones'). For the customers within the zone  $m_i = 25$ , and for the other customers  $m_i = 100$ . Third, we divide the total area in four quadrants ('quad'). If a customer is in the same quadrant as the depot  $m_i = 25$ , and  $m_i = 100$  otherwise.

To make a fair comparison between the DJRP and the DJRP-AT, the DJRP is extended with constraints on the customer's inventory capacity and a constraint on the number of customers served per period. Hence, the difference between this model and the DJRP-AT is the cost structure, i.e., fixed fees for individual replenishments versus transportation costs. Both models result in subsets of customers to be served in each period of the planning horizon and the corresponding delivery quantities. Importantly, it is not possible to directly compare the costs of both models. Therefore, the optimal traveling salesman tours for the resulting customer subsets are computed. The tour costs reflect the actual incurred routing costs. The total costs, for both models consisting of the inventory holding cost, the tour costs and the fixed costs per period, will be compared. First, the results of one instance are studied in more detail. Next, aggregated results over all tested instances are analyzed.

### 6.2.1. Illustrative result for one instance

In this section, one of the tested instances is studied in more detail and the results demonstrate the effect of the DJRP-AT compared with the DJRP. Consider an example containing 10 customers



**Fig. 2.** DJRP and DJRP-AT solutions for instance abs3n10 with  $T = 3$ ,  $k_M = 7$  and  $m_i = 25$  (black route in period 2, gray route in period 3).

(instance abs3n10), 3 time periods, a maximum of  $k_M = 7$  customers per period and an individual replenishment cost of  $m_i = 25$ . In [Fig. 2](#), the routes corresponding with the solution of the DJRP and the DJRP-AT are drawn. The depot is indicated by D and the customers are numbered according to the order of the customers in the instance. The black line represents the performed route in period two of the three periods and the grey line represents the route in period three; no customers are replenished in period one in both solutions. The DJRP solution shows that some customers are replenished multiple times in the planning horizon. Moreover, customers that are located in relatively close proximity, are not necessarily replenished in the same period, e.g., customers 7 and 8. The total costs of the DJRP solution are 4399 consisting of inventory holding cost of 123 and routing costs of 4276, including the fixed fee per period. The DJRP-AT solution is clearly more efficient from a routing point of view. The two routes cover distinct areas of the region in which the customers are located and all customers are replenished only once in the planning horizon. The holding cost, 159, is higher than in the solution of the DJRP. The routing costs, including the fixed fee per period, are 3914 which is lower than the DJRP routing costs. The total costs of DJRP-AT solution are 4073, which is 7.4% lower than the total costs of the DJRP solution.

### 6.2.2. Aggregated results

[Table 2](#) compares aggregated results of the DJRP-AT and the DJRP for the instances proposed by [Archetti et al. \(2007\)](#). The first columns indicate the number of customers ( $N$ ), the length of the time horizon ( $T$ ), the maximum number of customers ( $k_M$ ) and the number of instances solved out of the five instances ( $\#$ ). For each

**Table 2**  
Average and maximum improvement and maximum deterioration of DJRP-AT compared with DJRP.

N	T	k <sub>M</sub>	# <sup>a</sup>	Average cost improvement (%)					Maximum cost improvement (%)					Maximum cost deterioration (%)				
				m					m					m				
				25	100	prop	zones	quad	25	100	prop	zones	quad	25	100	prop	zones	quad
5	3	3	5	3.8	2.1	2.1	1.9	1.9	8.7	7.8	7.8	7.8	7.8	0.4	0.4	0.4	0.7	0.7
5	3	4	5	4.2	2.5	2.5	2.3	2.3	8.6	7.2	7.2	7.2	7.2	–	–	–	1.0	1.0
10	3	5	5	2.3	3.4	1.5	1.5	1.8	9.5	9.5	9.5	9.5	9.5	–	–	2.2	2.2	0.6
10	3	6	5	3.6	4.8	2.8	3.6	2.9	7.1	7.6	7.1	7.1	7.1	0.4	0.4	2.1	0.4	2.1
10	3	7	5	4.6	4.1	4.1	4.4	4.2	7.9	8.1	8.1	7.8	8.1	0.4	0.4	0.4	0.4	0.4
10	3	8	5	5.9	5.4	5.4	5.7	5.6	11.1	8.9	8.9	10.6	9.4	–	–	–	–	–
15	3	7	5	10.7	9.4	8.9	8.7	8.2	12.2	11.1	10.8	11.6	12.2	–	–	–	–	–
15	3	8	5	3.0	1.2	-0.5	0.0	1.1	6.7	3.4	0.4	2.7	5.3	0.5	–	3.0	2.6	2.6
15	3	9	5	3.6	0.7	0.6	1.7	2.5	5.6	4.6	4.1	4.7	5.7	–	1.5	1.5	0.2	0.2
15	3	10	5	6.1	3.4	3.3	4.3	5.1	11.0	9.6	9.1	9.6	11.1	–	1.0	1.0	0.1	0.1
15	3	11	5	8.2	5.6	5.5	6.2	7.0	14.4	14.3	13.9	14.4	13.9	–	2.7	2.7	3.2	3.2
20	3	10	5	6.8	4.9	1.9	4.3	2.8	8.7	8.7	8.7	8.7	8.7	–	–	0.8	–	0.4
20	3	11	4	2.2	2.4	2.4	2.7	2.7	6.6	7.4	7.4	8.2	8.2	0.5	–	–	–	–
20	3	12	5	2.6	1.5	1.5	2.2	2.2	8.5	5.2	5.2	8.1	8.1	0.9	–	–	–	–
20	3	13	5	4.4	3.4	3.4	3.6	3.6	10.1	8.8	8.8	8.8	8.8	0.9	–	–	–	–
5	6	3	5	5.8	2.6	4.4	4.5	4.5	12.3	5.9	6.6	6.6	6.6	–	–	–	–	–
5	6	4	5	3.2	1.8	1.2	1.1	1.4	6.2	4.2	4.1	4.2	4.2	–	0.2	1.5	1.5	1.5
10	6	5	5	5.3	4.8	4.8	4.8	5.3	10.4	7.9	7.9	8.2	10.4	–	–	–	–	–
10	6	6	5	5.7	5.7	5.9	5.7	5.8	9.5	8.4	8.1	8.9	8.0	–	–	–	–	–
10	6	5	5	3.0	2.5	2.2	2.1	2.9	6.9	5.3	5.1	4.2	5.8	0.6	–	–	–	–
10	6	8	5	4.3	2.6	3.2	3.0	4.5	7.6	4.9	5.4	5.0	7.7	–	–	–	–	–
15	6	8	3	8.6	5.4	6.2	7.7	7.7	11.3	6.1	7.4	11.3	11.3	–	–	–	–	–
15	6	9	4	8.1	7.1	6.6	7.1	7.9	11.7	9.5	9.3	9.3	11.6	–	–	–	–	–
15	6	10	5	3.7	2.4	2.4	3.2	3.4	6.1	6.1	6.3	5.2	6.2	–	–	–	–	–
15	6	11	5	5.1	3.8	3.1	4.3	4.2	7.6	5.3	4.8	5.9	5.5	–	–	–	–	–
Overall				5.0	3.7	3.4	3.9	4.1	14.4	14.3	13.9	14.4	13.9	0.9	2.7	3.0	3.2	3.2

<sup>a</sup> Number of instances solved out of 5 for each parameter combination.

combination of *N*, *T* and *k<sub>M</sub>* five different individual replenishment costs are considered for the DJRP (*m* = 25, 100, prop, zones, quad). The average percentage improvement in total costs, the maximum percentage improvement in total costs and the maximum percentage deterioration in any of the instances of the DJRP-AT, compared with the DJRP, are reported for each combination of parameter values. The percentage improvement in total cost of an instance is computed as (cost DJRP - cost DJRP-AT)/cost DJRP.

The results in Table 2 show that incorporating the approximated transportation costs in the DJRP reduces the total cost with 3.4%–5.0% on average for different schemes for *m* and 4% overall. Individual savings up to 14.4% are achieved. The average improvement between instances with different numbers of customers or periods is similar.

For 494 out of 605 optimally solved instances the DJRP-AT outperforms the DJRP and matches its costs for 51 instances. For only 60 instances the DJRP resulted in a slightly better solution than the DJRP-AT with a maximum cost difference of 3.2%, but only 0.96% on average over all *m* schemes. The cases in which the DJRP resulted in lower costs than the DJRP-AT can be explained by using the approximated transportation cost in the optimization, which does not always lead to the lowest routing and inventory holding cost. Moreover, the DJRP completely ignores customer location when determining replenishments and will mostly serve customers on the day their inventory is exhausted, provided all constraints are respected (except if one day's major cost can be saved). If not all customers can be served on the day their inventory is exhausted, the customers that have the lowest holding costs will be served a day earlier. In this case, the customer's holding costs exert substantial influence on the combination of customers served together in the DJRP solutions. This can, coincidentally, result in favorable combinations of customers regarding the actual routing cost, which can result in better DJRP solutions compared with the DJRP-AT. However, the results show that this scenario is unlikely, since this occurs in a limited number of the instances.

Table 2 suggests that the DJRP-AT performs better on problem instances with a longer planning horizon. The DJRP was only able to find a better solution than the DJRP-AT in five parameter settings of the instances with a six day planning horizon, and the savings were 0.2%–1.5%. Therefore, we did additional experiments in which we increased the planning horizon from three to six days of the instances with a three day planning horizon. The results do not show clearly that the DJRP-AT performs relatively better on instances with a longer planning horizon.

The improvement in total costs of incorporating the transportation costs in the DJRP slightly decreases if the individual ordering cost *m* in the DJRP increases from 25 to 100. This can be explained by the fact that if the individual replenishment cost is lower, then the number of replenishments is higher in the DJRP outcomes, resulting in higher actual total routing costs. This effect can also be observed in the number of instances for which an improvement, deterioration, or equal costs are reported (Table 3). The number of instances showing an improvement decreases as the value of the individual replenishment cost *m<sub>i</sub>* increases, compared with the DJRP. However, the percentage of instances for which deterioration must be reported does not increase as *m<sub>i</sub>* increases; instead, the percentage of instances with equal costs for both models increases.

