VU Research Portal

Teaching formal mathematics in primary education. Fraction learning as mathematising process.

Keijzer, R.

2003

citation for published version (APA)
Keijzer, R. (2003). *Teaching formal mathematics in primary education. Fraction learning as mathematising process.* CD-Beta Press, Center for Science and Mathematics Education.

Preface

A researcher's learning process

It is more than 30 years ago now that I learned fractions and fraction operations in primary school. I hardly experienced problems in learning mathematics and therefore have little recollection of this teaching and learning of fractions. However, there is one moment I never forgot; a moment which probably contributed to my (future) attitude toward mathematics and its teaching. My classmates and I learned how to divide fractions. We were – as I imagine – just told how to instrumentally perform the procedure. This, nevertheless, did not keep me from using the new procedure. I eagerly applied my new knowledge in the exercises the teacher presented. However, when I did so, I noticed that all exercises resulted in the same answer, namely '1'. I reckoned that there was something seriously wrong here, especially as I considered that my new procedure would lead to an answer '1' in all cases. Why then, I thought, would one learn how to divide fractions – the answer is '1' anyway.

I went to the teacher, who – I imagine – told me what I did wrong and what I should do to get the right answers. I certainly finally got the right answers. But getting the wrong answers first probably had a greater impact than reaching the right ones a little later. I made an important shift from following an instrumental procedure to imposing a sort of meaning on the (abstract and symbolic) situation; a shift to (intuitively) trying to understand structures underlying mathematical operations.

This doctoral thesis is a scientific treatise. However, it can also be considered as an account of the researcher's learning process. In this learning process many steps are planned and directed at the aim of relating research experiences to expectations from theoretical reflections; an explicit striving for understanding of mathematising processes. In this learning of the researcher, seemingly irrelevant experiences suddenly, years later, appear to be of importance. This was the case in learning how to divide fractions. This was also the case at university, after finishing my training as teacher and – again – in primary school.

Ben Knip at the University of Amsterdam, who guided me through my first experiences as a teacher, told me what to do with what I learned in the teacher training course he guided: 'You better write about teaching instead of going into teaching.' I later learned that writing about teaching is indeed worthwhile, but that being a teacher – at least for me – is a prerequisite for doing so. Furthermore, as a primary school student I learned what writing a thesis is about. My grade 5 teacher, Van Kooten, told his class that researchers research everything to base their thesis on: 'Some even research a topic like the learning of fractions in grade 6 and write a book when they finish their work.'

I think Van Kooten took fraction learning as a possible subject for a thesis because that would make his explanation on working on a thesis comprehensible. Coincidentally, more than 20 years later I started researching fraction learning in grade 6. I made it my personal assignment for a ten year period. Thirty years after his explanation I wrote the thesis Van Kooten suggested.

Teacher, the assignment wasn't easy and it took some time, but I finished it!

Mathematics as societal need

Halfway through my study in mathematics I recognised the importance of mathematics in a rapidly changing technological society. Mathematics changed from a subject that just attracted me to something relevant for solving societal needs. I became more and more interested in how human beings learn mathematics, and consequently took several courses in psychology and did a teacher training course at my university.

This shift in interest thus became a shift towards mathematics learning. And this next, two years later, made that I became a teacher trainer in mathematics myself. First at the secondary level and from 1988 at teacher training colleges for primary school. I, in some sense, re-entered primary school teaching and it was Wil Oonk who there introduced me to realistic mathematics education (RME). I liked (and still like) teaching in teacher training because of its eminent relation between mathematics learning from the perspective of unfolding primary school students' learning processes. RME attracted me because it provided a clear relationship between mathematics as a structure and its underlying phenomenology. I followed Ben Knip's advice and started to write about teaching mathematics.

Writing about mathematics education contributed to my development as a teacher trainer and I became well acquainted with mathematics education. Moreover, writing about student teachers' learning of mathematics and its didactics paved the way to my participation in many national groups and projects on mathematics education. One of these was the SLO/FI/Cito fraction project, which formed the actual starting point of the research reported in this book. In the first year of this fraction project, Adrian Treffers invited me to elaborate on the project's results, under his guidance, in developmental research. Working 'on fractions' for more than 10 years finally resulted in this book. In these ten years this study resulted in new ideas and even newer ideas on fraction learning and teaching, making new ideas old ones. The study changed its perspective several times, especially after I asked Jan Terwel to take over Adrian Treffers' role. Involvement in the developmental process and in teaching moved gradually in the direction of scientific distance and interpretation of students' outcomes. The fraction learning perspective changed to fraction learning as a typical example of a mathematising process. The focus turned to low achievers in an RME setting and re-turned to mathematics as societal need.

Acknowledgements

Learning processes do not proceed in a vacuum. Many people contributed to my development and therefore to what is set down in this book. I thank them all for helping me to write this thesis; fellow teacher trainers, people I met while working in the Panama project, members of the editorial board of 'Panama-Post', colleagues working at the Freudenthal Institute and many others in the field of mathematics education I have the privilege of working with. I especially wish to thank the colleagues that worked with me on the SLO/FI/Cito fraction project, Adrian Treffers, Anneke Noteboom, Anita Lek, Joop Bokhove and Kees Buys, as the ideas we shared and put down in 'De Breukenbode' [The Fraction gazette] formed the starting point of the study presented in this book.

This study could not have been done without the support of the Hogeschool van Amsterdam (Educatieve Faculteit Amsterdam), the Hogeschool IPABO Amsterdam/Alkmaar and the Freudenthal Institute. The Hogeschool van Amsterdam and the Hogeschool IPABO facilitated me to conduct the research and to write this thesis. The Freudenthal Institute helped me to realise this publication. I owe them my thanks.
I, moreover, thank Betty Heijman and Ellen Hanepen for editorial support and Nathalie Kuijpers, Paul van Buren and W.G. Topham for correcting the English in this book.

Experiments reported in this book were conducted at the Montessorischool Landsmeer. This school offered me the chance to work with their students. The school's teachers helped me in many ways to conduct the study reported upon here. I owe them my gratitude for the flexibility they expressed in sharing their school with an exacting researcher. I especially thank Ronald Steen, Ingeborg Brandenburg and Marjan Smid for their support.

The many discussions with my promotor Jan Terwel at the Vrije University Amsterdam finally made that this study got its present form. I thank him for his guidance, for his willing ear and sharp remarks. Jan taught me to combine commitment with scientific distance. He convinced me of the need to consider mathematics education from many perspectives. In introducing me to new topics he showed me examples of scaffolding. In discussing and explaining research results he exemplified knowledge construction. I enjoyed all his guidance and am grateful for the support it gave me to finally finish this book.

However, all this work also had its impact on the home front.
There, Twan and Vonne, who I love so much, had to share their father with mathematics education and researching fraction learning. They experienced that hard working is equivalent with staring at many letters on a computer screen and – from

time to time – adding yet another few. Betty, I hope finishing this book means that priorities are going to be changed somewhat. I thank you for all your support and patience.

Table of contents

1 Introduction

This first chapter provides an introduction of a research project on mathematics education.[1] Its main issue concerns the effects of a newly developed fraction programme on the learning processes of primary school students. The observations and analyses are in particular directed to the activity of 'mathematising' as an aim in mathematics education.

The two guiding research questions are: 'How are processes of learning formal fractions facilitated for 10-11 year old children?' and 'What main obstacles can be distinguished concerning the processes of learning formal fractions for 10-11 year olds?'

Traditionally the content of mathematics in primary school was mainly based on the product of formal academic mathematics, as it was advanced in the 16th and 17th century. In the past decades developers of mathematics teaching re-addressed teaching formal mathematical structures. Starting from the perspective of mathematics as a constructive activity, they tried to remodel mathematics teaching to their ideas. Thus, in the Netherlands realistic mathematics education (RME) was developed, in which mathematics teaching starts from exploring meaningful contexts followed by a process of mathematisation leading to generalised and abstract mathematical concepts.

The outcomes of our research will provide arguments that support the understanding that learning (formal) rational numbers in an RME teaching setting is feasible for the vast majority of students, but not for all. Moreover, we will consider how teaching can be organised to obtain the best chances for all students. This study is based on multiple perspectives, in particular cognitive theories and pedagogical content knowledge.

1.1 Research situation and conceptual model guiding the study

The research described here was conducted at a school in a small town north of Amsterdam. The students visiting the school generally have a middle class background. From two parallel grade 6 groups (9 to 10 years) we selected by a matching procedure an experimental group, which followed a newly developed programme, and a control group, which followed the school's more traditional fraction programme. We thoroughly observed and analysed the fraction learning processes of students in both the experimental and the control group during a whole school year. Our analyses focused on identifying and clarifying key-elements in fraction learning and on generalising findings to learning processes involved in formal mathematics acquisition. The conceptual model of the study schematises key-elements in the

research described here (figure 1.1). Pre-knowledge as measured by general mathematics tests (Janssen, Kraemer & Noteboom, 1995) and two fraction programmes – an experimental programme and a control programme – and the teaching thereof are variables that influence the fraction learning process. The post-tests in the study are used to establish students' general mathematical skills and students' strategies in operating with fractions.
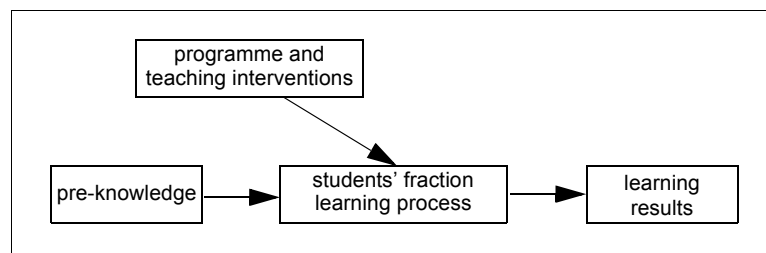
figure 1.1: conceptual model guiding the study

The scheme depicted in figure 1.1 forms the basis under the three studies reported in this book. In a quasi-experimental research design we compare student outcomes from students in experimental and control groups. Moreover, in two case studies we follow the development of two students with different characteristics who followed the experimental programme; one being an average student in mathematics and the other a low achieving pupil.

Our reflections on the observed fraction learning in the experimental programme lead to adaptations in the developed experimental programme. We hereby follow Gravemeijer's (1994) ideas on developmental research as a (cyclic) process of programme improvement, based on theoretical notions, student observations and teaching experiences. We, moreover, follow Freudenthal (1991), who described developmental research as 'experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience' (Freudenthal, 1991, 161).

1.2   Developmental research as perspective

This study evolved from the development of a fraction programme within the context of RME, as it is developed in the past thirty years in the Netherlands. In its theoretical analyses, however, it goes beyond RME to take the broad landscape of educational research, and especially the research landscape of mathematics education as its playground. From this broad perspective in the layered structure of this book sev-

eral topics related to the fraction learning-processes are elaborated upon to provide a deeper understanding of these processes. These layers include considerations for constructing the fraction programme. Moreover, they incorporate an analysis of mathematisation processes in general and provide additional arguments for considering learning as guided re-invention (Freudenthal, 1991).

The model presented in figure 1.2 provides an attempt to embed the main arguments that provide understanding on children's learning processes. It follows Gravemeijer (1994) in his ideas on developmental research as a cyclic process, where thought-experiments on designing teaching, based on an adequate knowledge base, precede classroom experiments and where subsequent reflections on these experiments lead to attuning the developed teaching.
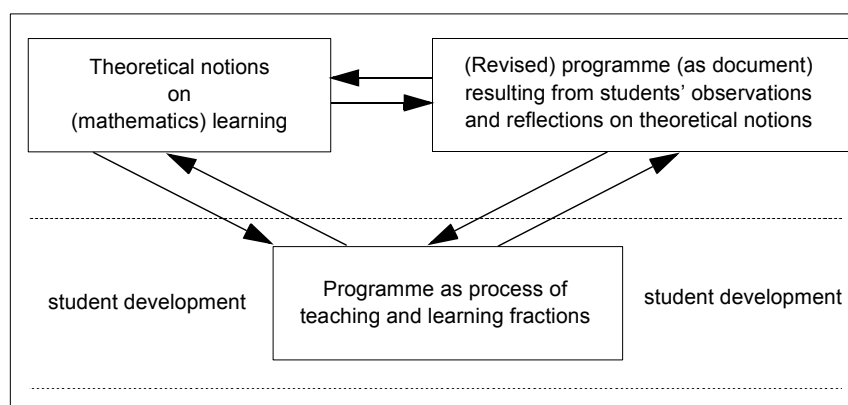
figure 1.2: framework for developing mathematics education

In the developmental model this cyclic character, as proposed by Gravemeijer, is replaced by a set of bi-directional arrows. These arrows indicate how 'theoretical notions on (mathematics) learning', 'the programme (as document)' and 'the programme as process of teaching and learning fractions' mutually influence each other. For example 'theoretical notions' (upper-left textbox) provide ideas for constructing the programme as document; as what we already know of learning fractions forms a basis in the construction process (arrow left to right). In a similar manner 'theoretical notions' direct the teaching and learning process, as these notions provide us with ideas that 'work out' in teaching (arrow down to the right). However, experiences in constructing the programme and (especially) observations of the teaching and learning process, form the basis for theoretical reflections, which provide new and enriched theoretical notions (arrows the other way around).

From another perspective, the model symbolizes the development of students in their interaction with the developed programme by positioning the programme in the

student development. In this sense, the model incorporates Simon's (1995) model of 'hypothetical learning trajectories'. Simon proposed his framework to model (local) teaching decisions, where teachers decide on their teaching and base their ideas on student development, the 'hypothetical trajectory' of the student's learning. In the conceptual model presented here, Simon's framework is extended to programme development and global teaching decisions. The 'hypothetical learning trajectories' thus become global (theoretical) notions on teaching and learning mathematics (and of fractions in particular).

1.3   Guided reinvention

In our present technological society human beings need to learn abstract concepts and formal relations. From these societal needs educational psychology strongly aimed at cognitive processes and problem solving strategies (Greeno, Collins and Resnick, 1996). In order to enable students to participate as competent members of their community it is important to guide children in the required strategies of symbolisation, modelling, abstraction, formalisation and generalisation; to cope with key elements in mathematising one's world (Sierpinska & Lerman, 1996; Streefland, 1997; Schoenfeld, 1994).
This task for educators naturally leads to the question of how to facilitate relevant learning processes. Should we focus on individual cognitive processes, should we regard learning as a social enterprise or take both perspectives into account (Anderson, Greeno, Reder & Simon, 2000)? Here, we will follow the argument that focussing on both these perspectives provides a means to better understand children's learning processes (De Corte, Greer & Verschaffel, 1996; Greeno, 1991; Mayer, 1999; Shulman & Quinlan, 1996).

In constructing and teaching mathematics, we follow in particular Freudenthal (1991) and take a phenomenological point of view, where mathematical strategy acquisition is considered as a process of *'guided reinvention'*. Freudenthal underlined the necessity of student guidance while learning mathematics. He used this argument to criticise constructivist ideas. He pleaded for guided *reconstruction*. Moreover, he recognized that mankind developed mathematics to solve all sorts of practical problems as mathematics evolved to the science of structures and became formal mathematics, which in some cases lost many of its obvious links to daily life (Struik, 1987; De Corte, Greer & Verschaffel, 1996), and that students should be guided to re-experience this lengthy process in only a few years
Formal mathematics as such forms a part of the mathematics curriculum in schools all over the world. It however, opposes Freudenthal's (1973) credo of 'mathematics as a human activity', which can be seen as an expression of a phenomenological theory of mathematics education with its point of departure in the practice of educa-

tion and teaching, and not in the transmission of mathematics as a pre-formed system (Gravemeijer & Terwel, 2000).

This thesis is grounded in the ongoing discussion on learning fractions in primary school. Should we, as is suggested by the Dutch curriculum standards, postpone learning fractions that are not directly related to daily life experiences to secondary education? Or, should the subject of fractions be considered as a relic from the past, that does not fit in present-day curricula (Goddijn, 1992)? This thesis focuses on the process of learning formal fractions in a newly developed programme. It brings forward and analyses several arguments to support the viewpoint that the vast majority of students in primary school can learn formal fractions in a meaningful manner. Moreover, it addresses the problems some students face in learning fractions.

1.4   Research questions and hypothesis

The central issue in this thesis concerns the effects of a recently developed fraction programme (see appendix A) on the learning processes and outcomes. In this experimental programme fractions are presented as folded bars and numbers on the number line, fractions are presented as (single) numbers between integers. On the other hand, the control programme emphasises fair sharing and dividing circles as fraction generating activities. There are strong indications that students are not prepared for uncovering fraction relations by dividing circles, as many divisions are difficult to obtain. Further, fair sharing – regarding $\frac{3}{5}$ as three pizza's divided by five children – does not clearly present a fraction as one number or entity, but rather presents a fraction as (a ratio of) two numbers (cf. Streefland, 1991). This effect analysis of the two programmes hands us on the manner in which students can benefit from the newly developed programme. Moreover, the research provides arguments for programme development. However, although this study thus could be considered a study on fraction learning processes in primary school, it is more than that. The school subject of fractions here also represents a typical example of the transition from recognisable situations to formal mathematical constructs, especially when focussed on the acquisition of equivalent fractions. More specifically, when reasoning is aimed at flexible use of an appropriate representative of one fraction equivalence class, this can be considered as an integrated process of generalisation, abstraction and formalisation; a process of mathematisation.

In line with what Anderson, Greeno, Reder & Simon (2000) suggest on attuning appropriate research approaches to relevant research issues, we will use several methodological approaches to analyse the aforementioned fraction programme and its student outcomes, with various instruments. However, before addressing these methodological issues, we will first outline the study's aim and its general research questions and hypothesis.

1.4.1 Aim of the study

This study researches the feasibility and effectiveness of a newly developed fraction programme in primary school as compared to a widely used RME programme (used in a somewhat traditional manner). The newly developed programme is reflected upon, to further improve it to the needs of present day education (Gravemeijer, 1994; cf. Gravemeijer, 2001). Moreover, the study is conducted to provide arguments on to what extent students can and should participate in a fraction programme aimed at formal fraction acquisition.

Although the learning of formal fractions is the main issue in this book, fraction learning is here seen also as a typical example of a (vertical) mathematising process. In other words, it is seen as process of modelling, symbolising, generalisation, abstraction and formalisation (Treffers, 1987; Freudenthal, 1991; Nelissen, 1998; Gravemeijer & Terwel, 2000). A more general treatment of this vertical mathematisation process can be considered to be a second layer in the study described. It is the by-product of considerations on learning fractions and, as such, an important result of the research.

1.4.2 General research questions

We will argue that using rich situational contexts (Arcavi, 1994; Trzcieniecka-Schneider, 1993; Streefland, 1990) and in particular using 'generic examples' that contain a prototype of the generalisable object and wherein the action of abstraction is still included (Harel & Tall, 1991) are necessary to prevent abstract ideas being disconnected from the physical world (Sfard, 1994; Greeno, 1997). Moreover, seeing abstracting as a cyclic process with several forms of constructing schemes for understanding concepts leaves room for different levels in strategies in one problem solution (Van Hiele, 1986; Dubinsky, 1991).

From this point of view we formulate the following general research questions:

1   How are mathematisation processes facilitated for 10-11 year old children, especially in the case of fraction acquisition?
2   What main obstacles can be distinguished concerning the processes of vertical mathematisation for 10-11 year olds, especially in the case of fraction acquisition?

1.4.3 Specific research question

We specify these general research questions into a specific research question:

How do student learning processes develop in an experimental curriculum in which the number line is used as a model for fractions and meanings are established by negotiation, and what are the learning outcomes of the developed programme as compared to a control group which learns fractions in a

more traditional programme, where the circle is the central model and where learning, to a large extent, is a solitary activity?

In elaborating the outcomes of the programme comparison, we will especially focus on possible differential effects on high and low achieving students (cf. Hoek, Terwel & Van den Eeden, 1997; Hoek, 1998; Hoek, Van den Eeden & Terwel, 1999; Terwel, Gillies, Van den Eeden & Hoek, 2001).

From this perspective our research questions lead us to the following hypotheses.

1.4.4 General hypothesis

A majority of 10-11 year old students can acquire abstract and formal mathematical concepts and strategies when these concepts and strategies are learned in an educational setting, where meanings are negotiated in meaningful and recognizable situations (Greeno, 1991) and where mathematical activities on different levels are closely connected (Van Hiele, 1986; Pirie & Kieren, 1989; Pirie & Kieren, 1994; Gravemeijer, 1994).

1.4.5 Specific hypothesis

Students in an experimental curriculum in which the number line is used as a model for fractions and meanings are established by negotiation, will outperform control students in a more traditional programme, where the circle is the central model and where learning, to a large extent, is a solitary activity.

1.5 Methods

From the perspective of theoretical notions development, the framework for the development of mathematics education – by following the arrows in the direction of the 'theoretical notions'-box – indicates that these notions are fed with 'experiences in constructing (a prototype of) the programme' and 'student observations' (see figure 1.2). However, generating theory, in this way, is a lengthy process, as it is closely linked to data and inevitably needs continuous modification and reformulation (Glaser & Strauss, 1977). In connection with Glaser and Strauss' ideas these notions will be closely linked to the process of teaching the developed fraction programme and to student observations – the data – and are therefore in need of further modification and reformulation.

Thus, although the main concern in the study described here is student development in two fraction programmes, we consider developmental research to be an overarching research perspective. In fact, the developmental research perspective could be considered to contain any concise form of active search or reflection leading to arguments to (a continuous) improvement of mathematics programmes (cf. Gravemeijer, 1994) and this study is conducted to provide those arguments. To visualise this over-

arching character of developmental research, we presented figure 1.2 to relate considerations in teaching and learning fractions. In the conceptual framework of the study (figure 1.1), that could be considered as a derivation of the developmental research scheme, we schematise key-elements in the research: the two programmes, teaching experiments, student development, pre-knowledge and learning results. Research interventions are embedded in a quasi-experimental research design (Cook & Campbell, 1979), more precisely a non-equivalent pre-test post-test control group design, where students in the experimental group follow the experimental curriculum, where the researcher took the teacher's role, and students in the control group follow a widely used fraction curriculum and where learning fractions is guided by the group's teacher (see chapter 3 and 6 for details on the study's setting).

In table 1.1 we provide a specification of the research design. In this design O_1, O_2 and O_3 are standardized tests to obtain students' general mathematical skills at the start, halfway through and at the end of the research year. Three standardised interviews, I_1, I_2 and I_3 were held to recover students' fraction knowledge at the start, halfway through and at the end of the research year, and thorough qualitative observations during fraction instruction in both experimental (Xe_1, Xe_2, Xe_3, Xe_4) and control group (Xc_1, Xc_2, Xc_3, Xc_4) were conducted to provide additional information on student development.

O_1	Xe_1	I_1	Xe_2	O_2	Xe_3	I_2	Xe_4	O_3	I_3
O_1	Xc_1	I_1	Xc_2	O_2	Xc_3	I_2	Xc_4	O_3	I_3

table 1.1: specification of the research design

Student observations provided us with a means to perform two case studies. In each of these case studies we describe the development in fraction learning of one of the students in the developed programme. To do so, we analysed all lessons in the experimental curriculum, to select the moments that clearly exhibit the students' development. Next the theory of RME helped us to interpret and – in some sense – generalise the findings (cf. Yin, 1984).

1.6 Outline of the book

The developed fraction programme (see appendix A) forms the object of study in this thesis. In the present chapter we formulate an aim for the study, general research questions, general hypotheses and specific research questions. In the following chapters we explicitly address the specific research questions.

Chapter 2 provides a general theoretical background for the study, as it describes how the development of teaching fractions, as is the main issue in this book, is embedded in recent ideas on learning and curriculum development.

It, moreover, provides an analysis of mathematising and presents fraction learning

as a typical example thereof.

Chapter 3, 4, 5 and 6 can be considered as empirical testing of one of the prototypes of the developed fraction programme. This prototype resulted from several years of developing and researching. In some sense it could be regarded as a temporary 'final product'. We consider the specific research questions, formulated in this chapter, as spotlights, enlightening key aspects of one overall process, to thus contribute towards formulating answers to this thesis' general research questions and find arguments to accept or reject its general hypothesis. Chapter 6 summarizes the quantitative results of our empirical study and presents an overall explanation for observed trends.

In chapter 3 we will direct our spotlight at the comparison of student development in the researched programme with the development of students in a traditional fraction programme. Observed differential effects of the experimental curriculum gave rise to analysis of the development of average students and low-achievers more closely. We will do so in chapter 4 and 5, where we report on two case studies. In chapter 4 we describe the learning process of an average student, Audrey. Shirley's fraction learning is the subject in chapter 5. Shirley is a low-achiever in mathematics. With Shirley's learning process as a reference point we will answer the specific research question on learning fractions by low achievers in the developed fraction programme: 'Do low-achieving students really benefit from a realistic problem solving approach for acquiring mathematical insights and proficiency in the domain of fractions, and what are the main obstacles in the formalisation process from real-life situations to mathematical number sense?'

Chapter 3 to 6 mainly focus on fraction acquisition, where the issue of mathematising is derived from. Theoretical analyses and questions of generalizing, abstracting and formalizing processes in 10-11 year old students are not fully and thoroughly addressed in these chapters. Such an analysis is included in chapter 2 and will be elaborated on in chapter 7, which can be considered as a theoretical reflection on the whole study. Finally chapter 8 re-addresses this study's research questions and hypotheses. This chapter will conclude with the formulation of answers to this study's questions.

This book's appendices provide for additional information on the study. These contain a description of the experimental programme and problem situations from the first and the second fraction interview. This information is presented in this form, as it helps to understand the analyses presented throughout this book.[2]

Notes

1 This project is situated in the research programme 'Strategic Learning in the Curriculum' at the Vrije University Amsterdam, Faculty of Psychology and Education, Department of Education.

2 Several chapters in this book are written in article form. To enable separate publication elsewhere these articles provide a overview over the study and its main research issues. This facilitates readers to read many chapters in this book without proper knowledge of the others. This, however, also causes some redundancy for readers of the whole work presented here.

References

Anderson, J.R., J.G. Greeno, L.M. Reder & H.A. Simon (2000). Perspectives on Learning, Thinking, and Activity. *Educational Researcher*, *29*(4), 11-13.

Arcavi, A. (1994). Symbol Sense: Informal Sense-making in Formal Mathematics. *For the Learning of Mathematics*, *14*(3), 24-35.

Cook, Th.D. & D.T. Campbell (1979). *Quasi-Experimentation. Design & Analysis Issues for Field Settings*. Chicago: Rand McNally College Publishing Company.

Corte, E. de, B. Greer & L. Verschaffel (1996). Mathematics Teaching and Learning. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology*. New York: Simon & Schuster Macmillan, 491-549.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In: D. Tall (ed.). *Advanced mathematical thinking*. Dordrecht/Boston/London: Kluwer Academic Publishers, 95-123.

Freudenthal, H. (1973). *Mathematics as an Educational Task.* Dordrecht: Reidel.

Freudenthal, H. (1991). *Revisiting Mathematics Education, China lectures*. Dordrecht: Kluwer Academic Publishers.

Glaser, B.G. & A.L. Strauss (1977). *The Discovery of Grounded Theory. Strategies for Qualitative Research*. Chicago: Aldine Publishing Company.

Goddijn, A.J. (1992). De oudste en de nieuwste breuken [The oldest and the newest fractions]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *11*(2), 18-31.

Gravemeijer, K.P.E. (1994). *Developing realistic mathematics education* (dissertation)*.* Utrecht: CD-ß Press.

Gravemeijer, K.P.E. & J. Terwel (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of Curriculum Studies*, *32*(6), 777-796.

Gravemeijer, K.P.E. (2001). *Reken-wiskundeonderwijs voor de 21^e eeuw* [Mathematics education for the 21th century]. Utrecht: Utrecht University.

Greeno, J.G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education*, *22*(3), 170-218.

Greeno, J.G., A.M. Collins & L. Resnick (1996). Cognition and Learning. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology*. New York: Simon & Schuster Macmillan, 15-46.

Greeno, J.G. (1997). On claims That Answer the Wrong Questions. *Educational Researcher*, *26*(1), 5-17.

Harel, G. & D. Tall (1991). The General, the Abstract, and the Generic in Advanced Mathematics. *For the Learning of Mathematics*, *11*(1), 38-42.

Hiele, P.M. van (1986). *Structure and Insight. A theory of Mathematics Education.* Orlando: Academic Press, Inc.

Hoek, D., J. Terwel & P. van den Eeden (1997). Effects of Training in the Use of Social and

cognitive Strategies: An Intervention Study in Secondary Mathematics in Co-operative Groups. *Educational Research and Evaluation*, *3*(4), 364-389.

Hoek, D.J. (1998). *Social and cognitive strategies in co-operative groups. Effects of strategy instruction in secondary mathematics* (dissertation). Amsterdam: University of Amsterdam.

Hoek D., P. van den Eeden & J. Terwel (1999). The effects of integrated social and cognitive strategy instruction on the mathematics achievement in secondary education. *Learning & Instruction*, *9*, 427-448.

Janssen, J., J.-M. Kraemer & A. Noteboom (1995). *Leerling Volg Systeem. Rekenen-Wiskunde 2* [Student Registration System. Mathematics 2]. Arnhem: Cito.

Mayer, R.E. (1999). Designing Instruction for Constructivist Learning. In: Charles M. Reigeluth (ed.). *Instructional-design theories and models. A new paradigm of instructional theory. Volume II.* Mahwah, NJ: Lawrence Erlbaum Associates, Publishers, 141-160.

Nelissen, J.M.C. (1998). Representaties in het reken-wiskundeonderwijs [Representations in mathematics education]. *Pedagogische Studiën*, *75*, 169-183.

Pirie, S.E.B. & T.E. Kieren (1989). A Recursive Theory of Mathematical Understanding. *For the Learning of Mathematics*, *9*(3), 7-11.

Pirie, S.E.B. & T.E. Kieren (1994). Beyond Metaphor: Formalising in Mathematical Understanding within Constructivist Environments. *For the Learning of Mathematics*, *14*(1), 39-43.

Schoenfeld, A.H. (1994). Reflections on Doing and Teaching Mathematics. In: Alan H. Schoenfeld (ed.). *Mathematical thinking and problem solving*. Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers, 53-70.

Shulman, L.S. & K.M. Quinlan (1996). The Comparative Psychology of School Subjects. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology.* New York: Simon & Schuster Macmillan, 399-423.

Sierpinska, A. & S. Lerman (1996). Epistemologies of Mathematics and of Mathematics Education. In: Alan J. Bishop, Ken Clements, Christine Keitel, Jeremy Kilpatrick & Colette Laborde (eds.). *International Handbook of Mathematics Education (part 2).* Dordrecht/Boston/London: Kluwer Academic Publishers, 826-876.

Simon, M.A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114-145.

Streefland, L. (1991). *Fractions in Realistic Mathematics Education, a Paradigm of Developmental Research.* Dordrecht: Kluwer Academic Publishers.

Streefland, L. (1997). Een geval van reflectief denken in ontwikkeling met verhoudingen als paradigma (slot) [A case of developing reflective thinking with ratio as paradigm]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *15*(3), 22-31.

Struik, D.J. (1987). *A Concise History of Mathematics (fourth revised edition).* New York: Dover Publications, Inc.

Terwel, J., R.M. Gillies, P. van den Eeden & D. Hoek (2001). Co-operative learning processes of students: A longitudinal multilevel perspective. *British Journal of Educational Psychology*, *71*, 619-645.

Treffers, A. (1987). *Three dimensions. A model of Goal and Theory Description in Mathematics Instruction – the Wiskobas Project.* Dordrecht: Reidel Publishing Company.

Trzcieniecka-Schneider, I. (1993). Some remarks on creating mathematical concepts. *Educational Studies in Mathematics*, *24*, 257-264.

Yin, R.K. (1984). *Case study research. Designs and Methods.* Beverly Hills, CA: SAGE Publications.

2 Theoretical background

This chapter surveys some theoretical notions that will embed the study described in this book in renewed paradigms in theories of learning and instruction. It touches upon the study's foundations and thus provides for means to generalise its findings beyond fraction learning.

'Constructivism', as learning theory, and 'situated knowledge' are considered paradigms that present 'the state of the art' in contemporary psychology. Both paradigms open the viewpoint of pedagogical content knowledge, which is here elaborated as mathematising; watching the world from a mathematical perspective to thus make it more mathematical (Freudenthal, 1968).

2.1 Paradigms in researching mathematics education

Three research paradigms played a role in psychology's recent history. The first paradigm of response strengthening based on stimulus-response mechanisms to establish learning was anchored in a behavioural tradition. Information processing, the second paradigm, found its basis in a knowledge acquisition view on learning. The third paradigm, the constructivist view, states that learners actively create their knowledge (Mayer, 1996; Greeno, Collins & Resnick, 1996; Mayer, 1999). Recently the notion of 'situated cognition' emerged from rethinking and criticising the information processing and constructivist paradigms (Greeno, 1991; Anderson, Greeno, Reder & Simon, 2000; Greeno, 1997; Shulman & Quinlan, 1996). 'Situated cognition' assumes that cognition is situated in contexts and learning and thinking are conceived of as interaction between the individual and the situation (Sierpinska, 1995; De Corte, Greer & Verschaffel, 1996).

In order to see the various paradigms in the right perspective it is helpful to realise that modern (cognitive) psychology was initially developed from attempts to tackle several issues concerning learning in different content areas (Reigeluth, 1999; Shulman & Quinlan, 1996). However, cognitive psychology and curriculum theory in the United States on the one hand and European didactical theories in the various subject areas on the other hand, have evolved more or less independently (De Corte, Greer & Verschaffel, 1996; Hopmann & Riquarts, 1995, Mayer, 2001; Shulman, 1995).

A dialogue between researchers involved in subject matter instruction – especially those involved in the development of mathematics curricula – and educational psychologists emerged from various developments. This dialog was initially restricted to the cognitive learning theories and curriculum theory in the United States (cf. Bruner, 1966). After the Sputnik shock various curriculum projects were developed in the United States. However the evaluation of the implementation and effects of

the newly developed curricula showed meagre effects. Until the late 80's, United States curricula in mathematics and science were mainly based on insights from cognitive learning theories and formal definitions of the subject matter e.g. the 'structure of the discipline' (cf. Huhse, 1968; Walker, 1990; Pinar, Reynolds, Slattery & Taubman, 1995). This curriculum introduction was usually followed by a discussion on its disappointing results. Moreover, the implementation of these new curricula seemed problematic (Schoenfeld, 1992; Terwel, Volman, Wardekker & Hameyer, 2000). For example, students were not motivated by the abstract and formal approaches in the curricula based on the ideas of the New Math movement. The teaching of general problem solving skills was not effective and transfer of knowledge formed a problem (cf. De Corte, Greer & Verschaffel, 1996; Greeno, Collins & Resnick, 1996; Greer & Harel, 1998).

One of the attempts to overcome these fundamental problems in mathematics education was to take students' notions as starting-point for teaching mathematics, for example by introducing realistic contexts that formed the basis for further mathematisation (Treffers, 1987). Moreover, 'Mathematics for all' (Freudenthal, 1968) became a credo for mathematics education and interaction and co-operative learning was promoted to support this goal (Bruner, 1966; Cobb & Steffe, 1983; Lawler, 1990; Schoenfeld, 1994; Perrenet, 1995; De Corte, Greer & Verschaffel, 1996; Hoek, Terwel & Van den Eeden, 1997; Brandt, 1998; Wood, 1998; Anderson, Greeno, Reder & Simon, 2000).

There were also developments within the cognitive learning theories. From its very start psychology focused on (general) problem solving and knowledge transfer; issues that are also deeply rooted in the learning of mathematics. However, it became clear that the cognitive 'information processing theory', mainly based on the computer metaphor and individual information processing, could not explain the complex processes and problems in mathematics education. The 'information processing' paradigm was gradually replaced by broader concepts of learning such as 'knowledge construction' and 'situated learning'. The 'learning as knowledge construction' paradigm now formed an excellent opportunity to examine the learning processes from multiple perspectives, including instructional psychology, curriculum theory and the didactics of mathematics education. The specific character of the subject matter knowledge construction, in this case 'mathematisation', could serve as a meeting point for the disciplines involved (Freudenthal, 1978; Shulman, 1995; Hopmann & Riquarts, 1995; De Corte, Greer & Verschaffel, 1996; Mayer, 2001).

Today there is a broad consensus that in considering learning and teaching processes one has to focus on the pedagogical content knowledge 'to identify the concepts and principles that [are] critical to the structure of a subject and to study how the most effective teachers organize them for instruction' (Shulman & Quinlan, 1996, 411). This links up with the German tradition of 'Didaktik', where the teacher both formulates and carries out educational decisions. It opposes traditional United States cur-

riculum theory, where teachers are only agents of the educational system to implement curricula (Westbury, 1995). In the 'Didaktik' tradition the teacher is mediator between content and learner. In this subject-matter 'Didaktik', the key characteristics of the subject-matter formed a basis to develop education (Hopmann & Riquarts, 1995).

This dialog between United States and European approaches set another stage for the development of a theory of mathematics teaching and learning to form a discipline that focuses on learning from the viewpoint of mathematising. This development parallels that of the development of constructivism (see e.g. Cobb & Steffe, 1983; Von Glaserfeld, 1987; Davis, 1992; Arcavi & Schoenfeld, 1992; Pirie & Kieren, 1994; Simon, 1995; Cobb & Whitenack, 1996; Forman & Fyfe, 1998; Mayer, 1999; Perkins & Unger, 1999; Reigeluth, 1999; Roelofs & Terwel, 1999; Terwel, 1999; Rasmussen, 2001). Constructivist learning environments give students opportunities to construct mathematics from meaningful problems (Greeno, Collins & Resnick, 1996). The students' task here is to make sense of the information presented in the environment and therefore depends on the learner's cognitive activity, rather than on that of the teacher (Mayer, 1999; Gravemeijer, 2001). In this way students are engaged in mathematising, as they are stimulated to regard the world from a mathematical perspective.

In these constructivist settings learning is considered to be a (mainly) social interactive enterprise, where mathematical constructions and meanings are negotiated (Hanna, 1989; Greeno, 1991; Greeno, 1997; Naujok, 1998; Lyle, 2000; Yackel, 2001). In constructing mathematics students learn to take others' opinions into consideration. In this educational context the 'didactical contract' needs to be reconsidered (cf. De Corte, Greer & Verschaffel, 1996), in that students become members of (mathematics) learning communities, who formulate and evaluate questions and problems, and construct and evaluate hypotheses, evidence, arguments, and conclusions (Greeno, Collins & Resnick, 1996; Van Oers, 2001). Therefore there is a need to discuss socio-mathematical norms in these classrooms to provide for an adequate learning environment for the learning of mathematics (Cobb, Yackel & Wood, 1989; Yackel & Cobb, 1996; Yackel, 2001).

In some sense the development of the Dutch RME can be considered as forerunner of the developments mentioned. One reason for this position lies in the embeddedness of Freudenthal's ideas in the European tradition. More than thirty years ago constructing mathematics as human activity formed RME's basis (Freudenthal, 1973; Gravemeijer & Terwel, 2000) and mathematising was established as a major learners' activity (Gravemeijer, 1994; De Corte, Greer & Verschaffel, 1996; Gravemeijer, 2001). A phenomenological analysis of mathematical structures and its learning lead to programmes based on the RME-principles (cf. Freudenthal, 1983). RME teaching and learning start with recognisable contexts. These meaningful sit-

uations, in time, are mathematised to form more formal relations and abstract structures (Van den Heuvel-Panhuizen, 1996). Treffers (1987), here made a distinction between horizontal and vertical mathematisation. The former involves converting a contextual problem into a mathematical problem, the latter involves taking mathematical matter onto a higher plane. Vertical mathematisation can be induced by setting problems which admit solutions on different mathematical levels (Freudenthal, 1991; Gravemeijer & Terwel, 2000). Moreover, embedding vertical mathematisation in RME gives it the potential to develop formal mathematics in students, rather than being limited to applying mathematics in recognisable contexts.