By using proportional costs, zones and quadrant based costs instead of the same individual replenishment costs for all customers, the routing costs are better reflected in the DJRP. Indeed, we can observe that on average the improvement of the DJRP-AT over the DJRP is lower than for the DJRP with *m<sub>i</sub>* = 25. However, for *m<sub>i</sub>* = 100 this is only the case for the proportional individual replenishment costs. The proportional costs best reflect the actual routing costs, in several works in the IRP literature the direct distance is used (as a starting point) to replace actual routing costs (see for example Cordeau et al. (2015) and Absi et al. (2015)). Still, from our results it shows that the DJRP-AT outperforms using proportional costs for replenishing a customer.

**Table 3**  
Number and percentage of instances that report improvement, deterioration and equal costs.

	Number of instances						Percentage					
	m					total	m					total
	25	100	prop	zones	quad		25	100	prop	zones	quad	
Improvement	108	97	92	99	98	494	89%	80%	76%	82%	81%	82%
Deterioration	8	9	14	14	15	60	7%	7%	12%	12%	12%	10%
Equal	5	15	15	8	8	51	4%	12%	12%	7%	7%	8%
Total	121	121	121	121	121		100%	100%	100%	100%	100%	

**Table 4**  
Average difference of DJRP-AT compared with IRP.

$N$	$T$	$k_M$	Number of instances solved	Number of instances equal result	Average difference (%)	Average time DJRP-AT (s)	Average time IRP (s)
5	3	3	5	3	-0.28	0.0	1.0
5	3	4	5	1	-1.30	0.2	1.6
10	3	5	5	4	-0.54	0.6	1155.8
10	3	6	4	1	-0.56	0.4	9348.8
5	6	3	5	4	-0.23	0.4	4.6
5	6	4	5	0	-1.47	0.4	13.4
10	6	5	5	3	-0.61	11.0	4944.0
10	6	6	1	0	-1.19	34.4	13738.0

In conclusion, these results show that in approximately 82% of all solved instances lower total costs can be achieved by using the DJRP-AT, instead of the DJRP. If the individual replenishment costs increase, the improvement that the DJRP-AT can achieve decreases, however, the number of instances with reported deteriorations does not increase. Also, for individual replenishment costs that better reflect the actual routing cost than the same cost for all customers, the DJRP-AT still outperforms the DJRP.

### 6.3. Comparison DJRP-AT and IRP

To gain further insight in the quality of the solutions of the DJRP-AT, we compare the results of the DJRP-AT to the results of a problem formulation that includes the actual routing problem. This formulation is a variant of the IRP with a constraint on the number of customers in a route, hence, this problem is different than the IRP often addressed in the literature (Coelho et al., 2014). Therefore, to solve this IRP, the labeling algorithm that solves the Pricing Problem of the DJRP-AT is replaced by an Integer Linear Program (ILP) which solves a resource constrained elementary shortest path problem. The resource is the number of customers in the route and note that there are arcs with negative cost, hence, negative cost cycles need to be prevented by adding subtour elimination constraints. A solution of the ILP is a route with corresponding costs that consists of the arc costs and the fixed costs  $B$ . We use Gurobi to solve the ILP and apply the 'Solution Pool' option to generate multiple solutions in one iteration. Since the solution method is not especially designed for solving this IRP, only some of the very small instances, with low values for  $k_M$ , can be solved within four hours of running time. The differences in computation time are therefore only indicative.

Table 4 gives the aggregated results of the comparison between the DJRP-AT and the IRP, the results per instance can be found in Appendix D. As in Section 6.2, the optimal traveling salesman tours are computed to compare the costs. For each combination of parameters, the number of instances solved, the number of instances with equal results for the DJRP-AT and the IRP, the average difference between the DJRP-AT and the IRP, and the average computation times of both models are indicated. For 10 customers, 3 time periods and  $k_M = 7$ , none of the instances could be solved within four hours of running time. For 10 customers, 6 time periods and  $k_M = 6$  only one instance was solved within four hours. In total

35 instances of the IRP are solved to optimality with computation times that are several orders of magnitude higher than those of the DJRP-AT. Out of the 35 instances, for 16 instances the DJRP-AT gives the same result as the IRP. On average, the costs of the solutions of the IRP are 0.77% lower than the costs of the DJRP-AT. Considering that solutions of the DJRP-AT are found by only using an approximation for the transportation costs, the results are quite close to the exact solutions.

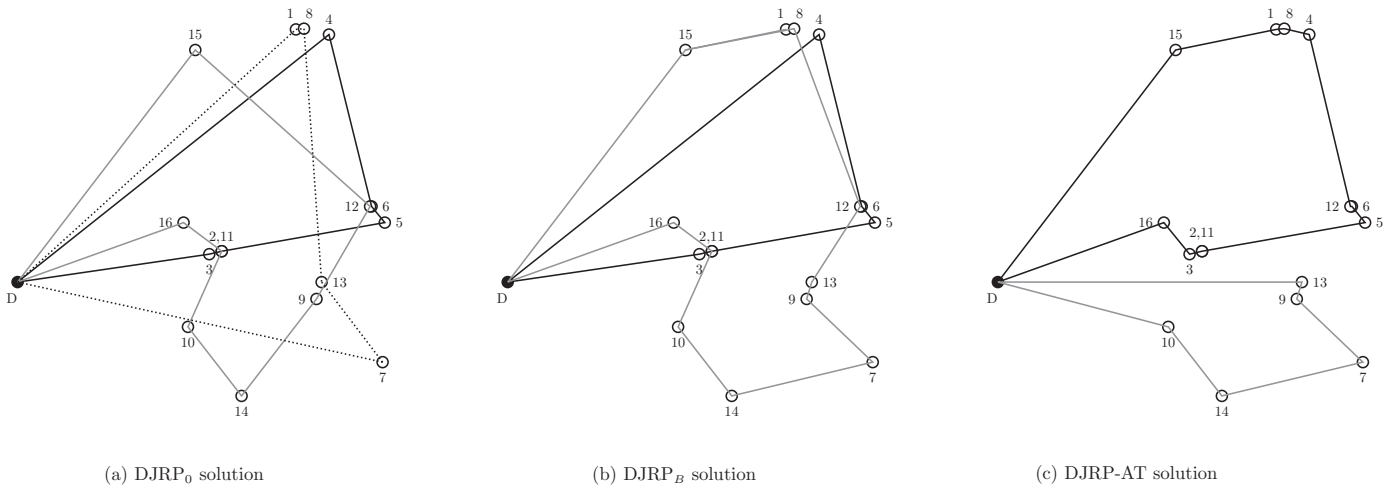
### 6.4. Case study ATM replenishment in Amsterdam

As described in the introduction, our research is motivated by a real-life case in ATM replenishment in the Netherlands, in which a supplier (vendor) decides on which ATMs to replenish per day and an LSP (CIT) designs the routes to perform the replenishments. Currently, the supplier only pays a 'minor' transportation cost to the LSP for each ATM replenishment and no 'major' cost. In this section we use company data to illustrate the benefit of alternative replenishment cost structures based on the DJRP and the proposed DJRP-AT.

Data on ATMs in Amsterdam and the depot location are provided by the supplier. The dataset contains per ATM, the address, storage capacity, dynamic daily demand, and initial inventory level. To use the existing solution framework, the ATM locations are mapped on the Euclidean plane and the demand data is expressed in thousands of Euros. Because the future cost structure parameters are not available and current numbers are not disclosed because of confidentiality reasons, the cost parameters are determined in consultation with the company to reflect the expected ratio between transportation and inventory holding costs. This includes minor ( $m_i$ ) and major ( $B$ ) transportation cost and inventory holding rates. A three-day planning horizon is considered appropriate and therefore we select the ATMs that need replenishment within the next three days, which results in 75 ATMs. Based on their geographical locations and postal codes, the set of ATMs is naturally split into four subsets of sizes 16, 19, and two of 20 customers. We let the holding rate vary from 0.1 to 0.3 to represent realistic cost ratios. Based on service times and travel times as observed by the company, we let the maximum number of customers served  $k_M$  range from 10 to 13. The same individual replenishment cost  $m_i$  is used for all customers, and tests are performed for

**Table 5**  
Percentage improvement DJRP<sub>B</sub> over DJRP<sub>0</sub> (left), and percentage improvement DJRP-AT over DJRP<sub>B</sub> (right) in Amsterdam case.

		Region 1		Region 2		Region 3		Region 4		Region 1		Region 2		Region 3		Region 4	
		<i>m</i>		<i>m</i>		<i>m</i>		<i>m</i>		<i>m</i>		<i>m</i>		<i>m</i>		<i>m</i>	
<i>h</i> = 0.1	<i>k<sub>M</sub></i>	25	100	25	100	25	100	25	100	25	100	25	100	25	100	25	100
	10	30.1	30.1	30.2	30.2	31.1	31.1	28.2	28.2	11.4	11.4	4.0	4.0	1.1	1.1	8.3	8.3
	11	30.1	30.1	30.3	30.3	29.3	29.3	27.8	27.8	11.4	11.4	4.4	4.4	4.3	4.3	8.1	8.1
	12	30.1	30.1	30.1	30.1	29.2	29.2	24.6	24.6	11.4	11.4	3.1	3.1	4.3	4.3	12.1	12.1
	13	30.1	30.1	32.4	32.4	29.3	29.3	25.0	25.0	11.4	11.4	-0.7	-0.7	4.6	4.6	11.5	11.5
<i>h</i> = 0.2	<i>k<sub>M</sub></i>	25	100	25	100	25	100	25	100	25	100	25	100	25	100	25	100
	10	29.6	29.6	29.1	29.4	30.3	30.3	27.7	27.7	10.8	10.8	3.6	3.6	0.9	0.9	7.7	7.7
	11	29.6	29.6	29.2	29.4	28.6	28.6	27.3	27.3	10.8	10.8	2.7	2.7	3.9	3.9	7.3	7.3
	12	29.6	29.6	29.0	29.3	28.5	28.5	24.1	24.1	10.8	10.8	2.6	2.6	4.0	4.0	11.3	11.3
	13	29.6	29.6	31.4	31.6	28.6	28.6	24.6	24.6	10.8	10.8	-1.2	-1.2	4.0	4.0	10.6	10.6
<i>h</i> = 0.3	<i>k<sub>M</sub></i>	25	100	25	100	25	100	25	100	25	100	25	100	25	100	25	100
	10	30.1	29.1	26.9	28.6	29.5	29.6	26.9	27.2	9.8	10.3	3.8	3.3	0.7	0.7	7.2	7.2
	11	29.8	29.1	27.3	28.7	29.5	27.9	26.5	26.8	10.3	10.3	2.4	2.4	1.4	3.6	6.8	6.8
	12	30.0	29.1	27.9	28.5	27.8	27.9	23.4	23.7	10.0	10.3	1.0	2.0	3.7	3.6	11.0	11.0
	13	30.0	29.1	27.8	30.8	27.8	28.0	23.8	24.2	10.0	10.3	0.4	-1.7	3.7	3.5	10.3	10.3



**Fig. 3.** Inventory based replenishment, DJRP<sub>B</sub> and DJRP-AT solutions for Region 1 (*k<sub>M</sub>* = 11, *m* = 100, *h* = 0.2, black route in period 1, gray route in period 2, dotted route in period 3).

values 25, 50 and 100. We only use the same values for *m<sub>i</sub>* for all customers to stay close to the real-life case.

We estimate the costs of the current situation at the company by solving the DJRP with the major cost set to zero (*B* = 0, denoted by DJRP<sub>0</sub>). To make a fair comparison with the DJRP with major cost (denoted by DJRP<sub>B</sub>) and the DJRP-AT, we subsequently add the major cost *B* = 1000 for each period in which a replenishment takes place. Note that in practice, the current minor cost should be higher than the future minor cost to cover fixed costs. After computing the solutions of the three models, the optimal traveling salesman tours are computed to compare the costs.

Table 5 reports the percentage cost improvements of the DJRP<sub>B</sub> over the DJRP<sub>0</sub>, and of the DJRP-AT over the DJRP<sub>B</sub>. We only report the results for *m* = 25 and *m* = 100 since the results for *m* = 50 are similar to those for *m* = 100. Comparing DJRP<sub>0</sub> and DJRP<sub>B</sub> shows that a substantial cost improvement of 28.6% on average can be obtained, caused by having a route in every period in the DJRP<sub>0</sub> while in the DJRP<sub>B</sub> often only two routes are used. For the comparison between DJRP<sub>B</sub> and DJRP-AT, the results show that for 136 out of 144 cases, the DJRP-AT results in lower total costs than the DJRP<sub>B</sub>, with a decrease in total costs up to 12.1%. For the remaining eight instances, the DJRP-AT results in slightly higher costs than

the DJRP<sub>B</sub>, with differences up to 1.7%. The DJRP-AT solutions are, on average, 6.4% better than the DJRP<sub>B</sub> solutions. Table 5 shows that the improvement of the DJRP-AT over the DJRP<sub>B</sub> decreases slightly for higher holding rates. For the DJRP-AT, it can be beneficial to serve customers earlier in the planning horizon than waiting until the customers run out of stock. However, if the holding rate is higher, serving customers earlier becomes more costly which results in higher costs for the DJRP-AT. Interestingly, varying the individual replenishment cost per ATM does not have a significant impact on the results. Furthermore, Table 5 shows that varying the maximum number of customers has a different impact per region. For Region 1 the results are similar for all values of *k<sub>M</sub>*, while for Region 2, increasing the maximum number of customers to *k<sub>M</sub>* = 13 leads to higher costs for the DJRP-AT, compared with the DJRP<sub>B</sub>. For Region 2, the lower cost solution for *k<sub>M</sub>* = 13 than for *k<sub>M</sub>* = 12 of the DJRP<sub>B</sub> can be explained by the fact that many customers in the region are located in close proximity and a few customers are located further away from these clustered customers. In the DJRP-AT solutions, the customers not in the cluster are served on the same day, for all values of *k<sub>M</sub>*. In the DJRP<sub>B</sub> solution, for *k<sub>M</sub>* = 12, the customers outside the cluster are not all served on one day, which results in high transportation cost, but if *k<sub>M</sub>* = 13,

these customers are served on the same day, which lowers the total costs of the DJRP<sub>B</sub> substantially, as opposed to  $k_M = 12$ .

The results of DJRP<sub>0</sub>, DJRP<sub>B</sub> and DJRP-AT are visualized for Region 1 in Fig. 3, with  $k_M = 11$ ,  $m = 100$ , and  $h = 0.2$ . The cost improvements for this instance are 29.6% and 10.8%, respectively. The results show that, as expected, the DJRP<sub>B</sub> gives cheaper solutions than the DJRP<sub>0</sub> and that the routes given by the DJRP-AT are much more efficient than the routes given by the DJRP<sub>B</sub>. Overall, the results of the case study show that using a DJRP cost structure could provide significant savings, and the DJRP-AT results in similar improvements for the real-life case as for the artificial benchmark instances.

## 7. Conclusion

In practice it is regularly the case that a supplier outsources customer deliveries to a Logistics Service Provider (LSP); the supplier often pays a fixed transportation fee to the LSP for this service. Hence, when deciding on the timing of customer replenishments, the supplier often does not take efficiency of the delivery routes into account. To optimize costs, suppliers can use joint replenishment models, such as the Dynamic-Demand Joint Replenishment Problem (DJRP) (Boctor et al., 2004) in case of dynamic demand. The DJRP minimizes inventory holding and replenishment costs while ensuring that customers do exhaust their stock. The replenishment costs consist of fixed fees that are independent of the actual routing costs. As a result, we argue that the DJRP is incapable of proposing efficient solutions from a transportation point of view. Although the fee per delivery is fixed, the costs of the inefficient routes are indirectly paid by the supplier via the negotiated delivery fees in the following contract. Inspired by the practical relevance of the DJRP, this paper proposes the Dynamic-Demand Joint Replenishment Problem with Approximated Transportation Costs (DJRP-AT) in which transportation costs are included by approximating the optimal tour length for given subsets of customers. Using the DJRP-AT will result in lower total costs and will increase resource utilization.

We propose a mathematical model for the DJRP-AT which is enriched with two different types of constraints. First, the tour duration is restricted to resemble the limitations encountered in practice. Second, the number of customers served per period is bounded, resulting in the opportunity to investigate the improvement of the proposed DJRP-AT compared with the existing DJRP. The Dantzig-Wolfe decomposition is applied to the proposed compact formulation and the resulting Master and Pricing Problems are solved in a Branch-and-Cut-and-Price framework. The Master Problem defines which subset of customers to replenish per period of the planning horizon and determines the delivery quantities of the customers served. The Pricing Problem generates, via a specially designed labeling algorithm, subsets of customers that can be served in one period and the transportation costs are then approximated for these customer subsets. To increase efficiency, labels need to be discarded during the labeling algorithm. However, existing sufficient dominance rules for shortest path problems are not adequate for discarding labels in the Pricing Problem of the DJRP-AT. Therefore, we introduce novel sufficient dominance conditions that make label discarding possible. Costs of the DJRP and DJRP-AT are compared by computing the optimal traveling salesman tours for the subsets of customers selected in the solutions of both models.

To assess the value of the DJRP-AT formulation and solution framework, existing problem instances from the Inventory Routing Problem (IRP) are adjusted for our experiments. The effectiveness of two types of valid inequalities that were proposed for the IRP (Archetti et al., 2007) is tested for both types of extra constraints.

The results show that both inequalities are effective for the DJRP-AT, which differs from the results obtained from the IRP. Computational experiments show average improvements of total transportation and inventory holding costs of 3.4% to 5%, respectively. Depending on the individual fixed fee charged in the DJRP, maximum improvements around 14.4% are obtained. The DJRP outperformed the DJRP-AT for only a few instances, due to the approximation of the tour length. The DJRP-AT solutions are also compared to the equivalent IRP solutions for which a different Pricing Problem is implemented. The results show that for the solved instances, the costs of the IRP solutions are on average only 0.77% lower than the costs of the DJRP-AT solutions. The computation times for the IRP are orders of magnitude higher than for the DJRP-AT. Analysis of a real-life case in ATM replenishment shows that significant cost reductions can be achieved for both the LSP and the supplier when using the DJRP-AT.

Computational results with the DJRP-AT show that calculating transportation costs, instead of using fixed fees, in joint replenishment is worthwhile and that approximation of transportation costs works well. Future research could focus on developing novel formulations for this problem, possibly inspired by formulations discussed in Narayanan and Robinson (2006), that may improve the integrality gaps. Moreover, new valid inequalities can be proposed to strengthen the linear relaxation of DJRP-AT models.

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## Appendix A. Tour length approximations

Several models to approximate the length of a traveling salesman tour have been proposed in the literature. The first type of model assumes that no information is available on the customer's exact location; the second type assumes that the locations are known. Within the first type of models, Beardwood, Halton, and Hammersley (1959) presented a simple formula to approximate the length of a tour and they showed that their approximation is asymptotically equal to the shortest traveling salesman tour for random points in a given area. The tour length  $D$  for an area  $A$  and  $M$  uniformly distributed points is approximated by:

$$D \approx \phi \sqrt{AM} \quad (14)$$

in which  $\phi$  is a constant, approximately 0.75 for the Euclidean space. Eilon, Watson-Gandy, and Christofides (1971) presented a more accessible proof for this formula. An extension was proposed by Daganzo (1984a) for the case in which the depot is not positioned in the same area as the customers. Therefore, a term for the line-haul distance from the depot to the customer's area is included. A variant of this formula was studied by Daganzo (1984b) who introduced a strip-strategy. In this method, non-overlapping strips of an optimized width cover the area in which the customers are located. The expected length of a route in one strip is easy to compute, hence, the routes for all strips together provide a tour length approximation.