In its thirty year history RME developed a theoretical basis that could be considered as pedagogical content knowledge – a theory of learning generated by key aspects of the subject matter (Treffers, De Moor & Feijs, 1989; Lampert, 1990; Shulman & Quinlan, 1996).

2.2 Mathematising as an aim in mathematics education

Learning mathematics here is considered as being engaged in a process of mathematising. Freudenthal (1968) considered mathematising as a strategy to make things more mathematical. Mathematising includes problem situations, problem solving processes and learning processes (Gravemeijer & Terwel, 2000). When discussing this mathematising process, we actually discuss processes of modelling, symbolising, generalising, formalising and abstracting. These aspects of the mathematising process are not meant to be regarded in isolation, as only the combination of all these aspects does justice to the mathematising process (Romberg, 1994; Schoenfeld, 1994; Streefland, 1997). For analytical reasons these aspects of mathematising are considered here separately but in connection. In our analysis of student strategies, we will however review the (coherent) mathematising process as a whole.

2.2.1 Modelling

'Modelling' is the process in which a model is initially constituted as a context-specific model of a situation (Gravemeijer, 1994). For example, if the sharing of pizzas constitutes a situation that generates fractions, a circle is the context-specific model as it provides for an image of the pizzas in the sharing process. 'Modelling' is, one could say, the process where irrelevant elements of the context are left out of consideration, to first obtain a context-specific representation which develops to a general entity, covering more situations, to finally become an entity on its own. The circle becomes any devisable object with that form. Models as though-objects are usually referred to as 'mental models'. A mental model 'is a special kind of mental representation, in that the properties and behaviour of symbolic objects in the model simulate the properties and behaviour of the objects they represent rather than stating

facts about them' (Greeno, 1991, p. 177).

A model, however, sometimes evolves away from the context it arose from, to become 'an entity on its own. In this new shape it can function as a basis, a model *for* mathematical reasoning on a formal level.' (Gravemeijer, 1994, 100).

2.2.2 Symbolising

Symbols can be used to refer to verbal labels and to inscriptions, like visual symbols on a piece of paper or a blackboard. On the one hand, they describe and organise an aspect of reality, on the other hand they create a new reality in itself (Gravemeijer, 1998). For example, in $\frac{2}{5}$ of a chocolate bar, the '5' symbolises that the bar should be divided in five parts and the '2' symbolises that two of these part should be taken. 'Symbolising' is the process in which symbols are constructed; it is the process of finding appropriate representations (Linchevski, 1995).

Considering both symbols and models as representations indicates a reflexive relation between symbols and models, namely, models can develop into symbols and models can consist of a collection of symbols (Gravemeijer, 1998). For example the '5' in $\frac{2}{5}$ of a chocolate bar represents dividing in five. It, however, also constitute the divided bar as model for fractions.

2.2.3 Generalising

Van Hiele (1986) considers 'generalising of rules' a level of mathematical activity. It is the level above the level of rule-constitution. At the 'generalising of rules' level students become aware of rule applicability to a (broad) range of comparable situations. For example, the division underlying $\frac{2}{5}$ pizza can be generalised to a (broad) range of objects; $\frac{2}{5}$ is generalised to divide in five and take two. Broaden applicability of rules, (mental) constructions or ideas is what 'generalising' is. This implies that 'generalising' mainly concerns pattern recognition (Linchevski, 1995) and involves the search for more efficient memory-use (Krutetskii, 1976).

2.2.4 Formalising

Hart (1987) argues that 'formalising' can be considered as extension of 'generalising', in the sense that 'formalisation' 'means a rule, a formula or general method, which can be applied to a variety of mathematical examples' (p. 409). Formalising thus assumes the use of symbols to describe mathematical relations (Bergeron, Herscovics & Bergeron, 1987).

2.2.5 Abstracting

Bergeron, Herscovics & Bergeron (1987) regard as abstraction the moment the learner becomes aware of the invariance of the mathematical object; they are in a position to connect 'higher' and 'lower' layers in thinking (Kohnstamm, 1948, 53). Mason (1989) considers abstracting as a shift of attention from the general to become a object or property in itself. In his view abstraction is …

> 'an extremely brief moment which happens in the twinkling of an eye; a delicate shift of attention from seeing an expression *as* an expression of generality, to seeing the expression *as* an object or property. Thus, abstraction lies between the expression of generality and the manipulation of that expression while, for example, constructing a convincing argument.' (p. 2)

This idea of shift in attention links up with Van Hiele's (1986) notion that on the abstract-symbolic level the concept has become a junction in a new network of relations and with Piaget's (1973) idea of reorganisation of thought.

2.2.6 Promoting mathematisation in teaching and learning

In this theoretical background we will further elaborate the role of modelling, symbolising, generalising, formalising and abstracting in mathematics education. We will view these highly releted aspects of mathematising from different angles: mathematics as constructive activity, meaning and reality, mathematics as a language and constructing mathematics education.

2.3 Meaning and decontextualisation

'Vertical mathematisation' is the process of taking mathematical matter onto a higher plane from well chosen contexts, in which the abstract object is embedded (Treffers, 1987; Freudenthal, 1991; Nelissen, 1998; Gravemeijer & Terwel, 2000). As in vertical mathematisation mathematics loses its direct links to recognisable situations and becomes an entity on its own, it can become a major obstacle in learning mathematics (Terwel, 1994). Terwel therefore suggests to explicitly emphasise the troublesome decontextualisation in teaching. Hart (1981) shows that this is especially needed for the subject of fractions as here there is a tremendous distance between meaningful situational contexts and formal manipulations.

If mathematics learning ends up in solving problems while performing meaningless and counter-intuitive formal manipulations, it might easily lead to meaningless mathematical artefacts and students disliking mathematics (McNeill, 1988). In searching for a solution, McNeill reflects on her early learning of mathematics. She analyses what made her like mathematics in primary school.

> 'Looking back, it seems to me that the class approached each of these topics in a basically playful and co-operative manner. My recollection is of discovery, doubtless expertly guided by the teacher, not the usual format of teacher's exposition followed by practice by the students.' (McNeill, 1988, 47).

McNeill pleas for facilitating this kind of discovery by the students in every teaching of mathematics. Freudenthal (1991) recognised this problematic situation and suggested to teach mathematics as a process of reconstruction and guided reinvention.

2.4   Constructing formal mathematics

Today, the formation of knowledge is considered to be a constructive activity (Von Glaserfeld, 1987) or a reconstructive activity (Freudenthal, 1973; Treffers, 1987; Lawler, 1990). Sfard (1991) argues that the construction of new concept evolves from a pre-conceptual phase, via an operational approach to a structural phase, where the new concept is recognised as such. Wittmann (1981) sketches how inter-weaving intuitive and deductive activities are needed to construct formal and abstract concepts (cf. Corwin, 1989 and Greer, 1987). Or, following Dubinsky (1991) in considering educational consequences of the proposed construction process:

> '(…) in or out of the classroom, the main concern should be with the students' construction of schemas for understanding concepts. Instruction should be dedicated to inducing students to make these constructions and helping them along in the process.' (Dubinsky, 1991, 119)

Moss and Case (1999), from their work on fractions, argue that 'one of the important roles that instruction can play is to refine and extend the naturally occurring process whereby new schemas are first constructed out of old ones, then gradually differentiated and integrated' (p. 125) Von Glaserfeld explains how next this construction process is facilitated by a succession of reflective efforts. In addition, Tall and Vinner (1981) offer a means to elicit reflection, namely by inducing cognitive conflicts.

Pirie and Kieren (1994) offer a scheme which shows the embeddedness of different levels in mathematical activity. They consider assigning abstract meaning to symbols, where mathematics is no longer based on concrete images, as the essential link between meaningful situations and the invention of formal mathematics (cf. Hiebert, 1988). Arvaci (1994) also addresses the symbolising process. He argues the need for rich situational contexts to stimulate students to acquire an attitude towards efficient symbol-manipulation. Many other authors also consider real world phenomena or recognisable situations as the starting point of learning processes leading to formal and abstract mathematics (e.g. Freudenthal, 1973; Treffers, 1987; Gravemeijer, 1994). Trzcieniecka-Schneider (1993) notices that daily-life concepts and more formal concepts often are unconnected or disconnected. She argues that these formal concepts emerge from isolating those elements which are similar in more familiar concepts. She herein finds a basis for a prolonged relation between these concepts on several levels of abstraction, which by Connell and Peck (1993) is considered as an active search for abstraction:

> 'Understanding in elementary mathematics must involve the active search for, creation of, and use of links between the powerful abstractions and generalisations of mathematics and the world of personal experiences from which they derive their application and utility.' (Connell & Peck, 1993, 348)

Hanna (1989) provides a means to promote this active search. She sees teaching abstract mathematical concepts as a process of negotiation. Along with for example Leron (1985) she considers interaction between the teacher and the students and between students an adequate means to achieve abstraction and formalisation. However, Behr et al. (1983) and Hart (1981 & 1987), reflecting on their fraction-learning work, show that the condition of promoting an active search to obtain the creation of powerful abstractions is not easy to fulfil. Well chosen situations can improve the situation. However, Behr et al. researched the relation between pictures, written symbols, spoken symbols, real world situations and manipulative aids in connection to fraction learning and showed that more is needed to acquire formal fraction addition and subtraction (cf. Pirie, 1988). Steinbring (1989) experienced a similar problem with the learning of decimals. Sfard (1994) offers an explaining argument. She argues that abstract ideas frequently do not have a counterpart in the physical world, which in some sense is true for abstract operations with fractions and decimals. In addition, Greeno (1997) argues that by exploring reality and personal experiences the road to abstraction is still not fully opened. He proposes to consider abstraction as a symbolic or iconic representation property that can be interpreted with and without its referents. Griffin, Case and Sandieson (1992) state that precise this representational properties can act as a constraint, when certain procedures are acquired, as 'they must somehow be brought into place before these procedures can be assembled and employed with effectiveness and flexibility.' (p. 97).

Greeno offers a solution to this problem. He pleads offering students standard conventions to limit the unguided search to find these. This links up with Freudenthal's (1973) idea of *guided* reinvention, by proposing teaching abstraction to be centred around a set of well chosen examples, which finally lead the students to recognition of the general and abstract. These examples should be *generic examples*, that contain a prototype of the generalisable object and wherein the action of abstraction is still included (Harel & Tall, 1991).

In the experimental fraction programme (see appendix A) a bar is used as a measuring device (Amsterdam foot) (cf. Bokhove et al., 1996). Students fold their bar to get precise measurements and discuss why measuring the same object yields different results. In this way these measuring activities naturally leads to positioning (abstract) fractions on a number line. Therefore this context could be considered as a well chosen generic example, as it contains a prototype of the fraction as position on the number line. Moreover, as fractions on the number line arise from measuring activities the situation provides for a way back; the action of abstraction is included in the abstract fraction on the number line.

2.5 Alternative angles in considering mathematisation processes

Greeno and Freudenthal thus provide possible solutions for the sometimes problematic process of acquiring formal and abstract mathematics. We will broaden our framework to possibly obtain more clarifications for this situation.

2.5.1 Mathematical language in formal mathematics acquisition

Language is generally considered to be an important vehicle in the mathematics learning process. In this process students are required to formalise natural language (Schoenfeld, 1994; Forman & Fyfe, 1998). Schoenfeld considers ambiguities in the use of language the heart of mathematical language, which in its turn is closely connected to abstraction in mathematics (p. 72). Forman and Fyfe provide for a specification of this connection. They regard language to be closely connected to inventing symbol systems (cf. Featherstone, 2000). They state that it is 'the nature of the relation among symbols that converts the medium into a message that motivates children to negotiate shared meanings and to construct knowledge.' (p. 249). However, Rasmussen (2001) argues that the hereby needed conversion from information to understanding by way of utterance makes communication frequently lead to miscommunication.

Also Hersh (1993), in his work on mathematical proof, considers language an important instrumentality in negotiating meanings. If we consider proving as clarifying conjectures, Hersh states that elaboration of these clarifications should be such that students learn underlying concepts, methods and applications (p. 398). And these clarifications as a rule do not require formal mathematical language. Language processing in humans cannot be compared with formal language processing by machines. Even when formal mathematical concepts are discussed, natural language is usually most appropriate (Davis & Hersh, 1984). However, if the use of formal mathematical language is the objective, machines – as programmable instruments – can serve as a context to formalise communication (Ainley, 1996).

2.5.2 Permanence principle

Sfard (1994) takes professional mathematicians learning as a starting-point in analysing possible approaches in mathematics acquisition. This links up with Hankel's permanence principle as a means in learning formal mathematics. Hankel established this principle to express the preservation of formal algebraic laws, for instance when shifting from natural numbers to integers (Streefland, 1996, 60; Freudenthal, 1989). Streefland showed how a student found a solution for $3 - 8$ as minus 5. The child argued that $3 - 8$ is meaningful and generalised its meaning (and solution) from subtractions leading to an answer equal or larger than 0. The extension of formal algebraic rules from natural numbers to integers is now at hand. We therefore may

regard Hankel's principle as pattern recognition in formal mathematics, which makes the principle a possible teaching strategy in fostering (vertical) mathematisation processes.

Also Kambartel (1961) shows how, starting from Hankel's permanence principle, algebraic rules lead to extending number systems. Moreover, he argues that Hankel's principle is the driving force behind exploring these extensions. Next, this driving force makes it suitable to consider the permanence principle as a means in developing mathematics education, when it comes to exploring formal number relations. Semadeni (1984) elaborated this idea and introduced the 'concretization permanence principle'. He considers this to be the most appropriate expedient to extend known properties. In addition Hefendehl-Hebeker (1991) argues that this process of building new structures from those that are known is of a heuristic nature, wherein meanings are enlarged and not detached from their sources. In other words, Hankel's permanence principle fits in the framework of RME and provides for a means to explore formal mathematical relations.

2.6 Conclusion

Today there is a broad consensus that in considering learning and teaching processes one has to focus on the pedagogical content knowledge. RME can be considered a typical example as it takes processes of mathematisation as its starting point in considering learning and teaching mathematics. We argued in this chapter that specific strategies such as modelling, symbolising, generalising, abstracting and formalising are essential to mathematising as a more general strategy. As these aspects of mathematising are closely connected, there is a need to consider these mathematical activities simultaneously. We discussed several approaches to connect these specific strategies in mathematics teaching and learning. We recognised that using rich situational contexts (Arcavi, 1994, Streefland, 1990) and in particular using 'generic examples', that contain a prototype of the generalisable object and wherein the action of abstraction is still included (Harel & Tall, 1991), is necessary to prevent that abstract ideas are disconnected from the physical world.

We argued that seeing abstraction as a cyclic process with several forms of constructing schemas for understanding concepts, leaves room for different levels in strategies in one problem solution. This offers students the opportunity to construct their own mathematical knowledge; in discussion with others and with appropriate guidance to prevent them from exploring tracks that offer no perspective.

Acquiring abstract mathematical structures can be stimulated by utilising intuition and reflection (Fischbein, 1982; Freudenthal, 1953) and reflection, as strategy embedded in mathematising can be elicited by inducing cognitive conflicts. Thereby building new structures from those that are known is of a heuristic nature, where meanings are enlarged and not detached from their sources (Hefendehl-Hebeker,

1991). This idea is embedded in Hankel's permanence principle, where professional mathematicians' approach to abstract mathematical phenomena – in a sense – is taken as the starting-point in analysing mathematical abstraction. And thus teaching abstract mathematical concepts can be considered as a process of negotiation, where natural language is an important vehicle.

References

Ainley, J. (1996). Purposeful Contexts for Formal Notation in a Spreadsheet Environment. *Journal of Mathematical Behavior*, *15*(4), 405-422.

Anderson, J.R., J.G. Greeno, L.M. Reder & H.A. Simon (2000). Perspectives on Learning, Thinking, and Activity. *Educational Researcher*, *29*(4), 11–13.

Arcavi, A. & A.H. Schoenfeld (1992). Mathematics Tutoring Through a Constructivist Lens: The Challenges of Sense-Making. *Journal of Mathematical Behavior*, *11*, 321-335.

Arcavi, A. (1994). Symbol Sense: Informal Sense-making in Formal Mathematics. *For the Learning of Mathematics*, *14*(3), 24-35.

Armstrong, B.E. & C. Novillis Larson (1995). Students' use of part-whole and direct comparison strategies for comparing partitioned rectangles. *Journal for Research in Mathematics Education*, *26*(1), 2-19.

Behr, M.J., R. Lesh, T.R. Post & E.A. Silver (1983). Redactional-Number Concepts. In: Richard Lesh & Marsha Landau (eds.). *Acquisition of Mathematical Concepts and Processes*. New York/London: Academic Press, 91-126.

Beishuizen, M. (1997). Development of mathematical strategies and procedures up to 100. In: M. Beishuizen, K.P.E. Gravemeijer & E.C.D.M. van Lieshout (eds.). *The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures.* Utrecht: Freudenthal Institute, 127-162.

Bergeron, A., N. Herscovics & J.C. Bergeron (1987). Kindergartners' knowledge of numbers: A longitudinal study. Part II: Abstraction and formalization. In: J.C. Bergeron, N. Herscovics & C. Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education, Volume II*. Montreal: s.n., 352-360.

Bokhove, J., K. Buys (ed.), R. Keijzer, A. Lek, A. Noteboom & A. Treffers (1996). *De Breukenbode. Een leergang voor de basisschool* (werkbladen en handleiding) [The Fraction gazette]. Enschede/Utrecht: SLO/FI/Cito.

Brandt, B. (1998). Forms of participation in elementary mathematics classroom interaction. In: P. Abrantes, J. Porfírio & M. Baía (eds.). *The interactions in the mathematics classroom*. Proceedings of the CIEAEM 49. Setúbal, Portugal (July, 1997): s.n., 157-163.

Brown, J.S., A. Collins & P. Duguid (1989). Situated cognition and the culture of learning. *Educational Researcher*, *18*(1), 32-42.

Bruner, Jerome S. (1966). *Toward a Theory of Instruction*. Cambridge, MA: The Belknap Press of Harvard University Press.

Bruner, J.S. (1966). On Cognitive Growth II. In: Jerome S. Bruner, Rose R. Olver & Patricia M. Greenfield (eds.). *Studies in Cognitive Growth. A Collaboration at the Center for Cognitive Studies*. New York/London/Sydney: John Wiley & Sons, Inc., 30-67.

Buys, K. & A. Noteboom (1995). Breuken: begrip en taalontwikkeling [Fractions: comprehension and language development]. In: C. van den Boer & M. Dolk (eds.). *Rekenen in de bovenbouw van de basisschool.* Utrecht: Panama/HvU, Freudenthal Institute, 39-52.

Carraher, D.W. & A.D. Schliemann (1991). Children's understanding of fractions as expressions of relative magnitude. In: Fulvia Furinghetti (ed.). *Proceedings of the fifteenth PME*

conference. *Volume 1*. Assisi, Italy: Dipartimento di Matematica dell'Università di Genova, 184-191.

Carraher, D.W. (1993). Lines of thought: A ratio and operator model of rational number. *Educational Studies in Mathematics*, *25*, 281-305.

Cobb, P. & L.P. Steffe (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education*, *14*(2), 83-94.

Cobb, P., E. Yackel & T. Wood (1989). Young Children's Emotional Acts While Engaged in Mathematical Problem Solving. In: D.B. McLeod & V.M. Adams (eds.). *Affect and Mathematical problem solving: A new perspective.* New York: Springer Verlag, 117-148.

Cobb, P. & J.W. Whitenack (1996). A Method for Conducting Longitudinal Analyses of Classroom Videorecordings and Transcripts. *Educational Studies in Mathematics*, *30*, 213-228.

Connell, M.L. & D.M. Peck (1993). Report of a Conceptual Intervention in Elementary Mathematics. *Journal of Mathematical Behavior*, *12*, 329-350.

Corte, E. de, B. Greer & L. Verschaffel (1996). Mathematics Teaching and Learning. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology.* New York: Simon & Schuster Macmillan, 491-549.

Corwin, R. B. (1989). Multiplication as Original Sin. *Journal of Mathematical Behavior*, *8*, 223-225.

Davis, P.J. & R. Hersh (1984). *The Mathematical Experience*. London: Penguin Books.

Davis, R.B. (1992), Understanding 'Understanding'. *Journal of Mathematical Behavior*, *11*, 225-241.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In: D. Tall (ed.). *Advanced mathematical thinking.* Dordrecht/Boston/London: Kluwer Academic Publishers, 95-123.

Easley Jr., J.A. (1981). Levels of abstraction and intellectual development. *The Journal of Children's Mathematical Behavior*, *1*(2), 76-83.

Featherstone, H. (2000). '− Pat + Pat = 0': Intellectual Play in Elementary Mathematics. *For the Learning of Mathematics*, *20*(2), 14-23.

Fischbein, E. (1982). Intuition and Proof. *For the Learning of Mathematics, 3*(2), 9-18 and 24.

Forman, G. & B. Fyfe (1998). Negotiated Learning Through Design, Documentation, and Discourse. In: C. Edwards, L. Gandini & G. Forman (eds.). *The hundred languages of children The Reggio Emilia Approach − Advanced Reflections (second edition).* Greenwich, Connecticut/London, England: Ablex Publishing Corporation, 239-260.

Freudenthal, H. (1953). Zur Geschichte der vollständigen Induktion. *Archives Internationales d'histoire des sciences*, *22*, 17-37.

Freudenthal, H. (1968). Why to teach mathematics as to be useful? *Educational Studies in Mathematic*s, *1*(1), 3-8.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics*, *3*(3/4), 413-435.

Freudenthal, H. (1973). *Mathematics as an Educational Task.* Dordrecht: Reidel.

Freudenthal, H. (1980). *Weeding and Sowing: Preface to a Science of Mathematics Education*. Dordrecht: Reidel.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures.* Dordrecht: Reidel-Kluwer.

Freudenthal, H. (1989). Einführung der negativen Zahlen nach dem geometrisch-algebraischen Permanenzprinzip. *Mathematiklehren, Heft 35*, 26-37.

Freudenthal, H. (1991). *Revisiting Mathematics Education, China lectures*. Dordrecht:

Kluwer Academic Publishers.

Glasersfeld, E. von (1987). Learning as a Constructive Activity. In: C. Janvier (ed.). *Problems of representation in the teaching and learning of mathematics.* Hillsdale, New Jersey/ London: Lawrence Erlbaum Associates, Publishers, 3-17.

Goldin, G.A. (1998). Representational Systems, Learning, and Problem Solving in Mathematics. *Journal of Mathematical Behavior*, *17*(2), 137-165.

Graeber, A.O. & E. Tanenhaus (1993). Multiplication and Division: From Whole Numbers to Rational Numbers. In: Douglas T. Owens (ed.). *Research Ideas for the Classroom − Middle Grades Mathematics*. New York: Macmillan, 99-117.

Gravemeijer, K.P.E. (1994). *Developing realistic mathematics education* (dissertation). Utrecht: CD-ß Press.

Gravemeijer, K. (1998). Symboliseren en modelleren als wiskundige activiteit [Symbolising and modelling as mathematical activity]. In: N. Boswinkel & M. Dolk (eds.). *Over rekenen gesproken − taal in/en rekenen −*. Utrecht: Panama/Freudenthal Institute, 35-52.

Gravemeijer, K.P.E. & J. Terwel (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of Curriculum Studies*, *32*(6), 777-796.

Gravemeijer, K.P.E. (2001). *Reken-wiskundeonderwijs voor de 21^e eeuw* [Mathematics education for the 21$^{\text{th}}$ century]. Utrecht: Utrecht University.

Greeno, J.G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education*, *22*(3), 170-218.

Greeno, J.G., A.M. Collins & L. Resnick (1996). Cognition and Learning. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology.* New York: Simon & Schuster Macmillan, 15-46.

Greer, B. (1987). Didactical obstacles in the development of the concepts of multiplication and division. In: Jacques C. Bergeron, Nicolas Herscovics & Carolyn Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Volume II*. Montreal: s.n., 402-407.

Greer, B. & G. Harel (1998). The Role of Isomorphisms in Mathematical Cognition. *Journal of Mathematical Behavior*, *17*(1), 5-24.

Griffin, S., R. Case & R. Sandieson (1992). Synchrony and Asynchrony in the Acquisition of Children's Everyday Mathematical Knowledge. In: R. Case (ed.). *The Mind's Staircase. Exploring the Conceptual Underpinnings of Children's thought and Knowledge.* Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers, 75-97.

Griffin, S. & R. Case (1997). Re-thinking the primary school math curriculum: An approach based on cognitive science. *Issues in Education*, *3*(1), 1-49.

Hanna, G. (1989). More than Formal Proof. *For the Learning of Mathematics*, *9*(1), 20-23.

Harel, G. & D. Tall (1991). The General, the Abstract, and the Generic in Advanced Mathematics. *For the Learning of Mathematics*, *11*(1), 38-42.

Hart, K.M. (1981). Fractions. In: K.M. Hart (ed.). *Children's understanding of mathematics: 11-16*. London: John Murray, 66-81.

Hart, K.M. (1987). Practical work and formalisation, too great a gap. In: Jacques C. Bergeron, Nicolas Herscovics & Carolyn Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Volume II*. Montreal: s.n., 408-415.

Hasemann, K. (1981). On difficulties with fractions. *Educational Studies in Mathematics*, *12*, 71-87.

Haylock, D.W. (1987). A framework for assessing mathematical creativity in schoolchildren. *Educational Studies in Mathematics*, *18*, 59-74.

Hefendehl-Hebeker, L. (1991). Negative Numbers: Obstacles in their Evolution from Intuitive to Intellectual Constructs. *For the Learning of Mathematics*, *11*(1), 26-32.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics, 24*, 389-399.

Heuvel-Panhuizen, M. van den (1996). *Assessment and realistic mathematics education* (dissertation)*.* Utrecht: CD-ß Press.

Hiebert, J. (1988). A theory of developing competence with written mathematical symbols. *Educational Studies in Mathematics*, *19*, 333-355.

Hiebert, J., Th.P. Carpenter, E. Fennema, K. Fuson, P. Human, H. Murray, A. Olivier & D. Wearne (1996). Problem Solving as a Basis for Reform in Curriculum and Instruction: The Case of Mathematics. *Educational Researcher*, *25*(4), 12-21.

Hiele, P.M. van (1986). *Structure and Insight. A theory of Mathematics Education.* Orlando: Academic Press, Inc.

Hoek, D., J. Terwel & P. van den Eeden (1997). Effects of Training in the Use of Social and cognitive Strategies: An Intervention Study in Secondary Mathematics in Co-operative Groups. *Educational Research and Evaluation*, *3*(4), 364-389.

Hopmann, S. & K. Riquarts (1995). Starting a dialogue: issues in a beginning conversation between *Didaktik* and the curriculum traditions. *Journal of Curriculum Studies*, *27*(1), 3-12.

Huhse, K. (1968). *Studien und Berichte. Theorie und Praxis der Curriculum Entwicklung.* Berlin: Institut für Bildungsforschung in der Max-Planck-Gesellschaft.

Hunting, R.P. (1983). Alan: A case study of knowledge of units and performance with fractions. *Journal for Research in Mathematics Education*, *14*(3), 182-197.

Hunting, R.P. (1986). Rachels schemes for constructing fraction knowledge. *Educational Studies in Mathematics*, *17*, 49-66.

Kambartel, F. (1961). Logische Stellung und konstruktive Bedeutung mathematischer Permanenzprinzipien. *Der Mathematikunterricht*, (Logische Probleme im Mathematikunterricht II), *7*(1), 57-78.

Kamii, C. & F.B. Clark (1995). Equivalent Fractions: Their Difficulty and Educational Implications. *Journal of Mathematical Behavior*, *14*, 365-378.

Keijzer, R. & K. Buys (1996). Groter of kleiner. Een doorkijkje door een nieuwe leergang breuken [Bigger or smaller. A close look at a new curriculum on fractions]. *Willem Bartjens*, *15*(3), 10-17.

Klein, A.S. (1998). *Flexibilization of mental arithmetic strategies on a different knowledge base* (dissertation). Utrecht: CD-ß Press.

Klein, A.S., M. Beishuizen & A. Treffers (1998). The Empty Number Line in Dutch Second Grades: Realistic Versus Gradual Program Design. *Journal for Research in Mathematics Education*, *29*(4), 443-465.

Kohnstamm, P.H. (1948). *Keur uit het didactisch werk* [Selection from the didactic work]. Groningen: J.B. Wolters' Uitgeversmaatschappij.

Krutetskii, V.A. (1976 (American edition)). *The psychology of Mathematical Abilities in Schoolchildren*. (Translated by Joan Teller, edited by Jeremy Kilpatrick and Izaak Wirszup). Chicago/London: The University of Chicago Press.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer. *American Educational Research Journal*, *27*(1), 29-63.

Lawler, R.W. (1990). Constructing Knowledge From Interactions. *Journal of Mathematical Behavior*, *9*, 177-192.

Lek, A.T.E. (1992). *Met repen begrepen* [Understood with chocolate bars] (unpublished doc-

toral thesis). Utrecht: Utrecht University.

Leron, U. (1985). A direct approach to indirect proofs. *Educational Studies in Mathematics*, *16*, 321-325.

Linchevski, L. (1995). Algebra With Numbers and Arithmetic With Letters: A Definition of Pre-Algebra. *Journal of Mathematical Behavior*, *14*, 113-120.

Lyle, S. (2000). Narrative understanding: developing a theoretical context for understanding how children make meaning in classroom settings. *Journal of Curriculum Studies*, *32*(1), 45-63.

Mason, J. (1989). Mathematical Abstraction as the Result of a Delicate Shift of Attention. *For the Learning of Mathematics*, *9*(2), 2-8.

Mayer, R.E. (1996). Learners as information processors: Legacies and limitations of educational psychology's second metaphor. *Educational Psychologist*, *31*(3/4), 151-161.

Mayer, R.E. & M.C. Wittrock (1996). Problem-Solving Transfer. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology*. New York: Simon & Schuster Macmillan, 47-62.

Mayer, R.E. (1999). Designing Instruction for Constructivist Learning. In: Charles M. Reigeluth (ed.). *Instructional-design theories and models Volume II. A new paradigm of instructional theory*. Mahwah, NJ: Lawrence Erlbaum Associates, Publishers, 141-160.

Mayer, R.E. (2001). What Good is Educational Psychology? The Case of Cognition and Instruction. *Educational Psychologist*, *36*(2), 83-88.

McIntosh, A., B.J. Reys & R.E. Reys (1992). A Proposed Framework for Examining Basic Number Sense. *For the Learning of Mathematics*, *12*(3), 2-8.

McNeill, R. (1988). A Reflection on When I Loved Math and How I Stopped. *The Journal of Mathematical Behavior*, *7*(1), 45-50.

Moss, J. & R. Case (1999). Developing Children's Understanding of the Rational Numbers: A New Model and an Experimental Curriculum. *Journal for Research in Mathematics Education*, *30*(2), 122-147.

Movshovits-Hadar, N. (1988). School Mathematics Theorems − an Endless Source of Surprise. *For the Learning of Mathematics*, *8*(3), 34-40.

Naujok, N. (1998). How Helpful Is Help among Students? Theoretical Basis of and Methodological Considerations for an Interaction Theory of Learning in School. In: P. Abrantes, J. Porfirio & M. Baía (eds.). *The interactions in the mathematics classroom*. Proceedings of the CIEAEM 49. Setúbal, Portugal (July, 1997): s.n., 52-62.

Nelissen, J.M.C. (1998). Representaties in het reken-wiskundeonderwijs [Representations in mathematics education]. *Pedagogische Studiën*, *75*, 169-183.

Newell, A. & H.A. Simon (1972). *Human problem solving*. Englewood Cliffs, NJ: Prentice-Hall.

Noteboom, A. (1994). De Kop van Jut. Een breukenactiviteit in groep 6 [The try-your-strength machine. An activity on fractions in grade 6]. *Willem Bartjens*, *14*(2), 9-13.

Novillis Larson, C. (1980). Locating Proper Fractions On Number Lines: Effect of Length and Equivalence. *School Science and Mathematics, Volume LXXX*(5), 423-428.

Oers, B. van (2001). Educational forms of initiation in mathematical culture. *Educational Studies in Mathematics*, *46*, 59-85.

Perrenet, J.Ch. (1995). *Leren probleemoplossen in het wiskunde-onderwijs: samen of alleen. Onderzoek van wiskunde leren bij 12- tot 16-jarigen* [Learning problem solving in mathematics education: together or alone. Research on learning mathematics of 12- to 16-year olds] (dissertation). Amsterdam: Universiteit van Amsterdam/Deventer: Sneldruk.

Perkins, D.N. & Ch. Unger (1999). Teaching and Learning for Understanding. In: Charles M.

Reigeluth (ed.). *Instructional-design theories and models, Volume II. A new paradigm of instructional theory.* Mahwah, NJ: Lawrence Erlbaum Associates, Publishers, 91-114.

Piaget, J. (1973). Comments on mathematical education. In: A.G. Howson (ed.). *Developments in Mathematical Education. Proceedings of the Second International Congress on Mathematical Education.* Cambridge: Cambridge University Press, 79-87.

Pinar, W.M., W.M. Reynolds, P. Slattery & P.M. Taubman (1995). *Understanding Curriculum.* New York: Peter Lang.

Pirie, S.E.B. (1988). Understanding: Instrumental, Relational, Intuitive, Constructed, Formalized...? How Can We Know? *For the Learning of Mathematics*, *8*(3), 2-6.

Pirie, S.E.B. & T.E. Kieren (1994). Beyond Metaphor: Formalising in Mathematical Understanding within Constructivist Environments. *For the Learning of Mathematics*, *14*(1), 39-43.

Rasmussen, J. (2001). The importance of communication in teaching: a system-theory approach to the scaffolding metaphor. *Journal of Curriculum Studies*, *33*(5), 569-582.

Reeuwijk, M. van (1991). Benoemde breuken: herinneringen van vroeger [Named fractions: memories of the past]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskunde-onderwijs*, *10*(1), 39-41.

Reigeluth, Ch.M. (1999). What Is Instructional-Design Theory and How Is It Changing. In: Charles M. Reigeluth (ed.). *Instructional-design theories and models, Volume II. A new paradigm of instructional theory.* Mahwah, NJ: Lawrence Erlbaum Associates, Publishers, 5-30.

Romberg, Th.A. (1994). Classroom Instruction That Fosters Mathematical Thinking and Problem Solving: Connections Between Theory and Practice. In: Alan H. Schoenfeld (ed.). *Mathematical thinking and problem solving*. Hilldale, NJ: Lawrence Erlbaum Associates, Publishers, 287-304.

Schoenfeld, A.H. (1992). Learning to think mathematically: problem solving, metacognition, and sense making in mathematics. In: Douglas A. Grouws (ed.). *Handbook of research on mathematics teaching and learning*. New York: Macmilian Publishing Company, 334-370.

Schoenfeld, A.H. (1994). Reflections on Doing and Teaching Mathematics. In: Alan H. Schoenfeld (ed.). *Mathematical thinking and problem solving*. Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers, 53-70.

Semadeni, Z. (1984). A principle of concretization permanence for the formation of arithmetical concepts. *Educational Studies in Mathematics*, *15*, 379-395.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different side of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics*, *14*(1), 44-56.

Shulman, L.S. (1995). Wisdom for Practice and Wisdom of Practice: Two Aspects of a Didactics Substance. In: Stefan Hopmann & Kurt Riquarts (eds.). *Didaktik and/or Curriculum.* Kiel: Institut für die Pädagogik der Naturwissenschaften, 201-204.

Shulman, L.S. & K.M. Quinlan (1996). The Comparative Psychology of School Subjects. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology.* New York: Simon & Schuster Macmillan, 399-423.

Sierpinska, A. (1995). Mathematics: "in Context", "Pure", or "with Applications"? A contribution to the question of transfer in the learning of mathematics. *For the Learning of Mathematics*, *15*(1), 2-15.

Simon, M.A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114-145.

Steinbring, H. (1989). Routine and Meaning in the Mathematics Classroom. *For the Learning of Mathematics*, *9*(1), 24-33.

Streefland, L. (1982). Subtracting fractions with different denominators. *Educational Studies in Mathematics*, *13*, 233-255.

Streefland, L. (1987). Free production of fraction monographs. In: Jacques C. Bergeron, Nicolas Herscovics & Carolyn Kieran (Eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Volume I*. Montreal: s.n., 405-410.

Streefland, L. (1991). *Fractions in Realistic Mathematics Education, a Paradigm of Developmental Research*. Dordrecht: Kluwer Academic Publishers.

Streefland, L. (1996). Negative Numbers: Reflections of a Learning Researcher. *Journal of Mathematical Behavior*, *15*, 57-77.

Streefland, L. (1997). Een geval van reflectief denken in ontwikkeling met verhoudingen als paradigma (slot) [A case of developing reflective thinking with ratio as paradigm]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs, 15*(3), 22-31.

Terwel, J. (1994). *Samen onderwijs maken. Over het ontwerpen van adaptief onderwijs* [Making education together. On developing adaptive education]. Groningen: Wolters-Noordhoff.

Terwel, J. (1999). Constructivism and its implications for curriculum theory and practice. *Journal of Curriculum Studies*, *31*(2), 195-199.

Terwel, J., M. Volman, W. Wardekker & U. Hameyer (2000). *Substantive trends in curriculum design and implementation. An analysis of innovations in the Netherlands*. Paper presented at the Annual meeting of the American Educational Research Association, New Orleans, April 22-29, 2000.

Treffers, A. (1987). *Three dimensions. A model of Goal and Theory Description in Mathematics Instruction – the Wiskobas Project*. Dordrecht: Reidel Publishing Company.

Treffers, A., E. de Moor & E. Feijs (1989). *Proeve van een nationaal programma voor het reken-wiskundeonderwijs op de basisschool. Deel 1: Overzicht einddoelen* [Specimen of a National Program for Primary Mathematics teaching. Part 1: Survey on Educational Aims]. Tilburg: Zwijsen.

Treffers, A. & E. de Moor (1990). *Proeve van een nationaal programma voor het reken-wiskundeonderwijs op de basisschool. Deel 2: Basisvaardigheden en cijferen* [Specimen of a National Program for Primary Mathematics teaching. Part 2: Basic Mental Strategies and Written Computation]. Tilburg: Zwijsen.

Trzcieniecka-Schneider, I. (1993). Some remarks on creating mathematical concepts. *Educational Studies in Mathematics*, *24*, 257-264.

Walker, D. (1990). *Fundamentals of Curriculum*. San Diego: Harcourt, Brace, Jovanovich.

Westbury, I. (1995). Didaktik and Curriculum Theory: Are They the Two Sides of the Same Coin. In: Stefan Hopmann & Kurt Riquarts (eds.). *Didaktik and/or Curriculum*. Kiel: Institut für die Pädagogik der Naturwissenschaften, 233-263.

Wittmann, E. (1981). The complementary roles of intuitive and reflective thinking in mathematics teaching. *Educational Studies in Mathematics*, *12*, 389-397.

Wood, T. (1998). Creating classroom interactions for mathematical reasoning: beyond "natural teaching". In: P. Abrantes, J. Porfírio & M. Baía (eds.). *The interactions in the mathematics classroom*. Proceedings of the CIEAEM 49. Setúbal, Portugal (July, 1997): s.n., 34-43.

Yackel, E. & P. Cobb (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458-477.

Yackel, E. (2001). Explanation, justification and argumentation in mathematics classrooms. In: Marja van den Heuvel-Panhuizen (ed.). *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education*, *Volume 1*. Utrecht: Freudenthal Institute, 9-24.

3 Learning for mathematical insight: a longitudinal comparative study on modelling[1]

Abstract

This chapter reports on a longitudinal study of teaching and learning the subject of fractions in two matched groups of ten 9- to 10-year-old students. In the experimental group fractions are introduced using the bar and the number line as (mental) models, in the control group the subject is introduced by fair sharing and the circle-model. In the experimental group students are invited to discuss, in the control group students work individually. The groups are compared on several occasions during one year. After one year, the experimental students show more proficiency in fractions than those in the control group.

3.1   Introduction

Mathematical insight is widely recognised as an important goal of education (Van Hiele, 1986; Perkins & Unger, 1999; Reigeluth, 1999). However, it is known from many studies that students have difficulty in applying this insight in meaningful ways in mathematical modelling and formal mathematics. Many educators and researchers confirm the problems which students encounter in learning fractions (Behr, Lesh, Post & Silver,1983; Carraher & Schliemann,1991; Hasemann, 1981; Kamii & Clark, 1995; Graeber & Tanenhaus, 1993), especially when fractions and fraction operations are not firmly connected to concrete experiences (Hart, 1981; Hiebert, 1988) or significant situations (Streefland, 1982, 1987 & 1991). Behr et al. (1983) attempt to seek the cause of students' difficulties in learning fractions in the necessary transition from concrete experiences to formal reasoning, and in the representation model of fractions, as well as in the many subsidiary concepts needed to obtain fraction proficiency.
In exploring the question of how to facilitate the transition process from concrete experiences via modelling fractions to formal reasoning and understanding several fraction-generating activities could be mentioned. However, we only will focus on two of these activities: fair sharing, as proposed by Streefland, and fraction learning in the context of number sense acquisition (Greeno, 1991; Mcintosh, Reys & Reys, 1992).

Although the two approaches both intend to pave the way to understanding and for-

malisation, there is little empirical evidence concerning the strengths and weaknesses of the two different strategies. The aim of this study is to explore the feasibility and effectiveness of a learning strategy for understanding through the use of the number line as a tool in the process of formalisation in a classroom environment, where meaning is negotiated in whole-class discussions (Greeno, 1991). Against this background, the following research question is central: How do student learning-processes develop in an experimental curriculum in which the number line is used as a model for fractions and meanings are established by negotiation, and what are the learning outcomes of the experimental program as compared to a control group, that learns fractions in a more traditional program, where the circle is the central model and where learning, to a large extent, is a solitary activity?

Both curricula discussed in this chapter, incorporating a program on fractions and teaching thereof, aim at the development of formal reasoning with fractions. By contrast, national curriculum standards in the Netherlands emphasise that this type of formal reasoning should not be an objective for all primary school students. However, we will show, many students in primary school grades 7 and 8 (10-12 year) seem to be capable, at a suitable level, of understanding the formal mathematics involved in such formal reasoning.

We consequently formulated the following two research hypotheses:

> ### Hypothesis 1: General mathematical proficiency hypothesis
> *Learning-processes in the experimental group progress such that a greater growth in general mathematical skills can be demonstrated, when this growth is compared to that of students in the control group, especially for those students who learn mathematics relatively easily.*

> ### Hypothesis 2: Proficiency in fractions hypothesis
> *Learning-processes in the experimental group progress in such a way that a greater growth in 'fraction numeracy' can be demonstrated, when this growth is compared to that of students in the control group, especially for those students who learn mathematics relatively easily.*

3.2   Theoretical background

Mental models such as the number line or the circle are important tools for mathematical problem solving and insight. However, according to Perkins & Unger (1999) the possession of a model, for example that of the number line, is not sufficient. In order to make good use of models also the ability to apply such models in various problem situations is required. Thus at least two important intrinsically linked questions are generated: (1) which model is most suitable for the representation of mathematical operations, and (2) how can mathematics education provide guidance in the

best use of these models? Mayer's studies (1989) have shown that conceptual models provided by curriculum developers and teachers may serve quite well in providing creative solutions to transfer problems.

These kinds of general questions also appear in the context of mathematics education, in particular in the teaching and learning of fractions. In past decades formal arithmetic with fractions in primary schools generally resulted in the great majority of students having to follow meaningless rules of calculation. Hart (1981, 1987) and others showed a considerable gap between practical experiences and formal calculations with fractions. As a consequence she recommends shifting the subject of formal reasoning with fractions from primary school to secondary education.

In the late 80s Streefland (1991) developed a new curriculum on fractions. With Bezuk and Bieck (1993) he emphasised the necessity of confronting students with meaningful situations in order to force them to generating their own fractions and language for fractions. In his developmental research on fractions (Freudenthal, 1991; Gravemeijer, 1994) he showed how thought-experiments followed by class-experiments leaded to a curriculum, where fair-sharing (e.g. 'divide three pizza's among four children') is the main fraction-generating activity.

In his research Streefland showed the importance of considering all curriculum-related elements as a whole in theorising about the development of a new curriculum; especially curriculum teaching and student models and activities are intrinsically connected in the described developmental process (cf. Greeno, 1991; Anderson, Greeno, Reder & Herbert, 2000). One of Streefland's main objectives was to stimulate students to develop a so called 'fraction language'. He therefore combined activities of fair sharing with classroom discussions, thus binding natural language related to the sharing process (e.g. 'each will get a quarter and a half') to formal fraction language ('each will get $\frac{3}{4}$').

Steefland's fraction curriculum addresses the issue of equivalent fractions, by considering equivalent sharing situations; the result of sharing three pizza's with four equals that of sharing six pizza's with eight. Streefland reports on the limitations of this approach to formal reasoning with fractions. The search for equivalent sharing situations masques fractions as numbers 'between whole numbers' and therefore limits global reasoning with fractions, which is considered essential in developing number sense (Greeno, 1991; Mcintosh, Reys & Reys, 1992). It, moreover, leaves little room for student approaches, other than multiplying numerator and denominator.

National Curriculum standards for primary education in the Netherlands indicate that learning fractions in primary school should aim at developing number sense (Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education], 1994). There is some evidence that using a bar as a model and a number line as abstraction of the bar can be

profitably incorporated into a curriculum that aims at number sense (Keijzer, 1997). Furthermore, Freudenthal (1973) considered the number line as 'the most valuable tool which modern ways of teaching arithmetic have borrowed from modern mathematics (…) it can be an excellent means of visualising the four main arithmetical operations' (1973, 211). However, the number line as a model for fractions hardly fits in with the approach Streefland suggested. In the experimental curriculum both the acquisition of fraction language and formal fractions will be addressed, in combining the bar and the number line as models with negotiation of meaning in whole class discussions.

3.3 The curricula

3.3.1 Realistic mathematics education

In the experimental setting the curriculum is an adaptation of the 'Fraction gazette' [De Breukenbode] (Bokhove, Buys, Keijzer, Lek, Noteboom & Treffers, 1996). The control setting uses the textbook series from the school where our experiments were conducted, entitled 'The world in numbers' [De wereld in getallen] (Huitema et al., n.d.). Both curricula have been developed in the Dutch tradition of realistic mathematics education (Treffers, 1987; Freudenthal, 1991; Gravemeijer, 1994; Van den Heuvel-Panhuizen, 1996). Consequently, both in the experimental curriculum and in the control group curriculum the focus of attention is on learning in context, in order to generate schemes and models. These, in turn, are used to support the application of mathematical knowledge in other situations, as well as in the development of formal mathematics. Moreover, in realistic mathematics education – and thus in both curricula – mathematics is considered to be a human activity, which is learned in interaction with others and is taught in such a way that various subjects are interwoven, to prevent the development of disconnected, inapplicable knowledge.

3.3.2 The experimental curriculum

Measuring to develop 'fraction language'

As in the case of the curriculum presented by Streefland, the experimental curriculum focuses on the meaning of fractions and the development of a language of fractions. However, Streefland mainly used situations of fair sharing to generate fractions as well as the language of fractions, while in the curriculum developed by Buys et al. (1996) that role is taken by measurement contexts (see also appendix A).
In one of the first lessons the students are given a bar, called the 'Amsterdam foot' (abbreviated av). They are subsequently invited to measure objects in the classroom with this new measuring instrument. The 'Amsterdam foot', however, is not suitable for making precise measurements. So the students are invited to fold the bar before

taking new measurements, resulting in various informal and semi-formal notations for the fractional parts; the students are free to develop there own fraction language. However, since the results needs to be communicated frequently, some consensus on the language is required. This finally results in a standard notation for fractions and a first investigation of fraction relations. And, by the use of a bar in this context, an extension of fractions to positions on a number line is at hand.

Comparing fractions

At the next curriculum stage the fraction language exploration provides a good reason for considering relations between fractions, such as '$\frac{3}{5}$ is greater than $\frac{1}{2}$'. As the number line is developed as a tool for making comparisons, students are free to decide on comparison strategies of their own. However, the fractions to be compared are selected in such a way that informal strategies are somewhat discouraged after some time, while other (more formal) strategies are encouraged (Armstrong & Novillis Larson, 1995; Keijzer & Buys, 1996).

One of the activities at this stage of the curriculum is working with the computer game 'treasure-digging'.[2] Here the students are first offered a fraction and subsequently invited to look for this fraction by clicking on the number line. Every attempt leads to the fraction being shown at the assigned position (figure 3.1). In this way the students are offered 'anchor points' facilitating the remainder of the searching process.

figure 3.1: treasure digging.
The first fraction found is $\frac{5}{9}$. This is greater than $\frac{2}{5}$, since it is greater than $\frac{1}{2}$.
Subsequently the smaller fractions $\frac{1}{3}$ and $\frac{3}{8}$ are tried. Finally, $\frac{2}{5}$ is found.

Equivalent fractions as a field of research for students

At a certain point, strategies for the comparison of fractions lead to the consideration of equivalent fractions. This forms the main topic in the third and final stage of the curriculum. Here, equivalence of fractions becomes a field of research for the students.
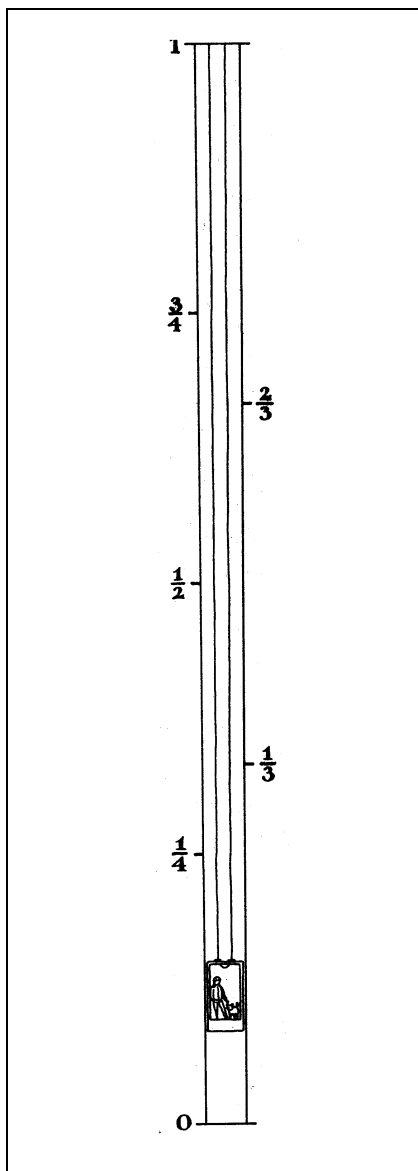

figure 3.2: the Fraction-lift (illustration adapted from the 'Fraction gazette')

By introducing the 'Fraction-lift' (figure 3.2) a start is made with the discussion of formal relations between fractions.[3] The 'Fraction-lift' transports fractions within a building of fractions; numbered lifts take fractions to the place where they live. But since fractions live on many different floors, many lifts are necessary. In this particular context the number of the lift corresponds to the number of stops that the lift makes. The 2-lift stops twice (at $\frac{1}{2}$ and 1), the 3-lift stops three times (at $\frac{1}{3}$, $\frac{2}{3}$ and 1), the 4-lift stops four times (at $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$ and 1), etc. This means, for example, that the 2-lift can transport $\frac{1}{2}$ to the appropriate position and the 3-lift can move $\frac{1}{3}$ and $\frac{2}{3}$ to their floors.

We ask students to look for lifts stopping at a specific floor, for instance at $\frac{6}{7}$. Initially, they find the 14-lift and 28-lift by doubling the number 7. Finally, one of the students discovers that every number in the 7-times table can be used. Equivalent fractions are now at hand.

3.3.3 The control group curriculum

Situational contexts to develop 'fraction language'

Although all curriculum subjects are connected in 'The world in numbers', the part of the curriculum that is aimed at fractions can be separated relatively easily. Thus isolated, the fractions curriculum can be considered in two stages. In the first stage we see part-whole situations, situations of fair sharing and situations in which the fraction operates on a quantity. These situations are used to learn and explore the language of fractions. At first simple unit-fractions are considered. The students are meant to make divisions in an intuitive manner. The different solutions that emerge form a basis for discussion of the various descriptions in fractions.

Exploring equivalent fractions

In the next curriculum stage students explore the equivalence of fractions. The situations used here include using (double indexed) bars to compare fractions. Thus students are challenged to find a proper quantity to allow for two or more fractions to operate upon. This enables them, among other things, to compare fractions. Moreover, fair sharing is used to explore formal relations between fractions.

When we observed students working on these problems in the control group, we never noticed any real interaction between students and between student and teacher. The teacher helped the students individually, and in doing so touched on formal relations, but never allowed them to reflect more deeply on these relations. However, many researchers have emphasised the importance of reflecting on approaches and solutions to reach higher-level knowledge (Wittmann, 1981; Van Hiele, 1986; Nesher, 1986; Von Glasersfeld, 1987; Treffers, 1987; Lawler, 1990; Dubinsky, 1991; Freudenthal, 1991; Herfs, Mertens, Perrenet & Terwel, 1991; Terwel, Herfs, Mertens & Perrenet, 1994; Terwel, 1994; Forman & Fyfe, 1998).

Other activities

In the control group two curricula on fractions were used side by side. Next to 'The world in numbers' curriculum, individual lessons with divided squares and circles were used to learn fraction language in part-whole situations, and to explore equivalence of fractions. This generally resulted in low-level solutions, as far as equivalent fractions were (or should be) concerned, since students usually got stuck in inching and pinching with concrete materials.

3.3.4   A comparison

In both curricula considered teaching and student's activities are intrinsically connected. However, we will view these two connected features of the curricula seperately. When we compare the teaching of fractions in the experimental curriculum with the control group curriculum, considerable differences emerge. These differences concern both the design and the contents of the curricula. As regards the design of the curricula, we found that the control group students in general work individually. Interaction takes place almost exclusively with the teacher, hardly ever with fellow students. The educational setting in the experimental group, on the other hand, may be characterised as negotiating meaning. Moreover, the tasks are designed in such a way that students are encouraged to look for meanings and (formal) relations between fractions. In the control group there is little explicit attention to level raising (cf. Dekker, 1991).

Comparing the content of the two curricula, we observe the central role of the number line in the experimental curriculum, whereas in the control group curriculum (divided) bars and circles are means to gain access to fractions. We also see that the number line is used in the experimental group to explore strategies for comparing fractions. Moreover, these strategies converge on the use of equivalent fractions for those students who are capable of understanding the formal mathematics involved, and in situations in which more informal approaches are inappropriate. By contrast, in the control group operator situations and manipulations with pre-divided circles are explored for the investigation of equivalent fractions.

3.4   Methods

A newly developed curriculum on fractions is considered and discussed here in a quasi-experimental setting (Cook & Campbell, 1979). An experimental group followed the newly developed curriculum and a control group followed a curriculum from the textbook series 'The world in numbers' (Huitema, et al., n.d.). Both groups and their teachers were followed for the duration of one school year.

Under these conditions this research may be characterised as a non-equivalent pretest post-test control group design (Cook & Campbell, 1979).

We schematised the research in table 3.1, where the first row shows the development of the experimental group, the second row that of the control group. In this setting O_1, O_2 and O_3 signify standardised tests used to examine the general mathematical skills of the students. As these tests are interrelated by scaling-techniques, the growth in mathematical skills both in the experimental group and the control group can be quantified over the year the experiment was conducted (Janssen, Kraemer & Noteboom, 1995).

O_1	$X\mathrm{e}_1$	I_1	$X\mathrm{e}_2$	O_2	$X\mathrm{e}_3$	I_2	$X\mathrm{e}_4$	O_3	I_3
O_1	$X\mathrm{c}_1$	I_1	$X\mathrm{c}_2$	O_2	$X\mathrm{c}_3$	I_2	$X\mathrm{c}_4$	O_3	I_3

table 3.1: specification of the research design

After the pre-test (O_1) a matching-process resulted in equivalent experimental and control groups. The experimental group followed the experimental program, Xe, while the control group followed the original program on fractions in the school, Xc.

Standardised individual interviews, inspired by the national curriculum standards on fractions, provide a more qualitative measure of the growth in what was earlier referred to as 'fraction numeracy' (cf. Paulos, 1988; Mcintosh, Reys & Reys, 1992; Bokhove, Buys (ed.), Keijzer, Lek, Noteboom & Treffers, 1996). During the experiment every student in the control group and the experimental group was interviewed three times (I_1, I_2 and I_3).

3.5 Data and analysis

3.5.1 Matching students

The experimental and control groups are constructed in such a way that they are both composed of grade 6 students (9 to 10 years) from two parallel groups, taken from a school near Amsterdam. Since the two groups involved in the experiments are mixed groups, with students from grades 6, 7 and 8 in both groups, there are only twelve students from grade 6 in each group. Only ten of these students participated in the pre-test. The matching procedure now consisted of the following steps:

– a pre-test to establish general mathematical skills of the students;
– making an initial one-to-one matching of students;
– interviews with the teachers of the students to retrieve general characteristics of the students (background, conduct, skills in mathematics and command of language);
– utilising this information to match the students from both groups one-to-one;
– random assignment of one of the groups as experimental and the other as control group.

Analysis of the test-results facilitated the initial matching, while the interviews with the teachers resulted in additional arguments for the matching.

3.5.2 General mathematical skills

During this one-year period the students were pre-tested, and two subsequent post-tests on general mathematical skills were administered. In addition, a series of three interviews on curriculum-specific topics were held with every student. The general mathematics tests consisted of a sub-test on 'numbers and operations' and on 'measuring and geometry'. We compared the scores, using a *paired-samples t-test.* This test was appropriate, as the observations for each matched pair were made under the same conditions.[4] We took the difference in skill as a measure of the difference in development of the matched students and also took the group-scores of the matched couples into account. In the process we did not find any significant differences between the experimental group and the control group. However, after restricting the paired t-test to those students who perform normally in mathematics, we did find significant differences ($p = .020$ resp. $.024$). From these observed differential effects, we decided to compare the two groups in this study, using a regression analyses.
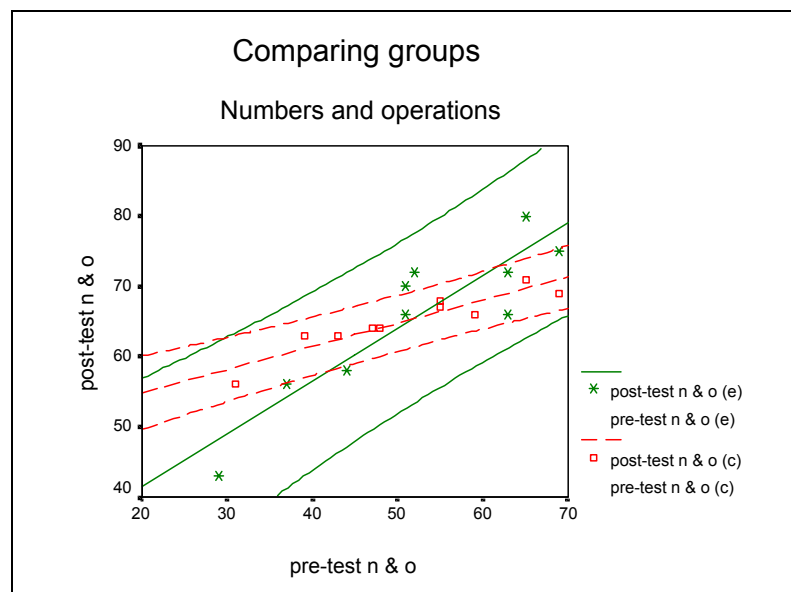
figure 3.3: development in skills in 'numbers and operations' for experimental group ($n = 10$) and control group ($n = 10$). The fit-method used is linear regression, with prediction intervals for single observations (confidence level = 95 %) for both control group (c) and experimental group (e). Regression equations: Experimental condition: post-test = .751 pre-test + 26.431; Control condition: post-test = .329 pre-test + 48.288

In figure 3.3 the individual results of the pre-test on 'numbers and operations' are plotted against those of the post-test in the same subject. Moreover for these tests we calculated the regression-lines and the prediction intervals for single observations (confidence level = 95 %).

The chart in this figure suggests a number of differential effects in the development of general mathematical skills for the experimental group as compared to the control group. The experimental-group students who perform better in the pre-test outperform their matched counterparts in the control group. In addition, the graphs suggest that the level of achievement of students who perform less well in the experimental group is slightly below that of the control group. The mean scores of the results of students with normal, respectively weak mathematical skills exhibit a similar tendency (weak students: mean scores 56.75 (experimental group) and 61.50 (control group); normally performing students: mean scores 71.83 (experimental group) and 67.50 (control group).

We hypothesised that the learning-processes of the students in the experimental group would progress in such a way that a greater increase in general mathematical skills would be demonstrated in relation to that of the control group students. In considering the whole groups, we found that, although the results are somewhat better for the students in the experimental group, they showed no significant difference between the (development of the) general mathematical skills of the experimental group compared to those of the control group.

We also hypothesised that the higher achievers in mathematics would perform better on general mathematical tasks. In comparing the pre-test and post-test scores by means of a regression analysis for the two conditions we found a tendency towards differential effects. Although no significant results could be established, this analysis of the data suggests that those students in the experimental group who acquire mathematics relatively easily, achieved better results on the post-tests, compared to their matched control group counterparts. In other words, we observed what is referred to as the 'Matthew-effect' for the experimental condition (Hoek, Terwel & Van den Eeden, 1997; Hoek, 1998; Hoek, Van den Eeden & Terwel, 1999). Thus we observed some indications that partially confirm our hypotheses on general mathematical skills.

3.5.3   Proficiency in fractions

Quantitative data

As the three general mathematical ability test were not appropriate to show the progress in the students' proficiency in fractions, three interviews were developed to reveal the students' skills in fractions. The problems in the first interview focused on

the knowledge of the 'language of fractions'. The subject in the second interview was comparison of fractions (see appendix B). By presenting open-ended problems we enabled students to show their mathematical attitude, for instance by manifesting flexibility in using their knowledge of fractions. In addition, these open-ended problems enabled students to come forward with solutions that suited their level and preferences. The third interview, aimed at applying fraction knowledge. Here students first solved a number of problems in a written test (Bokhove, Van der Schoot & Eggen, 1996), which formed the basis for the interview. The interviewees were subsequently asked to explain their approaches and here, too, were offered standardised help, if they did not provide a correct solution within a reasonable time.
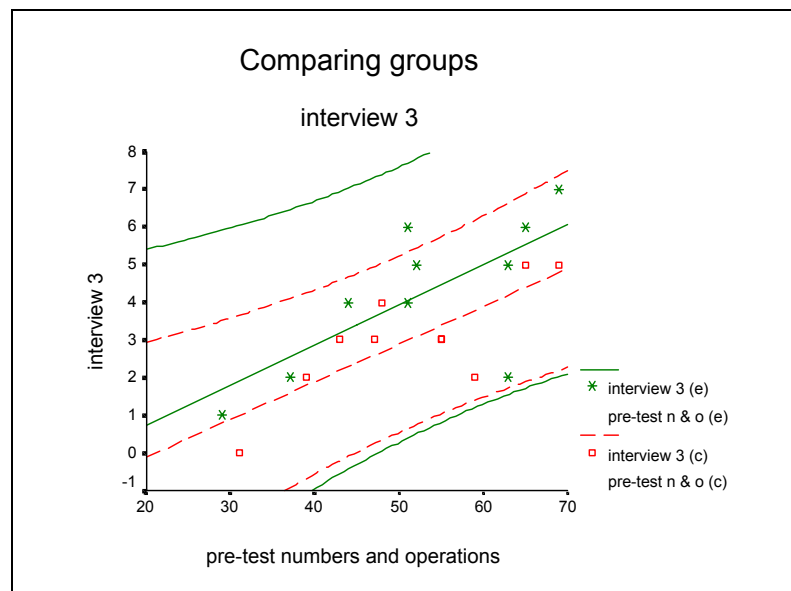
figure 3.4: results of the third interview in experimental group ($n = 10$) and control group ($n = 10$). The number of correct answers in the interview (with help) is plotted against the results of the pre-test in 'numbers and operations'. The fit-method used is linear regression, with prediction intervals for single observations (confidence level = 95 %) for both control group (c) and experimental group (e). Regression equations: Experimental condition: interview 3 = .107 pre-test – 1.398; Control condition: interview 3 = .100 pre-test – 2.115

This research setting offered us the opportunity to compare developments for the matched pairs. Moreover, we were able to show how both curricula helped the students in solving the problems, as well the extent to which they were prepared to use the help that was offered. In general, the help was constructed so as to aid the students in utilising and enhancing their flexibility in understanding fractions. As a consequence the students were encouraged in the interviews to revise their approaches

and to relate the situations to more familiar ones; the general effect being that mathematics comes across as a subject that can be discussed. In particular, the students' responses to the help offered may be regarded as indicators of fraction numeracy. Moreover, the help was intended to explore the Zone of Proximal Development (ZPD) (cf. Hedegaard, 1990; Van Oers, 1996; Tharp & Gallimore, 1998).

The results in the third interview may be regarded as an indicator of the differences in relevant skills between students in the two groups. The chart in figure 3.4 suggests how the students in the experimental group outperform their fellow students in the control group: the number of correct solutions in their interview on average exceeds that of their matched control group counterparts by one. The graphs consequently provide a first indication that the skills of the students in applying fractions in non-typical situations differ for the two groups.

In comparing the data from the interviews by using a *paired sample t-test* we also notice that in all but one case the students in the experimental group perform significantly better than their counterparts in the control group. In particular, we see that students in the experimental group in all cases achieve significantly better (sig. level 0.05 or better) when they receive standardised help.

The conclusion is warranted that the problems in the interviews enable the students to show their fraction numeracy, since they do not aim at standard procedures to solve problems. Moreover, showing number sense (in general) includes reflecting upon given solutions (Greeno, 1991). Since the students in general are invited to (re) consider their approach, we could consider the results from the interviews as a means of operationalising their fraction numeracy. Students in the experimental group showed to have a greater increase in fraction numeracy, which confirms our second hypothesis.

Qualitative analyses of students reactions

Since fraction numeracy is of a qualitative nature, we observed all students extensively during lessons and the three interviews. We observed how students in the experimental group, from the start, show a more developed fraction language than their control group counterparts. Moreover, experimental group students with good or normal mathematical skills exhibited more flexible ways of handling formal and informal relations between fractions than the matched control group students.

figure 3.5: problem in first interview: 'Irene has eaten $\frac{3}{5}$ of her chocolate bar.
Here you see how much is left.'

Within the scope of this chapter we present only one of the many observed reactions by a student in the experimental group compared to her control group counterpart. Roxanne (experimental group) and Gina (control group)[5] perform at an average level (compared to the national reference). In her first interview the interviewer asks Gina if Irene has eaten more than half a chocolate bar (figure 3.5). Her answer is affirmative.

> G.: 'Because she ate five thirds… Oh no, she ate three fifths.'
> I.: 'Why are three fifths more than a half?'
> G.: 'Because two fifths make a half.'
> G. (unsure): 'Oh, I don't know…'

When asked what fraction goes with the remaining piece of the chocolate bar, Gina replies: 'One fifth.' When next the interviewer asks her to complete the drawing of the bar, Gina does so by sketching a segment to the right of the piece of the bar on the work sheet. This drawn part of the bar is about twice the length of the piece already there. Gina divides the segment in two and adds another part. When asked to check her answer she concludes that she made $\frac{3}{4}$. She has no idea how she could make three-fifths. Gina decides to put the figure $\frac{3}{4}$ under her work (figure 3.6).
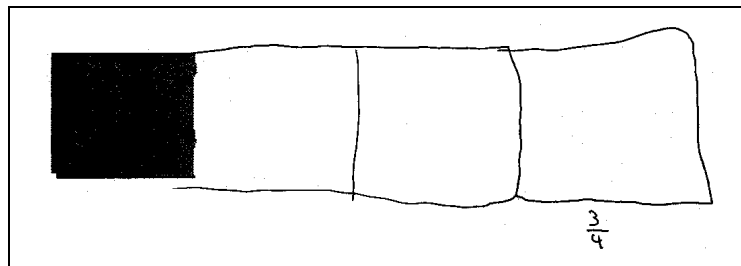

figure 3.6: Gina's drawing

Roxanne is Gina's counterpart in the experimental group. In her first interview she quickly solves the Irene-problem, by relating $\frac{2}{5}$ and $\frac{3}{5}$ in a correct way.
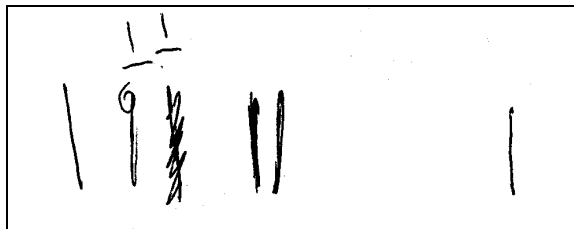
figure 3.7: Roxanne's drawing

One of the problems in the third interview is about Janita: 'At the beginning of her holidays Janita has 48 guilders, $\frac{1}{6}$ of which she spends on ice-cream and soft drinks, $\frac{1}{4}$ on picture postcards and the rest on presents'.

Roxanne, when doing the problem in a written test, writes that Janita spends $\frac{3}{4}$ of her money on presents. She explains why this answer cannot be correct: '$\frac{1}{6}$ of the money is 8 guilders and $\frac{1}{4}$ is ...'

When she gets stuck the interviewer asks her, by way of pre-designed standard help, to make a drawing. In her final product $\frac{1}{6}$ and $\frac{1}{4}$ end up over each other (figure 3.7). Roxanne interprets her sketch as follows: 'Together it is about half.' When asked to tell a little more about this, Roxanne shows in her drawing that the sum of $\frac{1}{4}$ and $\frac{1}{6}$ is a little less than half. She concludes: 'Janita has left a little more than half to spend on presents.'

Gina did not finish the sixth problem in the written test. Two days later, in her interview, she tells the interviewer the following about her problem: 'I don't get this!' The interviewer asks her to make a sketch of the situation and to calculate how much money Janita spent on ice cream and soft drinks. However, Gina cannot do anything with this help.

In the third interview we see how Roxanne's approach of using several levels at the same time, results in the reasonable solution of adding $\frac{1}{4}$ and $\frac{1}{6}$. Here Gina probably encounters the difficulty that she is incapable of applying her formal arithmetic to the fractions involved and drops out. We see how Roxanne, in contrast to Gina, has developed a repertoire to tackle the problems at an appropriate level. We consider this as a sign of acquired fraction numeracy.

Our main hypothesis was that the learning-processes of experimental group students will show a higher degree of fraction numeracy compared to the control group. This hypothesis is largely confirmed by the quantitative data from the interviews, while additional support is provided by observations made during the interviews.

3.6   Conclusions and discussion

In the previous sections we showed how the teaching of formal fractions is influenced by the inseparable combination of teaching and student activities. This dependent character of these two variables could be considered as a major limitation of this study, since we are unable to conclude to what extent each variable is responsible for the observed results. However, what remains is a valuable comparison of the two curricula, seen as well-considered combinations of teaching and student activities.

Doing so, we showed how the teaching of formal fractions is influenced by use of

the number line as a central model for fractions and by the creation of an educational setting in which formal mathematics is discussed in the classroom (cf. Greeno, 1991; Streefland & Elbers, 1995 & 1997). Students in this setting developed fraction numeracy, as they showed more proficiency in solving non-typical fraction problems than students in an educational setting in which formal fractions are mainly based on individual manipulation of predivided circles and bars, without any interaction between students. Our results also reveal some transfer of these skills to general mathematical strategies. However, when considering the development of their general mathematical skills, we noticed that the more gifted students benefited most from the discussion setting. However, if we look specifically at fraction proficiency, we see that all students in this setting perform better than their counterparts who work mainly on an individual basis. We showed that taking the number line as main model for fractions, combined with whole-class discussions facilitates the understanding of formal fractions. Students involved in these discussions, especially those who have no particular problems in mathematics, obtain skills in meaningful manipulating fractions. As a result, average performers in the experimental group of ten-year olds achieve greater proficiency than the curriculum standards require for twelve-year olds (Committee for the Reassessment of Curriculum Standards in Primary Education [Commissie Heroverweging Kerndoelen Basisonderwijs], 1994). However, in both research conditions, students with learning problems in mathematics also have problems with fractions. This matches the results of other researchers. From the present perspective one might wonder whether weak performers of this kind should be burdened with formal fractions.

In a sense, learning fractions in the experimental curriculum can be characterised as problem solving. That is, students are offered open-ended problem situations, for which various solutions are possible. Moreover, group discussion is a means of establishing the value of the approaches. Verschaffel (1995) mentions three types of knowledge necessary for problem solving:
1   the flexible use of a rich and well organised, domain-specific knowledge base;
2   skills in the use of heuristic methods;
3   metacognition.

From the analysis of the two curricula involved in this research we learned that the experimental programme offered students opportunities to build a rich knowledge base in respect of fractions and relations between fractions. Also, the problems posed forced students to explore their individual knowledge base, which constituted an important difference with the control curriculum. Although students in the latter programme also construct knowledge bases for fractions, they are not forced to reflect on relations between fractions.

Several activities in the experimental curriculum aim at the use of approximations in solutions to problems with fractions (cf. for example, Roxanne's approach to adding $\frac{1}{4}$ and $\frac{1}{6}$). In addition, in class discussions global reasoning is frequently used to test statements on relations between fractions. This allows for a powerful heuristic method in dealing with fractions, which has no equivalent in the control setting. Verschaffel states that the types of knowledge necessary for problem solving present major difficulties for those students who are not proficient in mathematics. Perhaps this offers an explanation for the observed differential effects concerning the development of the students' general mathematical skills. Particularly the students in the experimental group who are proficient in mathematics use the opportunity to move between fractions in flexible ways. They also know how to transfer this ability to other parts of mathematics, while students in the programme who are not proficient in mathematics do learn some (usually elementary and low-level) relations between fractions. These students are not capable of extending their approaches to other mathematical problems. By contrast, students from the control group are rarely offered the opportunity to acquire the types of knowledge required for problem solving mentioned by Verschaffel. This explains our observation that students who perform well in mathematics have difficulties in solving non-typical problems (both in the interviews on fractions as in the general proficiency tests in mathematics).

One of the main objectives in mathematics education is the stimulation of the students' 'number sense'. We argued that the experimental group programme contributed more to the development of number sense than the control group programme. Mcintosh, Reys & Reys (1992) show the relation between acquired number sense and heuristic approaches such as checking and discussing answers, relating answers to situations from which they originate and (more generally) reflecting on approaches. We showed how the experimental curriculum enabled students to acquire problem solving strategies.

Given these findings, we interpret the point of view expressed by Mcintosh, Reys and Reys as follows. The experimental programme aims at the acquisition of number sense by students. As a consequence students who perform *normally* in mathematics obtain skills in solving non-typical problems. They outperform their counterparts in the control group in both general mathematical skills and in domain-specific skills relating to fractions.

This is the point at which this study must leave some questions for future research. These questions mainly concern the prospects for students who are less proficient in mathematics. As we noticed, these students seem to benefit less from an educational setting in which formal mathematics is a topic in whole-class discussion. If we wish all students to benefit from class interaction in which formal mathematics is the topic of discussion, more research will be needed into the participation of weak students

in these class discussions, in order to establish the possibilities for these weak performers.

However, the idea that the skills of weak performers should limit the teaching of fractions in primary school to fraction language and simple applications, disregards not only the potentials and the rights of weak performers but also the formation of mathematics as an independent subject. It is our view that mathematics in primary school should not only cover applications in daily life, or otherwise recognisable situations, but also formal mathematics, where the latter is a synthesis of the first (Treffers, 1987). In other words, there is no real place for fractions in the curriculum if they are limited to simple applications only; that is to say, without any regard for level raising. In our study we found reasons to resist arguments advocating limitations on the scope of education with regard to fractions. The results presented here should be regarded as a plea for the teaching of fractions in meaningful ways, in which formal mathematics should be seen as a subject for whole-class discussion in primary schools.

Notes

1 Keijzer, R. & J. Terwel (in press). Learning for mathematical insight: a longitudinal comparative study on modelling. *Learning & Instruction.*
2 This program was created by Frans van Galen, who earlier created a similar program with whole-number tasks. One of the authors, Ronald Keijzer, proposed to extend the existing program to fractions.
3 Initially, the 'Fraction-lift' was Adrian Treffers's idea.
4 Moreover, we assume that the mean differences are normally distributed, which is also necessary for a proper application of this test.
5 For privacy reasons these names are not the real names of the students.

References

Anderson, J.R., J.G. Greeno, L.M. Reder & H.A. Simon (2000). Perspectives on Learning, Thinking, and Activity. *Educational Researcher*, *29*(4), 11-13.

Armstrong, B.E. & C. Novillis Larson (1995). Students' use of part-whole and direct comparison strategies for comparing partitioned rectangles. *Journal for Research in Mathematics Education*, *26*(1), 2-19.

Behr, M.J., R. Lesh, Th.R. Post & E.A. Silver (1983). Rational-Number Concepts. In: Richard Lesh & Marsha Landau (eds.). *Acquisition of Mathematical Concepts and Processes*. New York/London: Academic Press, 91-126.

Bezuk, N.S. & M. Bieck (1993). Current Research on Rational Numbers and Common Fractions: Summary and Implications for Teachers. In: Douglas T. Owens (ed.). *Research Ideas for the Classroom – Middle Grades Mathematics*. New York: Macmillan Publishing Company, 118-136.

Bokhove, J., F. van der Schoot & T. Eggen (1996). *Balans van het rekenonderwijs aan het einde van de basisschool 2* [Balance-sheet of mathematics education at the end of primaryschool 2]. Arnhem: Cito.

Bokhove, J., K. Buys (ed.), R. Keijzer, A. Lek, A. Noteboom & A. Treffers (1996). *De

Breukenbode. Een leergang voor de basisschool (werkbladen en handleiding) [The Fraction gazette]. Enschede/Utrecht: SLO/FI/Cito.

Carraher, D.W. & A.D. Schliemann (1991). Children's understanding of fractions as expressions of relative magnitude. In: Fulvia Furinghetti (ed.). *Proceedings of the fifteenth PME conference. Volume 1*. Assisi, Italy: Dipartimento di Matematica dell'Università di Genova, 184-191.

Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education] (1994). *Doelbewust leren; kerndoelen in maatschappelijk perspectief* [Purposeful learning; curriculum standards in social perspective]. Den Haag: SDU.

Cook, T.D. & D.T. Campbell (1979). *Quasi-Experimentation. Design & Analysis Issues for Field Settings*. Chicago: Rand McNally College Publishing Company.

Dekker, R. (1991). *Wiskunde leren in kleine heterogene groepen* [Learning mathematics in small heterogeneous groups] (dissertation). De Lier: Academisch Boeken Centrum.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In: D. Tall (ed.). *Advanced mathematical thinking*. Dordrecht/Boston/London: Kluwer Academic Publishers, 95-123.

Forman, G. & B. Fyfe (1998). Negotiated Learning Through Design, Documentation, and Discourse. In: Carolyn Edwards, Lella Gandini & George Forman (eds.). *The hundred languages of children The Reggio Emilia Approach – Advanced Reflections (second edition)*. Greenwich, Connecticut/London, England: Ablex Publishing Corporation, 239-260.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: Reidel.

Freudenthal, H. (1991). *Revisiting Mathematics Education, China lectures*. Dordrecht: Kluwer Academic Publishers.

Glasersfeld, E. von (1987). Learning as a Constructive Activity. In: C. Janvier (ed.). *Problems of representation in the teaching and learning of mathematics*. Hillsdale, New Jersey/ London: Lawrence Erlbaum Associates, Publishers.