Chien (1992) tested seven different approximations for the traveling salesman tour length that have the same functional form as (14). The author compared approximations that vary in the calculation of the area  $A$  and considered models both with and without an extra term for the depot. The considered shapes for the area were the smallest rectangle covering the customers, the smallest rectangle that covers both the depot and the customers, a circular sector

that covers both the depot and the customers and finally a lune shaped area covering all the customers. For these models the best values for the constants were derived by testing the models on instances up to 30 customers. The approximations of Chien (1992) allow for a comparison between subsets of customers that have the same cardinality. In comparison, the approximation by Beardwood et al. (1959) in Eq. (14) assumes the same area ( $A$ ) for each subset and therefore cannot be used to compare equally sized subsets of customers.

The parameters of formula (14) and Chien's model (1992), that includes a term for the depot, were reassessed by Kwon, Golden, and Wasil (1995) by considering instances with up to 80 customers located in rectangular areas having different length-to-width ratios. Two new models were introduced that include this ratio and the performance of these newly introduced models seems good. However, the models have only been tested for rectangular areas and very specific information on the input, the ratio between length and width of the area, has to be known for these tour length approximations limiting their practical use. Hindle and Worthington (2004) used simulation to refine Eq. (14) for a 100 x 100 square area by including a term with the natural logarithm of the number of customers. The results indicated that the formula by Beardwood et al. (1959) could be improved by using a different functional form, however, this new result was not generalized for other sizes and shapes of areas. More recently, Çavdar and Sokol (2015) tested several existing tour length approximations, including those of Beardwood et al. (1959) and Chien (1992), and introduced a new model incorporating the standard deviation of the horizontal and vertical customer coordinates. The tested instances have different node dispersions and the areas in which the customers are located have different shapes. The computations for the newly introduced approximation are more complicated than the models proposed by Beardwood et al. (1959) and Chien (1992). Moreover, the experiments showed that the new model accurately approximates the actual tour length for large numbers of customers, however, the approximation deviates significantly from the optimal tour length for small numbers of customers.

The previous mentioned models are all approximations for the length of a single tour. Extensions to multiple, capacitated, vehicles can be found in Daganzo (1984a); Figliozzi (2008); Langevin and Soumis (1989) and Turkensteen and Klose (2012). These publications include similar ideas as in the single tour approximation models and they are extended to handle multiple tours. Note that Langevin and Soumis (1989) also considered a time constraint on the tours.

Applications of the above mentioned tour length approximation models are mainly found in optimization of passenger transportation systems (Langevin, Mbaraga, & Campbell, 1996) and in location optimization models (Shen & Qi, 2007). Shen and Qi (2007) used an approximation for the Vehicle Routing Problem (VRP) by Haimovich and Rinnooy Kan (1985) that requires the length of a tour as input, which in turn is approximated by (14). Other applications include fleet composition models, e.g., Jabali, Gendreau, and Laporte (2012) who applied a VRP approximation, and production and distribution system design, e.g., Dasci and Verter (2001) who used the approximation in Eq. (14). For a more elaborate overview of continuous approximation models and applications we refer to Franceschetti, Jabali, and Laporte (2017).

## Appendix B. Complexity

Consider problem formulation (3a)–(3h) with the following function for the approximated transportation cost:

$$f(s) = B + \phi\sqrt{AM} \quad (15)$$

in which  $A$  is the area of the predefined rectangle in which all customers and the depot in the instance are located and  $M$  is the number of points in the tour (depot and customers). This means  $A$  is identical for every subset of customers  $s$ . Furthermore, assume the inventory holding rates are zero and that  $B = 0$ . The objective function of the DJRP-AT becomes  $\sum_{t \in T} \phi\sqrt{AM_t} = \sum_{t \in T} \phi'\sqrt{M_t}$  with  $M_t$  the number of points visited in period  $t \in T$  ( $\phi' = \phi\sqrt{A}$ ). This objective is minimized if  $\sum_t M_t$  is minimal.

Now, consider two different types of subset composition constraints: limited tour duration and a maximum on the number of customers in the subset. For this cost structure, these two side constraints are equivalent. The maximum number of customers in a set is imposed by  $M_t \leq k_M + 1$ . The duration is computed by the second part of (15), hence, given the fixed value of  $A$  the constraint  $\phi'\sqrt{M_t} \leq k'_D$  can be written as  $M_t \leq k_D = (k'_D/\phi')^2$ . Therefore the structure of the constraint is the same for a maximum on the tour duration and on the number of customers.

Considering the functional forms of the objective and the constraints, the problem is to minimize the sum of the number of visits per period under the constraints that a maximum number of customers can be visited per period and that customers cannot run out of stock. A special case of this problem is when only one customer can be visited per period ( $k_M = 1$ ), also known as the so-called Pinwheel Scheduling Problem (or Windows Scheduling Problem). In the Pinwheel Scheduling Problem, a feasible schedule must be found to repeatedly process a set of jobs; for each job  $j$  a time limit between two executions is given which is the period of a job  $p_j$ . This is similar to replenishing customers in such a way that they do not run out of stock: after a replenishment, calculations can determine the latest possible timing of the next delivery. That Pinwheel Scheduling is a special case of the DJRP-AT can be seen by assuming that inventory holding costs are zero,  $B = 0$  and by introducing customers who must be replenished every period, whose locations define the rectangle containing all customers. Then, the area in the transportation cost function is the same in every period, which means that selecting which customers to replenish is based solely on inventory levels and the maximum number of customers to replenish. This requires finding a feasible replenishment schedule, which is equivalent to the Pinwheel Scheduling Problem.

It was recently shown by Jacobs and Longo (2014) that the Pinwheel Scheduling Problem cannot be solved in pseudopolynomial time, unless there is a randomized algorithm solving the classical problem Satisfiability in quasipolynomial time. Since this is unlikely, it is plausible that the Pinwheel Scheduling Problem is not solvable in polynomial time. Therefore, a final conclusion on the complexity of the DJRP-AT with cost function (15) cannot be given, but it is very unlikely that this problem is mathematically easy.

Now consider the case in which the area in the cost function  $R(s)$  is the smallest rectangle that covers the depot and the customers in the subset. Then  $R(s)$  is dependent on the specific subset  $s$  that must be visited. The problem with  $A$  equal to the complete area, is a special case of the problem with the smallest rectangle  $R(s)$ . This can be easily seen by creating some customers in the corners of the complete area requiring replenishment every period; then the smallest rectangle is equal to the complete area in every period, resulting in a reduction from one problem to the other. Therefore, if the problem is NP-complete with  $A$  (the whole area), then the problem is also NP-complete with  $R(s)$  (the smallest rectangle covering the points).

## Appendix C. Derivation $\Delta(L, L')$

The value of  $\Delta(L, L')$  in condition P.3 of Proposition 1 should be such that for every possible extension  $P \subseteq \mathcal{N} \setminus s'$  condition D.2

holds. Therefore, we examine  $\bar{c}_t(L \oplus P)$  and  $\bar{c}_t(L' \oplus P)$ . For ease of notation denote  $s(L)$  by  $s$  and let  $R(s)$  denote the area of the smallest rectangle covering the depot and the customers in  $s$ . Furthermore, define  $\pi_j := \pi_j^1$  and  $\pi_t := \pi_t^2$ .

$$\begin{aligned} \bar{c}_t(L \oplus P) &= \phi\sqrt{R(s \cup P)|s \cup P|} + \sum_{j \in s} u_j \pi_j + \sum_{j \in P} u_j \pi_j - \pi_t \\ &= \phi\sqrt{(R(s \cup P) - R(s))|s \cup P| + R(s)|s| + R(s)|P|} \\ &\quad + \sum_{j \in s} u_j \pi_j + \sum_{j \in P} u_j \pi_j - \pi_t \\ &\leq \phi\sqrt{(R(s \cup P) - R(s))|s \cup P|} + \phi\sqrt{R(s)|s|} \\ &\quad + \phi\sqrt{R(s)|P|} + \sum_{j \in s} u_j \pi_j + \sum_{j \in P} u_j \pi_j - \pi_t \\ &= \bar{c}_t(L) + \phi\sqrt{(R(s \cup P) - R(s))|s \cup P|} \\ &\quad + \phi\sqrt{R(s)|P|} + \sum_{j \in P} u_j \pi_j \end{aligned} \tag{16}$$

and similarly

$$\begin{aligned} \bar{c}_t(L' \oplus P) &\leq \bar{c}_t(L') + \phi\sqrt{(R(s' \cup P) - R(s'))|s' \cup P|} \\ &\quad + \phi\sqrt{R(s')|P|} + \sum_{j \in P} u_j \pi_j \end{aligned} \tag{17}$$

Hence, we can express  $\bar{c}_t(L \oplus P)$  and  $\bar{c}_t(L' \oplus P)$  in terms of  $\bar{c}_t(L)$  and  $\bar{c}_t(L')$ . It is already known that  $\bar{c}_t(L) < \bar{c}_t(L')$ , for dominance also  $\bar{c}_t(L \oplus P) \leq \bar{c}_t(L' \oplus P)$  has to hold.

$$\begin{aligned} \bar{c}_t(L \oplus P) - \bar{c}_t(L' \oplus P) &= \bar{c}_t(L) + \phi\sqrt{(R(s \cup P) - R(s))|s \cup P|} + \phi\sqrt{R(s)|P|} + \sum_{j \in P} u_j \pi_j \\ &\quad - \bar{c}_t(L') - \phi\sqrt{(R(s' \cup P) - R(s'))|s' \cup P|} - \phi\sqrt{R(s')|P|} - \sum_{j \in P} u_j \pi_j \\ &= \bar{c}_t(L) - \bar{c}_t(L') + \phi\sqrt{(R(s \cup P) - R(s))|s \cup P|} \\ &\quad - \phi\sqrt{(R(s' \cup P) - R(s'))|s' \cup P|} \\ &\quad + \phi\sqrt{R(s)|P|} - \phi\sqrt{R(s')|P|} \end{aligned} \tag{18}$$

To find a dominance rule, an upper bound (UB) on the bold part in the last expression must be determined to guarantee

dominance.