Graeber, A.O. & E. Tanenhaus (1993). Multiplication and Division: From Whole Numbers to Rational Numbers. In: Douglas T. Owens (ed.). *Research Ideas for the Classroom – Middle Grades Mathematics*. New York: Macmillan Publishing Company, 99-117.

Gravemeijer, K.P.E. (1994). *Developing realistic mathematics education* (dissertation). Utrecht: CD-ß Press.

Greeno, J.G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education*, *22*(3), 170-218.

Hart, K.M. (1981). Fractions. In: K.M. Hart (ed.). *Children's understanding of mathematics: 11-16*. London: John Murray, 66-81.

Hart, K.M. (1987). Practical work and formalisation, too great a gap. In: Jacques C. Bergeron, Nicolas Herscovics & Carolyn Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Volume II*. Montreal: s.n., 408-415.

Hasemann, K. (1981). On difficulties with fractions. *Educational Studies in Mathematics*, *12*, 71-87.

Hedegaard, M. (1990). The zone of proximal development as basis for instruction. In: L.C. Moll. *Vygotsky and education*. Cambridge: Cambridge University Press, 349-371.

Herfs, P.G.P., E.H.M. Mertens, J.Chr. Perrenet & J. Terwel (1991). *Leren door samenwerken. Adaptief groepsonderwijs (Ago) als een curriculum-innovatie in het voortgezet onderwijs* [Learning by co-operation. Adaptive co-operative education as a curriculum-innovation

in secondary education]. Amsterdam/Lisse: Swets & Zeitlinger BV.

Heuvel-Panhuizen, M. van den (1996). *Assessment and realistic mathematics education* (dissertation). Utrecht: CD-ß Press.

Hiebert, J. (1988). A theory of developing competence with written mathematical symbols. *Educational Studies in Mathematics*, *19*, 333-355.

Hiele, P.M. van (1986). *Structure and insight, a theory of mathematics education.* Orlando: Academic Press.

Hoek, D., J. Terwel & P. van den Eeden (1997). Effects of Training in the Use of Social and cognitive Strategies: An Intervention Study in Secondary Mathematics in Co-operative Groups. *Educational Research and Evaluation*, *3*(4), 364-389.

Hoek, D.J. (1998). *Social and cognitive strategies in co-operative groups. Effects of strategy instruction in secondary mathematics* (dissertation). Amsterdam: University of Amsterdam.

Hoek, D., P. van den Eeden & J. Terwel (1999). The effects of integrated social and cognitive strategy instruction on the mathematics achievement in secondary education. *Learning and Instruction*, *9*(5), 427-448.

Huitema, S., A. van der Klis, F. van de Molengraaf, M. Timmermans & L. Erich (n.y.). *De wereld in getallen 6. Handleiding a & b* [The world in numbers 6. Teacherguide a & b]. Den Bosch: Malmberg.

Janssen, J., J.-M. Kraemer & A. Noteboom (1995). *Leerling Volg Systeem. Rekenen-Wiskunde 2* [Student Registration System. Mathematics 2]. Arnhem: Cito.

Kamii, C. & F.B. Clark (1995). Equivalent Fractions: Their Difficulty and Educational Implications. *Journal of Mathematical Behavior*, *14*, 365-378.

Keijzer, R. (1994). De ontwikkeling van een breukenleergang – een voorbeeld van ontwikkelingsonderzoek [The development of a curriculum on fractions – an example of developmental research]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *13*(1), 24-31.

Keijzer, R. & K. Buys (1996). Groter of kleiner. Een doorkijkje door een nieuwe leergang breuken. [Bigger or smaller. A close look at a new curriculum on fractions.] *Willem Bartjens*, *15*(3), 10-17.

Keijzer, R. (1997). Formeel rekenen met breuken [Formal arithmetic with fractions]. In: C. van den Boer & M. Dolk (eds.). *Naar een balans in de reken-wiskundeles – interactie, oefenen, uitleggen en zelfstandig werken –*. Utrecht: Panama/Freudenthal Instituut, 101-116.

Lawler, R.W. (1990). Constructing Knowledge From Interactions. *Journal of Mathematical Behavior*, *9*, 177-192.

Mayer, R.E. (1989). Models for Understanding. *Review of Educational Research*, *59*(1), 43-64.

McIntosh, A., B.J. Reys & R.E. Reys (1992). A Proposed Framework for Examining Basic Number Sense. *For the Learning of Mathematics*, *12*(3), 2-8.

Nesher, P. (1986). Are Mathematical Understanding and Algorithmic Performance Related? *For the Learning of Mathematics*, *6*(3), 2-9.

Oers, B. van (1996). Learning mathematics as a meaningful activity. In: L.P. Steffe, P. Nesher, P. Cobb, G.A. Goldin, B. Greer (eds.). *Theories of mathematical learning*. Mahwah, NJ: Lawrence Erlbaum Associates, Publishers, 91-113.

Paulos, J.A. (1988). *Innumeracy, mathematical illiteracy and its consequences*. New York: Hill and Wang.

Perkins, D.N. & Chr. Unger (1999). Teaching and Learning for Understanding. In C.M.

Reigeluth (ed.). *Instructional-design theories and models. Volume II. A new paradigm of instructional theory*. Mahwah, NJ: Lawrence Erlbaum Associates, Publishers, 91-114.

Reigeluth, C.M. (1999). *Instructional-design theories and models. Volume II. A new paradigm of instructional theory*. Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Streefland, L. (1982). Subtracting fractions with different denominators. *Educational Studies in Mathematics*, *13*, 233-255.

Streefland, L. (1987). Free production of fraction monographs. In: Jacques C. Bergeron, Nicolas Herscovics & Carolyn Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Volume I.* Montreal: s.n., 405-410.

Streefland, L. (1991). *Fractions in Realistic Mathematics Education: a Paradigm of Developmental Research*. Dordrecht: Kluwer Academic Publishers.

Streefland, L. & E. Elbers (1995). Interactief realistisch reken-wiskundeonderwijs werkt [Interactive realistic mathematics education works]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *14*(1), 12-21.

Streefland, L. & E. Elbers (1997). De klas als onderzoeksgemeenschap [Class as research community]. In: C. van den Boer & M. Dolk (eds.). *Naar een balans in de reken-wiskundeles – interactie, oefenen, uitleggen en zelfstandig werken –*. Utrecht: Freudenthal institute, 11-24.

Terwel, J. (1994). *Samen onderwijs maken. Over het ontwerpen van adaptief onderwijs* [Making education together. On developing of adaptive education]. Groningen: Wolters-Noordhoff.

Terwel, J., P.G.P. Herfs, M. Mertens & J.Chr. Perrenet (1994). Co-operative learning and adaptive instruction in a mathematics curriculum. *Journal of Curriculum Studies*, *26*(2), 217-233.

Tharp, R. & R. Gallimore (1998). The theory of teaching as assisted performance. In: D. Faulkner, K. Littleton & M. Woodhead (eds.). *Learning relationships in the classroom*. London: Routledge, 93-111.

Treffers, A. (1987). *Three dimensions. A model of Goal and Theory Description in Mathematics Instruction – the Wiskobas Project*. Dordrecht: Reidel Publishing Company.

Verschaffel, L. (1995). Beïnvloeden van leerprocessen [Influencing learningprocesses]. In: Joost Lowyck & Nico Verloop (eds.). *Onderwijskunde. Een kennisbasis voor professionals*. Groningen: Wolters-Noordhoff, 153-187.

Wijnstra, J.N. (ed.) (1988). *Balans van het rekenonderwijs aan het einde van de basisschool* [Balance-sheet of mathematics education at the end of primaryschool]. Arnhem: Cito.

Wittmann, E. (1981). The complementary roles of intuitive and reflective thinking in mathematics teaching. *Educational Studies in Mathematics*, *12*, 389-397.

4 Audrey's acquisition of fractions: a case study into the learning of formal mathematics[1]

Abstract

National standards for teaching mathematics in primary schools in the Netherlands leave little room for formal fractions. However, a newly developed programme in fractions aims at learning formal fractions. The starting point in the development of this curriculum is the students' acquisition of 'numeracy in fractions'. In this case study we describe the growth in reasoning ability with fractions of one student in this newly developed programme of 30 lessons during one whole school year. In the study we found indications that the programme and its teaching stimulated the progress of an average performer in mathematics. Moreover we found arguments as to what extent formal operations with fractions suits as an educational goal.

4.1 Introduction

Students differ in many aspects. Restricted to the learning of fractions, one observes how some ten year olds relatively easy acquire rational numbers, while others experience difficulties in the most simple manipulations with fractions. Many researchers reported on the differences in students' fraction learning (cf. e.g. Holt, 1964; Behr, Lesh, Post, & Silver, 1983; Hunting, 1984; Streefland, 1987; Behr, Harel, Post & Lesh, 1992; Kamii & Clark, 1995). In addition the Dutch National Testing Institute for primary and secondary education researches the development of the mathematical ability of students in grade 8 in primary schools (11 – 12 year olds). The three nation wide investigations performed so far show, again and again, tremendous differences in skills between the students, especially in fractions (Wijnstra, 1988; Bokhove, Van der Schoot & Eggen, 1996; Janssen, Van der Schoot, Hemker & Verhelst, 1999). These investigations showed that the p75-level student skills suggest that a significant number of students should be able to acquire the ability for formal reasoning with fractions rather easily.

However, national curriculum standards for primary education establish that teaching should be aimed at acquiring competence in using fractions in simple contexts or supported by models (Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education], 1994; cf. Principles and Standards for School Mathematics, 2000). We thus ascertain a discrepancy between the potentialities of groups of students and what is decided upon as curriculum standards. This discrepancy forms the starting point of the

research described here. In the Dutch tradition of realistic mathematics education (Treffers, 1987; Freudenthal, 1991; Streefland, 1991b; Gravemeijer, 1994; Van den Heuvel-Panhuizen, 1996), we developed an experimental curriculum on fractions as an extension of the newly developed curriculum, 'the Fraction gazette' [De Breukenbode] (Bokhove, Buys, Keijzer, Lek, Noteboom & Treffers, 1996). This experimental curriculum is constructed to seize upon potentialities of that group of students that is able to obtain meaningful formal reasoning with fractions with relative ease, where we will refer to this formal level in reasoning as 'formal fractions'. This level of reasoning 'is characterised by a sense that one's mathematical methods work "for all" relevant examples' (Pirie & Kieren, 1994, 43). Or, following Hart (1987), we consider this 'formalisation' to be '(…) a rule, formula or general method which can be applied to a variety of mathematical examples' (p. 409). In the study reported on here, we will observe a process of learning formal mathematics. Freudenthal (1973) considers this to be a process where a general principle emerges from a series of well chosen examples. Here we will report upon this process leading to the generalised notion of equivalent fractions, which on their turn facilitate fraction operations.

During one school year we observed the development of one of the students, Audrey, in the experimental programme. We describe her fraction learning here as a case-study, and analyse Audrey's development in learning fractions. Audrey is a student with average skills in mathematics.[2] Elsewhere we report on the development of students involved in the experiments in a quasi-experimental research design (Keijzer & Terwel, 2000). Our analyses here finally lead to assessing general characteristics on learning meaningful formal fractions.

4.2 Designing a programme

Mathematical insight is widely recognised as an important educational goal. Mathematics education should promote learning for understanding (Freudenthal, 1968; Van Hiele, 1986; Sfard, 1994; Perkins & Unger, 1999; Reigeluth, 1999; Van Dijk, Van Oers & Terwel, 2000). However, it is known from many studies that students have difficulty in applying their mathematical knowledge in meaningful ways in formal mathematics. Moreover history proves that teaching mathematics often results in imitative, meaningless following of rules of calculation. This is especially so for the learning of fractions.

For this reason, various researchers in the 1990's and 1980's plea for constructivist approaches of the difficult problem of teaching fractions. Graeber and Tanenhaus (1993) suggest naming fractions for example by making them results of measuring. With Bednarz and Janvier (1988) and Mack (1990) Graeber and Tanenhaus choose to explicitly build on informal knowledge of students.[3]

In the Netherlands Streefland (1982, 1983, 1987 & 1991a) developed a new curriculum on fractions in the Dutch tradition of realistic mathematics education. By choosing fair-sharing as main fraction-generating activity, Streefland also takes informal knowledge of the students into account. Through these activities of fair-sharing, Streefland at first stimulates the development of a fraction-language by the students. For example students develop their language of fractions, when they have to divide three pizzas among four children. Each child gets three pieces of a quarter of a pizza or half a pizza and a quarter of a pizza, etc. Later Streefland elaborates the sharing-situation so that equivalent fractions emerge; equivalent sharing situations are observed. Sharing three pizzas among four children results in the same amount of pizza for each child as sharing six pizzas among eight. Moreover by comparing results of fair-sharing formal operations with fractions are facilitated.

As we mentioned, in the late 1990's a new fraction programme, 'The Fraction gazette' [De Breukenbode] was developed (Bokhove, Buys, Keijzer, Lek, Noteboom & Treffers, 1996). This programme was created to link up with new curriculum standards for teaching fractions. It, moreover, provided explicitly for students' acquisition of 'numeracy' or number sense (Greeno, 1991; Mcintosh, Reys & Reys, 1992; Keijzer & Buys, 1996a). Our programme, which we describe here and which we mark for reasons of convenience as 'experimental programme', is an extension of 'The Fraction gazette', and emphasises formal reasoning with fractions.

As in Streefland's curriculum, in the experimental programme at first there is explicit focus on the learning of fraction-language. However, unlike Streefland in this programme situations of measuring are mainly used. Bezuk and Bieck (1993) emphasise the importance of this kind of situation in teaching fractions; in this manner fractions are seen as a length, which helps students in making estimations and thus facilitates reflection on one's work. Connell and Peck (1993) provide arguments for using a bar as measuring instrument as forerunner for the number line. They observed students' preference for the bar (in the context of a rectangular cake) as a model:

> 'The students (…) universally selected a 'cake' model for dealing with fractions because they seemed to sense its general applicability.' (Connell & Peck, 1993, 336)

In the next stages of the experimental programme from the situations of measuring the number line is developed. Moreover, this model forms the key-instrument in comparing fractions. Many strategies for comparing fractions are discussed with the students. Furthermore, the situations presented encourage students to use equivalent fractions more and more when comparing. And next these equivalent fractions form a base for formal operations with fractions.

Several researchers (e.g. Behr, Lesh, Post & Silver (1983) and Novillis Larson (1980)) reported on students' difficulties with the number line. Behr, Lesh, Post and

Silver observed three kinds of problems:

'1. Children differed in how they identified the unit on the number line.
2. Problems in which the subdivisions of the unit did not equal the denominator of the fraction were harder to solve than were problems in which subdivisions equalled the denominator.
3. Problems with perceptual distractors (inconsistent cues) were harder to solve than were problems in which subdivisions of the unit were factors or multiples of the denominator or the fraction (incomplete cues or irrelevant cues).'
   (Behr, Lesh, Post & Silver, 1983, 118)

To overcome these problems, in the experimental curriculum using a bar as a measuring-device, and then making measurements is closely connected with the development of the number line. Moreover, the number line is used extensively to compare fractions (Keijzer & Buys, 1996b). The following observation by Mack (1990) provides additional support for relating the development of the number line and comparing fractions. Students, she observed, spontaneously find and use a comparison-strategy, that clearly is supported by the number line:

'One common characteristic of all student-invented algorithms, with the exception of the alternative algorithm for comparing fractions (via 1, RK/JT), was that in general, they were not utilised for an extended period of time. Students soon discovered quicker ways of solving the problems. As soon as they discovered these quicker algorithms, they abandoned their alternative algorithms in favour of the more efficient ones, which often reflected those that are traditionally taught in schools.' (Mack, 1990, 26-27)

In the final stage of the programme formal operations with fractions become a field of exploration for the students (cf. Streefland & Elbers, 1995 & 1997). However, as formal fractions are difficult (Hart, 1981; Hasemann, 1981; Hiebert, 1988; Kamii & Clark, 1995), in the experimental program formal approaches are used next to informal ones. Moreover, dealing with formal and informal fractions occurs next to each other, intending to facilitate students to switch from one strategy to another.

Thus the finalised programme has the following key features:
– As whole class discussions are used to negotiate and construct meanings in learning fractions, the teaching can be characterised as interactive.
– The curriculum is directed towards the acquisition of number sense: students learn to give meaning to fractions in various kinds of situations, develop a good notion of the size of fractions, and learn to handle fractions in simple applications.
– The curriculum contains a teaching strategy in four stages in which number sense is developed: (1) a language of fractions, (2) developing the number line for fractions, (3) comparing fractions, (4) learning formal fractions.
– Different situational contexts and models are used: two types of situations, divid-

ing and measuring, lead to the bar and the number line as central models for fractions.

– Students are offered the opportunity to present their approaches at several levels: initially, when confronted with fraction problems, they opt for informal approaches. These are followed by semi-formal and formal solutions, which are embedded in the informal approaches. Thus the students are challenged to reach approaches at higher levels.

4.3 Aim

Our objective is to describe, analyse and explain the complex fraction learning process of an average student, Audrey. Yin (1984) clearly considers a case study appropriate here, as the study is sustained by a theoretical framework. Moreover, he states on case studies:

> 'The most important is to *explain* the causal links in real-life interventions that are too complex for the survey or experimental strategies. A second application is to *describe* the real-life context in which an intervention has occurred. Third, an evaluation can benefit, again in a descriptive mode, from an illustrative case study (...) of the intervention itself. Finally, the case study strategy may be used to *explore* those situations in which the intervention being evaluated has no clear, single set of outcomes.' (Yin, 1984, 25)

Yin thus offers us a methodology to design the case-study. We describe Audrey's fraction learning process in the newly developed programme to obtain a description of the real-life context where the intervention took place. Moreover, Greeno (1991) offers us a theoretical framework that, together with the theoretical notions mentioned, we will use to analyse and explain the relation between the fraction-programme, the style of teaching and Audrey's strategies in handling formal and informal fractions. We developed what Greeno refers to as an 'environment that fosters curiosity and exploration' (p. 173) and will analyse relevant relations within this environment; those between teaching and learning fractions, between the programme and the teaching thereof and between activities in the programme and Audrey's fraction learning process. We will show relevant elements of the developed environment and will show how Audrey's learning is supported by a process of 'negotiation of meaning'.

Against this background we will argue how students with average mathematical skills can acquire formal fractions, in a programme that aims at students gaining number sense and which reaches formal approaches in situations where the number line is a central model and where comparing fractions is a key-activity.

4.4 Audrey's learning of fractions

4.4.1 Obtaining a language of fractions

Many researchers emphasise the importance of gaining competence in a language of fractions (Bezuk & Bieck,1993; Connell & Peck, 1993; Streefland, 1991a). This acquisition of fraction language therefore marks the beginning of the fraction program discussed here. Audrey's first lessons in fractions aim at dividing square-shaped and circle-shaped objects in parts of equal size. Moreover the pieces are named in both informal and formal manners and are symbolised as unit-fractions. For instance in the first lesson we ask Audrey to make a square cake with four different toppings. She does so with remarkable ease (figure 4.1).
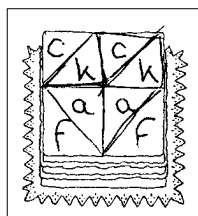
figure 4.1: Audrey's cake with four toppings

In the next lessons Audrey quickly starts to use unit-fractions in a correct manner. Then, in the fourth lesson a new context is presented to introduce other than unit fractions. We ask the students to measure their table with a bar, representing an 'Amsterdam foot' (abbreviated to av). Folding the bar leads to accurate measuring results, but generates the problem of naming the pieces of the bar. At first students use informal and long names for the fractions that arise, like a quarter or three pieces of the bar that is folded in eight. Audrey soon shortens these fraction-names to the formal notation.

This situation shows what Greeno (1991) refers to as 'a social construction in which students interact with the teacher and with each other about quantities and numbers' (p. 173). Here the meaning of fraction is 'negotiated' as precise names are needed to communicate measuring results.

However, Audrey does not yet fully understand the fraction-language. In the fourth lesson she also introduces $\frac{1}{3}$ av as alternative name for three quarter av. However, Audrey continuously uses correct interpretations of the fractions and gradually improves her names and notations of the fractions.

After ten lessons we interviewed Audrey, to assess her knowledge of the language of fractions. One of the problems we present her here is about Irene's chocolate bar. We tell Audrey Irene ate $\frac{3}{5}$ of her chocolate bar and show her what is left.

In solving this problem, Audrey shows her mastery of fraction language (I = Interviewer; A = Audrey):

I: 'Do you think Irene ate more than half the bar?'
A: 'Yes…'
I: 'Can you explain?'
A: 'The bar has five pieces and she ate three. Two are left.' (indicates two pieces in the drawn part of the bar)
I: 'Can you name these pieces?'
A: 'Eh…one-second… Oh no, that is wrong…'
I: 'Can you draw the whole chocolate bar?'
A (after drawing the bar): 'Three pieces are out and two left.'

Audrey now knows how to name the pieces as fractions. She writes what part is left (figure 4.2).
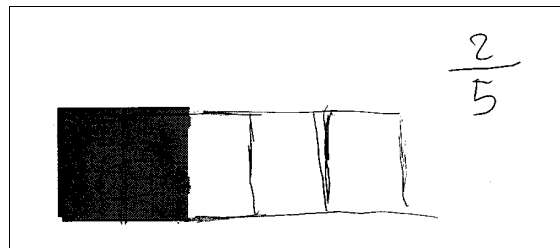
figure 4.2: Audrey's solution of the Irene-problem

We thus observe how Audrey in about ten lessons in fractions obtains a firm grip on the language involved in working with fractions. Her experiences in dividing objects and measuring with divided bars and, moreover, discussing the outcomes of these activities, resulted in the development of the use of fractions as descriptors in this kind of situation.

4.4.2   Developing the number line for fractions

In the experimental programme, the number line is an important tool to reach formal (operations with) fractions. Equivalent fractions emerge as fractions on the same position on the line. Moreover, introducing the number line here enables us to exploit students' knowledge of operations with whole numbers and stimulates students to make rough calculations with fractions, such as $\frac{3}{4} + \frac{1}{3}$ makes approximately 1. However, it takes students some effort to grasp the number line (Novillis Larson, 1980). Measuring with the 'Amsterdam foot' is one of the activities to help students to develop the number line for fractions. In general the bar, as a model for fractions, can be seen as means to form the number line.

In the following weeks the bar is further developed as a model for fractions. Sometimes the connection with the number line is made explicit. In lesson eleven we

introduce the try-your-strength machine (Noteboom, 1994). If one hits the machine, water starts running through a pipe. When the water in the machine reaches the top, you are the strongest of all. Many children however, cannot reach the top. Audrey here in a free production compares the efforts of two children hitting to $\frac{1}{2}$ and $\frac{1}{3}$. Audrey's, in comparing these results, moves her finger alongside the water pipe and concludes that $\frac{1}{2}$ is higher than $\frac{1}{3}$. Her bar has flattened to a line (figure 4.3).
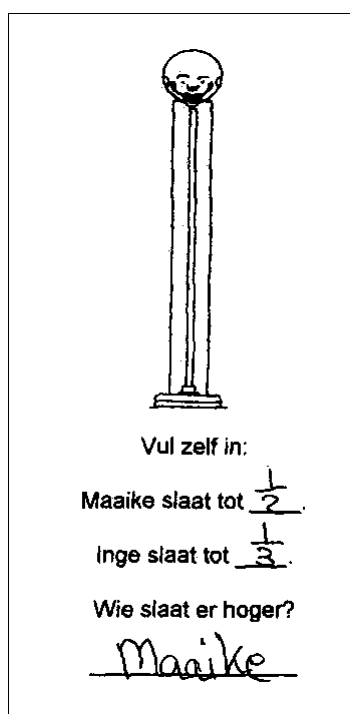
figure 4.3: Audrey's free production with the try-your-strength machine

In the thirteenth lesson we use a drawing-contest as context. Approximately 600 children are participating in the contest. $\frac{1}{5}$ of those are in the youngest (4 and 5 year) group. We ask the students, among other things, how many children in the contest were 4 or 5 years old. To solve this problem, we suggest the students to represent the 600 participants in a bar. Audrey, in doing so, more or less constructs a double indexed number line, where the participants in the contest are on the one side and fractions on the other. Audrey explains what she did: 'If you take this five times you arrive at 600.' On the whole we observe that using a double indexed bar or number line becomes Audrey's approach in solving problems, where a fraction is operating on a large number such as 600.

4.4.3 Comparing fractions

In the next lessons the number line is developed further. We thus try students to form various strategies to compare fractions to finally reach equivalent fractions (Keijzer & Buys, 1996b). These equivalent fractions provide a way to compare fractions in an algorithmic manner, for instance by transforming the two fractions involved in equivalents, that have an equal numerator of denominator. Moreover equivalent fractions form the key to formal reasoning with fractions.

However, considerable effort is needed to grasp equivalent fractions in order to use them in the described way. In one of the first lessons, lesson 6, we observe how Audrey misinterprets the equivalence of fractions. In the context of the 'Amsterdam foot' we ask the students to compare the heights of Christel, $2\frac{1}{3}$ av, and Tessa, $2\frac{2}{6}$ av. Audrey thinks that Christel is taller.

In the same lesson we find a clue for this comparison strategy. Audrey compares with ease the lengths of Melle, $6\frac{1}{3}$ av, and Auke, $6\frac{1}{4}$ av. In explaining her approach Audrey points at the lengths of the pieces $\frac{1}{3}$ and $\frac{1}{4}$ and thus concludes that Auke is a bit shorter. We think that Audrey, in comparing fractions, at this stage of her fraction learning, concentrates on the denominator only. The larger the denominator the smaller the pieces. Reasoning this way a larger denominator always leads to a smaller fraction, independent of the size of the numerator of the fractions involved (cf. Noelting, 1980).

figure 4.4: the fraction-lift, with 2-lift, 3-lift and 4-lift.
The fractions $\frac{1}{2}$ and $\frac{2}{4}$ are at the same level.

We saw how the context of the try-your-strength machine in the eleventh lesson again resulted in Audrey comparing fractions by looking at the size of the pieces. In the fift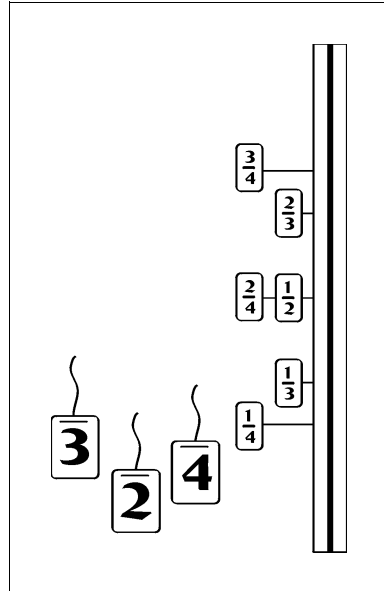eenth lesson we introduce the context of the fraction-lift.[4] Here we again introduce a metaphor for learning fractions. Sfard (1994) from several interviews with mathematicians, shows how experts and novices use metaphors to construct mathematical knowledge. One of her interviewees tells her how he uses personification to perform manipulations on the concept (Sfard, 1994, 48). We see this personification in the context of the fraction-lift. Moreover, Greeno's (1991) notion of situated knowledge portrays the fraction-lift to interact 'with the environment in its own terms – exploring the territory, appreciating its scenery, and understanding how its various components interact.' (p. 175).

Here a vertical number line houses fractions; the fractions live in a fraction-building. Lifts connect the different floors in the building. The numbers of the lifts indicate the stops they make: for instance the 3-lift stops three times, at $\frac{1}{3}$, $\frac{2}{3}$ and at the top of the building (at 1). Similarly the 4-lift stops at $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$ and at 1, the 2-lift stops at $\frac{1}{2}$ and at 1, et cetera (figure 4.4). This context thus makes explicit that different fractions belong to the same position on the number line. In other words the fraction-lift, by personalising the fractions, becomes a metaphor for fractions on the number line.

In the next lesson Audrey uses this newly introduced context, when comparing $\frac{3}{10}$ and $\frac{3}{13}$, to show how she extended her initial comparison strategy to fractions that are not unit-fractions. Moreover in this lesson we observe Audrey using 1 as anchor-point to compare fractions.

The teacher and the students discuss finding fractions in the fraction-building higher than $\frac{99}{100}$. One of the students mentions $\frac{199}{200}$ and $\frac{299}{300}$ as candidates. Audrey explains how this last result could be established: 'Only $\frac{1}{300}$ is needed to reach the top of the building.'

In the nineteenth lesson equivalent fractions are again approached as fractions on the same position on the number line. In the lesson we discuss with the students which fractions occupy the same place on the line as $\frac{2}{3}$. Many students here choose to (repeatedly) double both numerator and denominator of the fraction to make equivalent fractions (cf. Streefland, 1991a). Audrey thus constructs the fraction $\frac{16}{24}$. Next we discuss how these 'room-mates' of $\frac{2}{3}$ can be used to find fractions between $\frac{1}{3}$ and $\frac{2}{3}$ (figure 4.5).
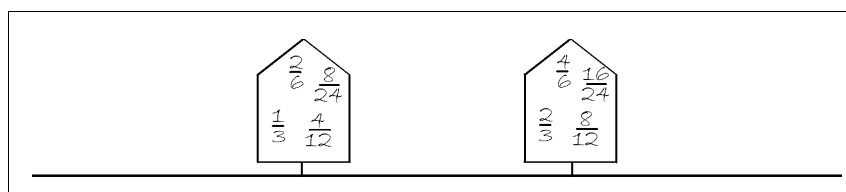
figure 4.5: 'room-mates' of $\frac{1}{3}$ and $\frac{2}{3}$

Here again we see the metaphor of living in connection with positioning fractions on the number line (cf. Sfard, 1994). As $\frac{4}{12}$ is a room-mate of $\frac{1}{3}$ and $\frac{8}{12}$ is one of $\frac{2}{3}$, the fractions $\frac{5}{12}$, $\frac{6}{12}$, etc. are between $\frac{1}{3}$ and $\frac{2}{3}$.

In her individual work following this class discussion, Audrey shows at least three different strategies to compare the fractions involved:

– she compares the fractions 'by the look of it';
– she reasons with the size of the pieces, e.g. $\frac{1}{3}$ is bigger than $\frac{1}{4}$;
– she reasons with equivalent fractions.

In the twentieth lesson we present the students the computer game, 'treasure-digging'.[5] Here the students are offered a fraction and next they are invited to look for this fraction by clicking on the number line. Every attempt leads to showing the fraction at the appointed position (figure 4.6). In this way the students are offered 'anchor-points' to assist them in the searching process.[6]
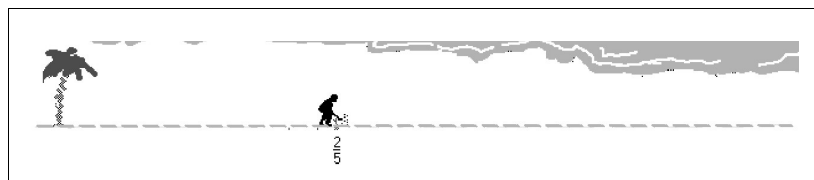
figure 4.6: a fragment of the screen of 'treasure-digging'.
The fraction $\frac{2}{5}$ is found.

Audrey plays the game with Ines. Their fourth task is to find the fraction $\frac{3}{8}$. In doing so they first dig up the fraction $\frac{2}{5}$. Audrey uses the position of this fraction to construct the position of $\frac{3}{8}$. She therefore estimates the distance to $\frac{1}{2}$, by the look of it. Next, she uses the equivalence of $\frac{1}{2}$ and $\frac{4}{8}$ to indicate where $\frac{3}{8}$ should be: 'Left of $\frac{2}{5}$.' Audrey thus shows how she uses $\frac{1}{2}$ as an reference point to compare $\frac{2}{5}$ and $\frac{3}{8}$. Audrey's use of the fraction $\frac{1}{2}$ like a whole number corresponds with findings of Hunting (1986) and Hart (1981). Both Hart and Hunting state that students relatively easily extend whole numbers to $\frac{1}{2}$. Kieren, Nelson and Smith (1985) emphasise the importance of this kind of findings in learning fractions, as the fraction $\frac{1}{2}$ can support fraction generating activities. In a similar way, when looking for $\frac{9}{10}$ in the tenth game, Audrey shows she is able to compare fractions like, $\frac{3}{4}$, $\frac{5}{6}$ and $\frac{7}{8}$ by referring to the distance between the fraction and 1.

After twenty one lessons (seven months after the start of her fraction programme) we interview Audrey a second time. If we examine Audrey's increasing ability to compare fractions, on the whole, we observe that Audrey gradually developed various approaches. At first her strategies are restricted to unit-fractions. She compared these fractions by reasoning the relation between the size of the denominator and that

of the fraction. Later she extended her approach to non-unit-fractions. Moreover she learned how to compare fractions by using both $\frac{1}{2}$ and 1 as anchor-points. Finally using equivalent fractions is seen as a means of comparing fractions. After about twenty lessons in fractions, Audrey is about to conquer these formal relations between fractions.

4.4.4 Learning formal fractions

However, reaching a full understanding of formal fractions is a long process. From the start of the programme Audrey explores formal relations between fractions. At first, these relations are closely connected to situational contexts. For example in the seventh lesson, where fractions emerge when a part of a lighthouse is to be painted. When asked what part still needs to be painted, the students need to find the complement of fractions. Moreover, in this context fractions with denominator 8 are constructed from fractions with denominator 2 and 4, when students are asked to fold the bar-like lighthouse for example to 'paint' $\frac{5}{8}$. Audrey here explains how she constructed $\frac{5}{8}$: 'You have to fold in two and again in two and again in two.' There are then eight pieces and Audrey knows five of those are needed to make $\frac{5}{8}$.

In the next lessons Audrey frequently shows how she relates fractions with denominator 2, 4 and 8. It, however, takes some time to extend this knowledge to other fractions. In the fourteenth lesson we observe this for the first time. We ask Audrey to select two fractions from $\frac{4}{10}$, $\frac{2}{10}$ and $\frac{3}{5}$ that together make 1. As Audrey finds this difficult, we advise her to make a sketch. When she does so, she discovers the equivalence of $\frac{4}{10}$ and $\frac{2}{5}$ (figure 4.7).
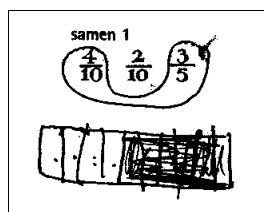
figure 4.7: make 1. Audrey finds equivalent fractions by halving parts.

In the next lessons Audrey repeatedly finds equivalent fractions by doubling both numerator and denominator of the fractions. Sometimes she also uses other strategies. For example in the seventeenth lesson we present the students the fraction-lift in the form of a computer-game. Here students use the lifts to move fractions through the building. In one of their games Audrey and Ines need to shift a fraction from $\frac{1}{5}$ to $\frac{1}{3}$. Audrey suggests the multiplication tables of 3 and 5 can be used to find an appropriate lift. She thus soon finds that the 15-lift can be used.

In the twenty second lesson we introduce 'difficult problems' and 'easy problems' for adding and subtracting fractions. 'Easy problems' are those problems, where the denominators of the fractions that need to be added or subtracted are equal, as with $\frac{4}{6} + \frac{1}{6}$. In 'difficult problems' the denominators of the fractions are unequal, as with $\frac{1}{3} + \frac{1}{2}$. We discuss with the students how we can turn $\frac{4}{6} + \frac{1}{6}$ into a 'difficult problem'. The students seem eager to do so. Soon the fraction $\frac{4}{6}$ is turned into $\frac{8}{12}$ and $\frac{1}{6}$ into $\frac{6}{36}$. We ask Audrey to suggest still another fraction to replace $\frac{4}{6}$ or $\frac{1}{6}$. Audrey chooses to change $\frac{4}{6}$ into $\frac{16}{24}$, by doubling numerator and denominator of $\frac{8}{12}$, as mentioned a little earlier.

Later that lesson Audrey abandons the approach of doubling or halving both numbers in the fraction, to change fractions in equivalent ones. She, for example, replaces the 'easy problem' $\frac{5}{15} + \frac{3}{15}$ with the difficult one $\frac{1}{3} + \frac{1}{5}$.
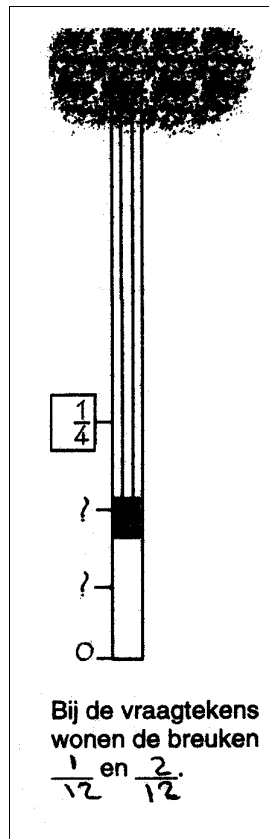
figure 4.8: searching for fractions that live at the question-marks.
Audrey finds $\frac{1}{12}$ and $\frac{2}{12}$

Some time later, in the twenty eighth lesson, Audrey shows how she uses equivalent fractions. In this lesson we again present the fraction-lift context. We now ask the students to use this context to divide fractions by two, three or more. When doing this, we see Audrey struggling with the problem of dividing $\frac{1}{4}$ by three. She wonders what could be done here. We advise her to search for fractions that are 'at the same floor of the fraction building'. Audrey does so and finds an answer (figure 4.8).

We consider formal arithmetic with fractions to be the ability to use equivalent fractions in a proper manner, as equivalent fractions facilitate fraction operations. When we thus analyse Audrey's grasp of equivalent fractions, we see that initially manipulating bars results in only one strategy of obtaining equivalent fractions. In the beginning she tends to only double numerator and denominator. However, when the situation forces Audrey to use other equivalence relations, she does so. This though causes her some difficulty. After 30 lessons Audrey still needs some assistance in finding a general approach to find and use equivalent fractions. If, subsequently, Audrey uses this general strategy, equivalent fractions are used in a creative and flexible manner.

4.4.5 Overview

We conducted a case study on Audrey's fraction learning and found various signs of acquired 'numeracy in fractions' clearly related to the experimental programme and its teaching. Moreover, we found Audrey's flexible strategies in managing equivalent fractions. In this overview we will here outline a few of the observations that typify Audrey's growing 'numeracy in fractions'.

We observed how Audrey needed only a few lessons to fully grasp unit-fractions. Moreover, reasoning with unit-fractions made her develop a way to compare fractions, by considering the denominator. Soon she found more relations between fractions. For example, in the seventh lesson, she constructed $\frac{3}{8}$ by repeated halving and in her first interview she easily related $\frac{2}{5}$ and $\frac{3}{5}$. Later on we saw that her knowledge in fractions is sufficient for her to be able to compare fractions with 1. Moreover she shows creative use of the fraction $\frac{1}{2}$ in various situations.

When we consider Audrey's formal reasoning with fractions, we see that her preferred strategy is the doubling of numerator and denominator. After 30 lessons in fractions she still needs some support in using other approaches to find equivalent fractions. However, when she finds an appropriate equivalent fraction, she is proficient in using this fraction.

4.5 Conclusions and discussion

One of the limitations of a case-study – like the one we described here – is its difficulty to obtain generalisable results. Yin (1984), however, provides a tool to gain some generalisation, namely by explaining case study events in a theoretical frame-

work. That is what we did in the case study described. We first showed how we selected Audrey from her group. Being an average student, Audrey sets an example of how ordinary students can gain proficiency in formal fractions. In line with Yin's (1984) ideas in performing case study research, we ordered our information on Audrey as such we observed signals of causal relations between the constructed programme, and the teaching thereof, and Audrey's learning-process. We thus found the programme and the teaching thereof could be held responsible for Audrey's observed growth in 'numeracy in fractions' and her proficiency in using equivalent fractions.