$$\begin{aligned} UB &= \phi\sqrt{(R(s \cup P) - R(s))|s \cup P|} - \phi\sqrt{(R(s' \cup P) - R(s'))|s' \cup P|} \\ &\quad + \phi\sqrt{R(s)|P|} - \phi\sqrt{R(s')|P|} \\ &\leq \phi\sqrt{(R(s \cup P) - R(s))|s \cup P|} - \phi\sqrt{(R(s' \cup P) - R(s'))|s' \cup P|} \\ &\leq \phi\sqrt{(R(s \cup P) - R(s))|s \cup P|} - (R(s' \cup P) - R(s'))|s' \cup P| \\ &= \phi\sqrt{R(s \cup P)|s \cup P| - R(s' \cup P)|s' \cup P| + R(s')|s' \cup P| - R(s)|s \cup P|} \\ &\leq \phi\sqrt{R(s')|s' \cup P| - R(s)|s \cup P|} \end{aligned} \tag{19}$$

The first inequality follows from  $R(s) - R(s') < 0$  and the second inequality follows from  $\sqrt{a} - \sqrt{b} \leq \sqrt{a - b}$  given that  $a \geq b \geq 0$ . Then, by rearranging terms the equality is found and the last inequality follows from  $R(s \cup P) \leq R(s' \cup P)$  and  $|s \cup P| < |s' \cup P|$ . Hence,

$$\Delta(L, L') = \phi\sqrt{R(s')|s' \cup P| - R(s)|s \cup P|}. \tag{20}$$

### Appendix D. Results per instance

Tables 6–12 show the results per instance for the DJRP-AT. In Tables 6–9 results are presented for the model with a constraint on the duration; the remaining tables show the results of the model with limit on the number of customers served per period. In all tables, the instance number, number of customers  $N$ , number of periods  $T$ , and the upper bound  $k$  on the extra constraint are given. Furthermore, the solution time ('Time (s)') in seconds, the objective value ('DJRP-AT Solution'), the size of the branch-and-bound tree ('Tree'), the number of columns in the final model ('Cols') and the integrality gap ('Gap (%)') are presented. For all tables, the total costs computed with the traveling salesman tour solution are given in the column indicated by 'TSP Solution'. For Tables 6–9 the final column 'Difference' indicates the percentage difference between the model objective value and the actual costs with the tour costs. This difference indicates the cost underestimation of the route length approximation, as opposed to the actual shortest tour. In Tables 10–12 a comparison with the solution of the DJRP is shown for individual replenishment costs  $m = 25, 100$ , prop. zones, quad by giving the total costs and the percentage difference with the DJRP-AT solution ('TSP Sol'). In Table 13 the results per instance of the comparison between the DJRP-AT and the equivalent IRP are presented.

**Table 6**  
Results per instance for duration constraint for  $T = 3$  and  $N = 5, 10$ .

Instance	$N$	$T$	$k_D$	Time (s)	DJRP-AT Solution	Tree	Columns	Gap (%)	TSP Solution	Difference (%)
abs1n5	5	3	600	0	2868	12	65	15	3471	-17.4
abs2n5	5	3	600	0	2628	6	58	24	3146	-16.5
abs3n5	5	3	600	0	4310	2	30	21	5250	-17.9
abs4n5	5	3	600	0	2657	4	52	21	3050	-12.9
abs5n5	5	3	600	0	1753	0	40	0	2030	-13.7
abs1n5	5	3	800	0	2798	8	68	22	3391	-17.5
abs2n5	5	3	800	0	1704	0	44	0	1975	-13.7
abs3n5	5	3	800	1	3330	6	53	12	4169	-20.1
abs4n5	5	3	800	0	1746	0	30	0	1983	-11.9
abs5n5	5	3	800	0	1753	0	40	0	2030	-13.7
abs1n5	5	3	1000	0	1892	0	59	0	2225	-15.0
abs2n5	5	3	1000	0	1704	0	44	0	1975	-13.7
abs3n5	5	3	1000	0	3330	20	64	24	4169	-20.1
abs4n5	5	3	1000	0	1746	0	30	0	1983	-11.9
abs5n5	5	3	1000	0	1753	0	40	0	2030	-13.7
abs1n5	5	3	1200	0	1892	0	59	0	2225	-15.0
abs2n5	5	3	1200	0	1704	0	44	0	1975	-13.7
abs3n5	5	3	1200	0	2240	0	37	0	2592	-13.6
abs4n5	5	3	1200	0	1746	0	30	0	1983	-11.9
abs5n5	5	3	1200	0	1753	0	40	0	2030	-13.7

(continued on next page)

**Table 6** (continued)

Instance	$N$	$T$	$k_D$	Time (s)	DJRP-AT Solution	Tree	Columns	Gap (%)	TSP Solution	Difference (%)
abs1n10	10	3	600	1	4802	18	311	12	5935	-19.1
abs2n10	10	3	600	0	x	2	144	-	-	-
abs3n10	10	3	600	0	4272	6	205	9	5485	-22.1
abs4n10	10	3	600	0	x	2	199	-	-	-
abs5n10	10	3	600	1	4736	78	468	23	5481	-13.6
abs1n10	10	3	800	0	3642	6	306	3	4332	-15.9
abs2n10	10	3	800	2	5192	40	401	25	6422	-19.1
abs3n10	10	3	800	0	3429	0	304	0	4073	-15.8
abs4n10	10	3	800	1	4974	94	660	29	6128	-18.8
abs5n10	10	3	800	1	4678	24	668	30	5442	-14.0
abs1n10	10	3	1000	0	3642	14	808	13	4332	-15.9
abs2n10	10	3	1000	1	3938	12	750	9	4585	-14.1
abs3n10	10	3	1000	0	3429	6	760	8	4073	-15.8
abs4n10	10	3	1000	1	4820	38	1210	34	5783	-16.7
abs5n10	10	3	1000	1	3591	12	847	19	4107	-12.6
abs1n10	10	3	1200	0	3474	8	1077	18	4406	-21.2
abs2n10	10	3	1200	1	3830	20	1051	16	4452	-14.0
abs3n10	10	3	1200	0	3429	6	1238	12	4073	-15.8
abs4n10	10	3	1200	1	3572	20	1350	22	4237	-15.7
abs5n10	10	3	1200	0	2520	0	1034	0	2728	-7.6

x: instance infeasible.

**Table 7**

Results per instance for duration constraint for  $T = 3$  and  $N = 15, 20$ .

Instance	$N$	$T$	$k_D$	Time (s)	DJRP-AT Solution	Tree	Columns	Gap (%)	TSP Solution	Difference (%)
abs1n15	15	3	600	0	x	2	607	-	-	-
abs2n15	15	3	600	1	x	2	541	-	-	-
abs3n15	15	3	600	0	x	2	531	-	-	-
abs4n15	15	3	600	0	x	2	484	-	-	-
abs5n15	15	3	600	1	x	2	491	-	-	-
abs1n15	15	3	800	10	5361	108	2901	19	6208	-13.6
abs2n15	15	3	800	5	5498	50	1991	17	6182	-11.1
abs3n15	15	3	800	1	x	14	1082	-	-	-
abs4n15	15	3	800	1	x	6	1047	-	-	-
abs5n15	15	3	800	3	5411	46	1888	15	6067	-10.8
abs1n15	15	3	1000	6	5159	28	6395	26	5698	-9.5
abs2n15	15	3	1000	8	5387	124	6223	25	5972	-9.8
abs3n15	15	3	1000	7	5480	52	2389	15	6580	-16.7
abs4n15	15	3	1000	11	5438	200	3914	27	6061	-10.3
abs5n15	15	3	1000	12	5401	164	6278	24	6038	-10.5
abs1n15	15	3	1200	11	4194	80	21348	18	4873	-13.9
abs2n15	15	3	1200	3	4477	14	12174	16	4753	-5.8
abs3n15	15	3	1200	1	4394	10	4647	6	5035	-12.7
abs4n15	15	3	1200	12	5401	78	13773	31	5824	-7.3
abs5n15	15	3	1200	6	4234	24	15278	15	4588	-7.7
abs1n20	20	3	600	1	x	2	490	-	-	-
abs2n20	20	3	600	0	x	2	1393	-	-	-
abs3n20	20	3	600	0	x	2	776	-	-	-
abs4n20	20	3	600	1	x	2	603	-	-	-
abs5n20	20	3	600	0	x	2	450	-	-	-
abs1n20	20	3	800	1	x	2	1373	-	-	-
abs2n20	20	3	800	22	5755	110	7714	10	6518	-11.7
abs3n20	20	3	800	1	x	2	1909	-	-	-
abs4n20	20	3	800	1	x	2	2137	-	-	-
abs5n20	20	3	800	1	x	2	1520	-	-	-
abs1n20	20	3	1000	34	x	94	8085	-	-	-
abs2n20	20	3	1000	268	5648	530	33518	19	6228	-9.3
abs3n20	20	3	1000	18	6161	36	12315	16	7073	-12.9
abs4n20	20	3	1000	4	x	10	9090	-	-	-
abs5n20	20	3	1000	1	x	2	9487	-	-	-
abs1n20	20	3	1200	2122	6312	984	107421	27	6944	-9.1
abs2n20	20	3	1200	138	5546	34	72993	20	6160	-10.0
abs3n20	20	3	1200	248	5979	148	63819	25	6649	-10.1
abs4n20	20	3	1200	1235	6532	984	71117	26	7341	-11.0
abs5n20	20	3	1200	4479	6714	3540	145401	20	7592	-11.6

x: instance infeasible.