Simon (1995) here hands us another means to further analyse Audrey's progress. As in our study, Simon performed the roles of both teacher and researcher. He states he based his local teaching decisions on the assumed learning of the students. 'A hypothetical learning trajectory provides the teacher with a rationale for choosing a particular instructional design; thus, I make my design decisions based on my best guess of how learning might proceed.' (Simon, 1995, 135). We now used this decision-making design in teaching in analysing key-elements in Audrey's formal fraction learning process, in order to little by little reconstruct the development of Audrey's fraction learning.

We took Greeno's (1991) notion of situated knowledge to develop an environment that fostered classroom discussion on fraction meanings and relations in order to develop number sense within the fraction domain. We constructed the first activities in the programme so that a language of fractions would be elicited by the students. We therefore used problems where the students had to divide objects. This resulted in Audrey using unit-fractions in a proper way. However, Audrey scarcely used other than unit-fractions here. For that reason we introduced the bar as measuring instrument. In the activities the bar presented a length of several parts. We predicted that the students now would turn over to other than unit-fractions, as the measuring activities would give rise to counting. And that is what we observed with Audrey, for example when she constructed $\frac{7}{8}$ as being seven pieces of $\frac{1}{8}$. Next, we aimed our teaching activities at developing a number line for fractions. We anticipated the developed bar serving as a model for fractions. In the situational contexts we used here the number line became an measuring scale on a bar. We observed how this made Audrey shift from the bar to the line and vice versa, for instance when 'hitting' on the try-your-strength machine.

We expected that considering fractions as parts of folded bars or points on a number line would support students to compare fractions on several levels. Namely, laying two bars side by side, would give a way of comparing the constructed fractions visually and folding bars would generate a few simple relations between fractions, like $\frac{1}{2} = \frac{2}{4}$, $\frac{3}{4} = \frac{6}{8}$, et cetera. Next well chosen situations were developed to elicited other strategies of comparing fractions, such as comparing with 1 and with $\frac{1}{2}$. We

observed how Audrey soon acquired several fraction comparison strategies. However, we initially observed Audrey having problems in comparing the equivalent fractions $\frac{1}{3}$ and $\frac{2}{6}$. We therefore used the fraction-lift to clarify equivalent fractions, by making them fractions living at the same floor of the fraction-building and by introducing the metaphor of 'roommates' for fractions at the same position on the number line. We now observed how Audrey used the strategy of doubling both numerator and denominator to generate equivalent fractions, for example by replacing $\frac{2}{3}$ by $\frac{4}{6}$ to compare the latter fraction with $\frac{5}{6}$.

Folding bars, as we mentioned, cleared the way for the construction of equivalent fractions. We saw how this resulted in Audrey relating fractions with denominator 2, 4 and 8. Moreover Audrey developed doubling both numerator and denominator as an approach to make equivalent fractions. This turned out to be her favoured (and persisting) strategy. We concluded that by using the bar as a manipulative tool we actually were encouraging this approach (cf. Gravemeijer, 1994). To overcome this one-sided strategy in obtaining equivalent fractions, we introduced several problems, where other approaches are needed, such as constructing a 'difficult problem' by the sum $\frac{3}{15} + \frac{5}{15}$. Moreover we again used the fraction-lift to construct other relations between fractions, for example dividing $\frac{1}{4}$ by three. We observed how Audrey slowly started to consider other approaches than doubling both numerator and denominator to construct equivalent fractions.

Simon's (1995) idea of constructing hypothetical learning trajectories as mini-theories of the learning of a student, offered us a useful instrument to analyse Audrey's learning of fractions. Looking at Audrey's progress in gaining proficiency in fractions, the projected learning trajectories became the means of observing and valuing the learning. Teaching interventions following from the analyses were essential in this scheme. They provided possibilities to follow Audrey's progress over an extended period of time and to draw conclusions concerning the potential of Audrey and that of students like her to learn formal fractions.

Audrey is not a special student in any aspect, and consequently her learning activities teach us about average students like her. Therefore this study supports the view that the teaching we described, developing the number line takes a special place and where comparing fractions forms a natural introduction to formal reasoning, could offer many students the prospect of learning formal fractions in a meaningful manner.

Notes

1 Keijzer, R. & J. Terwel (2001). Audrey's acquisition of fractions: a case study into the learning of formal mathematics. *Educational Studies in Mathematics, 47,* 53-73.
2 We used the standardised LVS-tests (Janssen, Kraemer and Noteboom, 1995) to obtain the general mathematical skills of the student. Audrey's pre-test score shows she performs at the nation-wide average.

3 Padberg (1989) in an inventory study summarises all possible approaches for fractions in relation to the four operations: addition, subtraction, division and multiplication.

4 The fraction-lift was an idea of Adrian Treffers.

5 This program is made by Frans van Galen, who earlier made a similar program with whole-number tasks. Ronald Keijzer proposed to extend the existing program to fractions.

6 The program only shows fractions that facilitate reasoning on several levels. Therefore only fractions with denominator 2, 3, 4, 5, 6, 8, 9, 10 and 12 get visible.

References

Bednarz, N. & B. Janvier (1988). A constructivist approach in primary school: results of a three year intervention with the same group of children. *Educational Studies in Mathematics*, *19*, 299-331.

Behr, M.J., R. Lesh, Th.R. Post & E.A. Silver (1983). Rational-Number Concepts. In: Richard Lesh & Marsha Landau (eds.). *Acquisition of Mathematical Concepts and Processes*. New York/London: Academic Press, 91-126.

Behr, M.J., G. Harel, Th.R. Post & R. Lesh (1992). Rational number, ratio, and proportion. In: Douglas A. Grouws (ed.). *Handbook of research on mathematics teaching and learning*. New York: Macmillan Publishing Company, 296-332.

Bezuk, N.S. & M. Bieck (1993). Current Research on Rational Numbers and Common Fractions: Summary and Implications for Teachers. In: Douglas T. Owens (ed.). *Research Ideas for the Classroom – Middle Grades Mathematics*. New York: Macmillan Publishing Company, 118-136.

Bokhove, J., K. Buys (ed.), R. Keijzer, A. Lek, A. Noteboom & A. Treffers (1996). *De Breukenbode. Een leergang voor de basisschool* (werkbladen en handleiding) [The Fraction gazette]. Enschede/Utrecht: SLO/FI/Cito.

Bokhove, J., F. van der Schoot & Th. Eggen (1996). *Balans van het rekenonderwijs in de basisschool 2* [Balance-sheet of arithmetics education in primary school 2]. Arnhem: Cito.

Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education] (1994). *Doelbewust leren; kerndoelen in maatschappelijk perspectief* [Purposeful learning; curriculum standards in social perspective]. Den Haag: SDU.

Connell, M.L. & D.M. Peck (1993). Report of a Conceptual Intervention in Elementary Mathematics. *Journal of Mathematical Behavior*, *12*, 329-350.

Dijk, I.M.A.W. van, B. van Oers & J. Terwel (2000). Aanreiken of ontwerpen? Leren modelleren als strategie voor het werken met problemen in contexten in het reken-wiskundeonderwijs [Handing or developing? Learning to model as a strategy in working with problems in context in mathematics education]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *18*(2), 38-51.

Freudenthal, H. (1968). Why to teach mathematics as to be useful. *Educational Studies in Mathematics*, *1*, 3-8.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: Reidel.

Freudenthal, H. (1991). *Revisiting Mathematics Education, China lectures*. Dordrecht: Kluwer Academic Publishers.

Graeber, A.O. & E. Tanenhaus (1993). Multiplication and division: from whole numbers to rational numbers. In: Douglas T. Owens (ed.). *Research Ideas for the Classroom – Middle Grades Mathematics*. New York: Macmillan Publishing Company, 99-117.

Gravemeijer, K.P.E. (1994). *Developing realistic mathematics education* (dissertation). Utrecht: CD-ß Press.

Greeno, J.G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22*(3), 170-218.

Hart, K.M. (1981). Fractions. In: K.M. Hart (ed.). *Children's understanding of mathematics: 11-16*. London: John Murray, 66-81.

Hart, K.M. (1987). Practical work and formalisation, too great a gap. In: Jacques C. Bergeron, Nicolas Herscovics & Carolyn Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Volume II*. Montreal: s.n., 408-415.

Hasemann, K. (1981). On difficulties with fractions. *Educational Studies in Mathematics*, *12*, 71-87.

Heuvel-Panhuizen, M. van den (1996). *Assessment and realistic mathematics education* (dissertation). Utrecht: CD-ß Press.

Hiebert, J. (1988). A theory of developing competence with written mathematical symbols. *Educational Studies in Mathematics*, *19*, 333-355.

Hiele, P.M. van (1986). *Structure and insight, a theory of mathematics education*. Orlando: Academic Press.

Holt, J. (1964). *How Children Fail*. New York: Dell Publishing Co.

Hunting, R.P. (1984). Understanding equivalent fractions. *Journal of Science and Mathematics Education in S.E. Asia*, *Volume VII* (1), 26-33.

Hunting, R.P. (1986). Rachels schemes for constructing fraction knowledge. *Educational Studies in Mathematics*, *17*, 49-66.

Janssen, J., J.-M. Kraemer & A. Noteboom (1995). *Leerling Volg Systeem. Rekenen-Wiskunde 2* [Student Registration System. Mathematics 2]. Arnhem: Cito.

Janssen, J., F. van der Schoot, B. Hemker & N. Verhelst (1999). *Balans van het reken-wiskundeonderwijs in de basisschool 3* [Balance-sheet of mathematics education in primary school 2]. Arnhem: Cito.

Kamii, C. & F.B. Clark (1995). Equivalent Fractions: Their Difficulty and Educational Implications. *Journal of Mathematical Behavior*, *14*, 365-378.

Keijzer, R. & K. Buys (1996a). *Making sense of mathematics in the newspaper*. Paper presented at ICME 8, Seville, Spain (July 1996).

Keijzer, R. & K. Buys (1996b). Groter of kleiner. Een doorkijkje door een nieuwe leergang breuken [Bigger or smaller. A close look at a new curriculum on fractions]. *Willem Bartjens*, *15*(3), 10-17.

Keijzer, R. & J. Terwel (2000). *Learning for mathematical insight: a longitudinal comparison of two Dutch curricula on fractions*. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, April, 2000.

Kieren, T., D. Nelson & G. Smith (1985). Graphical Algorithms in Partitioning Tasks. *Journal of Mathematical Behavior*, *4*(1), 25-36.

Mack, N.K. (1990). Learning fractions with understanding: building on informal knowledge. *Journal for Research in Mathematics Education*, *21*(1), 16-32.

McIntosh, A., B.J. Reys & R.E. Reys (1992). A Proposed Framework for Examining Basic Number Sense. *For the Learning of Mathematics*, *12*(3), 2-8.

Noteboom, A. (1994). De Kop van Jut. Een breukenactiviteit in groep 6 [The try-your-strength machine. A fraction-activity in grade 6]. *Willem Bartjens*, *14*(2), 9-13.

Novillis Larson, C. (1980). Locating Proper Fractions On Number Lines: Effect of Length and Equivalence. *School Science and Mathematics*, *Volume LXXX*(5), 423-428.

Noelting, G. (1980). The development of proportional reasoning and the ratio concept. Part I – differentiation of stages. *Educational Studies in Mathematics*, *11*, 217-253.

Padberg, F. (1989). *Didaktik der Bruchrechnung: Gemeine Brüche – Dezimalbrüche*. Lehrbücher und Monographien zur Didaktik der Mathematik. Band 11. Mannheim/Wien/ Zürich: Wissenschaftsverlag.

Perkins, D.N. & Chr. Unger (1999). Teaching and Learning for Understanding. In: C.M. Reigeluth (ed.). *Instructional-design theories and models. Volume II. A new paradigm of instructional theory*. Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Pirie, S.E.B. & Th.E. Kieren (1994). Beyond Metaphor: Formalising in Mathematical Understanding within Constructivist Environments. *For the Learning of Mathematics*, *14*(1), 39-43.

Principles and Standards for School Mathematics (2000). Reston, VI: NCTM.

Reigeluth, Ch.M. (1999). *Instructional-design theories and models. Volume II. A new paradigm of instructional theory*. Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics*, *14*(1), 44-56.

Simon, M.A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114-145.

Streefland, L. (1982). Subtracting fractions with different denominators. *Educational Studies in Mathematics*, *13*, 233-255.

Streefland, L. (1983). *Aanzet tot een nieuwe breukendidactiek volgens wiskobas* [Impuls for a new pedagogy on fractions according to wiskobas]. Utrecht: OW&OC.

Streefland, L. (1987). Free production of fraction monographs. In: Jacques C. Bergeron, Nicolas Herscovics & Carolyn Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Volume I*. Montreal: s.n., 405-410.

Streefland, L. (1991a). *Fractions in Realistic Mathematics Education: a Paradigm of Developmental Research*. Dordrecht: Kluwer Academic Publishers.

Streefland, L. (1991b). *Realistic Mathematics Education in Primary School*. Utrecht: CD-ß Press/Freudenthal Institute.

Streefland, L. & E. Elbers (1995). Interactief realistisch reken-wiskundeonderwijs werkt [Interactive realistic mathematics education works]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *14*(1), 12-21.

Streefland, L. & E. Elbers (1997). De klas als onderzoeksgemeenschap [Class as research community]. In: C. van den Boer & M. Dolk (eds.). *Naar een balans in de reken-wiskundeles – interactie, oefenen, uitleggen en zelfstandig werken –*. Utrecht: Freudenthal Institute, 11-24.

Treffers, A. (1987). *Three dimensions. A model of Goal and Theory Description in Mathematics Instruction – the Wiskobas Project*. Dordrecht: Reidel Publishing Company.

Wijnstra, J.M. (ed.) (1988). *Balans van het rekenonderwijs in de basisschool* [Balance-sheet of arithmetics education in primary school]. Arnhem: Cito.

Yin, R.K. (1984). *Case study research. Designs and Methods*. Beverly Hills, CA: SAGE Publications.

5 The fraction learning process in low-achieving students: Shirley's choice and use of strategies in primary mathematics

Abstract

Research in mathematics education offers a considerable body of evidence that both high and low-achievers can benefit from learning mathematics in meaningful contexts. This case study offers an in-depth analysis of the learning process of a low-achieving student in the context of Realistic Mathematics Education (RME). The focus is on the use of productive and counter productive strategies in the solution of problems with fractions. We found support for our idea that low-achievers do benefit from RME, but experience difficulties in the formalisation process with regard to fractions. We seize upon the observed difficulties by discussing the implications of uniform standards in mathematics education.

5.1 Introduction

Developing students' problem-solving strategies is generally accepted as a legitimate educational goal. Many developers in mathematics education therefore have taken problem solving as a starting point for their work (Schoenfeld, 1992; Schoenfeld, 1994; Romberg, 1994; Brenner, et al., 1997). Schoenfeld (1992) embeds problem solving in a general description of mathematical thinking. In characterising aspects of mathematical thinking, he distinguishes knowledge base, problem-solving strategies, monitoring and control, beliefs and affects, and practices (Schoenfeld, 1992, 348). Romberg (1994) likewise connects problem solving to general mathematical activities. In his overview of literature on the subject of problem solving he states:

> 'It is also argued that problem-solving ability and encoding of information are enhanced when schema are interrelated and form a hierarchical arrangement analogous to the way knowledge is used.' (Romberg, 1994, 298)

Krutetskii (1976), in reviewing problem-solving strategies, analyses solving processes of both low- and high-achieving students. He states that high-achieving students are able to shorten their solution process, by quickly seeing appropriate connections, whereas low-achievers often undertake a long and complex search.

Moreover, he shows that low-achievers seem unable to escape from solutions once found, while high-achievers easily change from one approach to another. Finally, Krutetskii found that high-achieving students are much more efficient in managing their memories by choosing effective problem representations.

Krutetskii's observations therefore could easily lead to the conclusion that low-achievers are disadvantaged (more than they already are) in a fraction programme that aims at students constructing formal fractions in a learning environment in which knowledge on fractions is evoked by the solution of complex problems in meaningful contexts, and in which processes and results are discussed to achieve the construction of higher level fraction relationships.

In the present chapter we describe a case study of a low performing student in mathematics, who learned fractions in an experimental programme. We reported elsewhere on the development of the programme and its impact on the learning processes and outcomes of students in grade 6 (9-10 years) (Keijzer, 1994; Keijzer & Lek, 1995; Keijzer & Buys, 1996; Keijzer, 1997; Keijzer, 1999; Keijzer & Terwel, 2000; Keijzer & Terwel, 2001). We found that students in the experimental programme outperformed students in a control group (Keijzer & Terwel, 2000; Keijzer & Terwel, in press).

The student we write about in the present study, Shirley, belongs to the 25 per cent weakest in mathematics in her age group, as judged within a national context (Janssen, Kraemer & Noteboom, 1995). Janssen, Kraemer & Noteboom consider students in this group to be low-achievers, that need special teacher's attention. In the present case study we observe low-achiever Shirley having major difficulties in learning fractions, and answer the question why fractions are as difficult as they are for Shirley. Moreover, regarding the limitations of case-study research, we consider to what extent findings concerning Shirley can be generalised for low-achievers like her.

This case study is part of a larger research project, in which we analyse the development of grade 6 students (9-10 years) learning fractions in two different programmes (Keijzer & Terwel, 2000; Keijzer & Terwel, in press). All students in this project were observed meticulously during their fraction lessons. In these observations we noticed a so-called 'Matthew-effect'; the strong students grew stronger, while the poor performers in mathematics stayed behind (Kerckhoff & Glennie, 1999). For this reason we try here to uncover the processes by which low-achievers in mathematics end up with disadvantaged outcomes.

5.2 Theoretical background

5.2.1 Low-achievers in mathematics education

Many researchers in mathematics education focus on learning processes of low-achievers in mathematics. There is some evidence from the literature that the process of learning arithmetic for these students is different from that of students who perform normally (Van Lieshout, 1997). Others (e.g. Kraemer & Janssen, 2000; Kraemer, 2000) argue that the learning of arithmetic of low-achievers in mathematics is different from their more advanced classmates because these low achieving students lack a repertoire of context-bound mathematical relations and therefore experience difficulties when there is a need for considering numbers as formal objects, as they miss the reference to the contexts which embedded these more formal objects and relations. Hoek, Terwel and Van den Eeden (1997) found similar results in their research of interaction processes in co-operative groups in secondary mathematics. Moreover they found an additional mechanism that disadvantaged low-achievers as help seekers:

> 'Low-achievers are not always able to ask for the right help, because it is difficult for them to explain what they do not understand.' (Hoek, Terwel & Van den Eeden, 1997, 366).

According to Sweller's 'cognitive load' theory, it is conceivable that low-achievers may also have problems with memory capacity, especially in solving complex problems in real-life situations. These problem situations are essential 'ill structured' and require flexible problem solving strategies (Reigeluth, 1999). Therefore the learning of a new mathematical topic, e.g. fractions, as it necessitates the use of new problem solving strategies and the many factors represented in an 'ill structured' problem situation exceed the limits of their memory capacity. These memory problems are even increased by inefficient management of their memory capacity especially in the context of a less efficient problem representation and a more complex road to the solution (Sweller, 1994; Krutetskii, 1976; Hoek, Van den Eeden & Terwel, 1999).
However, Kraemer and Janssen (2000) indicate that low-achievers can achieve number relations if these are given sufficiently lengthy explanations in meaningful, recognisable and identifiable contexts.

5.2.2 Fractions and realistic mathematics education

We now turn from this general description of low-achievers mathematics learning to fraction learning. Hasemann (1981) indicates why the subject of fractions is one of the most difficult in primary education. His arguments show why fractions can easily become an obstacle for students with learning difficulties in mathematics:

> '– fractions are used less often in daily life and are less easily described than natural numbers,

  – the written form of fractions is comparatively complicated,
  – it is difficult to put the fractions in order of size on the number line,
  – for the arithmetic of fractions there exist many rules, which are more complicated than those for natural numbers.' (Hasemann, 1981, 71)

The fractions curriculum considered here was developed as an extension of the 'Fraction gazette' [De Breukenbode (Bokhove, et al., 1996)]. Both the 'Fraction gazette' and its extension were developed within the Dutch context of realistic mathematics education (RME). This implies that recognisable and meaningful contexts are used to help students build upon their informal knowledge. Moreover, these contexts lead to modelling, schematising and hence to the construction of formal relations between numbers and other mathematical objects (Treffers, 1987; Freudenthal, 1991; Gravemeijer, 1994; Van den Heuvel-Panhuizen, 1996).

Starting from the RME paradigm, the constructed extension of the 'Fraction gazette' is marked by whole class discussions, in which meaningful fractions are negotiated and constructed. Thus the teaching may be characterised as interactive. The curriculum is directed towards the acquisition of number sense (Greeno, 1991; Mcintosh, Reys & Reys, 1992), that is, students learn to attach meaning to fractions in various kinds of situations, develop a good notion of the size of fractions, and learn to handle fractions in simple applications.

Recent research in the Dutch context of RME offers a considerable body of evidence that both high- and low-achievers are helped by acquiring mathematics through problem solving in meaningful contexts (Van den Heuvel-Panhuizen, 1996; Van Luit & Van de Rijt, 1997; Hoek, Terwel & Van den Eeden, 1997; Kraemer & Janssen, 2000; Kraemer, 2000). By learning fractions in meaningful situations in this RME-context, the fraction programme discussed here might remove some of the difficulties mentioned by Hasemann. By using meaningful contexts, the students develop 'fraction language', which also elicits a notation for fractions. Furthermore, the contexts are chosen in such a way that the number line as a model for fractions comes into sight (cf. Moss & Case, 1999). As regards the fractions that belong in the same position on the number line, equivalent fractions are highlighted, laying a base for formal manipulations with fractions.

Although results of RME in this respect are remarkable, it obviously has its limitations. For instance, one could argue that these contexts, which are meaningful to most students, might well be meaningless to many low-achieving students. And, if these low-achievers come to regard a context which is meant-to-be meaningful as just another confusing mathematical artefact, learning fractions could easily degenerate into a mechanical application of the rules of arithmetic (cf. Erlwanger, 1973).

5.2.3 An experimental curriculum

The curriculum considered here contains a four-stage teaching strategy in which

number sense is developed by: (1) providing a language of fractions, (2) developing the number line for fractions, (3) comparing fractions, and (4) learning formal fractions. In the curriculum different situational contexts and models are used. Two types of situations, dividing and measuring, lead to the bar and the number line as central fraction models. In the teaching of the curriculum students are offered the opportunity to present their approaches at several levels. Initially, when confronted with fraction problems, they opt for informal approaches. These are followed by semi-formal and formal solutions, which are embedded in the informal approaches. In this way the students are challenged to achieve approaches at higher levels.

5.2.4   Problem solving

Several researchers (e.g. Behr, et al., 1983) show that teaching fractions in the way described, with students constructing their own higher level fraction relationships, inevitably leads to problem solving, where 'problem solving' may be characterised as consisting of all those heuristic approaches to mathematical solutions in which the problem-solver has no direct algorithmic approach available. Verschaffel (1995) points to three types of knowledge needed for problem solving:

1   The flexible use of a rich and well organised, domain-specific knowledge base.
2   The ability to use heuristic methods.
3   Metacognition.

Verschaffel emphasises that these points represent difficulties especially for those students who are weak in mathematics. In combination with the argument advanced by Behr et al. that constructing higher level fraction relationships can be regarded as problem solving, Verschaffel explains why fractions are so difficult for low-achievers. A similar argument is presented in Nelissen (1998a). He takes the formation of representations as the starting-point for his argument:

> 'By reflecting on their own actions, children can construct representations on a higher level, requiring critical testing.'[1] (Nelissen, 1998a, 175)

Goldin (1998) considers teaching design, permitting students a flexible use of appropriate representations. He argues that developers of mathematics curricula should elicit multiple representations for each concept. Nelissen (1998a) specifies the problem solving process for both high- and low-achievers. In solving mathematical problems, low-achievers, once they have found a (usually standard) approach to a solution, in general hold on to it. By contrast, students who solve problems at a high level, that is, high-achievers, in general dare to change their strategy and abandon a chosen solution where appropriate. In addition, Lemoyne and Tremblay (1986) characterise good problem solvers as students with rich and precise associations. They argue that problem-solving strategies largely depend on linguistic and heuristic strategies. This links up with Nelissen (1998b) who argues that both the learning of lan-

guage and the learning of mathematics are characterised by the use of representations. He shows what makes learning mathematical language so difficult:

> 'In daily life ambiguities [in natural language] do not trouble us, because there are many situational cues. In mathematics classes, however, children are often unprepared when confronted with words such as: table, times, angle, magnitude, power, set, small number, operation, match, dividing etc.'[2] (Nelissen, 1998b, 15/6)

We also learn from Nelissen (1998a) that it is precisely the low-achievers who are disadvantaged when it comes to using heuristic strategies, since such strategies presuppose that the student is able and dares to review his/her solution in order to shift to another if necessary. Booth and Thomas (2000) add yet another argument to the difficulty of problem-solving tasks for low-achievers in mathematics. They state that it is easier for students to use visual representations of problems that are context-near, such as drawings, than more developed representations such as schemata and models. They argue that more developed (formal) representations of this kind might cause problems for low-achievers.

5.2.5 Turning tide for low-achievers

In summary the aforementioned researchers observed the following characteristics of low-achievers:

– low-achievers often undertake long and complex searches, and lack the metacognitional strategies to escape from solutions that work elsewhere (Krutetskii, 1976; Verschaffel, 1995);
– low-achievers have problems with cognitive overload, especially in solving complex problems in real life situations (Krutetskii, 1976; Sweller, 1994);
– low-achievers lack the flexible use of a rich and well organised, domain-specific knowledge base (Verschaffel, 1995; Lemoyne & Tremblay, 1998; Kraemer, 2000);
– low-achievers lack the ability to use heuristic methods (Verschaffel, 1995; Nelissen, 1998a);
– low-achievers have difficulties in understanding more developed representations like schemata and models (Nelissen, 1998a; Booth & Thomas, 2000); they also experience difficulties in developing mathematical language (Nelissen, 1998b);
– low-achieving students experience difficulties when there is a need for considering numbers as formal objects, and no clear reference to the contexts that produced them (Kraemer & Janssen, 2000);
– as a consequence, low-achievers are not always able to ask for the right form of assistance, because it is difficult for them to explain what it is they do not understand (Hoek, Terwel & Van den Eeden, 1997);
– finally, they face dropping out (Holt, 1964).

Although these researchers provide a considerable body of evidence that low-achievers experience major difficulties in problem-solving situations, many others find arguments in the nature of problem solving to make this the starting-point of their mathematics education. Burton (1980b) argues that a focus on problem solving in the teaching of mathematics values the differences between students. Burton (1980a) suggests that (mathematical) puzzles provide an appropriate way for placing problem solving in the centre of student learning. Kraemer (2000) found that low-achievers in mathematics are best helped if they have a chance to explore well-chosen contexts thoroughly. Schoenfeld (1994) and Streefland & Elbers (1995 & 1997) show another inviting manner. They made the classroom '(…) a community of mathematical judgement which, to the best of its ability, employs appropriate mathematical standards to judge the claims made before it.' (Schoenfeld, 1994, 62). In the case of fractions, for example, Behr et al. (1983), Streefland (1982 & 1991), Watson, Cambell and Collis (1993), Tzur (1999) and Mack (1990 & 2000) show how the construction of fractions by students can originate in the solving of meaningful problems. Watson, Cambell and Collis (1993) show that offering students open problem situations, where no solution is at hand, may result in several approaches on various levels. Watson et al. point out that, in this way, justice is done to the potentials of all students (cf. Freudenthal, 1973). Streefland (1991) demonstrates how situations of fair-sharing may lead to the naming of the sharing process results as fractions. Mack (2000) in her paper describes a two-year case-study of a group of students who learned to multiply fractions in an algorithmic manner from their teacher and received weekly lessons in fractions by the researcher. The researchers' lessons were aimed at learning to multiply fractions in meaningful contexts and subsequently to generalise the acquired knowledge to bare fraction multiplication. Mack found that, after two years, the students had more confidence in the approaches that were learned in problem solving situations than in algorithmic approaches.

Although meaningful situations help low-achievers to acquire mathematical knowledge, their high achieving peers do a much better job (cf. Keijzer & Terwel, 2000). The 'landscape of learning' metaphor used by Fosnot and Dolk (2001) provides an appropriate way of depicting the way students learn. Fosnot and Dolk consider learning as making a journey through the landscape. Different students take different routes on their way to the horizon, each having their individual experiences. In this metaphor, the low-achieving students are the ones who leave the main route, and get lost in the knowledge that they are without adequate means to find the way back and that some parts of the terrain will remain closed to them.

5.3   Research questions

By and large, we may conclude that there is a considerable body of evidence to justify the conclusion that a problem solving approach in teaching creates difficulties for low-achievers. This effect is strengthened by the fact that the teaching here concerns fractions, one of the most difficult subjects in primary school (Hasemann, 1981). However, other researchers show that, when mathematics education is aimed at learning in meaningful contexts, there is little room for a curriculum design in which mathematics consists in merely following incomprehensible rules (Treffers, 1987; Freudenthal, 1991; Gravemeijer, 1994; Romberg, 1994; Schoenfeld, 1994). The basic issue in the development of mathematics education can therefore be stated as follows: how can we adopt a problem-solving approach to teaching fractions in such a way that both low and high-achievers benefit from this approach? This developmental question was answered elsewhere (Keijzer, 1994; Keijzer & Lek, 1995; Keijzer & Buys, 1996; Keijzer, 1997; Keijzer & Terwel, 2000) and resulted in the curriculum we described earlier, in which well-chosen contexts mainly elicit the number line as a model for fractions, which subsequently establishes a basis for formal manipulations with fractions (see also appendix A).

Here we focus on the learning processes of Shirley, a low-achieving student in mathematics, and formulate our main research question starting from the perspective discussed in the previous paragraph. However, we are not merely interested in Shirley's learning. In analysing her learning, within the theoretical framework presented, we look for indications for possible generalisations of Shirley's learning to the fraction learning other low-achievers. Thus, although the case-study design enables us to carefully analysing Shirley's learning, its generalisability to other students' learning is limited. Considering this, we formulate our main research question in a general sense and specify this question to Shirley's learning in two additional questions:

Do low-achieving students really benefit from a realistic problem solving approach in acquiring mathematical insights and proficiency in the domain of fractions, and what are the main obstacles in the formalisation process from real-life situations to mathematical number sense?

From this overall question we formulated two questions concerning the Shirley's learning:

1   *What are the characteristics of Shirley's learning process in the acquisition of formal fractions?*
    The programme, Shirley is involved in, is aimed at the acquisition of 'numeracy in fractions'; that is, learning how to attach meaning to fractions in various kinds of situations, developing a sound notion of the size of fractions, and learning how to handle fractions in simple applications. We try to recover the relation between Shirley's learning process and the programme.

2 *What are the key processes showing that Shirley's learning process develops less well or not at all, in particular with regard to formal fractions?*

Put differently, we will look at Shirley's choice and use of strategies in problem solving and how she copes as a low achiever in the mathematics classroom. From other research conducted in this field we learn that low-achievers in mathematics are in an unfavourable position when learning formal fractions. They seem to have quite a lot of difficulty in acquiring 'numeracy' or 'number sense' (Mcintosh, Reys & Reys, 1992; Greeno 1991). Moreover, they experience difficulties in problem solving tasks (Verschaffel, 1995). This means that there is a real chance that they are failing to learn fractions and are desperately trying to cope with the situation by using counter-productive strategies (Holt, 1964).

5.4   Methods

In this study we closely follow the learning process of one student. Several researchers show the impact of a case-study design as a means to reveal the effects of fraction teaching programmes (e.g. Erlwanger, 1973; Carraher & Schliemann, 1991; Mack, 1990; Mack, 1995; Mack, 2000; Hart, 1981; Hunting, 1983, Tzur, 1999). Yin (1984), in his general work on case studies, thinks a case-study design is appropriate here. He addresses the problem of replication of case studies, to obtain generalisable research results. He pleads for the development of a rich, theoretical framework:

> 'The framework needs to state the conditions under which a particular phenomenon is likely to be found (a literal replication) as well as the conditions when it is not likely to be found (a theoretical replication).' (Yin, 1984, 49)

In the previous paragraphs we provided several elements of the theoretical framework underlying the present study. This framework includes the theory of realistic mathematics education (RME) (Treffers, 1987; Van den Heuvel-Panhuizen, 1996), as it is developed in the Netherlands in the past 30 year. Moreover it includes the notion of learning mathematics as a social enterprise (Schoenfeld, 1992; Schoenfeld, 1994; Romberg, 1994; Cobb & Whitenack, 1996; Greeno, 1997). And finally it includes theoretical notions on the learning of fractions by the way of modelling well-chosen contexts (Streefland, 1983; Streefland, 1991; Treffers, 1987; Freudenthal, 1991).

We gave an overview of the problems low-achievers experience in problem solving situations. We especially found that low-achievers lack the flexible use of a rich and well organised, domain-specific knowledge base and the ability to use heuristic methods (Verschaffel, 1995; Lemoyne & Tremblay, 1998; Nelissen, 1998a; Kraemer, 2000). However, we also found that, in particular, programmes consisting of meaningful, but open, contexts – which provide for problem solving situations – are

essential if fractions are to be taught in a meaningful manner. This situation, where means to make mathematics education more meaningful on the other hand form an obstacle for many low-achievers, led to the formulation of two research questions for this case study.

As subject for our study we selected Shirley, as a student belonging to the 25 per cent weakest in mathematics in her age group, as judged within a national context. Shirley participated in an experimental fraction programme, where the number line and the bar are central models and where fraction relations are negotiated in whole class discussions. As we search for explanations for low-achievers' difficulties in learning fractions we here study Shirley's fraction learning in the experimental programme, which proved beneficial for normal performing and high achieving students. In the analysis of the observations of Shirley's learning process, and of her written work, we looked for explanations of her development.

Yin (1984) states that one of the main problems in performing a case study is to organise the large amount of research data that is generated during the inquiry. In this study our data include reports of all the fraction lessons in grade 6, results of general mathematics tests, and accounts and analyses of standardised interviews. To make it presentable we adapted this material in several steps. First, we rewrote the observations made during the lessons and our analyses of the interviews as narratives (cf. Gudmundsdottir, 1995) which tell the story of Shirley's progress with fractions. We then ordered these narratives in such a way, that they show key moments in Shirley's development. After having ordered these key elements we labelled the stages in Shirley learning of formal fractions as follows: 'acquisition of fraction language', 'process of formalisation' and 'dropping out'. In addition, we used the labels of the stages to reduce the number of narratives. From the narratives in the first stage of the programme we selected those that clearly show the improvements in Shirley's mastery of fraction language. Similarly, from the second stage we selected the narratives that give a clear picture of Shirley's struggle with formal fractions, from the third stage we chose the narratives which show how Shirley signals that she is dropping out.

The development of a new fractions curriculum is one of the objectives in the research project described here. In this context, the researcher takes on three roles:
– as developer of the fractions programme;
– as teacher of the developed programme;
– as an researcher of the student's development.

Gravemeijer (1994) considers the development of mathematics education as a cyclic process where thought-experiments precede field-experiments. These classroom experiments lead to reflections and another prototype of the curriculum (Keijzer, 1994). Gravemeijer thus emphasises the importance of this combination of roles in

developing mathematics education. Yin (1984), however, warns against this participant role of the researcher:

> 'The major problems related to participant-observation have to do with the potential biases produced. First, the investigator has less ability to work as an external observer and may, at times, have to assume positions or advocacy roles contrary to the interest of good scientific practices. Second, the participant-observer is likely to follow a commonly known phenomenon and become a supporter of the group or organisation being studied, if such support did not already exist. Third, the participant role may simply require too much attention relative to the observer role.
> Thus, the participant-observer may not have sufficient time to take notes or to raise questions about events from different perspectives, as a good observer might.' (Yin, 1984, 87)

Our manner of reporting in this study provides one of the ways to overcome these objections. Both lessons and interviews in this inquiry are audio-taped. These tapes, in turn, form the basis for the reports of the lessons and interviews. In this way our narratives provide accurate and unbiased descriptions of the events observed. A second means of achieving objectivity was to arrange regular discussions and co-authorship with a co-researcher who was not directly involved in the teaching process (cf. Mead, 1959; Ten Have, 1977). In addition, we follow Freudenthal (1991) in overcoming Yin's objections to a research design in which the researcher is participant. Freudenthal (with Gravemeijer (1994)) refers to 'developmental research' in this context:

> 'Developmental research means experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others in such a way as to become like their own experience.' (Freudenthal, 1991, 161)

In this chapter, as elsewhere (Keijzer & Terwel, 2000; Keijzer & Terwel, in press), we report on the researcher's activities. In his teaching and development activities he is bound to influence the results of the research, seeing that this is partly the object of the research! That is, the curriculum being investigated is the result of the researcher's developmental activities. We also consider the style of teaching of the researcher to be an integral element of the curriculum. This entitles the researcher (in his role as teacher) to take a position and support the developed curriculum.

In short, we collected our data on the development of Shirley in three different ways. First, we audio taped and observed all the lesson on fractions Shirley attended during the year the research took place. We elaborated the data for each lesson into a report containing both essential narratives about the things that went on during the lesson, as well as protocols of relevant student-student and student-teacher interactions. In addition, we tested Shirley three times using standardised tests to establish her skill

in 'numbers and operations' and 'measurement and geometry' (Janssen, Kraemer & Noteboom, 1995). Furthermore, in three (standardised) interviews we determined Shirley's knowledge of fractions. The first interview thus aimed at establishing her fraction language, the second interview focussed on comparing fractions and the third on applying fractions in simple contexts. All interviews were audio-taped and transcribed for further elaboration. Elsewhere we provided an extensive description of the interviews and the measures we took to standardise the interviews (Keijzer & Terwel, 2000; Keijzer & Terwel, in press).

5.5 Data and analysis

5.5.1 Introduction

As stated before, Shirley, being a low-achiever, belongs to the 25 per cent weakest in mathematics in her age group, as judged within a national context. From the diagram that displays the pre- and post-test scores on general mathematics test, recorded at the start and the end of the case study we read Shirley's progression in this one-year period (figure 5.1).
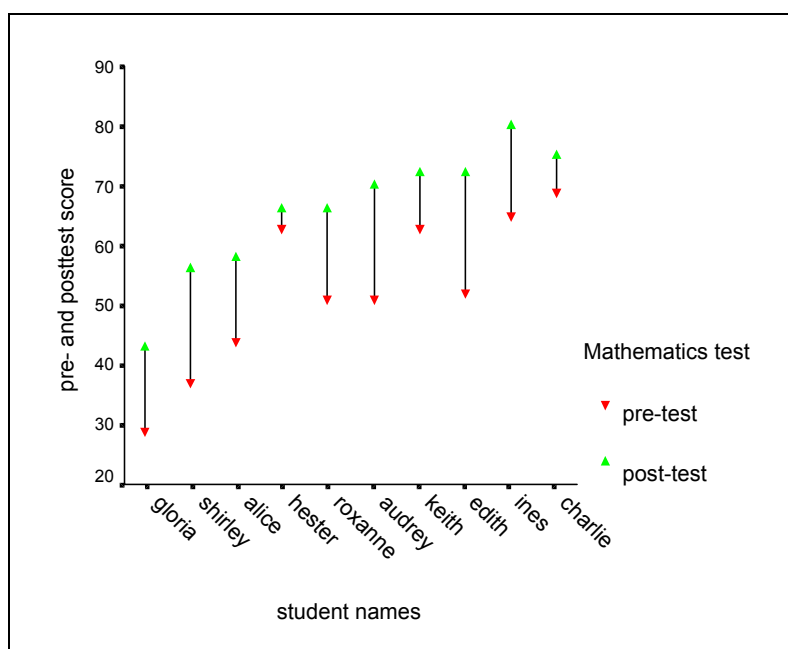

figure 5.1: development of number skills of Shirley and her classmates over the year reported upon in this case-study. Scores adapted from Janssen, Kraemer & Noteboom (1995)

Here we present a description of Shirley's learning of fractions in the sixth grade (9-10 year). Since we wanted to disclose the learning process of a low-achiever in the sixth-grade curriculum, Shirley participated in the programme for the whole school year. Her test-scores, that did not differed that much from the average student, suggested us that Shirley should be capable of following at least a major part of the developed programme. However, our choice implied that Shirley would remain in the programme, even if that would not be the case in normal teaching. We did not offer Shirley special treatment or extensive help, since we wanted to find out how she would develop in the fraction programme without such treatment or help.[3]

We decided to conduct our research in this way in order to provide arguments concerning the extent to which low-achievers should have to learn formal fractions at all in primary school. In normal school practice there is only a limited amount of time for fraction programmes. Bokhove et al. (1996) in their programme 'the Fraction gazette' suggest about 80 fraction lessons in grade 6, 7 and 8 (9-12 year). The most recent Dutch textbook series spends about this time on the teaching of fractions (e.g. Huitema (ed.), n.y.). Moreover, this limited attention for fractions is in line with Dutch curriculum standards (Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education], 1994). And, if this limited time is not sufficient to teach fractions in a meaningful manner, there would seem to be a good reason for reconsidering educational priorities.