**Table 8**  
Results per instance for duration constraint for  $T = 6$  and  $N = 5, 10$ .

Instance	$N$	$T$	$k_D$	Time (s)	DJRP-AT Solution	Tree	Columns	Gap (%)	TSP Solution	Difference (%)
abs1n5	5	6	600	1	6252	28	106	3	7518	-16.8
abs2n5	5	6	600	1	5527	52	145	16	6694	-17.4
abs3n5	5	6	600	0	9178	0	36	0	11474	-20.0
abs4n5	5	6	600	0	6061	18	82	5	7255	-16.5
abs5n5	5	6	600	0	4481	2	109	0	5281	-15.1
abs1n5	5	6	800	1	6032	26	117	12	7305	-17.4
abs2n5	5	6	800	0	4565	0	74	0	5462	-16.4
abs3n5	5	6	800	0	8163	46	97	9	10328	-21.0
abs4n5	5	6	800	0	5126	2	91	1	5916	-13.3
abs5n5	5	6	800	0	4481	2	109	0	5281	-15.1
abs1n5	5	6	1000	0	5156	2	109	1	6124	-15.8
abs2n5	5	6	1000	0	4565	0	74	0	5462	-16.4
abs3n5	5	6	1000	1	7223	58	129	11	9033	-20.0
abs4n5	5	6	1000	0	5126	2	91	1	5916	-13.3
abs5n5	5	6	1000	0	4481	2	109	0	5281	-15.1
abs1n5	5	6	1200	0	5156	2	109	1	6124	-15.8
abs2n5	5	6	1200	0	4565	0	74	0	5462	-16.4
abs3n5	5	6	1200	0	6149	0	69	0	7235	-15.0
abs4n5	5	6	1200	0	5126	2	91	1	5916	-13.3
abs5n5	5	6	1200	1	4481	2	109	0	5281	-15.1
abs1n10	10	6	600	6	x	28	479	-	-	-
abs2n10	10	6	600	1	x	2	194	-	-	-
abs3n10	10	6	600	39	x	280	759	-	-	-
abs4n10	10	6	600	1	x	2	255	-	-	-
abs5n10	10	6	600	39	9785	266	899	8	11330	-13.6
abs1n10	10	6	800	16	8699	170	974	7	10203	-14.7
abs2n10	10	6	800	20	10435	132	854	12	12832	-18.7
abs3n10	10	6	800	3	8456	34	752	5	9997	-15.4
abs4n10	10	6	800	4	10044	36	653	5	12388	-18.9
abs5n10	10	6	800	105	9681	678	1546	17	11303	-14.3
abs1n10	10	6	1000	8	7832	66	1384	7	8828	-11.3
abs2n10	10	6	1000	26	9225	208	1343	11	11047	-16.5
abs3n10	10	6	1000	9	7674	58	2090	6	8790	-12.7
abs4n10	10	6	1000	13	9988	102	1307	16	12187	-18.0
abs5n10	10	6	1000	30	7904	184	2207	10	8885	-11.0
abs1n10	10	6	1200	16	7672	152	2182	14	8951	-14.3
abs2n10	10	6	1200	20	8174	130	2084	9	9510	-14.0
abs3n10	10	6	1200	5	7467	72	2729	11	8670	-13.9
abs4n10	10	6	1200	19	8095	148	2179	9	9334	-13.3
abs5n10	10	6	1200	1	6785	10	1698	3	7526	-9.8

x: instance infeasible.

**Table 9**  
Results per instance for duration constraint for  $T = 6$  and  $N = 15, 20$ .

Instance	$N$	$T$	$k_D$	Time (s)	DJRP-AT Solution	Tree	Columns	Gap (%)	TSP Solution	Difference (%)
abs1n15	15	6	600	2	x	2	906	-	-	-
abs2n15	15	6	600	0	x	2	665	-	-	-
abs3n15	15	6	600	2	x	2	650	-	-	-
abs4n15	15	6	600	1	x	2	707	-	-	-
abs5n15	15	6	600	1	x	2	565	-	-	-
abs1n15	15	6	800	7201	+	12454	29997	-	-	-
abs2n15	15	6	800	123	x	392	4239	-	-	-
abs3n15	15	6	800	1	x	2	1338	-	-	-
abs4n15	15	6	800	2	x	2	1499	-	-	-
abs5n15	15	6	800	23	x	76	3169	-	-	-
abs1n15	15	6	1000	7201	+	17038	32513	-	-	-
abs2n15	15	6	1000	154	11027	532	7814	11	12274	-10.2
abs3n15	15	6	1000	7201	+	19738	18391	-	-	-
abs4n15	15	6	1000	48	11018	156	6099	9	12132	-9.2
abs5n15	15	6	1000	259	11106	602	11452	11	12379	-10.3
abs1n15	15	6	1200	1851	9194	3610	47858	13	10180	-9.7
abs2n15	15	6	1200	7201	+	15552	42386	-	-	-
abs3n15	15	6	1200	7203	+	11598	46260	-	-	-
abs4n15	15	6	1200	3312	11018	7030	37617	15	12132	-9.2
abs5n15	15	6	1200	132	9322	334	23432	9	9682	-3.7
abs1n20	20	6	600	1	x	2	591	-	-	-
abs2n20	20	6	600	3	x	2	2306	-	-	-
abs3n20	20	6	600	2	x	2	758	-	-	-
abs4n20	20	6	600	1	x	2	605	-	-	-

(continued on next page)

Table 9 (continued)

Instance	N	T	k <sub>D</sub>	Time (s)	DJRP-AT Solution	Tree	Columns	Gap (%)	TSP Solution	Difference (%)
abs5n20	20	6	600	0	x	2	342	–	–	–
abs1n20	20	6	800	2	x	2	1617	–	–	–
abs2n20	20	6	800	127	x	170	21133	–	–	–
abs3n20	20	6	800	3	x	2	2933	–	–	–
abs4n20	20	6	800	3	x	2	2822	–	–	–
abs5n20	20	6	800	2	x	2	1550	–	–	–
abs1n20	20	6	1000	4	x	2	8483	–	–	–
abs2n20	20	6	1000	7201	+	4702	121980	–	–	–
abs3n20	20	6	1000	1850	x	1716	35683	–	–	–
abs4n20	20	6	1000	4	x	2	9973	–	–	–
abs5n20	20	6	1000	6	x	2	14757	–	–	–
abs1n20	20	6	1200	7202	+	1596	151670	–	–	–
abs2n20	20	6	1200	7203	+	1010	240387	–	–	–
abs3n20	20	6	1200	7202	+	1862	233434	–	–	–
abs4n20	20	6	1200	7201	+	2894	145444	–	–	–
abs5n20	20	6	1200	7201	+	2788	239743	–	–	–

x: instance infeasible, +: instance not solved within two hours.

Table 10 Results per instance for maximum on number of customers constraint for T = 3 and N = 5, 10.

Instance	N	T	k <sub>M</sub>	Time (s)	DJRP-AT Solution	Tree	Cols	Gap (%)	TSP Sol	DJRP Solution					Difference (%)				
										m					m				
										25	100	prop	zones	quad	25	100	prop	zones	quad
abs1n5	5	3	3	0	2868	12	62	11	3471	3581	3581	3581	3581	3581	3.1	3.1	3.1	3.1	3.1
abs2n5	5	3	3	0	2628	6	59	9	3146	3413	3413	3413	3413	3413	7.8	7.8	7.8	7.8	7.8
abs3n5	5	3	3	0	3330	8	53	10	4169	4568	4169	4169	4140	4140	8.7	0.0	0.0	−0.7	−0.7
abs4n5	5	3	3	0	3740	10	63	36	4335	4335	4335	4335	4335	4335	0.0	0.0	0.0	0.0	0.0
abs5n5	5	3	3	0	2673	6	57	11	3260	3246	3246	3246	3246	3246	−0.4	−0.4	−0.4	−0.4	−0.4
abs1n5	5	3	4	1	2798	8	68	22	3391	3581	3581	3583	3581	3581	5.3	5.3	5.4	5.3	5.3
abs2n5	5	3	4	0	2628	6	58	24	3146	3392	3392	3392	3392	3392	7.2	7.2	7.2	7.2	7.2
abs3n5	5	3	4	0	3330	20	64	24	4169	4563	4169	4169	4127	4127	8.6	0.0	0.0	−1.0	−1.0
abs4n5	5	3	4	0	2657	4	55	24	3050	3050	3050	3050	3050	3050	0.0	0.0	0.0	0.0	0.0
abs5n5	5	3	4	0	2668	6	67	25	3187	3187	3187	3187	3187	3187	0.0	0.0	0.0	0.0	0.0
abs1n10	10	3	5	0	3737	4	531	2	4315	4315	4315	4315	4315	4315	0.0	0.0	0.0	0.0	0.0
abs2n10	10	3	5	1	4045	6	675	5	4842	4842	4842	4842	4842	4842	0.0	0.0	0.0	0.0	0.0
abs3n10	10	3	5	0	3429	0	676	0	4073	4502	4502	4502	4502	4502	9.5	9.5	9.5	9.5	9.5
abs4n10	10	3	5	1	5003	58	979	28	6163	6297	6654	6032	6032	6124	2.1	7.4	−2.2	−2.2	−0.6
abs5n10	10	3	5	1	3859	8	545	6	4351	4351	4351	4351	4351	4351	0.0	0.0	0.0	0.0	0.0
abs1n10	10	3	6	0	3642	8	707	8	4332	4315	4315	4315	4315	4315	−0.4	−0.4	−0.4	−0.4	−0.4
abs2n10	10	3	6	1	3939	16	788	11	4559	4830	4842	4842	4830	4842	5.6	5.8	5.8	5.6	5.8
abs3n10	10	3	6	0	3429	6	962	3	4073	4386	4386	4386	4386	4386	7.1	7.1	7.1	7.1	7.1
abs4n10	10	3	6	1	4820	18	1061	32	5783	5785	6258	5665	5785	5665	0.0	7.6	−2.1	0.0	−2.1
abs5n10	10	3	6	0	3652	12	820	10	4192	4436	4351	4351	4431	4374	5.5	3.7	3.7	5.4	4.2
abs1n10	10	3	7	1	3642	16	1035	14	4332	4315	4315	4315	4315	4315	−0.4	−0.4	−0.4	−0.4	−0.4
abs2n10	10	3	7	1	3830	16	1018	15	4452	4830	4842	4842	4830	4842	7.8	8.1	8.1	7.8	8.1
abs3n10	10	3	7	0	3429	6	1598	9	4073	4399	4386	4386	4386	4386	7.4	7.1	7.1	7.1	7.1
abs4n10	10	3	7	1	3716	16	1178	18	4336	4339	4336	4336	4339	4339	0.1	0.0	0.0	0.1	0.1
abs5n10	10	3	7	0	3591	10	961	15	4107	4458	4351	4351	4431	4374	7.9	5.6	5.6	7.3	6.1
abs1n10	10	3	8	0	3545	8	1117	18	4299	4315	4315	4315	4315	4315	0.4	0.4	0.4	0.4	0.4
abs2n10	10	3	8	1	3719	14	1080	18	4430	4830	4842	4842	4830	4842	8.3	8.5	8.5	8.3	8.5
abs3n10	10	3	8	0	3429	6	1366	13	4073	4399	4386	4386	4386	4386	7.4	7.1	7.1	7.1	7.1
abs4n10	10	3	8	1	3572	22	1262	20	4237	4339	4336	4336	4339	4339	2.3	2.3	2.3	2.3	2.3
abs5n10	10	3	8	0	3543	8	1080	19	3962	4458	4351	4351	4431	4374	11.1	8.9	8.9	10.6	9.4

Table 11 Results per instance for maximum on number of customers constraint for T = 3 and N = 15, 20.