Following Yin (1984) we established three recognisable stages in the development in Shirley's learning of fractions. Initially, Shirley is involved in the process of learning the language of fractions. Subsequently, relations between fractions are explored. Here we see, however, that failure leads to a drop-out process.

5.5.2 Acquisition of fraction language

From the beginning, the fraction programme in which Shirley was involved in paid considerable attention to the learning of fraction language. Many researchers underline the importance of the knowledge of fraction language as a basis for forming proper and extended fraction concepts (Bezuk & Bieck, 1993; Connell & Peck, 1993; Mack, 1995; Mack, 2000; Streefland, 1983; Streefland, 1991). Others show how students get stuck when their fraction language is not firmly based during (formal) operations with fractions (Clements, 1980; Hunting, 1983).

When we observe Shirley's acquisition of fraction language, the first thing we notice is the difficulty she experiences in dividing an object into equal parts, or parts equal in size. In Shirley's first lesson we introduced a context in which cakes have different toppings. One of the students' tasks is to make a cake with four toppings of equal size. Though the equal size of the different parts of the cake is emphasised by the teacher, Shirley produces the division depicted in figure 5.2.

In her attempt to divide into four equal parts, Shirley at first separates topping 'K', then marks the position of taste 'A', and finally makes 'C' and 'S'. Unlike many of her classmates, Shirley does not divide by drawing two perpendicular lines. This indicates the first clear distinction between her approach and that of most of her peers. Drawing two lines, as most students did, can be seen as a prelude to understanding the numerical relation between a half and a quarter, $\frac{1}{2} = \frac{2}{4}$.
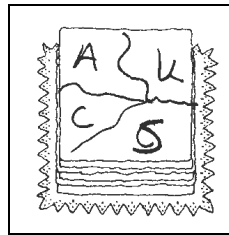
figure 5.2: Shirley's cake with four toppings, which should be of equal size

In the following lessons, we see how Shirley meets with considerable difficulties in naming fraction parts. In her fourth lesson she names most parts as a quarter (figure 5.3) or just piece (figure 5.4). By that time, when most of Shirley's classmates know how to name fractions which move forward as parts of a folded bar, Shirley denominates part-wholes in terms of halves and quarters only.
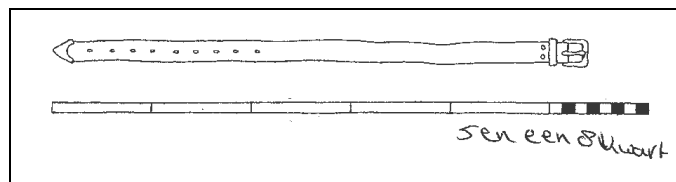
figure 5.3: the length of the belt is '5 and an 8 quarter'
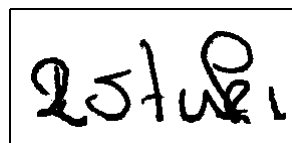
figure 5.4: Shirley in her fifth lesson: '2 piece 1'.

Shirley's first ten lessons in fractions were mainly devoted to helping her to develop a fraction language. After these ten lessons she was interviewed. One of the interview problems is about a partly painted wall (see appendix B). In a picture we showed Shirley, $\frac{5}{6}$ ths of the rectangular object were shaded. We asked Shirley to

name the part painted in terms of fractions. Shirley restricts herself to an informal fraction-name and calls the shaded part 'three quarters'. When we ask her to write this down, she gets a little confused. At first she names the fraction '3 quarter', but Shirley is not sure this is the right answer, so she next tries '$\frac{3}{1}$' and '$\frac{1}{3}$' (figure 5.5).
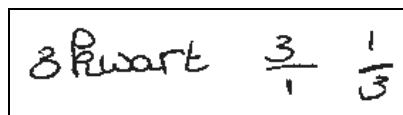
figure 5.5: Shirley's attempts to write 'three quarter'

After this interview, the lessons gradually focus less on the acquisition of fraction language. The aim in the next ten lessons is to develop the number line as a model for fractions and to form several strategies in comparing fractions. These comparison strategies for most of Shirley's classmates gradually develop into formal reasoning with fractions. Shirley, however, experiences difficulties in grasping the strategies involved in comparing fractions and more and more use uncomprehended numerical relations and starts guessing answers. After these ten lessons we interviewed Shirley a second time.

In her second interview we use a skating-tour context to ask Shirley to compare the fractions $\frac{2}{3}$ and $\frac{5}{6}$ (see appendix B). Here we observe how Shirley is unable to position $\frac{2}{3}$ on the line depicting the tour. Moreover, her strategy in comparing $\frac{2}{3}$ and $\frac{5}{6}$ shows that Shirley has a poor understanding of fractions. According to Shirley, $\frac{5}{6}$ is greater since the pieces in $\frac{5}{6}$ are bigger than those in $\frac{2}{3}$. Later in this interview we observe two approaches to compare fractions side by side. First, Shirley, the same way she compared $\frac{2}{3}$ and $\frac{5}{6}$, focuses only (and wrongly) on the denominator of the fractions involved. Secondly, she decides on the size of the fraction by considering both numerator and denominator. The greater these two numbers are, the greater the fraction. When she is searching for fractions close to $\frac{3}{4}$, her approach becomes even clearer. According to Shirley $\frac{3}{3}$ and $\frac{3}{5}$ are equally far from $\frac{3}{4}$, as they are both 'one away from $\frac{3}{4}$'.

The following lessons were aimed at further development of the number line. This served two objectives. By considering fractions at the same position on the number line, equivalent fractions emerged, clearing the way to reaching more formal relations between fractions. On the other hand, positioning fractions on the number line provided the students with an opportunity to reconsider strategies in comparing fractions. Thus, on several occasions after her second interview we discussed comparing strategies with Shirley. However, at the end of one year of fraction learning, after 30 forty-five minutes lessons in fractions, Shirley is still encounting difficulties in comparing simple fractions. Moreover, she suffers some difficulties in dividing objects when the division is described in terms of fractions, and in the (exact) positioning of

fractions on a number line. On the whole, we observe that emphatic attention to the learning of fraction language did not result in Shirley's developing a firm grip on the language of fractions. She used informal fraction-names for an extended period of time and it took her a very long time to use formal fraction-names next to the informal ones. Moreover, Shirley continuously struggled with the meaning of fractions and from time to time treated them as two whole number pairs.

5.5.3 Process of formalisation

In our programme of fractions, well-chosen contexts elicit fraction language, followed by operations with fractions. Fractions hereby transform from describers of recognisable situations to formally embedded mathematical objects (Bergeron, Herscovics & Bergeron, 1987; Davis & Hersh, 1984; Dubinsky, 1991; Easley Jr., 1981; Freudenthal, 1973; Hart, 1987; Krutetskii, 1976; Moore, 1994; Piaget, 1973; Streefland, 1987; Streefland, 1997). The learning of fractions may thus be regarded as a process of formalisation.

For Shirley, too, interpreting recognisable contexts forms the start of the fraction learning process. In her second lesson we ask Shirley to divide a sausage into four parts. Shirley does so by halving the sausage twice. In her third lesson, the context of baker Bas is reintroduced. This baker prepares fruit-tarts with different toppings. Shirley works on the problem of preparing 'Joop's fruit-tart' with $\frac{1}{3}$ pineapple, $\frac{1}{3}$ berries and $\frac{1}{3}$ kiwi. She divides the tart into nine pieces and makes three pieces of kiwi (K), three pieces of berries (B) and three pieces of pineapple (A) (figure 5.6).
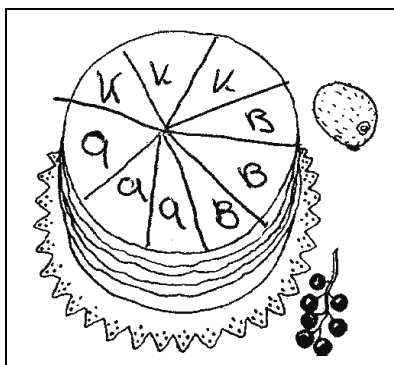
figure 5.6: 'Joop's fruit-tart' with $\frac{1}{3}$ pineapple (A),
$\frac{1}{3}$ berries (B) and $\frac{1}{3}$ kiwi (K)

These constructions in Shirley's second and third lessons look promising, in view of the intended process of formalisation.[4] Let us therefore turn to more formal situations, to see how Shirley uses and generalises these potential equivalence relations.

As we saw, after about twenty lessons Shirley did not use the equivalence of $\frac{2}{3}$ and $\frac{4}{6}$ to compare $\frac{2}{3}$ and $\frac{5}{6}$. In her second interview we asked Shirley to arrange the fractions in three groups: smaller than one half, exactly one half and greater than one half. Unlike her classmates, Shirley did not halve the denominator to compare this result with the numerator, although this strategy could be expected from Shirley's approach when dividing a sausage or fruit-tart. The context of the fruit-tart probably helped her to understand the fractions involved, and supported her in visualising the fractions, which, in turn, created an overview of the situation. However, in comparing bare fractions, she seems to place fractions with large denominators and numerators in the group exceeding one half (figure 5.7). More generally, we observed that Shirley compared fractions by looking at the size of the denominator and the numerator.

figure 5.7: arranging fractions: smaller than a half,
exactly a half and greater than a half

We further observed that Shirley had difficulties with fractions in situations that are less familiar to her. In the fifteenth lesson we introduced the context of the fraction-lift.[5] Here the vertical number line represents a so-called 'fraction house'; which houses a number of fractions

Lifts connect the different floors in the building. The numbers of the lifts indicate the stops they make: for instance, the 3-lift stops three times, at $\frac{1}{3}$, $\frac{2}{3}$ and at the top of the building (at 1). Similarly, the 4-lift stops at $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$ and at 1, the 2-lift stops at $\frac{1}{2}$ and at 1, et cetera (figure 5.8). This context thus makes explicit the different fractions belonging to the same position on the number line.

The fraction-lift was developed as context, where numerator and denominator could be considered separately and where fraction positioning on a number line – by means of the lifts – made considering equivalent fractions necessary. Moreover, the fraction-lift provides for a language to consider fraction positions and therefore facilitates fraction operations (cf. Sfard, 1994). However, the result of this context-construction is also rather abstract, in the sense that it is hardly embedded in meaningful

experiences, apart from those within mathematics as abstract structure itself. We saw that especially low-achievers could be disadvantaged in this situation..
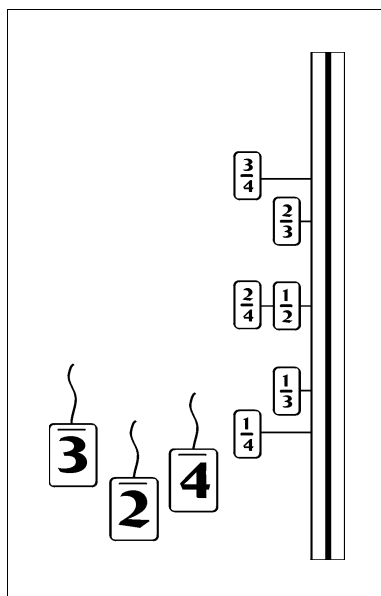
figure 5.8: the fraction-lift, with 2-lift, 3-lift and 4-lift.
The fractions $\frac{1}{2}$ and $\frac{2}{4}$ belong on the same floor.

In the fifteenth lesson we discuss fractions which live at the top of the building. All students know that $\frac{4}{4}$ lives at that highest position. Shirley is eager to name another fraction at the top: $\frac{8}{8}$. She shows she sees regularities and tells the 8-lift will go to $\frac{1}{2}$ and so does the 16-lift. Later in a similar situation we ask Shirley where the 3-lift stops. She thinks the first stop is at $\frac{1}{4}$. To give Shirley some kind of clue, we draw the 9-lift on the blackboard and ask Shirley if this lift stops at $\frac{1}{3}$. Shirley hesitates and finally thinks it does not.

What we see here typifies Shirley's way of dealing with fractions in non-familiar or artificial contexts. If she is asked to show the meaning of fractions within a flexible context, she fails. However, if she recognises regularities in the numbers, she is eager to bring in several other examples. A strategy, which in general can be described as 'doubling', is Shirley's favourite.

In the seventeenth lesson Shirley shows that a few of the fraction relations she constructed by doubling became 'fraction-facts', to be used in suitable situations. In this lesson we introduce a computer version of the fraction-lift. While playing the game, Shirley shows she knows that $\frac{5}{10}$ is halfway the lift-line and that placing a fraction there can be helpful. Furthermore she knows that $\frac{1}{5}$ and $\frac{2}{10}$ are equivalent fractions

and is able to use this knowledge in a meaningful way. Another observation, however, shows the important disadvantage of relying solely on memorised 'fraction-facts'. In the fourteenth lesson we discuss a school-journey with the students. In this context the children are allowed to choose between two attractions. We tell the students that $\frac{3}{5}$ of the students chose the first attraction and $\frac{2}{5}$ the second. Shirley explains why she thinks more children chose the first attraction. She points at the fraction $\frac{2}{5}$: 'This is a half and the other one is more than a half.' Shirley memorised the equivalence of $\frac{2}{5}$ and a half and then used this 'fact' in her reasoning.

In the discussion following Shirley's explanation, Shirley's classmates try to convince her that $\frac{2}{5}$ does not equal $\frac{1}{2}$. Charley explains: '$\frac{2}{5}$ is less than a half. $\frac{2\frac{1}{2}}{5}$ is a half.' He draws a bar on the blackboard to clarify his conclusion. However, it is very difficult for Shirley to use this reasoning in reviewing her answer. Moreover, Shirley has not developed a mechanism to check her answer by using her knowledge of fraction-language ($\frac{2}{5}$ meaning 2 of 5 pieces) or by drawing the situation. In addition, she does not seem to be willing to review her answer, once she has found a way to solve a problem.

Holt (1964) in his numerous observations of children having trouble while learning mathematics, shows a similar misunderstanding by Pat, in adding $\frac{1}{2}$ and $\frac{1}{3}$:

'Pat had the problem $\frac{1}{2} + \frac{1}{3} = ?$ She thought about it a while, then drew two rectangles, each divided in thirds. She shaded two sections of one rectangle, and wrote, "This is $\frac{1}{2}$." Then she shaded one section of the other, and wrote, "This is $\frac{1}{3}$." She looked at them a bit; then she wrote "$\frac{1}{2} + \frac{1}{3} = 1$ whole." And she sat back with a pleased and satisfied look on her face.' (p. 83)

Holt blames the schools – as institutions – in not helping students to attach meaning to such simple addition problems. Similarly, we can blame the fractions programme and its teacher in not being willing or able to help Shirley to understand the fraction language involved in comparing $\frac{2}{5}$ and $\frac{1}{2}$. In spite of all our efforts to develop appropriate and meaningful contexts, at this point in the curriculum we seemed unable to help Shirley develop her fraction language sufficiently to be able to compare other fractions with $\frac{1}{2}$.
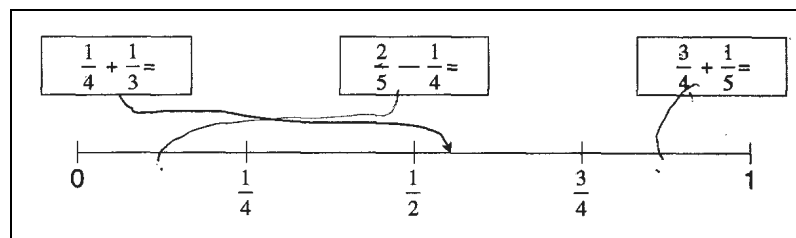
figure 5.9: placing $\frac{2}{5} - \frac{1}{4}$ and $\frac{3}{4} + \frac{1}{5}$ on a number line

After about twenty-five lessons Shirley among other things considers fractions as formal objects, where the denominator decides the kind of object. This supports her in adding and subtracting fractions with equal denominator. She uses this knowledge, together with known relations between simple fractions like $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$, to roughly position the result of sums like $\frac{2}{5} - \frac{1}{4}$ and $\frac{3}{4} + \frac{1}{5}$ on a number line (figure 5.9). In doing this, Shirley, depending on her strategy, shows a reasonable knowledge of the size of the fractions involved.

On the whole we see that Shirley has major difficulties in explaining her approaches and inclines towards instrumental understanding. Booth and Thomas (2000), in their research, found that this is typical for weak performers in mathematics.

> 'The findings of our problem-solving interviews show that some disadvantaged students may encounter difficulties with visual presentation of information, especially when this involves interpretation on their behalf.' (Booth & Thomas, 2000, p. 186)

In familiar contexts, presented visually, we saw that Shirley was able to interpret the situation and could solve connected problems. We found how Shirley experienced difficulties in interpreting the visual presented fraction-lift. Our observations and those of other researchers suggest that familiarity with the context is a more important key to success than the manner in which the problems are visualised (Kraemer, 2000; Featherstone, 2000; Greer & Harel, 1998; Kick & Li, 1996; Mack, 1990; Streefland, 1982; Hart, 1981).

5.5.4 Facing problems

In the thirteenth lesson we present the context of a painting contest. We tell the students that 600 children participated in the contest. We use a bar, to depict the 600 children, when we discuss what number of children used a felt pen ($\frac{1}{2}$ of the participants), what number of children were 4 and 5 year old ($\frac{1}{5}$ of the participants in the contest), et cetera.
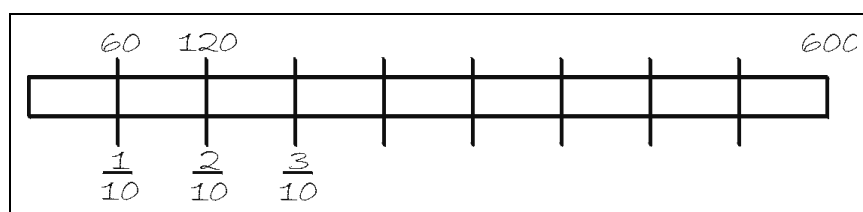
figure 5.10: double-indexed bar

At the end of the lesson we explore divisions in a bar while dealing with fractions with denominator 10. The results are represented on a double-indexed bar. On the bar we write the number of children that fits with $\frac{1}{10}$ and $\frac{2}{10}$, when it is Shirley's turn

(figure 5.10). She is asked what number of children is $\frac{3}{10}$ of the group of 600. Shirley knows she has to add 120 and 60 to solve this problem, but experiences great difficulties in doing so.

> Shirley: 'Ah…240.'
> Teacher: 'Please try one more time.'
> Shirley: '…'
> Teacher (points at the drawn bar): '$\frac{1}{10}$ of the children equals 60 children, $\frac{2}{10}$ equals 120, $\frac{3}{10}$ equals …'
> Shirley: 'It's 240!'
> Other student (whispering): '180.'
> Shirley (repeats): '180.'

From this moment onwards, Shirley experiences difficulties in working with fractions as a result of her weakness in basic number strategies, and especially as a result of her incapacity to see multiplicative number relations. Moreover, from this moment too, there is a shift from developing models from situations to using models for situations where formal manipulations with fractions are needed (Gravemeijer, 1994). These two elements of the fraction programme in which Shirley is involved signify the start of facing major problems in learning fractions. In the next lessons Shirley signals her dislike of the fraction programme. Frequently she scamps her work and on some occasions, when she is working individually, she does not want to be helped by her fellow students or the teacher, she copies her answers from her neighbours and yells out numbers at random to answer questions during the lessons. Our observations of Shirley are consistent in this respect with findings by Deal et al. (2000). In their case study they describe the development in reasoning of Reed, a low achiever in mathematics. Deal et al. found that Reed was unable to construct reasoning on more formal levels. Moreover they state: 'Reed remained hesitant throughout the study, despite the play-like atmosphere and the research team's frequent visits to the school.' (Deal et al., 2000, 25). Under similar conditions we observed the same hesitant reactions from Shirley.

5.5.5 Summary

The fraction programme followed by Shirley started with situational contexts which evoked fractions. Shirley seemed able to deal with the problems as long as informal answers were a possibility. At the same time she started to learn formal fraction language. While doing so, Shirley experienced her first difficulties in the acquisition of fraction language. It took her quite some effort to get the idea of pieces of equal size. However, her proficiency gradually developed and after 30 lessons was able to (instrumentally) add fractions with identical denominators, while still encountering problems in translating fraction-symbols into divisions on bars, circles or a number line. More generally Shirley experienced obstacles in situations where she was asked to order (mental, schematic or physical) objects.

At the next stage of the curriculum, with situational contexts aimed at positioning fractions on a number line and on comparing fractions, Shirley developed strategies that can be characterised as instrumental understanding. In manipulating fractions, she generalised number-patterns, without realising how these referred to the situational contexts underlying the number-patterns. This answers our research question (1) concerning the characteristics of Shirley's formal fraction learning process. When unfamiliar situations are applied, in which real understanding of the nature of fractions and fraction language is needed, Shirley starts dropping out. The process is strengthened by Shirley's limited number strategies and also by her lack of enthusiasm for the topic of fractions. Shirley gradually developed several coping strategies in handling fractions. These strategies constitute an answer to research question (2) concerning key processes showing how Shirley's learning process develops less well, or not at all. These strategies include:

– use drawings and informal approaches to deal with fractions;
– when no way can be found to reflect on her answer, state that the given answer is correct;
– when mathematical connections cannot be made, yell out answers;
– when problems do not make sense, copy answers from others or use the back of the worksheet to make drawings, especially when the context encourages you to do so (figure 5.11) (cf. Schoenfeld, 1992, 359).
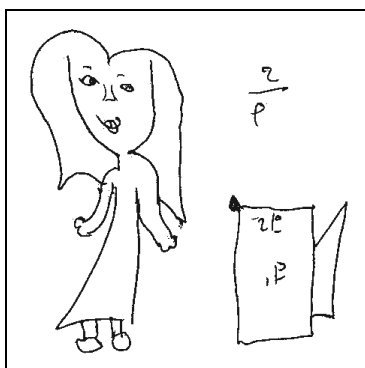
figure 5.11: drawing by Shirley

During the research year we found that Shirley gradually and consequently grew in her knowledge of fractions and observed her – in some sense adequate – coping strategies. This convinced us that we could go along with Shirley, to see how she liked to draw at the back of her worksheet, how she made others laugh, when yelling an answer and how her friends liked to help her with their answers, to protect her from failure.

Shirley – in some sense – failed to cope with our fraction programme, but was able

to survive in her group (cf. Holt, 1964). We therefore found a more or less negative answer to our main research question. We were not able to adapt a problem solving approach in teaching fractions in such a way that both low and high-achievers could benefit from the approach. However, the criteria used to conclude benefiting or not are to be discussed.

5.6 Discussion and conclusion

The long tradition of mathematics teaching has shown that learning formal fractions is not easy, especially for low-achievers in mathematics (Hasemann, 1981). We argued that if we want students to learn (formal) fractions in a meaningful manner, teaching should include discussions between students and teacher to establish meanings (Romberg, 1994). Schoenfeld (1994) provides arguments not to restrict mathematics teaching to rote-learning:

> 'In general, when mathematics is taught as received knowledge rather than as a system that (a) should fit together meaningfully, and (b) should be shared with others, students neither attempt to use it in order to make sense, nor develop it as a means of communication.' (Schoenfeld, 1994, 57)

We indicated that a fraction programme should include meaningful contexts abstracted into useful models, which, in turn, should support the ongoing process of formalisation (Streefland, 1991; Freudenthal, 1991; Gravemeijer, 1994).

We developed a fraction-teaching programme as an extension to an existing programme, the 'Fraction gazette' (Bokhove et al., 1994). In our programme we established classroom discussions and found contexts that generated the number line as a useful model to support the process of formalisation. In our previous studies, we found that in the experimental programme medium and high achieving students were able to develop formal relations between fractions. However, low-achieving students were confronted with serious difficulties (Keijzer & Terwel, 2000; Keijzer & Terwel, in press). In order to get more insight into the underlying factors, the process of a student was described and analysed in a case study. Our point of departure was the following question:

> *Do low-achieving students really benefit from a realistic problem solving approach in acquiring mathematical insights and proficiency in the domain of fractions, and what are the main obstacles in the formalisation process from real life situations to mathematical number sense?*

In the case study described here, we found that Shirley, a low achiever in mathematics, experienced several difficulties in learning fractions as part of our programme.

We analysed the obstacles Shirley found on her way and established from the extensive literature on learning mathematics that Shirley's development is typical for low-achievers. In this way we found indications that low-achieving students use at least two kinds of strategies: constructive and disruptive strategies i.e. productive and counterproductive strategies. The constructive strategies are the ones mentioned by Alexander, Graham and Harris (1998) and Siegler (1991) in their analysis of cognitive strategies. In this category we encounter, for example, the strategy of drawing the situation and using a model, drawing a bar or dividing a circle. However, Shirley, like many low-achievers, often uses disruptive, counterproductive, self-defeating and (even) self-handicapping strategies. These strategies include taking a wild guess and see what happens, make the teacher do the job, cheating and copying answers without understanding. The students thus express anxiety and a fear of failing, which, in turn, is a threat to their self-esteem. In the course of Shirley's year-long learning process we see a kind of shift from productive strategies towards counterproductive ones; and even disruptive strategies as described so vividly by Holt (1964). There is a limit to the effectiveness of providing low-achieving students with strategies and models for handling problems with fractions. The transmission of techniques for thinking and problem solving falls on barren ground, unless anxiety can be reduced and children are given more time to explore fractions in familiar contexts, in a more relaxed pace, under the guidance of the teacher and in interaction with more able peers (Schoenfeld,1985 & 1992; Greeno & Goldman, 1998). We argued that the limits inherent in teaching primary school mathematics to low-achievers provide a convincing argument for setting different priorities in the teaching context. Furthermore, spending more time on the teaching of fractions, for example by devoting special attention to low-achievers, in this case, appears not to be an appropriate choice. Since we are dealing with low-achievers in mathematics, there are probably other topics requiring more serious consideration with regard to how educational time should be spent. Davis (1994) in commenting Romberg (1994) uses a similar argument to differentiate subject-matter in American classrooms:

> 'I do not see how we can achieve really good education for young Americans as long as we insist that nearly all students must learn nearly the same thing. We have created schools that are ineffective, in part because they are *designed* to be ineffective. We need to take a fundamental look at our individual young people and make decisions that are appropriate to each individual one of them.' (Davis, 1994, 318)

In analysing Shirley's learning process we observed her fraction-learning process in depth. We saw how difficulties arose due to Shirley's limited knowledge of number relations, her uncertainty in representing problems and her lack of reflection on her work. In this connection, let us look at Van Streun's (1989) schematised problem solving. In his schematic representation (figure 5.12) we can track Shirley's approach in solving problems with fractions. When her first inspection leads to the

conclusion that she does not recognise the problem, she drops out. However when the first inspection leads to the recognition of a known problem – that is, in Shirley's perception – she then assumes she knows the answer at once, or starts some (usually erratic) algorithmic approach. In other words, Shirley ends up on the left side of Van Streun's diagram, while teaching was concentrated on the right side, i.e., on heuristic approaches.
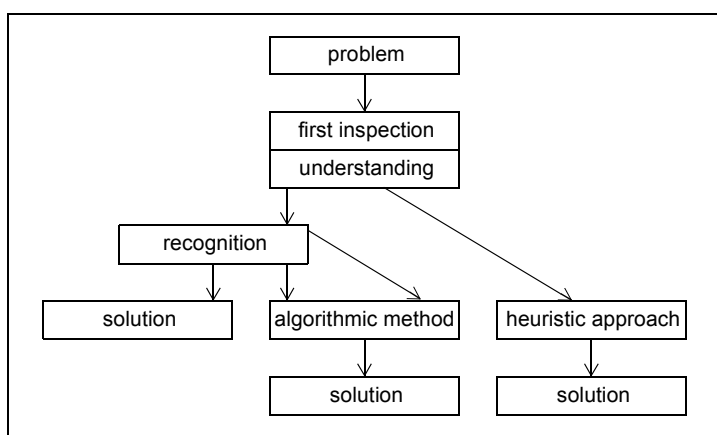
figure 5.12: scheme for solving problems, adapted from Van Streun (1989, 17)

This adds another answer to the question of why low-achievers should experience so much difficulty in the programme described.

Sweller (1994) adds yet another explanation for why fractions should be so troublesome for Shirley and low-achievers like her. Sweller describes how a programme such as the one developed could easily cause 'cognitive overload' in low-achievers. He himself, for example, in focusing on representations of problems and interconnectivities in subject areas, states:

> 'If, as in some areas, interactions between many elements must be learned, then intrinsic cognitive load will be high.' (Sweller, 1994, p. 295)

Sweller proposes to reduce cognitive load by means of improved isolating skills and strategies. However, as Schoenfeld (1994) points out, the interconnectivity is inherent in learning mathematics since it is about mathematisation, abstraction and understanding structure:

> 'Learning to think mathematically means (a) developing a mathematical point of view – valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade and using those tools in the service of the goal of understanding structure – mathematical sense-making.' (Schoenfeld, 1994, 60)

We only analysed the learning of one student. We argued that therefore our conclusions cannot be easily generalised to the whole group of low-achievers. However, as the observed patterns in Shirley's learning are typical for many low-achievers, we are convinced that Shirley and low-achievers like her are trapped, on the one hand, between the nature of the mathematics learning as proposed by Schoenfeld and as elaborated by us in our fraction programme, and, on the other hand, the limits of their mathematical abilities.

If we want to establish 'mathematics for all' we should set priorities for all students. And, in the case of low-achievers, this might well lead to limited attention to fractions, in order to enable these students to develop the mathematics that suit their aptitudes (cf. Kraemer, Van der Schoot & Engelen, 2000). Or, as Doornbos (1997) pleads:

> 'In primary education (…) an exhaustive list of unequivocally formulated standards – aims for all students to be pursued – is superfluous and mistaken. We are talking about the education of children of school age. Also, children who experience temporary learning difficulties, or whose ability to learn is limited should be made to feel welcome, without being discriminated against.' (Doornbos, 1997, p. 26)[6]

Shirley and low-achievers like her should be able to feel accepted at their school. Teaching her formal fractions, in which she is required to discuss formal relations that are obscure to her, and forcing her to construct models that do not help her to gain the required insights should not be part of her curriculum.

We propose that uniform standards should be reconsidered, and that we should abandon the idea in primary education that, with a very small number of exceptions, all students should be required to learn the same things. Students who cannot learn formal mathematics should be made to feel welcome, since they have a right to experience mathematics on a level they can understand and use in daily life. The policies advocated by those in the public arena who almost obsessively talk of uniform standards should be regarded as unrealistic and even counter productive.

We need to look for a 'sounder' model of learner growth and academic development, especially in mathematics education (Alexander, 2000). This point of view is not only based on the experience of the large differences in the acquisition of mathematics observed in various studies in mathematics education (Terwel, 1990; Hoek, Terwel & Van den Eeden, 1997; Hoek, Van den Eeden & Terwel, 1999), but also on the theories and views of scholars who have encountered individual students like Shirley in their research, in their classes or in their tutorial interactions (Davis, 1994; Freudenthal, 1973, 1991; Doornbos, 1997; Gravemeijer & Terwel, 2000).

It is important to point out here that our views should not be understood as a plea for early selection, ability grouping or streaming. On the contrary, in our opinion the issue of how to organise teaching in such a way that all students can benefit is still

very much open to resolution (we will address this issue later). Finally, we agree with Freudenthal, (1973) who was strongly committed to mathematics as a human activity, that both high and low-achieving students should be included in the community of learners.

Notes

1 Translated from the original Dutch text.
2 Translated from the original Dutch text.
3 We were well aware that in the research we were responsible for Shirley's development. We, of course, did not want to do anything that might harm her. In that case we would have removed her from the programme immediately.
4 However, the division of the fruit-tart could well be the result of a misconception of the problem. Shirley here could have interpreted that $\frac{1}{3}$ pineapple, $\frac{1}{3}$ berries and $\frac{1}{3}$ kiwi meant 3 pieces of each, since three pieces is her meaning of $\frac{1}{3}$.
5 The fraction-lift is an idea of Adrian Treffers.
6 Translated from the original Dutch text.

References

Alexander, P.A., S. Graham & K.R. Harris (1998). A Perspective on Strategy Research: Progress and Prospects. *Educational Psychology Review*, *10*(2), 129-151.

Alexander, P.A. (2000). Toward a Model of Academic Development: Schooling and the Acquisition of Knowledge. *Educational researcher*, *29*(2), 28-34.

Behr, M.J., R. Lesh, T.R. Post & E.A. Silver (1983). Rational-Number Concepts. In: Richard Lesh and Marsha Landau (eds.). *Acquisition of Mathematical Concepts and Processes*. New York/London: Academic Press, 91-126.

Bergeron, A., N. Herscovics & J.C. Bergeron (1987). Kindergartners' knowledge of numbers: A longitudinal study. Part II: Abstraction and formalization. In: J.C. Bergeron, N. Herscovics & C. Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education, Volume II*. Montreal: s.n., 352-360.

Bezuk, N.S. & M. Bieck (1993). Current Research on Rational Numbers and Common Fractions: Summary and Implications for Teachers. In: Douglas T. Owens (eds.). *Research Ideas for the Classroom – Middle Grades Mathematics*. New York: Macmillan Publishing Company, 118-136.

Bokhove, J., K. Buys (ed.), R. Keijzer, A. Lek, A. Noteboom & A. Treffers (1996). *De Breukenbode. Een leergang voor de basisschool* (werkbladen en handleiding) [The Fraction gazette]. Enschede/Utrecht: SLO/FI/Cito.

Booth, R.D.L. & M.O.J. Thomas (2000). Visualization in Mathematics Learning: Arithmetic Problem-solving and Student Difficulties. *Journal of Mathematical Behavior*, *18*(2), 169-190.

Brenner, M.E., R.E. Mayer, B. Moseley, T. Brar, R. Durán, B. Smith Reed & D. Webb (1997). Learning by Understanding: The Role of Multiple Representations in Learning Algebra. *American Educational Research Journal*, *34*(4), 663-689.

Burton, L. (with the help of Mark Burton) (1980a). Problems and Puzzles. *For the Learning of Mathematics*, *1*(2), 20-23.

Burton, L. (1980b). The teaching of mathematics to young children using a problem solving approach. *Educational Studies in Mathematics*, *11*, 43-58.

Carraher, D.W. & A.D. Schliemann (1991). Children's understanding of fractions as expressions of relative magnitude. In: Fulvia Furinghetti (ed.). *Proceedings of the fifteenth PME conference*, *Volume 1*. Assisi, Italy: Dipartimento di Matematica dell'Università di Genova, 184-191.

Clements, M.A. (1980). Analyzing children's errors on written mathematical tasks. *Educational Studies in Mathematics*, *11*, 1-21.

Cobb, P. & J.W. Whitenack (1996). A Method for Conducting Longitudinal Analyses of Classroom Videorecordings and Transcripts. *Educational Studies in Mathematics*, *30*, 213-228.

Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education] (1994). *Doelbewust leren; kerndoelen in maatschappelijk perspectief* [Purposeful learning; curriculum standards in social perspective]. Den Haag: SDU.

Connell, M.L. & D.M. Peck (1993). Report of a Conceptual Intervention in Elementary Mathematics. *Journal of Mathematical Behavior*, *12*, 329-350.

Davis, P.J. & R. Hersh (1984). *The Mathematical Experience*. London: Penguin Books.

Davis, R.B. (1994). Comments on Thomas Romberg's Chapter. In: Alan H. Schoenfeld (ed.). *Mathematical thinking and problem solving*. Hilldale, NJ: Lawrence Erlbaum Associates, Publishers, 312-321.

Deal, D., M.M. Buehl, M.G.O. Tron, L. Harris, S. Clements & S. Hardy (2000). *Portraying Mathematical and Analogical Reasoning in the Young: Three Case Studies.* Paper presented at the annual meeting of the American Educational Research Association, New Orleans, April 2000.

Doornbos, K. (1997). *De pedagogische balans* [The pedagogical balance]. Groningen: Wolters-Noordhoff.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In: David Tall (ed.). *Advanced mathematical thinking*. Dordrecht/Boston/London: Kluwer Academic Publishers, 95-123.

Easley Jr., J.A. (1981). Levels of abstraction and intellectual development. *The Journal of Children's Mathematical Behavior*, *1*(2), 76-83.

Erlwanger, S.H. (1973). Benny's conception of Rules and Answers in IPI Mathematics. *Childern's Mathematical Behavior*, *1*(2), 7-26.

Featherstone, H. (2000). '− Pat + Pat = 0': Intellectual Play in Elementary Mathematics. *For the Learning of Mathematics*, *20*(2), 14-23.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: Reidel.

Freudenthal, H. (1991). *Revisiting Mathematics Education, China lectures*. Dordrecht: Kluwer Academic Publishers.

Fosnot, C.T. & M. Dolk, M. (2001). *Young Mathematicians at Work: Constructing the Number System, Addition and Subtraction*. Portsmouth, NH: Heinemann Press.

Goldin, G.A. (1998). Representational Systems, Learning, and Problem Solving in Mathematics. *Journal of Mathematical Behavior*, *17*(2), 137-165.

Gravemeijer, K.P.E. (1994). *Developing realistic mathematics education* (dissertation). Utrecht: CD-ß Press.

Gravemeijer, K.P.E. & J. Terwel (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of Curriculum Studies*, *32*(6), 777-796.

Greeno, J.G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education*, *22*(3), 170-218.

Greeno, J.G. (1997). On claims That Answer the Wrong Questions. *Educational Researcher*,

26(1), 5-17.

Greeno, J. G. & S.V. Goldman (1998). *Thinking Practices in Mathematics and Science Learning*. Mahwah, New Jersey/London: Lawrence Erlbaum Associates, Publishers.

Greer, B. & G. Harel (1998). The Role of isomorphisms in Mathematical Cognition. *Journal of Mathematical Behavior*, *17*(1), 5-24.

Gudmundsdottir, S. (1995). The narrative nature of pedagogical content knowledge. In: H. McEwan & K. Egan (eds.). *Narrative in teaching, learning and, research*. New York: Teachers College, Columbia University, 24-38.

Hart, K.M. (1981). Fractions. In: K.M. Hart (ed.). *Children's understanding of mathematics: 11-16*. London: John Murray, 66-81.

Hart, K.M. (1987). Practical work and formalisation, too great a gap. In: Jacques C. Bergeron, Nicolas Herscovics & Carolyn Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Volume II*. Montreal: s.n., 408-415.

Hasemann, K. (1981). On difficulties with fractions. *Educational Studies in Mathematics*, *12*, 71-87.

Have, P. ten (1977). *Sociologisch veldonderzoek* [Sociological field-research]. Meppel: Boom.

Heuvel-Panhuizen, M. van den (1996). *Assessment and realistic mathematics education* (dissertation). Utrecht: CD-ß Press.

Hoek, D., J. Terwel & P. van den Eeden (1997). Effects of Training in the Use of Social and Cognitive Strategies: An Intervention Study in Secondary Mathematics in Co-operative Groups. *Educational Research and Evaluation*, *3*(4), 364-389.

Hoek, D., P. van den Eeden & J. Terwel (1999). The effects of integrated social and cognitive strategy instruction on the mathematics achievement in secondary education. *Learning and Instruction*, *9*(5), 427-448.

Holt, J. (1964). *How Children Fail*. New York: Dell Publishing Co.

Huitema, S., A. van der Klis, F. van de Molengraaf, M. Timmermans & L. Erich (n.y.). *De wereld in getallen 6. Handleiding a & b* [The world in numbers 6. Teacherguide a & b]. Den Bosch: Malmberg.

Hunting, R.P. (1983). Alan: A case study of knowledge of units and performance with fractions. *Journal for Research in Mathematics Education*, *14*(3), 182-197.

Janssen, J., J.-M. Kraemer & A. Noteboom, (1995). *Leerling Volg Systeem. Rekenen-Wiskunde 2* [Student Registration System. Mathematics 2]. Arnhem: Cito.

Janssen, J., F. van der Schoot, B. Hemker & N. Verhelst (1999). *Balans van het reken-wiskundeonderwijs in de basisschool 3* [Balance-sheet of mathematics education in primary school 3]. Arnhem: Cito.

Keijzer, R. (1994). De ontwikkeling van een breukenleergang – een voorbeeld van ontwikkelingsonderzoek [The development of a curriculum on fractions – an example of developmental research]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *13*(1), 24-31.

Keijzer, R. & A. Lek (1995). Breuken: kerndoelen en formaliseren [Fractions: curriculum standards and formalising]. In: C. van den Boer & M. Dolk (eds.). *Rekenen in de bovenbouw van de basisschool*. Utrecht: Panama/HvU, Freudenthal Institute, 53-64.

Keijzer, R. & K. Buys(1996). Groter of kleiner. Een doorkijkje door een nieuwe leergang breuken [Bigger or smaller. A close look at a new curriculum on fractions]. *Willem Bartjens*, *15*(3), 10-17.

Keijzer, R. (1997). Formeel rekenen met breuken [Formal arithmetic with fractions]. In: C.

van den Boer & M. Dolk (eds.). *Naar een balans in de reken-wiskundeles – interactie, oefenen, uitleggen en zelfstandig werken –*, Utrecht: Panama/Freudenthal Institute, 101-116.

Keijzer, R. (1999). Een klein verschil? [A small difference?]. In: W. Faes & W. Oonk (eds.). *Van Rekenend Nederland voor Fred Goffree. 65/2 Onderwijsverhalen voor pabostudenten.* Groningen: Wolters-Noordhoff, 37-40.

Keijzer, R. & J. Terwel (2000). *Learning for mathematical insight: a longitudinal comparison of two Dutch curricula on fractions.* Paper presented at the annual meeting of the American Educational Research Association, New Orleans, April, 2000.

Keijzer, R. & J. Terwel (2001). Audrey's acquisition of fractions: a case study into the learning of formal mathematics. *Educational Studies in Mathematics*, *47*(1), 53-73.

Keijzer, R. & J. Terwel (in press). Learning for mathematical insight: a longitudinal comparative study on modelling. *Learning & Instruction.*

Kerckhoff, A.C. & E. Glennie (1999). The Matthew effect in American education. In: Aaron M. Pallas (ed.). *Research in sociology of education and socialization, Volume 12.* Stamford: Connecticut: Jai Press Inc., 35-66.

Koch, L.C. & X. Li (1996). Students' Understanding of Computation-Related Rational Number Skills. *Journal of Mathematical Behavior*, *15*, 193-205.

Kraemer, J.-M. & J. Janssen (2000). Druppels op een gloeiende plaat – aandachtspunten voor een grondige discussie over de continue ontwikkeling van rekenzwakke leerlingen tussen vier en zestien jaar [Drops in the ocean – points of attention for a thorough discussion on the continuous development of low-attainers in arithmetic between four and sixteen years of age]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *18*(3), 3-14.

Kraemer, J.-M. (2000). Met handen en voeten rekenen. Naar een natuurlijke benadering van rekenzwakke kinderen [Arithmetic with hands and feet. Towards a natural approach for low arithmetic achievers]. *Vernieuwing. Tijdschrift voor onderwijs en opvoeding*, *59*(1/2), 29-32.

Kraemer, J.-M., F. van der Schoot & R. Engelen (2000). *Balans van het reken-wiskunde-onderwijs op LOM- en MLK-scholen 2: uitkomsten van de tweede peiling in 1997* [Balance-sheet of mathematics education in special schools 2: results of the second gauging in 1997]. Arnhem: Cito.

Krutetskii, V.A. (1976 (English edition)). *The psychology of Mathematical Abilities in Schoolchildren* (translated by Joan Teller, edited by Jeremy Kilpatrick and Izaak Wirszup). Chicago/London: The University of Chicago Press.

Lemoyne, G. & C. Tremblay (1986). Addition and multiplication: problem-solving and interpretation of relevant data. *Educational Studies in Mathematics*, *17*, 97-123.

Lieshout, E.C.D.M. van (1997). What can research on word and context problems tell about effective strategies to solve subtraction problems? In: M. Beishuizen, K.P.E. Gravemeijer & E.C.D.M van Lieshout (eds.). *The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures*. Utrecht: CD-ß Press, 79-111.

Luit, J.E.H. van & B.A.M. van de Rijt (1997). Stimulation of early mathematical competence. In: M. Beishuizen, K.P.E. Gravemeijer & E.C.D.M. van Lieshout (eds.). *The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures*. Utrecht: CD-ß Press, 215-237.

Mack, N.K. (1990). Learning fractions with understanding: building on informal knowledge. *Journal for Research in Mathematics Education*, *21*(1), 16-32.

Mack, N.K. (1995). Confounding whole-number and fraction concepts when building on

informal knowledge. *Journal for Research in Mathematics Education*, *26*(5), 422-441.

Mack, N.K. (2000). *Long-term effects of building on informal knowledge in a complex content domain: the case of multiplication of fractions*. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, April 2000.

McIntosh, A., B.J. Reys & R.E. Reys (1992). A Proposed Framework for Examining Basic Number Sense. *For the Learning of Mathematics*, *12*(3), 2-8.

Mead, M. (1959). *Coming of age in Samoa: a psychological study of primitive youth for Western civilisation*. New York: New American Library.

Moss, J. & R. Case (1999). Developing Children's Understanding of the Rational Numbers: A New Model and an Experimental Curriculum. *Journal for Research in Mathematics Education*, *30*(2), 122-147.

Nelissen, J.M.C. (1998a). Representaties in het reken-wiskundeonderwijs [Representations in mathematics education]. *Pedagogische Studiën*, *75*, 169-183.

Nelissen, J.M.C. (1998b). Taal en betekenis in het realistisch reken-wiskundeonderwijs [Language and meaning in realistic mathematics education]. In: N. Boswinkel & M. Dolk (eds.). *Over rekenen gesproken – taal in/en rekenen –* . Utrecht: Panama/Freudenthal Institute, 11-34.

Piaget, J. (1973). Comments on mathematical education. In: A.G. Howson (ed.). *Developments in Mathematical Education. Proceedings of the Second International Congress on Mathematical Education*. Cambridge: Cambridge University Press, 79-87.

Reigeluth, Ch.M. (1999). What Is Instructional-Design Theory and How Is It Changing. In: Charles M. Reigeluth (ed.). *Instructional-design theories and models Volume II. A new paradigm of instructional theory*. Mahwah, NJ: Lawrence Erlbaum Associates, Publishers, 5-30.

Romberg, T.A. (1994). Classroom Instruction That Fosters Mathematical Thinking and Problem Solving: Connections Between Theory and Practice. In: Alan H. Schoenfeld (ed.). *Mathematical thinking and problem solving*. Hilldale, NJ: Lawrence Erlbaum Associates, Publishers, 287-304.

Schoenfeld, A.H. (1985). *Mathematical problem solving*. San Diego, CA: Academic Press.

Schoenfeld, A.H. (1992). Learning to think mathematically: problem solving, metacognition, and sense making in mathematics. In: Douglas A. Grouws (ed.). *Handbook of research on mathematics teaching and learning*. New York: Macmillan Publishing Company, 334-370.

Schoenfeld, A.H. (1994). Reflections on Doing and Teaching Mathematics. In: Alan H. Schoenfeld. *Mathematical thinking and problem solving*. Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers, 53-70.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics*, *14*(1), 44-56.

Siegler, R.S. (1991). Strategy choice and strategy discovery. *Learning and Instruction*, *1*(1), 89-102.

Streefland, L. (1982). Subtracting fractions with different denominators. *Educational Studies in Mathematics*, *13*, 233-255.

Streefland, L. (1983). *Aanzet tot een nieuwe breukendidactiek volgens wiskobas* [Impuls for a new pedagogy on fractions according to wiskobas]. Utrecht: OW&OC.

Streefland, L. (1987). Free production of fraction monographs. In: J.C. Bergeron, N. Herscovics & C. Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Volume I*. Montreal: s.n., 405-410.

Streefland, L. (1991). *Fractions in Realistic Mathematics Education, a Paradigm of Develop-*

mental Research. Dordrecht: Kluwer Academic Publishers.

Streefland, L. & E. Elbers (1995). Interactief realistisch reken-wiskundeonderwijs werkt [Interactive realistic mathematics education works]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *14*(1), 12-21.

Streefland, L. & E. Elbers (1997). De klas als onderzoeksgemeenschap [Class as research community]. In: C. van den Boer & M. Dolk (eds.). *Naar een balans in de reken-wiskundeles – interactie, oefenen, uitleggen en zelfstandig werken – *. Utrecht: Freudenthal Institute, 11-24.

Streefland, L. (1997). Een geval van reflectief denken in ontwikkeling met verhoudingen als paradigma (slot) [A case of developing reflective thinking with ratio as paradigm]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *15*(3), 22-31.

Streun, A. van (1989). *Heuristisch wiskunde-onderwijs – verslag van een onderwijsexperiment* [Heuristic Mathematics Education] (dissertation). Groningen: Rijksuniversiteit Groningen.

Sweller, J. (1994). Cognitive load theory, learning difficulty, and instructional design. *Learning and Instruction*, *4*, 295-312.

Terwel, J. (1990). Real Maths in Cooperative Groups in Secondary Education. In: N. Davidson (ed.). *Cooperative Learning in Mathematics*. Menlo Park: Addison-Wesley, 228-264.

Treffers, A. (1987). *Three dimensions. A model of Goal and Theory Description in Mathematics Instruction – the Wiskobas Project*. Dordrecht: Reidel Publishing Company.

Tzur, R. (1999). An Integrated Study of Children's Construction of Improper Fractions and the Teacher's Role in Promoting That Learning. *Journal for Research in Mathematics Education*, *30*(4), 390-416.

Verschaffel, L. (1995). Beïnvloeden van leerprocessen [Influencing of learning processes]. In: J. Lowyck & N. Verloop (eds.). *Onderwijskunde. Een kennisbasis voor professionals*. Groningen: Wolters-Noordhoff, 153-187.

Watson, J.M., K.J. Cambell & K.F. Collis (1993). Multimodal Functioning in Understanding Fractions. *Journal of Mathematical Behavior*, *12*, 45-62.

Yin, R.K. (1984). *Case study research. Designs and Methods*. Beverly Hills, CA: SAGE Publications.

6 Effects of an experimental fraction programme in primary mathematics: a longitudinal analysis

Abstract

This chapter describes a fraction programme's learning effects in primary school mathematics during one whole school year. These effects were established in a longitudinal control group design. The experimental programme aims at model development by constructing a bar and empty number line. Moreover, in the group working with the experimental programme meanings of mathematical constructions are negotiated in whole class discussions. In contrast, in the control programme, a regular widely used fraction programme, dividing circles and fair sharing are fraction generating activities. In the control group, the group working with the control programme, students work individually.

The study is a small scale study, as it aims at a thorough analysis of students' learning processes. However, in comparing learning processes and outcomes in the two fraction programmes we also conducted quantitative analyses. These analyses on the one hand enabled us to determine the effects of the researched fraction programmes, and on the other hand provided us with means to interpret our more qualitative results.

This chapter offers the study's quantitative results and discusses the effects of the experimental programme as compared to the control condition. No significant general effect on the learning outcomes were found. However, some clear longitudinal trends in the hypothesised direction could be determined. In addition significant interaction effects were found between pre-knowledge and condition. A clarifying theoretical analysis is provided to explain these programme effects.

The study reported on here is a small scale study ($n = 20$). Quantitative results that were found are discussed in relation to the study's scale.

6.1 Theoretical background

From the 1960's the New Math movement developed and disseminated its educational paradigm world-wide. Mathematics was presented as formal construction. Students were to learn mathematics within formal structures, without clear references to recognisable situations (cf. Bruner, 1966). It was Freudenthal (1968) who opposed this movement and made that New Math hardly reached Dutch schools. With the Wiskobas-group he developed what was going to be named realistic mathematics education (RME) (Treffers, 1987).

More than thirty years ago, learning of mathematics in 'real-life' contexts, guided reinvention and constructing mathematics as human activity formed RME's basis (Freudenthal, 1973; Gravemeijer & Terwel, 2000) and mathematising was established as a major learners' activity (Gravemeijer, 1994; De Corte, Greer & Verschaffel, 1996; Gravemeijer, 2001). RME teaching and learning starts with recognisable contexts. These meaningful situations, in time, are mathematised to form more formal relations and abstract structures (Van den Heuvel-Panhuizen, 1996). Moreover, discussions among students and between students and teacher make that mathematical constructions are explicated to form more efficient approaches and more general notions (Treffers, De Moor & Feijs, 1989). In this respect RME shares important similarities with ideas in constructivism (Greeno, 1991).

Two fraction programmes are involved in the study reported upon here. Both these programmes were constructed as RME programmes. However, there are important differences between the programmes, both in teaching the topic of fractions and in teaching styles. In the programme in the experimental group (see appendix A), students are asked to measure objects with a paper bar, to then discuss the students' measurements in terms of fractions. From these activities the bar and number line are developed as models for fractions. Equivalent fractions, which form the basis of fraction operations, now come forward as fractions that share their position on the number line. The control programme emphasises fair sharing and dividing circles as fraction generating activities. Equivalent fractions are presented as equivalent sharing results or equivalent subdivisions of the circle (Keijzer & Terwel, 2000; Keijzer & Terwel, in press; see also chapter 3). The researcher takes the teacher role in the fraction programme in the experimental group. He stimulates whole class discussions to establish shared understanding of fractions. The control group uses a widely used fraction programme in a frequently used text-book series. In the control group students work individually and hardly have the chance to discuss constructions with each other.

Although there is a substantial body of evidence from qualitative case studies that many students profit from RME programmes (e.g. Streefland, 1991; Menne, 2001), there is a lack of quantitative data supporting this claim. Moreover, there are indications that low achievers in some cases do not profit from the RME approach (Van den Eeden & Terwel, 1994; Janssen, Van der Schoot, Hemker, Verhelst, 1999; Vedder, 2002). This chapter will provide the outcomes of a study on fraction learning in a RME-setting. The quantitative data clarify the development of students in the experimental and control groups. The focus is on acquired fraction strategies as well as on general mathematics strategies. Moreover, differences in student development are discussed.

6.2   Research question

This study researches the feasibility and effectiveness of a newly developed fraction programme in primary school as compared to a widely used RME programme (used in a somewhat traditional manner).

The control programme emphasises fair sharing and dividing circles as fraction generating activities. However, there are strong indications that students are not prepared for uncovering fraction relations by dividing circles, as many divisions are difficult to obtain. Further, fair sharing – regarding $\frac{3}{5}$ as three pizza's divided by five children – does not clearly present a fraction as one number or entity, but rather presents a fraction as (a ratio of) two numbers (cf. Streefland, 1991). Moreover, as both dividing circles and fair sharing do not present fractions as numbers 'between the integer numbers', it is difficult for students to relate their knowledge on integers to fractions.

Therefore a new experimental programme was developed to overcome these problems. As in this experimental programme fractions are presented as folded bars and numbers on the number line, fractions are presented as (single) numbers between integers. As a bar is easier divided than a circle, many divisions are obtained and important fraction relations come forward rather easily (cf. Connell & Peck, 1993, Moss & Case, 1999). This facilitates comparing fractions and – later – fraction operations (Armstrong & Novillis Larson, 1995).

In this light the study's specific research question is:

> *What are the effects in terms of fraction learning and in terms of learning mathematics in general of an experimental programme in fractions, where the bar and the number line are central models and where understanding is established in whole class discussions, as compared to a more traditional control programme, where the circle and fair sharing are the main models for fractions and where students mainly work individually?*

6.3   Hypothesis

As we mentioned, the development of the empty number line and the bar as models for fractions is one of the experimental programme's key-aspects. These models make that fraction learning is related to the learning of whole numbers (cf. Klein, 1998; Klein, Beishuizen & Treffers, 1998). Furthermore, as the number line and bar facilitate global operating with fractions, these models support number sense acquisition (Mcintosh, Reys & Reys, 1992). Moreover, the experimental group learned fractions in a educational setting where meanings are negotiated in whole class discussions. The bar and empty number line models were introduced and discussed to elicit relevant dialogues between students and between students and teacher to thus

promote fraction learning. In this way choices made in the experimental condition concerning the educational setting and the fraction programme form an entity.

If we reckon discourse between students and teacher and between students to be essential for learning mathematics, there might be a transfer from fraction learning to learning other topics in the mathematics curriculum in the experimental group. For example, if a student learns to explain his or her approach to clarify the situation from the discussions on fractions, this student might use this in other situations. Furthermore, as fraction learning in the experimental fraction programme relates fractions and measuring, the programme might support students in measuring situations or – more generally – situations where a linear structure is underlying. Moreover, as making estimations, global reasoning and explaining formal relations is embedded in the experimental programme there could be a transfer of these strategies from fraction learning to other subjects in mathematics.

As a consequence our hypothesis not only aims at fraction learning in the two programmes, but also on the development of general mathematical strategies. Against this background the following hypothesis was formulated:

> *Students in the experimental condition outperform students in the control condition in fraction strategies and in general mathematical strategies.*

6.4 Methods

In this research project we followed two groups of students from grade 6 (10-11 year) during one school-year. At that time the students attended a school north of Amsterdam. To compare the mathematisation processes in the two groups and to characterise obstacles in learning mathematics, a quasi-experimental research design was used (Cook & Campbell, 1979), more precisely a non-equivalent pre-test post-test control group design, where students in the experimental group follow the experimental curriculum and students in the control group follow a widely used fraction curriculum. Moreover, the experimental group's regular teacher took care of other topics in the mathematics curriculum. In doing so, the students worked on these topics individually from the same widely used text book as was used in the control group.

O_1	Xe_1	I_1	Xe_2	O_2	Xe_3	I_2	Xe_4	O_3	I_3
O_1	Xc_1	I_1	Xc_2	O_2	Xc_3	I_2	Xc_4	O_3	I_3

table 6.1: specification of the research design

In table 6.1 a specification of the research design is presented. In this design O_1, O_2 and O_3 are standardised tests to obtain students' general mathematical strategies at

the start, halfway through and at the end of the research year. Three standardised interviews, I_1, I_2 and I_3 were held to determine students' fraction knowledge at the start, halfway through and at the end of the research year. In addition, direct observations during fraction instruction in both the experimental (Xe$_1$, Xe$_2$, Xe$_3$, Xe$_4$) and control group (Xc$_1$, Xc$_2$, Xc$_3$, Xc$_4$) were conducted to provide additional information on student development.

On the basis of a pre-test, we matched students from the experimental group one-on-one with students in the control group. The results from the pre-tests in 'numbers and operations' and in 'measuring and geometry' were used to do so. Moreover, the regular teachers of the two groups provided arguments for making a fair match. These arguments included attitude towards mathematics, general behaviour in class and communicative skills. The matching procedure resulted in two equivalent groups at the start of the study.

We used several instruments to compare students' outcomes in the experimental and control group. Six of these instruments aimed at uncovering students' general mathematical strategies. Three instruments were used to establishing students' fraction strategies.

The following instruments (Janssen, Kraemer & Noteboom, 1995) were used to measure students' general mathematics strategies:
– a pre-test 'numbers and operations' (O_1, at the start of the experiment);
– a pre-test 'measuring and geometry' (O_1, at the start of the experiment);
– a medio-test 'numbers and operations' (O_2, five months after the start of the experiment, after 15 weekly lessons in fractions);
– a medio-test 'measuring and geometry' (O_2, five months after the start of the experiment, after 15 weekly lessons in fractions);
– a post-test 'numbers and operations' (O_3, ten months after the start of the experiment, after 30 weekly lessons in fractions);
– a post-test 'measuring and geometry' (O_3, ten months after the start of the experiment, after 30 weekly lessons in fractions)

The medio- and post-tests O_2 and O_3 are transfer tests as they are used to determine to what extent fraction learning in the experimental and control condition influences mathematics learning in general. The two tests in O_1 function as co-variables, as these tests provide a measure of how the students gain general mathematical strategies from O_1 to O_2 and from O_1 to O_3.

Three interviews were developed and conducted to uncover students' fraction strategies. The following interviews were held:

– interview 'fraction language' (I_1, three months after the start of the experiment, after 10 weekly lessons in fractions);

- interview 'comparing fractions' (I_2, six months after the start of the experiment, after 20 weekly lessons in fractions);
- interview 'operating with fractions' (I_3, ten months after the start of the experiment, after 30 weekly lessons in fractions).

The interview-setting enabled us to provide students with standardised help, if the student needed this. This help was of a heuristic nature. If help was offered, students were asked to explain their approach and they were stimulated to check answers given, for example by making a sketch of the situation. Moreover, interviews offer opportunities to perform qualitative analyses of student approaches. We described these qualitative analyses elsewhere (Keijzer & Terwel, 2000, 2001 and in press; see also chapter 3, 4 and 5). This chapter mainly focuses on quantitative results of the interviews and the pre-, medio- and post-tests.

In the next paragraph, quantitative data analyses from the study reported on here will be presented. In these analyses we will focus on differences between the experimental and control condition in paired t-tests. In order to form student pairs where the two students in one pair share essential characteristics concerning learning mathematics, pre-tests were used in the matching procedure. Moreover, the focus will be on the development of low achieving students in comparison to their high achieving peers. To allow us to do so, we partitioned the groups in two: five pairs of low achievers and five pairs of high achievers. We next analysed the developments of the student pairs that were the lowest performers (in the pre-tests) and – analogously – the student pairs that were the highest performers (in the pre-tests). Due to space limitations these analyses are not presented here, but will be reported on in comments on overall results.

In order to control for small, non-significant initial differences in general mathematical strategies at the start of the experiment, regression analysis was used. Regression analysis therefore is more accurate then the paired t-tests in determine programme effects. Because of our interest in possible differential effects on high and low achieving students, regression analysis was also used to analyse possible interaction effects between pre-test and condition: how high and low achieving students benefit differently from learning mathematics in a setting where meaning is negotiated and in which the bar and empty number line are used as models in a process of mathematising and formalisation. At the end of the results section we will report on these analyses.

6.5 Results

6.5.1 Characteristics of the distribution of general mathematics strategies

Six instruments are used to determine the characteristics of the students' general

mathematics strategies, two pre-tests, two medio-tests and two post-tests. The first tests are the two pre-tests, one in 'numbers and operations' and one in 'measuring and geometry'. The pre-tests are used to match students in the two student groups. Moreover, these tests provide a reference point for a longitudinal analysis of student development. The second and third tests also concern 'numbers and operations' and 'measuring and geometry', and serve as medio- and post-tests.

First general mathematics tests (pre-tests)

Pre-tests provide characteristics of general mathematical strategies at the start of the experiment. Table 6.2 shows the characteristics of these distributions of general mathematical strategies.

	Mean	SD	Min	Max	Sig.
Control programme (*n* = 10)					
pre-test 'numbers and operations'	51.10	11.78	31	69	
pre-test 'measuring and geometry'	52.30	10.03	38	69	
Experimental programme (*n* = 10)					
pre-test 'numbers and operations'	52.40	12.99	29	69	
pre-test 'measuring and geometry'	54.40	12.32	33	73	
Paired differences, cont. – exp. (*n* = 10)					
pre-test 'numbers and operations'	-1.30	5.52	-12	6	n.s. (0.48)
pre-test 'measuring and geometry'	-2.10	7.03	-10	8	n.s. (0.37)

Note: Cronbach Alphas for internal consistency for these instruments are 0.9204 for the test 'numbers and operations' and 0.8277 for the test 'measuring and geometry'. Number of items in the tests: 48 (numbers and operations) and 36 (measuring and geometry). We followed the guidelines for test scoring that were issued with the test (Janssen, Kraemer & Noteboom, 1995).

table 6.2: characteristics of the distributions of general mathematical strategies and paired t-test, pre-tests

The figures in the table show that there seems to be a slight difference in favour of the experimental group, due to little differences between high performers. However, these data indicate that the selection procedure resulted in two groups that do not differ significantly on the pre-test and, furthermore, are similar in many respects.

Medio- and post-test general mathematics strategies

The medio-tests and post-tests were used to follow the general mathematical devel-

opment of the students in the two groups. We presented these test as transfer tests to measure to what extent students use strategies developed while learning fractions in other (mathematical) situations. Differences in teaching in the two groups provide arguments for a possible transfer. While the regular teachers stimulated students to work for themselves, the teacher-researcher in the fraction lessons emphasised on reflecting on one's work and on discussing the results of others (cf. Keijzer & Terwel, in press). And the strategies thus focused on in the experimental group, when used in other fields in mathematics, might lead to deeper understanding.

	Mean	SD	Min	Max	Sig.
Control programme (*n* = 10)					
medio-test 'numbers and operations'	57.50	7.66	43	65	
medio-test 'measuring and geometry'	53.10	7.16	36	64	
Experimental programme (*n* = 10)					
medio-test 'numbers and operations'	54.90	17.33	18	80	
medio-test 'measuring and geometry'	54.10	14.72	27	78	
Paired differences, cont. – exp. (*n* = 10)					
medio-test 'numbers and operations'	2.60	12.72	-17	25	n.s. (0.53)
medio-test 'measuring and geometry'	-1.00	10.25	-21	11	n.s. (0.76)

Note: Cronbach Alphas for internal consistency for these instruments are 0.9468 for the test 'numbers and operations' and 0.8838 for the test 'measuring and geometry'. Number of items in the tests: 68 (numbers and operations) and 51 (measuring and geometry). We followed the guidelines for test scoring that were issued with the test (Janssen, Kraemer & Noteboom, 1995).

table 6.3: characteristics of the distributions of general mathematical strategies
and paired t-test, medio-tests

Table 6.3 displays the scores on the medio-tests 'numbers and operations' and 'measuring and geometry'. From this table we learn that the hypothesised transfer did not occur: no significant differences between student outcomes on the medio-test in the two conditions could be found. The figures in the table, however, suggest another unanticipated effect. The experimental programme seems to enlarge differences between students as compared to the control condition. This effect is clarified more if only half the groups (lowest and highest performers) are taken into consideration. Clear trends show that high achieving students in the experimental group outperform their matched peers on these general mathematics tests (where we consider trends to be non-significant results, that otherwise provide indications that sup-

port the hypothesised outcomes). Further, low achieving students in the experimental condition have lower scores than students in the control group.

Table 6.4 provides the scores in the two post-tests. Although we here observe a minor trend in favour of the experimental group on the post-test 'measuring and geometry', the figures in the table show no significant effects. Like the figures in table 6.3, the figures in this table again suggest that the experimental condition enlarges differences between the experimental students as compared to the control students. These differences are confirmed by separate analyses of low and high achieving students.

	Mean	SD	Min	Max	Sig.
Control programme (*n* = 10)					
post-test 'numbers and operations'	65.10	4.18	56	71	
post-test 'measuring and geometry'	60.60	4.45	53	68	
Experimental programme (*n* = 10)					
post-test 'numbers and operations'	65.80	10.84	43	80	
post-test 'measuring and geometry'	66.90	15.93	32	93	
Paired differences, cont. – exp. (*n* = 10)					
post-test 'numbers and operations'	-0.70	7.15	-9	13	n.s. (0.76)
post-test 'measuring and geometry'	-6.30	12.94	-28	21	n.s. (0.15)

Note: Cronbach Alphas for internal consistency for these instruments are 0.9171 for the test 'numbers and operations' and 0.8822 for the test 'measuring and geometry'. Number of items in the tests: 70 (numbers and operations) and 50 (measuring and geometry). We followed the guidelines for test scoring that were issued with the test (Janssen, Kraemer & Noteboom, 1995).

table 6.4: characteristics of the distributions of general mathematical strategies
and paired t-test, post-tests

Graphical representations general mathematics tests

Figure 6.1 shows the mean scores on the tests 'numbers and operations' of experimental and control group. The graphs in this figure illuminate what we read from table 6.3 and 6.4. The mean scores of the two groups on these tests are about the same. This is also true for the development of strategies in 'measuring and geometry'. Figure 6.2 displays the mean scores of these tests. Like the figures in the table 6.3 and 6.4, figure 6.2 shows that the scores of students in the experimental and control group are about the same. Figure 6.3 shows the development of students in the control group in 'numbers and operations'. The graphs displaying students' development indicate that especially low achieving students in the control group gain pro-

ficiency in 'numbers and operations' over the experimental year. As the other students develop in a much slower pace, the differences between students in the control group are diminished.
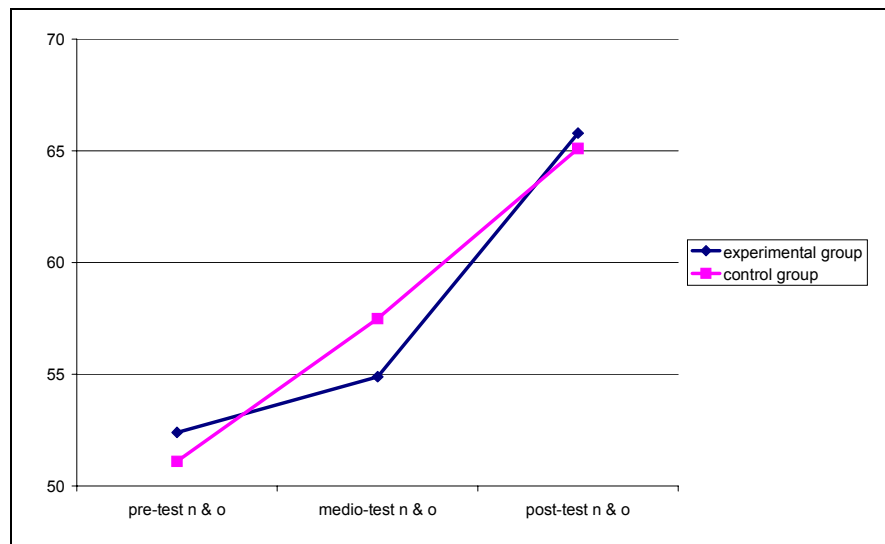
figure 6.1: graphs of (scaled) mean scores for the development in 'numbers and operations' in experimental and control group
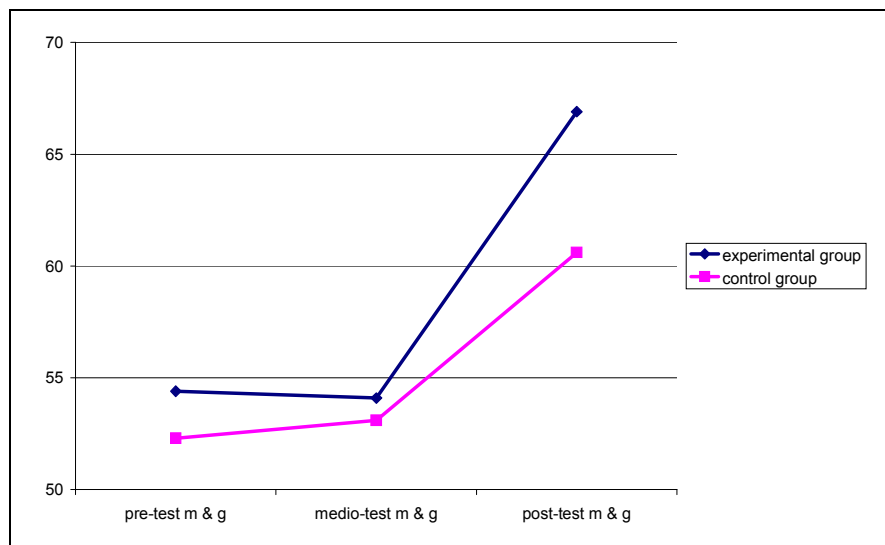
figure 6.2: graphs of (scaled) mean scores for the development in 'measuring and geometry' in experimental and control group

Although there is no effect on the mean scores, table 6.3 and 6.4 suggest another form of effect, namely the differences in scores in the experimental condition are enlarged, while the differences in scores in the control condition are reduced. We again use a graphical representation of the student scores on the general mathematics tests to illuminate these effects of the experimental programme. Figure 6.3 and figure 6.4 show all students' developments. In these figures the development of each student is represented by a line connecting (scaled) pre-test scores to (scaled) medio-test scores and (scaled) post-test scores (Janssen, Kraemer & Noteboom, 1995).
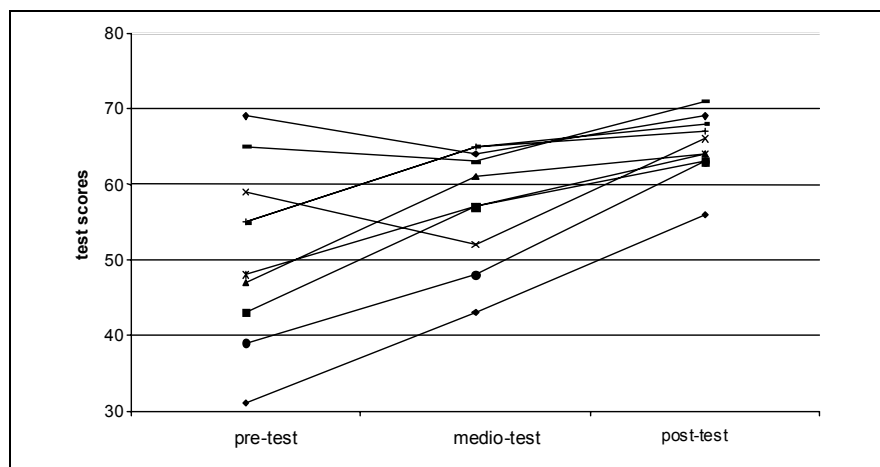
figure 6.3: individual development of strategies in 'numbers and operations'
in the control group (*n* = 10)
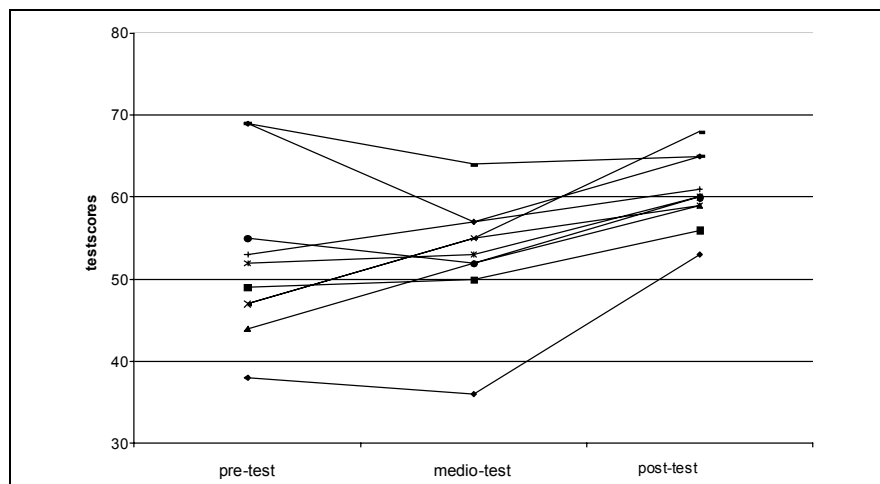
figure 6.4: individual development of strategies in 'numbers and operations'
in the experimental group (*n* = 10)

Figure 6.4 displays the development of students in the experimental group in 'numbers and operations' over the experimental year. This graphical representation of students' development indicates differences between the experimental and control condition.

On the whole, all students in the experimental group gain in 'numbers and operations'-strategies. As the high performers gain a little more than the low achievers, we here observe a so-called Matthew-effect (Kerckhoff & Glennie, 1999) in the experimental condition. Further analysis will be provided to substantiate these indications from the visual representations in figure 6.3 and figure 6.4.

Figure 6.5 shows student development of students in the control condition in 'measuring and geometry'. This figure is similar to figure 6.3. By comparing these figures, we again observe how low achieving students in the control group gain more proficiency than high achievers. As a result differences between students in this group for 'measuring and geometry' are reduced.
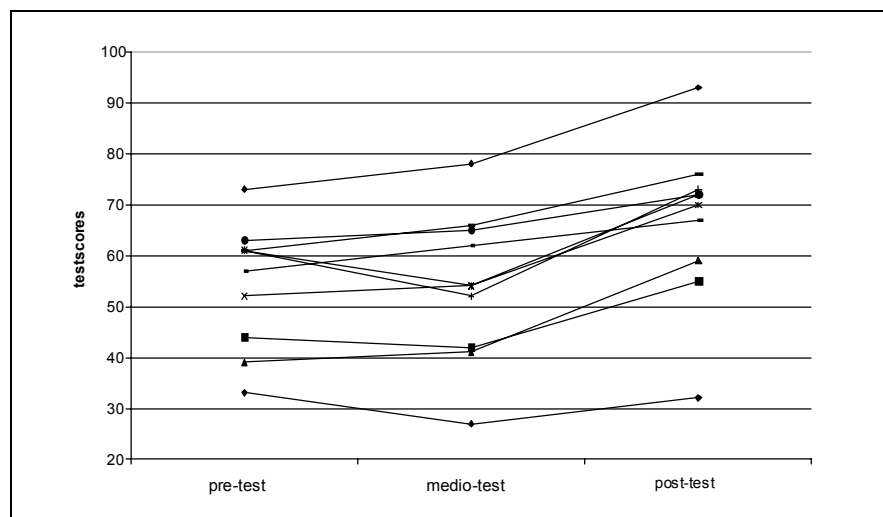
figure 6.5: individual development of strategies in 'measuring and geometry'
in the experimental group ($n = 10$)

Figure 6.6 displays the development of the ten students in the experimental group for 'measuring and geometry'. Here we observe a development that is similar to what we saw in figure 6.4. All students in the experimental condition gain in 'measuring and geometry'; however, normal and high performers gain a little more than low achieving students. In that sense there is a remarkable difference between the two groups. While in the control condition the low achievers especially gain proficiency in 'measuring and geometry', in the experimental group the normal and high achievers benefit – a trend that is confirmed by separate analyses of low and high achieving students.