Instance	N	T	k <sub>M</sub>	Time (s)	DJRP-AT Solution	Tree	Cols	Gap (%)	TSP Sol	DJRP Solution					Difference (%)				
										m					m				
										25	100	prop	zones	quad	25	100	prop	zones	quad
abs1n15	15	3	7	20	5228	178	5765	19	5843	6422	6571	6108	6225	6108	9.0	11.1	4.3	6.1	4.3
abs2n15	15	3	7	15	5396	146	5061	19	5965	6793	6316	6674	6405	6397	12.2	5.6	10.6	6.9	6.8
abs3n15	15	3	7	7	5419	66	2974	16	6006	6729	6729	6729	6729	6729	10.8	10.8	10.8	10.8	10.8
abs4n15	15	3	7	22	5499	154	6378	21	5972	6781	6703	6645	6756	6799	11.9	10.9	10.1	11.6	12.2
abs5n15	15	3	7	24	5465	186	10481	21	5996	6646	6555	6561	6521	6455	9.8	8.5	8.6	8.1	7.1
abs1n15	15	3	8	7	4198	46	9429	10	4862	4836	4862	4836	4836	4836	−0.5	0.0	−0.5	−0.5	−0.5
abs2n15	15	3	8	4	4477	18	12731	9	4753	4763	4753	4753	4753	5019	0.2	0.0	0.0	0.0	5.3
abs3n15	15	3	8	2	4416	14	7909	3	4967	5323	4989	4989	4989	4989	6.7	0.4	0.4	0.4	0.4

(continued on next page)

Table 11 (continued)

Instance	N	T	k <sub>M</sub>	Time (s)	DJRP-AT		Gap (%)	TSP Sol	DJRP Solution					Difference (%)					
					Solution	Tree			Cols	m					m				
										25	100	prop	zones	quad	25	100	prop	zones	quad
abs4n15	15	3	8	25	5453	188	14584	27	5957	6112	6164	5978	6120	6120	2.5	3.4	0.4	2.7	2.7
abs5n15	15	3	8	12	5282	102	11395	23	6155	6550	6303	5975	6000	6000	6.0	2.3	-3.0	-2.6	-2.6
abs1n15	15	3	9	14	4198	100	17735	15	4636	4863	4862	4836	4863	4836	4.7	4.6	4.1	4.7	4.1
abs2n15	15	3	9	5	4403	24	14836	13	4735	5017	4753	4753	4753	5019	5.6	0.4	0.4	0.4	5.7
abs3n15	15	3	9	5	4400	16	12516	8	4994	5193	4921	4921	4984	4984	3.8	-1.5	-1.5	-0.2	-0.2
abs4n15	15	3	9	6	4478	26	17787	16	4813	4918	4813	4813	4927	4927	2.1	0.0	0.0	2.3	2.3
abs5n15	15	3	9	5	4265	16	16694	9	4655	4729	4655	4655	4719	4692	1.6	0.0	0.0	1.4	0.8
abs1n15	15	3	10	2	3911	16	17735	13	4394	4863	4862	4836	4863	4836	9.6	9.6	9.1	9.6	9.1
abs2n15	15	3	10	3	4272	16	22755	14	4463	5017	4753	4753	4753	5019	11.0	6.1	6.1	6.1	11.1
abs3n15	15	3	10	4	4370	16	14612	11	4970	5188	4921	4921	4965	4965	4.2	-1.0	-1.0	-0.1	-0.1
abs4n15	15	3	10	14	4451	60	24897	19	4767	4918	4813	4813	4927	4927	3.1	1.0	1.0	3.2	3.2
abs5n15	15	3	10	8	4234	32	26594	14	4588	4706	4655	4655	4719	4692	2.5	1.4	1.4	2.8	2.2
abs1n15	15	3	11	4	3894	26	24961	15	4164	4863	4862	4836	4863	4836	14.4	14.3	13.9	14.4	13.9
abs2n15	15	3	11	4	4221	20	27849	16	4417	5017	4753	4753	4753	5019	12.0	7.1	7.1	7.1	12.0
abs3n15	15	3	11	7	4348	24	19962	15	5054	5169	4921	4921	4897	4897	2.2	-2.7	-2.7	-3.2	-3.2
abs4n15	15	3	11	10	4285	42	21741	20	4627	4918	4813	4813	4927	4927	5.9	3.9	3.9	6.1	6.1
abs5n15	15	3	11	6	4135	26	21417	16	4410	4706	4655	4655	4719	4692	6.3	5.3	5.3	6.5	6.0
abs1n20	20	3	10	1536	5980	196	257354	27	6257	6837	6837	6257	6696	6257	8.5	8.5	0.0	6.6	0.0
abs2n20	20	3	10	55	4661	4	159085	3	4760	5211	5211	5211	5211	5211	8.7	8.7	8.7	8.7	8.7
abs3n20	20	3	10	207	4886	22	321432	8	5418	5496	5496	5496	5496	5496	1.4	1.4	1.4	1.4	1.4
abs4n20	20	3	10	4121	6033	640	282808	21	7157	7768	7604	7100	7209	7129	7.9	5.9	-0.8	0.7	-0.4
abs5n20	20	3	10	2116	6202	296	671812	16	6929	7502	6929	6929	7247	7247	7.6	0.0	0.0	4.4	4.4
abs1n20	20	3	11	2060	5090	146	294071	9	5158	5172	5158	5158	5172	5172	0.3	0.0	0.0	0.3	0.3
abs2n20	20	3	11	322	4655	18	160968	8	4815	5157	5202	5202	5247	5247	6.6	7.4	7.4	8.2	8.2
abs3n20	20	3	11	612	4870	38	237531	9	5416	5550	5543	5543	5543	5543	2.4	2.3	2.3	2.3	2.3
abs4n20	20	3	11	7206	+	492	506631	-	-	-	-	-	-	-	-	-	-	-	-
abs5n20	20	3	11	425	5108	28	425524	5	5746	5717	5746	5746	5746	5746	-0.5	0.0	0.0	0.0	0.0
abs1n20	20	3	12	3686	5038	146	723553	12	5150	5172	5158	5158	5172	5172	0.4	0.1	0.1	0.4	0.4
abs2n20	20	3	12	899	4655	38	244041	8	4815	5261	5082	5082	5238	5238	8.5	5.2	5.2	8.1	8.1
abs3n20	20	3	12	1283	4870	48	397099	11	5416	5597	5540	5540	5540	5540	3.2	2.2	2.2	2.2	2.2
abs4n20	20	3	12	1550	5153	58	713459	15	5707	5819	5707	5707	5707	5707	1.9	0.0	0.0	0.0	0.0
abs5n20	20	3	12	1454	5106	64	833293	9	5743	5693	5746	5746	5746	5746	-0.9	0.0	0.0	0.0	0.0
abs1n20	20	3	13	3388	4991	112	990314	14	5151	5172	5158	5158	5172	5172	0.4	0.1	0.1	0.4	0.4
abs2n20	20	3	13	559	4632	16	253365	10	4721	5252	5082	5082	5118	5118	10.1	7.1	7.1	7.8	7.8
abs3n20	20	3	13	423	4690	14	280999	10	5051	5594	5540	5540	5540	5540	9.7	8.8	8.8	8.8	8.8
abs4n20	20	3	13	1673	5068	58	486108	17	5654	5819	5707	5707	5707	5707	2.8	0.9	0.9	0.9	0.9
abs5n20	20	3	13	3730	5106	136	901660	13	5743	5693	5746	5746	5746	5746	-0.9	0.0	0.0	0.0	0.0

+ : instance not solved within two hours.

Table 12

Results per instance for maximum on number of customers constraint for T = 6 and N = 5, 10, 15.

Instance	N	T	k <sub>M</sub>	Time (s)	DJRP-AT		Gap (%)	TSP Sol	DJRP Solution					Difference (%)					
					Solution	Tree			Cols	m					m				
										25	100	prop	zones	quad	25	100	prop	zones	quad
abs1n5	5	6	3	0	6161	6	91	1	7430	7783	7518	7783	7783	7783	4.5	1.2	4.5	4.5	4.5
abs2n5	5	6	3	0	5527	18	99	5	6694	7052	7052	7052	7052	7052	5.1	5.1	5.1	5.1	5.1
abs3n5	5	6	3	0	8059	28	94	6	9866	11246	9866	10396	10398	10398	12.3	0.0	5.1	5.1	5.1
abs4n5	5	6	3	1	8031	26	113	16	9510	10125	10107	10182	10182	10182	6.1	5.9	6.6	6.6	6.6
abs5n5	5	6	3	1	5460	20	112	6	6680	6750	6721	6721	6750	6750	1.0	0.6	0.6	1.0	1.0
abs1n5	5	6	4	0	6032	26	117	12	7305	7789	7623	7619	7623	7623	6.2	4.2	4.1	4.2	4.2
abs2n5	5	6	4	0	5527	52	145	16	6694	6932	6941	6941	6932	6941	3.4	3.6	3.6	3.4	3.6
abs3n5	5	6	4	1	7223	58	129	11	9033	9444	9019	9045	9020	9020	4.3	-0.2	0.1	-0.1	-0.1
abs4n5	5	6	4	1	6061	26	113	7	7255	7352	7352	7150	7150	7150	1.3	1.3	-1.5	-1.5	-1.5
abs5n5	5	6	4	0	5459	24	140	17	6566	6602	6558	6558	6527	6633	0.5	-0.1	-0.1	-0.6	1.0
abs1n10	10	6	5	11	8980	54	1022	4	10421	10631	10631	10631	10631	10631	2.0	2.0	2.0	2.0	2.0
abs2n10	10	6	5	11	9547	48	1162	4	11283	11731	11731	11731	11731	11731	3.8	3.8	3.8	3.8	3.8
abs3n10	10	6	5	6	8456	52	1395	4	9997	11155	10856	10856	10889	11155	10.4	7.9	7.9	8.2	10.4
abs4n10	10	6	5	5	10116	38	771	6	12517	13317	13317	13317	13317	13317	6.0	6.0	6.0	6.0	6.0
abs5n10	10	6	5	22	9243	124	1206	8	10529	10992	10992	10992	10992	10992	4.2	4.2	4.2	4.2	4.2
abs1n10	10	6	6	10	8629	102	1475	10	10126	10674	10631	10631	10589	10631	5.1	4.8	4.8	4.4	4.8
abs2n10	10	6	6	34	9211	254	1947	11	10875	11655	11738	11646	11655	11740	6.7	7.4	6.6	6.7	7.4
abs3n10	10	6	6	20	7846	146	2019	3	8956	9116	9116	9255	9255	9305	1.8	1.8	3.2	3.2	3.8
abs4n10	10	6	6	40	9988	260	1675	13	12187	12896	12980	13076	12896	12858	5.5	6.1	6.8	5.5	5.2
abs5n10	10	6	6	68	8761	484	2173	15	10097	11160	11018	10992	11086	10971	9.5	8.4	8.1	8.9	8.0
abs1n10	10	6	7	16	7774	120	1753	7	8773	8877	8871	8854	8877	8854	1.2	1.1	0.9	1.2	0.9
abs2n10	10	6	7	13	8174	84	1985	6	9510	9457	9701	9564	9552	9731	-0.6	2.0	0.6	0.4	2.3
abs3n10	10	6	7	8	7662	52	2360	6	8805	9461	9027	9104	9134	9344	6.9	2.5	3.3	3.6	5.8
abs4n10	10	6	7	12	8321	74	1600	4	9483	9620	9650	9588	9586	9586	1.4	1.7	1.1	1.1	1.1
abs5n10	10	6	7	21	7904	136	2011	10	8885	9471	9386	9360	9279	9291	6.2	5.3	5.1	4.2	4.4

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Table 12 (continued)

Instance	N	T	k <sub>M</sub>	Time (s)	DJRP-AT		Gap (%)	TSP Sol	DJRP Solution					Difference (%)					
					Solution	Tree			Cols	m					m				
										25	100	prop	zones	quad	25	100	prop	zones	quad
abs1n10	10	6	8	25	7698	200	2271	13	8826	9001	8865	8919	8922	8930	1.9	0.4	1.0	1.1	1.2
abs2n10	10	6	8	18	8114	120	2504	11	9294	9599	9773	9828	9783	9801	3.2	4.9	5.4	5.0	5.2
abs3n10	10	6	8	14	7583	126	2572	11	8727	9441	8907	8950	8888	9451	7.6	2.0	2.5	1.8	7.7
abs4n10	10	6	8	17	8095	124	2010	8	9334	9528	9420	9583	9527	9493	2.0	0.9	2.6	2.0	1.7
abs5n10	10	6	8	33	7809	244	2336	13	8857	9526	9278	9278	9313	9495	7.0	4.5	4.5	4.9	6.7
abs1n15	15	6	8	3592	10140	7730	35017	9	11601	12163	12215	12163	12163	12163	4.6	5.0	4.6	4.6	4.6
abs2n15	15	6	8	7201	+	736	110588	–	–	–	–	–	–	–	–	–	–	–	–
abs3n15	15	6	8	7201	+	13558	45565	–	–	–	–	–	–	–	–	–	–	–	–
abs4n15	15	6	8	5014	11146	8704	41826	12	12287	13846	12933	13276	13846	13846	11.3	5.0	7.4	11.3	11.3
abs5n15	15	6	8	722	11084	1568	26574	15	12269	13613	13069	13136	13231	13231	9.9	6.1	6.6	7.3	7.3
abs1n15	15	6	9	7201	+	10642	61282	–	–	–	–	–	–	–	–	–	–	–	–
abs2n15	15	6	9	3313	10281	6384	46828	11	10993	12444	12111	11749	11953	12434	11.7	9.2	6.4	8.0	11.6
abs3n15	15	6	9	247	9536	512	27663	7	10833	11231	11143	11176	11231	11231	3.5	2.8	3.1	3.5	3.5
abs4n15	15	6	9	134	10133	286	29297	10	10920	12309	12072	12035	12036	12035	11.3	9.5	9.3	9.3	9.3
abs5n15	15	6	9	934	10145	1864	40232	14	10846	11535	11662	11724	11708	11699	6.0	7.0	7.5	7.4	7.3
abs1n15	15	6	10	193	8895	390	43861	8	9734	10334	10361	10384	10266	10378	5.8	6.1	6.3	5.2	6.2
abs2n15	15	6	10	191	9422	388	35056	7	9859	10270	10207	10212	10203	10176	4.0	3.4	3.5	3.4	3.1
abs3n15	15	6	10	1472	9504	2476	54081	13	10690	11385	10775	10808	11264	11264	6.1	0.8	1.1	5.1	5.1
abs4n15	15	6	10	6231	9728	9122	72982	8	10206	10367	10336	10252	10368	10382	1.6	1.3	0.4	1.6	1.7
abs5n15	15	6	10	795	9311	1536	33943	9	9664	9785	9732	9716	9732	9741	1.2	0.7	0.5	0.7	0.8
abs1n15	15	6	11	592	8848	1096	53819	13	9735	10306	10279	10114	10351	10236	5.5	5.3	3.7	5.9	4.9
abs2n15	15	6	11	231	9300	478	33939	10	9830	10414	10330	10324	10326	10407	5.6	4.8	4.8	4.8	5.5
abs3n15	15	6	11	1851	9407	3214	62292	12	10569	11443	10965	10898	11015	11015	7.6	3.6	3.0	4.0	4.0
abs4n15	15	6	11	2479	9370	4216	58445	10	9776	10180	10017	10008	10189	10158	4.0	2.4	2.3	4.1	3.8
abs5n15	15	6	11	546	9160	932	41385	12	9520	9781	9817	9682	9785	9799	2.7	3.0	1.7	2.7	2.8

+: instance not solved within two hours.

Table 13  
Results per instance comparison DJRP-AT and IRP.

Instance	N	T	k <sub>M</sub>	DJRP-AT Solution	IRP Solution	Difference (%)	Time DJRP-AT (s)	Time IRP (s)
abs1n5	5	3	3	3471	3471	0.00	0	1
abs2n5	5	3	3	3146	3146	0.00	0	1
abs3n5	5	3	3	4169	4140	−0.70	0	1
abs4n5	5	3	3	4335	4335	0.00	0	1
abs5n5	5	3	3	3260	3237	−0.69	0	1
abs1n5	5	3	4	3391	3358	−0.96	1	2
abs2n5	5	3	4	3146	3031	−3.67	0	1
abs3n5	5	3	4	4169	4127	−1.01	0	4
abs4n5	5	3	4	3050	3050	0.00	0	0
abs5n5	5	3	4	3187	3160	−0.85	0	1
abs1n10	10	3	5	4315	4315	0.00	0	324
abs2n10	10	3	5	4842	4842	0.00	1	960
abs3n10	10	3	5	4073	4073	0.00	0	81
abs4n10	10	3	5	6163	5996	−2.70	1	3800
abs5n10	10	3	5	4351	4351	0.00	1	614
abs1n10	10	3	6	4332	4296	−0.84	0	11567
abs2n10	10	3	6	4559	4533	−0.56	1	9625
abs3n10	10	3	6	4073	4038	−0.85	0	5763
abs4n10	10	3	6	5783	+	–	1	–
abs5n10	10	3	6	4192	4192	0.00	0	10440
abs1n10	10	3	7	4332	+	–	1	–
abs2n10	10	3	7	4452	+	–	1	–
abs3n10	10	3	7	4073	+	–	0	–
abs4n10	10	3	7	4336	+	–	1	–
abs5n10	10	3	7	4107	+	–	0	–
abs1n5	5	6	3	7430	7430	0.00	0	1
abs2n5	5	6	3	6694	6694	0.00	0	4
abs3n5	5	6	3	9866	9866	0.00	0	4
abs4n5	5	6	3	9510	9510	0.00	1	5
abs5n5	5	6	3	6680	6603	−1.16	1	9
abs1n5	5	6	4	7305	7299	−0.07	0	25
abs2n5	5	6	4	6694	6556	−2.07	0	10
abs3n5	5	6	4	9033	8844	−2.09	1	18
abs4n5	5	6	4	7255	7150	−1.44	1	5
abs5n5	5	6	4	6566	6456	−1.69	0	9
abs1n10	10	6	5	10421	10421	0.00	11	2577
abs2n10	10	6	5	11283	11283	0.00	11	12021

(continued on next page)

Table 13 (continued)

Instance	N	T	k <sub>M</sub>	DJRP-AT Solution	IRP Solution	Difference (%)	Time DJRP-AT (s)	Time IRP (s)
abs3n10	10	6	5	9997	9944	−0.54	6	1759
abs4n10	10	6	5	12517	12201	−2.53	5	3422
abs5n10	10	6	5	10529	10529	0.00	22	4941
abs1n10	10	6	6	10126	+	–	10	–
abs2n10	10	6	6	10875	+	–	34	–
abs3n10	10	6	6	8956	8849	−1.19	20	13738
abs4n10	10	6	6	12187	+	–	40	–
abs5n10	10	6	6	10097	+	–	68	–

+: instance not solved within four hours.

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