We will return to these impressionistic findings from graphs in the last part of this section where we, by means of regression analyses, will determine whether or not these trends represent significant interaction effects between pre-test and intervention.
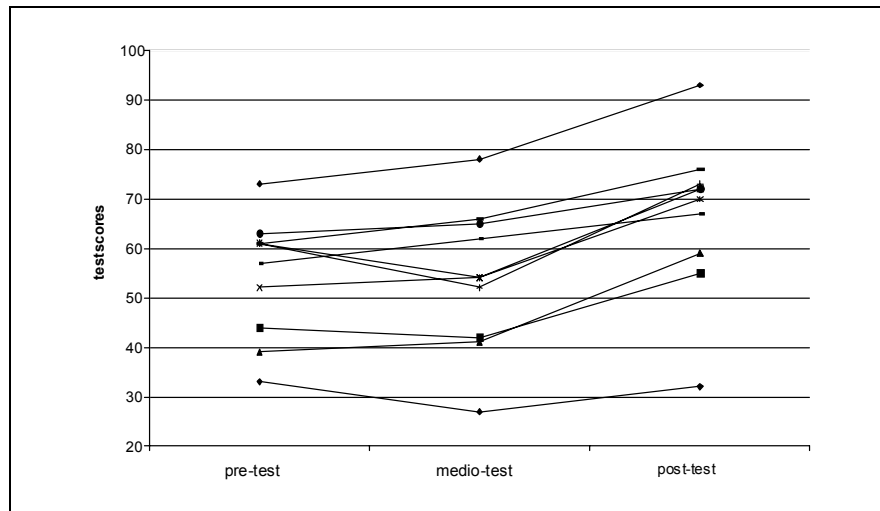
figure 6.6: individual development of strategies in 'measuring and geometry'
in the experimental group ($n = 10$)

If we combine our findings from table 6.3 and table 6.4 and figure 6.3 – figure 6.6, we see that the individual student graphs illuminate the remarkable differences in standard deviations, as displayed in table 6.3 and table 6.4. This seems to indicate that there is an effect of the experimental program on general mathematical strategies, where the experimental programme enlarged the differences between students in the experimental condition as compared to the control condition. Elsewhere (Keijzer & Terwel, 2000; Keijzer & Terwel, in press; see also chapter 3), we found that normal and high achieving students in the experimental condition perform better than their matched peers in the control group. This finding is confirmed here.

6.5.2 Characteristics of the distribution of the fraction strategy interview scores

Three interviews were conducted to recover students' fraction strategies. The first interview consisted of three problems, the second interview contained four problems and the third interview seven.

We explained that interviews were chosen as research instrument to enable both qualitative and quantitative analyses of students' strategies. Moreover, these fraction interviews gave us the opportunity to provide students with standardised help, if the students needed this help. With this method, we link up with, for example, Hamers

and Ruijssenaars (1984) who, referring to Vygotsky's theory of zone of proximal development, argue that tests should be learning ability tests and aim more at the potential developmental level, which can be reached with some help, and less on the actual developmental level (p. 5) (cf. Van Parreren, 1975). Fraction learning will be regarded here from both perspectives. Scores on problems in the interviews where no help is offered provides a measure of the students' actual development. These scores will be presented and analysed here. We will also present the interview scores in the condition where help is offered, illuminating the students' potentials. Finally, for each interview a constructed score will be considered. This score combines the scores 'with help' and 'no help provided' in the following manner. Giving a correct answer without help offered will result in two points. If the correct answer was given after having received standardised help, only one point was administrated. In this way we constructed a score that provides an overall picture of the students' development.

First fraction interview

The first fraction interview was held after 10 weekly lessons in fractions and aimed at uncovering the students' 'fraction language'. Flexible fraction language knowledge is considered to be a important element in fraction learning (Streefland, 1991; Bezuk & Bieck, 1993; Mack, 1995). Moreover, fraction language acquisition is embedded in teaching standards for mathematics in primary school (Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education], 1994). The experimental programme explicitly emphasises fraction language development, whereas there is no explicit attention for learning of fraction language in the control condition.

	Mean	SD	Min	Max	Sig.
Control programme (*n* = 10)					
first interview without help	0.50	0.53	0	1	
Experimental programme (*n* = 10)					
first interview without help	1.60	0.97	0	3	
Paired differences, cont. – exp. (*n* = 10)					
first interview without help	-1.10	0.88	-2	0	0.003

Note: Cronbach Alpha for internal consistency for this instrument is 0.6130. Number of items in the tests: 3. ES (effect size) = 2.08

table 6.5: characteristics of the distributions of the fraction scores and paired t-test first interview, without help

Three fraction problems in the first interview were developed by the researcher (see appendix B). In the interview the students were invited to use their fraction language to clarify the problem situations. Table 6.5 displays the students' actual development in fraction language. After 10 weekly lessons students in the experimental group significantly outperform their matched peers in fraction language. While students in the control condition solved no more than one of the problems without help, some students in the experimental group solved all three problems without help.

Those students that had difficulties in solving the problems, received standardised help. This help was of a heuristic nature. If help was offered, students were asked to explain their approach and they were stimulated to check answers given by, for example, making a sketch of the situation.

	Mean	SD	Min	Max	Sig.
Control programme (*n* = 10)					
first interview with help	1.30	0.82	0	2	
Experimental programme (*n* = 10)					
first interview with help	2.70	0.48	2	3	
Paired differences, cont. – exp. (*n* = 10)					
first interview with help	-1.40	0.84	-3	0	0.001

Note: Cronbach Alpha for internal consistency for this instrument is 0.6583. Number of items in the tests: 3. ES (effect size) = 1.71

table 6.6: characteristics of the distributions of the fraction scores and paired t-test first interview, with help

Table 6.6 shows the interview scores in the first interview after giving standardised help. Again we see how students in the experimental condition outperform their matched peers in the control condition. We see that offering help reduced the effect size (from 2.08 to 1.71). This indicates that mainly students in the control condition benefited from the help offered. However, considering the high scores of students in the experimental condition in the situation where no help was offered, it is obvious that students in the control group gain more from this help than their peers in the experimental group. Thus, in this first interview high achieving students in the experimental condition are possibly hindered by a ceiling effect, as they were often unable to improve their already correct answers by means of the offered help.

Since all scores go up, when comparing the 'no help' condition with the 'help'-condition, we can justify the conclusion that all students appear to gain from the standardised help in this interview. Especially students in the experimental group after ten weeks in the experiment have a well developed fraction language within reach.

Table 6.7 shows the constructed scores from interview 1. If a student gave a correct answer without standardised help offered he of she would receive two points. If the student needed standardised help to reach a correct solution he or she would receive one point. This constructed score thus represents both the potential developmental level as the actual developmental level, where actual developmental levels are scored a little higher than potential developmental levels.

	Mean	SD	Min	Max	Sig.
Control programme (*n* = 10)					
interview 1	1.80	1.03	0	3	
Experimental programme (*n* = 10)					
interview 1	4.30	1.34	2	6	
Paired differences, cont. – exp. (*n* = 10)					
interview 1	-2.40	1.51	-5	0	0.001

Note: Cronbach Alpha for internal consistency for this instrument is 0.6939. Number of items in the tests: 3. We scored 2 points for a correct answer without help offered and 1 point if a correct answer was presented after giving standardised help. ES (effect size) = 2.33

table 6.7: characteristics of the distributions of the constructed interview
scores and paired t-test, first interview

We see from table 6.7 that students in the experimental group score significantly higher when the interview scores of the first interview are combined as indicated. The data in this table confirms what we concluded from table 6.5 and table 6.6. Probably because of the explicit attention on the language of fractions, students in the experimental group after 10 weeks developed more fluency in fraction language than their matched peers. It follows that the experimental programme offered more chances to develop fraction language than the programme in the control group.

Second fraction interview
The second interview was held after 20 weekly lessons in fractions and consisted of four fraction situations (see appendix B). The problems for this interview were also developed by the researcher. In this interview the students were asked to compare fractions. As is the case with fraction language acquisition, ability to compare fractions is embedded in the Dutch teaching standards for fractions. Moreover, as comparing fractions in many cases is related to global reasoning and estimation, it contributes to number sense acquisition and therefore aims at more general goals in mathematics education. The experimental programme explicitly focuses on these

strategies, whereas the control programme does so only in an implicit way. Again following arguments to distinguish between 'no help' scores, as indicators of actual development in comparing fractions, and 'with help' scores, to uncover students' potentials in development (Hamers & Ruijssenaars, 1984), we will here also present student outcomes from this second interview in the situation where no help was offered and in the situation where standardised help was offered. As we did with the first interview, in the end the scores will be combined.

	Mean	SD	Min	Max	Sig.
Control programme (*n* = 10)					
second interview without help	1.00	0.94	0	3	
Experimental programme (*n* = 10)					
second interview without help	2.10	1.73	0	4	
Paired differences, cont. – exp. (*n* = 10)					
second interview without help	-1.10	1.10	-3	0	0.012

Note: Cronbach Alpha for internal consistency for this instrument is 0.8199. Number of items in the tests: 4. ES (effect size) = 1.17

table 6.8: characteristics of the distributions of the fraction scores and paired t-test second interview, without help

Table 6.8 shows the students' outcomes in the second interview in the situation where no help was offered. We argued that this score represents the actual developmental level of the students. We again observe a significant programme effect. Students in the experimental group outperform their matched peers in the control condition in their actual knowledge of comparison strategies for fractions.

	Mean	SD	Min	Max	Sig.
Control programme (*n* = 10)					
second interview with help	1.20	0.92	0	3	
Experimental programme (*n* = 10)					
second interview with help	2.40	1.90	0	4	
Paired differences, cont. – exp. (*n* = 10)					
second interview with help	-1.20	1.40	-3	1	0.024

Note: Cronbach Alpha for internal consistency for this instrument is 0.8568. Number of items in the tests: 4. ES (effect size) = 1.30

table 6.9: characteristics of the distributions of the fraction scores and paired t-test second interview, with help

Table 6.9 displays the potential in developmental level as it shows the students' outcomes of the second interview in the situation where standardised help was offered. Also in this situation students in the experimental group significantly outperform their matched peers in the control group. Moreover, the figures in the table indicate that students in both groups equally benefit from the offered help. However, like we argued for the first interview, the effects of the standardised help could well be flattened by a ceiling effect, especially for high achieving students in the experimental condition.

	Mean	SD	Min	Max	Sig.
Control programme (*n* = 10)					
interview 2	2.20	1.81	0	6	
Experimental programme (*n* = 10)					
interview 2	4.50	3.57	0	8	
Paired differences, cont. – exp. (*n* = 10)					
interview 2	-2.30	2.36	-5	1	0.013

Note: Cronbach Alpha for internal consistency for this instrument is 0.8663. Number of items in the tests: 4. We scored 2 points for a correct answer without help offered and 1 point if a correct answer was presented after giving standardised help. ES (effect size) = 1.27

table 6.10: characteristics of the distributions of the constructed interview scores and paired t-test, second interview

Table 6.10 provides the constructed scores for the second interview. The figures confirm what was found in considering 'no help' scores and 'help' scores independently. Again we observe that students in the experimental group outperform their matched peers in the control group. We established that the experimental programme after 20 weeks better prepared students for fraction comparing strategies than the programme in the control group.

Another feature of table 6.10 that can also be observed in table 6.8 and table 6.9 is the substantial differences in standard deviations (SD) between the control and experimental condition. The large SD of the experimental group as compared to the control group is similar to what we found for the general mathematics strategies. The figures presented in table table 6.8 – table 6.10 therefore confirm differential effects of the experimental programme as compared to the control programme. We clarified these differential effects by separate analyses of high and low achievers' results. The highest achievers in the experimental group are responsible for the observed differences between experimental and control group. We found no programme effect

when we compared low achievers in both groups. We thus see that these differential effects are not limited to general mathematical strategies, but also can be found in the fraction learning. Further, we established that these differential effects seem to be independent of the help offered.

Third interview

Seven problems in the third interview aimed at operating with fractions. These problems were taken from a national mathematics survey at the end of primary school (Bokhove, Van der Schoot & Eggen, 1996). Operating with fractions is considered an aim in primary school as long as the fractions are 'elementary' and operating takes place in recognisable contexts (Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education], 1994). Bokhove et al. worked this out in problems where 'elementary fractions' are fractions with denominator '2', '3', '4', '5', '6', '8', '9', '10', '12' or '15' and where operations with fractions take place in recognisable contexts. The third interview was held after 30 lessons in fractions, at the end of the experiment.

Students solved the problems that played a part in the third interview individually, as a pen-and-paper test, before discussing their approach in the interview. The scores in this test provided the 'no help' scores of the third interview. These scores are displayed in table 6.11.

	Mean	SD	Min	Max	Sig.
Control programme (*n* = 10)					
third interview without help	1.90	1.91	0	5	
Experimental programme (*n* = 10)					
third interview without help	2.40	1.51	0	5	
Paired differences, cont. – exp. (*n* = 10)					
third interview without help	-0.50	1.43	-3	1	n.s. (0.30)

Note: Cronbach Alpha for internal consistency for this instrument is 0.6415. Number of items in the tests: 7.

table 6.11: characteristics of the distributions of the fraction scores and paired t-test third interview, without help

The paper-and-pencil test resulted in a non significant trend in favour of the experimental group. If we compare this with 'no help'-scores in the first and second interview, we see that differences between control group and experimental group diminished. We argued that we considered this as the students' actual developmental level.

This suggests that students in the control group developed at a faster pace than their matched peers in the experimental group, to reach – at the end of the experimental year – the level of students in the experimental group.

After the students did the problems individually, they were asked to explain their answers in the third interview. If the students needed help they were offered standardised help as in the previous interviews. The results of these interviews in terms of student outcomes are shown in table 6.12. In this situation we again determine a significant difference in favour of the experimental group. Students in the experimental condition seem to benefit more from the standardised help offered during the interview than their matched peers in the control group. Our separate analyses of low and high achievers show that especially high achievers gain from the help offered.

	Mean	SD	Min	Max	Sig.
Control programme (*n* = 10)					
third interview with help	3.00	1.49	0	5	
Experimental programme (*n* = 10)					
third interview with help	4.20	1.99	1	7	
Paired differences, cont. – exp. (*n* = 10)					
third interview with help	-1.20	1.03	-2	1	0.005

Note: Cronbach Alpha for internal consistency for this instrument is 0.7189. Number of items in the tests: 7. ES (effect size) = 0.81

table 6.12: characteristics of the distributions of the fraction scores
in third interview, with help

	Mean	SD	Min	Max	Sig.
Control programme (*n* = 10)					
interview 3	4.90	3.25	0	10	
Experimental programme (*n* = 10)					
interview 3	6.60	3.34	1	11	
Paired differences, cont. – exp. (*n* = 10)					
interview 3	-1.70	1.83	-5	2	0.016

Note: Cronbach Alpha for internal consistency for this instrument is 0.7608. Number of items in the tests: 7. We scored 2 points for a correct answer without help offered and 1 point if a correct answer was presented after giving standardised help. ES (effect size) = 0.52

table 6.13: characteristics of the distributions of the constructed interview scores
for the third interview

In the constructed scores in table 6.13 we again see that students in the experimental condition outperform their matched peers in the control group. If we, however, focus on the development of the effect size in the constructed scores (in paired t-tests) from the first interview to the third, we see that, although the experimental group in all cases does a better job, the control group seems to grow at a faster pace, and more and more approaches the experimental group. However, the results of the experimental group in the first two interviews could well be influenced by the specific attention for fraction language learning and fraction comparison strategies in the experimental programme – which is in line with what is written in the Dutch curriculum standards. The problems in the third interview are chosen so that this kind of effect of programmes are not present. Over the interviews especially the lowest achievers are caught up, but the high achievers lose terrain also (figure 6.7).
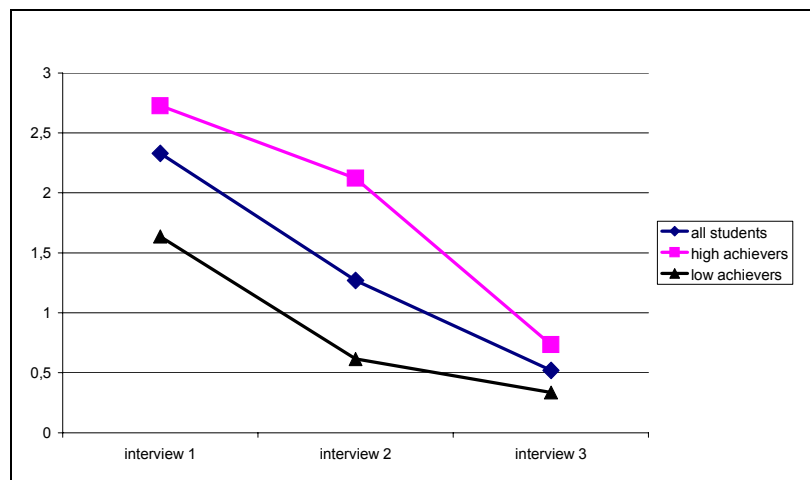
figure 6.7: effect sizes of the constructed interview scores

Overview over interviews

In the first two interviews we established a possible ceiling effect. High performers in (especially) the experimental group were unable to use the standardised help as they often provided correct answers without help. We therefore in these first interviews did not offer high performers in (especially) the experimental group the chance to improve their scores. When the problems were chosen so that these students needed help, as was the case in the third interview, students in the experimental condition benefited more from the standardised help than their matched peers in the control condition. It is just this situation that caused the large difference between the 'no help' and 'help offered' conditions in the third interview. We therefore may conclude in general that high performing students in the experimental condition benefit

most from the standardised help offered, when the problems are such that they need help. If we focus on the constructed interview scores, we observe significant differences between the experimental and the control group concerning operating with fractions all through the experimental year. It follows that the experimental programme offers more chances to learn fraction and fraction operations in meaningful situations than the programme in the control group. However, we noticed that the control group caught up. As the development of this trend after the experimental year is unclear, we can only guess how the groups will develop further. For example, it can be argued that, as the third interview covered all elements in the fraction programme in primary school (Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education], 1994), and this interview was not 'prepared for' by specific programme elements in the experimental condition, students in the experimental group will keep their advanced position. In other words: the effect sizes in the first two interviews are possibly influenced by specific elements in the experimental programme, whereas the effect size of the third interview is not. However, because the control programme also reached a reasonable score on the third interview, students in this group might have reached a suitable position to further extend their fraction strategies.

6.5.3 Correlational and regression analyses

Correlation between the instruments' outcomes

Before introducing the outcomes of the regression analyses, the Pearson correlations between the variables are presented here. Pearson's correlation coefficients are measures of linear association.

Table 6.14 presents the Pearson correlation matrix for the instruments (six general mathematics tests and three interviews) used in this study. The figures in this table show that the data generated by the study's instruments correlate highly.

Table 6.14 shows that all general mathematical tests are highly related (with values between 0.944 and 0.718). We interpret this as a signal of cross-test reliability; student scores on one test predict scores on the other. This cross-test reliability is claimed by composers of the test (Janssen, Kraemer & Noteboom, 1995) and is confirmed by our observations. Similarly the figures in table 6.14 signal cross-test reliability for the three fraction interviews (constructed scores). This indicates that if a student has developed any fluency in fraction language, he or she, about three months later, will find little difficulty in applying this knowledge to comparing fractions and, again about three months later, will have developed a means to operate with fractions. In other words, choices made in the experimental curriculum, where fraction language is a means to develop fraction comparison strategies, which next form the basis for operating with fractions, are supported by the observed correlations.

instruments	pre-test n&o	pre-test m&g	medio-test n&o	medio-test m&g	post-test n&o	post-test m&g	Int. 1	Int. 2
pre-test m&g	.795							
medio-test n&o	.753	.718						
medio-test m&g	.806	.863	.858					
post-test n&o	.847	.815	.944	.917				
post-test m&g	.782	.789	.785	.877	.874			
Interview 1	.438	.455	.442	.470	.511	.607		
Interview 2	.575	.827	.656	.705	.722	.696	.737	
Interview 3	.717	.775	.696	.697	.743	.701	.643	.797

Note: Almost all correlations are significant at the 0.01 level (2-tailed). The correlation between interview 1 and the pre-test and the medio test 'measuring and geometry' and the correlation between interview 1 and the post-test 'numbers and operations' are significant at the 0.05 level (2-tailed). The correlation between the first interview and the pre-test and medio test 'numbers and operations' is not significant.

table 6.14: Pearson correlation matrix for the study's instruments ($n = 20$)

In table 6.14 we also find a high correlation between general mathematical tests and (especially) interview 2 and 3. We consider this an indication that general mathematical strategies to a large extent determine students' chances to be proficient in fraction learning. Fractions, in this sense, come forward as a typical example of mathematics acquisition; learning fractions forms a typical example of mathematising one's world.

A longitudinal perspective: regression analyses

In the previous paragraphs we elaborated on the student matching, which was performed at the start of the experiment, to compare student outcomes for several tests during the experimental year. In this way small differences between matched students at the start of the experiments were not accounted for. Moreover, these analyses do not provide a full longitudinal perspective over the study and its results. Regression analysis is a means to study student development over the experimental year. Variable outcomes get an 'explaining' power, since by regression analysis we can determine how each independent variable contributes to the observed effects.

Curriculum specific tests: effects on fraction learning

For the fraction interviews we chose the pre-test score on the test 'numbers and operations' and the condition (experimental or control) as independent variables. Moreover, we tested for interaction effects by constructing a variable that is the product of pre-test score on 'numbers and operations' and condition. Namely, if interaction between the independent variables is established, statements about main effects become hazardous (Pedhazur, 1982).

For reasons of space limitations we will not present all possible tables from the regression analyses here. Instead, we present only one of the tables for the second fraction interview (table 6.15). We will elaborate on the figures in this table. Moreover, we will present and explain similar findings for the other interviews.

	R	R²	std. error	R² change	F change	df1	df2	sig. F change
pre-test n & o	.575	.331	2.52	.331	8.899	1	18	.008
condition	.680	.462	2.32	.131	4.158	1	17	.057
interaction	.688	.473	2.37	.011	.319	1	16	.580

table 6.15: regression analysis of the effects of condition on fraction learning
outcomes (second interview, dependent variable = constructed score; $n = 20$)

From table 6.15 we may conclude that the pre-test contributes highly to the differences in the post-test. Condition and interaction do not show significant effects. The non-significant 'effect' of condition in table 6.15 can be seen as a main effect (Pedhazur, 1982).

These findings for the second interview are typical for all the interviews. From the regression analysis therefore the following conclusions concerning the learning of fractions:

1 Pre-knowledge in general mathematical strategies at the start of the experiment contributes significantly to the learning of fractions; it 'explains' from about 20 to 50 percent of the differences in the outcomes on fraction learning as measured by the interviews.

2 The contribution of condition ranges from 5 in the third interview to 50 percent for the first interview, which gives a significant effect for the first interview and clear trends in the hypothesised direction for the second and third interview.

3 No interaction effect could be found. Thus, no differential effects of the program on the learning of fractions between high and low students could be detected, meaning that our earlier conclusions regarding differential effects for high and low achieving students should be considered as non-significant trends.

The extreme high contribution of the condition for the first interview can be

explained from the clear differences in attention for fraction language in the two conditions. While there was explicit attention for fraction language in the experimental condition, there was no such attention in the control condition. Moreover, these findings are in line with what we saw earlier. Although students in the experimental group outperform their peers in the control group in all the interviews, students from the control group seem to catch up (cf. figure 6.7).

General mathematics strategy tests: effects on general mathematics strategy learning
The outcomes of the general mathematical tests were analysed in a similar regression analysis. We established the following results. In contrast with the outcomes of the regression analyses on fraction learning, no main effects of condition could be found on the general mathematics medio and post-tests. Therefore we conclude that the fraction program had no (transfer) effect on general mathematical strategies. However, in all cases interaction effects were found between the condition and the pre-test scores. This means that the effects of the experimental programme on general mathematical strategies depend on the pre-test scores of the students. In other words: the experimental programme's effect is different for low achievers and high achievers, in favour of the latter. This finding is consistent with what we found earlier (cf. Keijzer & Terwel, 2000; Keijzer & Terwel, in press; see also chapter 3).
The interaction in this case is 'disordinal' (Pedhazur, 1982). This means that the separate regression lines for the two conditions intersect at the range of interest; they intersect where there are relevant scores. Johnson and Neyman provide formulas to calculate the 'regions of significance' (Pedhazur, 1982), score intervals where one condition is significantly better than the other. We calculated these regions of significance for the post-test in 'numbers and operations' and the post-test in 'measuring and geometry' (as dependent variable, with the associated pre-test as independent variable). We found a relevant region of significance for 'numbers and operations'; four low achieving students ($n = 20$) that belong to the weakest 25 percent group in 'numbers and operations' (within a national reference) profited significantly more from the control programme, as compared to the experimental programme. Although we observed a clear trend, we found no significant effects for the higher achieving students in favour of the experimental condition for 'numbers and operations'.
In addition the relevant region of significance could be calculated for 'measuring and geometry'. In this case we determined that the highest achieving students benefit significantly more from the experimental programme as compared to the control programme in the sense that their strategies in measuring and geometry during the experimental year are better developed. Moreover, the lowest achievers here benefit significantly more from the control programme as compared to the experimental programme.

6.5.4 Summary

The aforementioned findings could be summarised as follows. For general mathematical tests the performed t-tests, as well as the regression analysis, do not reveal any significant effects. From graphical representations of the students' outcomes and standard deviations, we found indications for differential effects of the experimental programme as compared to the control condition. Regression analysis confirmed these differential programme effects on students' general mathematical strategies. We established that low achievers benefited more from the control condition for 'numbers and operations' and 'measuring and geometry' than their counterparts in the experimental condition. High achievers benefited from the experimental condition for 'measuring and geometry'.

In our analyses of programme effects on fraction learning through paired t-tests, we established that students in the experimental group outperformed their matched peers in the control group in fractions. However, a regression analysis which corrects for initial differences, showed only significant programme effects for the first interview. However, as the Cronbach Alpha for this first interview was 0.61, stating conclusions on this outcome is hazardous (cf. table 6.5). For the second and third interview only clear trends in the hypothesised direction appeared.

From an analysis of effect sizes we observed that the distance between fraction interview scores in the experimental and control groups diminished over the experimental year. We argued that this observed development probably is due to characteristics of the programmes in relation to the interviews, and that the effect size in the third interview therefore could be considered as an accurate measure of the trend established by the experimental programme.

6.6 Explaining model

In explaining the observed trends in student development in the two groups we distinguish between fraction learning as a direct consequence of the experimental and control programmes, and the learning of other mathematical topics as transfer from fraction learning.

We observed transfer for high achieving students from the experimental programme to 'measuring and geometry'. Transfer from the experimental condition to general mathematical strategies can be described as using mathematising elements, like looking at a problem from several perspectives, an active search for (numerical) relations and looking for the most efficient visualisation for a problem (Schoenfeld, 1992). As in the experimental condition common understanding is negotiated between students, these strategies link up with the programme in this condition. Especially in solving problems in 'measuring and geometry' these strategies are rewarding, explaining why high achieving students in the experimental condition here show transfer from the programme to other topics in mathematics. Moreover,

these strategies are absent in the control condition, as here the students work mainly individually and experience mathematics as an already present structure instead of one to be constructed (cf. Van Oers & Wardekker, 1999).

Van Hiele (1986) argues that, in order to use these high level strategies, the student should understand the language used to negotiate the common understanding well. Only high performing students in the experimental group learned to use the mathematical language well enough to be effective in transferring this strategies to the topic of 'measuring and geometry'. This is reflected in the observed differential effects of the experimental programme regarding 'measuring and geometry' as compared to the control condition. In other words: high achieving students were in a position to extend their mathematical language to form symbolic, formal and abstract mathematics so that these strategies could be transferred to new situations. This is consistent with the Vygotskian view that language development precedes the formation of useful representations (Van Parreren, 1975).

Discussions in the experimental group were aimed at revealing formal and abstract relations between fractions. Although informal approaches were stimulated in the experimental condition, low achievers could be obstructed by the perspective of formal and abstract fractions. In that sense the experimental programme did not take low achievers' needs into account (cf. Keijzer, Baltussen, Ter Heege, Kaskens & Veldhuis, 2001). Moreover, from the perspective of cognitive load theory (Sweller, 1994) we assume that low achieving students in the experimental group were hindered by the two different approaches towards mathematics. They learned fractions in a setting where the teacher-researcher discussed meanings with the class, while other topics in mathematics where presented in a more structured manner (cf. Milo & Ruijssenaars, 2002). This confusing situation for low achieving students in the experimental group presumably resulted in extra cognitive load and therefore in the observed advanced position for students in the control group for general mathematical strategies. Dar and Resh (1994) add yet another argument why low achievers in the experimental condition do not benefit from the rich learning environment created. These low achievers lack the knowledge and the strategies to appropriately grasp the situation, where 'appropriately' here should be read as 'such that the knowledge can be applied in new situations'.

As there is no explicit attention for the learning of fraction language in the control group, students in the control group learn their fraction language with some delay and to a lesser extent. As we argued, this explains why students in the experimental condition do much better on the first two interviews and why the size of the observed effects diminishes in the third interview. As a consequence, we stated that we consider the third interview's effect size an adequate measure for the effect of the experimental programme in the domain of fractions. And although separate interview analyses suggest differential programme effects on fractions, as no significant interaction between the condition and any of the fraction interviews could be established

in the regression analysis, we assume that all students in the experimental condition gain equally from the experimental programme.

We reckon that we see no differential effects here, as the situation is not confusing for low achievers. Fractions are presented in an educational setting, where fractions are developed in whole class discussions. Moreover, fraction knowledge is supported by contexts that are aimed at developing the bar and the number line models. For most students the bar and the number line become models *for* fractions (Gravemeijer, 1994) that can be applied in rich situational contexts as were presented in interview 3; especially in the setting where standardised help is presented. In this third interview (when help is offered) students in the control condition do not exhibit equal flexibility in applying their fraction knowledge based on circle divisions and fair sharing (Keijzer & Terwel, in press; see also chapter 3). In this way the focus on the growth of mathematical language as an instrument to develop models for fractions, to thus support fraction symbolising and formalisation also explains the observed programme effects.

6.7   Conclusions and discussion

This chapter aimed at determining the effects of a recent developed fraction programme in primary school. We hypothesised that students in the experimental condition outperform students in the control condition in fraction strategies and in general mathematical strategies.

This study is a small scale study ($n = 20$). That makes it difficult to establish significant results as small fluctuations tremendously influence analyses' outcomes, which especially holds for examining differential effects. We indeed found clear trends in the expected direction. However, strictly speaking we cannot substantiate a significant general effect of the program on the learning gains of the students. This conclusion holds true for both the effects on general mathematical strategies as on the outcomes of fraction learning. Therefore, in general the hypothesis has to be rejected. Although no general effects for all students were found, the study showed significant differential effects. As far as the effects on general mathematics strategies is concerned, high achieving students clearly benefitted from the programme while low achieving students showed significantly less gains as compared to their counterparts in the control programme. Thus there is an effect of the program which is not only statistical significant but also relevant for both educational theory and practice.

To put it differently, students in the experimental condition significantly outperform their counterparts in a paired t-test, while a regression analysis which controls for initial differences shows clear trends in this direction. In addition to these main effects in fraction learning, no differential (interaction) effects for high and low achieving students could be found.

The outcomes of the analyses regarding the general mathematical tests show that there is no direct (main-)effect of the experimental programme in the t-test or in the regression analysis. Thus the fraction learning programme did not show the expected effect on transfer to general mathematical strategies. However, the outcomes indicate differential effects of the experimental programme. Low achieving students in the experimental condition end up with lower outcomes on general mathematical tests, while their better performing peers seem to profit from the experimental condition in 'measuring and geometry'. In other words, we observed a so-called Matthew-effect (Kerckhoff & Glennie, 1999).

It is argued that educational settings where meanings are negotiated in whole class discussions – especially when these discussions are aimed at formal and abstract mathematical relations – offer opportunities for normal and high performing students, but may set obstacles for low achievers, possibly because of the lack of structure in the learning environment (Vedder, 2002; cf. Menne, 2001). The observed differential effects suggests that similar obstacles play a role in this study. This effect is clarified in an explaining model for the study. High performers profit from their mathematical language, both in fraction learning and in general mathematical strategies. The t-tests on fraction interview scores indicate that low achievers in the experimental condition also profit from the interactive nature of teaching in learning fractions. In this condition, fraction language is formed as a means to negotiate fraction relations. Furthermore, by discussing well-chosen contexts (see appendix A), fraction language generates the two main fraction models in the experimental condition. In this way, both low achievers and high achievers in the experimental condition gained advanced positions as compared to their matched peers in the control condition.

Moreover, table 6.14 shows how acquired fraction strategies correlate highly with acquired general mathematical strategies. We interpret this as follows: mathematics is a connected structure constructed by each student. This construction process leads to various student mental schemes, some well developed and others filled with highly disconnected information. It goes without speaking that new relations are difficult embedded in poorly developed mental schemes. This is especially so if this process of connecting new knowledge in an existing framework is not supported by pre-structured or highly structured information. While an educational setting as in the experimental condition sets the challenge to form their own mathematical knowledge for most students, this setting sets an extra cognitive load for low achievers (cf. Sweller, 1994).

Elsewhere we reported on qualitative analyses (Keijzer & Terwel, 2000; see also chapter 3). In line with the findings here, we have established that low achievers experience difficulty in participating in whole class discussions and mostly develop isolated strategies that are not easily expanded to other situations. And this explains

why we found indications that these students enlarged their fraction knowledge in a setting where meanings are negotiated, but are unable to generalise these strategies to other situations.

References

Armstrong, B.E. & C. Novillis Larson (1995). Students' use of part-whole and direct comparison strategies for comparing partitioned rectangles. *Journal for Research in Mathematics Education, 26*(1), 2-19.

Bezuk, N.S. & M. Bieck (1993). Current Research on Rational Numbers and Common Fractions: Summary and Implications for Teachers. In: Douglas T. Owens (ed.). *Research Ideas for the Classroom – Middle Grades Mathematics.* New York: Macmillan Publishing Company, 118-136.

Bokhove, J., F. van der Schoot & Th. Eggen (1996). *Balans van het rekenonderwijs in de basisschool 2* [Balance-sheet of arithmetics education in primary school 2]. Arnhem: Cito.

Bruner, J.S. (1966). *Toward a Theory of Instruction.* Cambridge, MA: The Belknap Press of Harvard University Press.

Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education] (1994). *Doelbewust leren; kerndoelen in maatschappelijk perspectief* [Purposeful learning; curriculum standards in social perspective]. Den Haag: SDU.

Connell, M.L. & D.M. Peck (1993). Report of a Conceptual Intervention in Elementary Mathematics. *Journal of Mathematical Behavior, 12*, 329-350.

Cook, Th.D. & D.T. Campbell (1979). *Quasi-Experimentation. Design & Analysis Issues for Field Settings.* Chicago: Rand McNally College Publishing Company.

Corte, E. de, B. Greer & L. Verschaffel (1996). Mathematics Teaching and Learning. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology.* New York: Simon & Schuster Macmillan, 491-549.

Dar, Y. & N. Resh (1994). Separating and mixing students for learning: Concepts and research. *Pedagogisch Tijdschrift, 19*, 109-126.

Eeden, P. van den & J. Terwel (1994). Evaluation of a mathematics curriculum: differential effects. *Studies in Educational Evaluation, 20*, 457-475.

Freudenthal, H. (1968). Why to teach mathematics as to be useful? *Educational Studies in Mathematics, 1*(1), 3-8.

Freudenthal, H. (1973). *Mathematics as an Educational Task.* Dordrecht: Reidel.

Gravemeijer, K.P.E. (1994). *Developing realistic mathematics education* (dissertation). Utrecht: CD-ß Press.

Gravemeijer, K.P.E. & J. Terwel (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of Curriculum Studies, 32*(6), 777-796.

Gravemeijer, K.P.E. (2001). *Reken-wiskundeonderwijs voor de 21^e eeuw* [Mathematics education for the 21th century]. Utrecht: Utrecht University.

Greeno, J.G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22*(3), 170-218.

Hamers, J.H.M. & A.J.J.M. Ruijssenaars (1984). *Leergeschiktheid en Leertests. Een leertestonderzoek bij eersteklassers in het gewoon lager onderwijs* [The concept of learning ability and Learning ability tests. A research on learning ability tests in first graders in primary school] (dissertation). Harlingen: Flevodruk.

Heuvel-Panhuizen, M. van den (1996). *Assessment and realistic mathematics education* (dissertation). Utrecht: CD-ß Press.

Hiele, P.M. van (1986). *Structure and Insight. A theory of Mathematics Education.* Orlando: Academic Press, Inc.

Janssen, J., J.-M. Kraemer & A. Noteboom (1995). *Leerling Volg Systeem. Rekenen-Wiskunde 2* [Student Registration System. Mathematics 2]. Arnhem: Cito.

Janssen, J., F. van der Schoot, B. Hemker & N. Verhelst. (1999). *Balans van het reken-wiskundeonderwijs in de basisschool 3* [Balance-sheet of mathematics education in primary school 3]. Arnhem: Cito.

Keijzer, R. & J. Terwel (2000). *Learning for mathematical insight: a longitudinal comparison of two Dutch curricula on fractions.* Paper presented at the annual meeting of the American Educational Research Association, New Orleans, April, 2000.

Keijzer, R. & J. Terwel (2001). Audrey's acquisition of fractions: a case study into the learning of formal mathematics. *Educational Studies in Mathematics*, *47*(1), 53-73.

Keijzer, R., M. Baltussen, H. ter Heege, J.M.M. Kaskens & E.C. Veldhuis (2001). Rekenen in het speciaal onderwijs [Arithmetic in special education]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *19*(3), 14-18.

Keijzer, R. & J. Terwel (in press). Learning for mathematical insight: a longitudinal comparative study on modelling. *Learning & Instruction.*

Kerckhoff, A.C. & E. Glennie (1999). The Matthew effect in American education. In: Aaron M. Pallas (ed.). *Research in sociology of education and socialization (Volume 12).* Stamford, Connecticut: Jai Press Inc., 35-66.

Klein, A.S. (1998). *Flexibilization of mental arithmetic strategies on a different knowledge base* (dissertation). Utrecht: CD-ß Press.

Klein, A.S., M. Beishuizen & A. Treffers (1998). The Empty Number Line in Dutch Second Grades: Realistic Versus Gradual Program Design. *Journal for Research in Mathematics Education*, *29*(4), 443-465.

Mack, N.K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education, 26*(5), 422-441.

McIntosh, A., B.J. Reys & R.E. Reys (1992). A Proposed Framework for Examining Basic Number Sense. *For the Learning of Mathematics*, *12*(3), 2-8.

Menne, J.J.M. (2001). *Met Sprongen Vooruit: Een productief oefenprogramma voor zwakke rekenaars in het getallengebied tot 100 – een onderwijsexperiment* [By leaps and bounds ahead: a productive training-programme in arithmetic up to 100 for low achievers – a teaching experiment] (dissertation). Utrecht: CD-ß Press.

Milo, B.F. & A.J.J.M. Ruijssenaars (2002). Interactie in het speciaal basisonderwijs [Interaction in special education]. In: R. Keijzer & W. Uittenbogaard (eds.). *Interactie in het reken-wiskundeonderwijs*. Utrecht: Panama/Freudenthal Institute, 77-88.

Moss, J. & R. Case (1999). Developing Children's Understanding of the Rational Numbers: A New Model and an Experimental Curriculum. *Journal for Research in Mathematics Education*, *30*(2), 122-147.

Oers, B. van & W. Wardekker (1999). On becoming an authentic learner: semiotic activity in the early grades. *Journal of Curriculum Studies*, *31*(2), 229-249.

Pedhazur, E.J. (1982). *Multiple regression in behavioral research – explanation and prediction*. New York: Holt, Rinehart and Winston.

Parreren, C.F. van (1975). De relatie onderwijs – cognitieve ontwikkeling in de Russische psychologie [The relation education – cognitive development in Russian psychology]. In: C.F. van Parreren & W.A. van Loon-Vervoorn (eds.). *Teksten en Analyses Sovjetpsycho-*

logie 1: Denken. Groningen: H.D. Tjeenk Willink BV, 10-31.

Schoenfeld, A.H. (1992). Learning to think mathematically: problem solving, metacognition, and sense making in mathematics. In: Douglas A. Grouws (ed.). *Handbook of research on mathematics teaching and learning*. New York: Macmillan Publishing Company, 334-370.

Streefland, L. (1991). *Fractions in Realistic Mathematics Education, a Paradigm of Developmental Research.* Dordrecht: Kluwer Academic Publishers.

Sweller, J. (1994). Cognitive load theory, learning difficulty, and instructional design. *Learning and Instruction*, *4*, 295-312.

Treffers, A. (1987). *Three dimensions. A model of Goal and Theory Description in Mathematics Instruction – the Wiskobas Project*. Dordrecht: Reidel Publishing Company.

Treffers, A., E. de Moor & E. Feijs (1989). *Proeve van een nationaal programma voor het reken-wiskundeonderwijs op de basisschool. Deel 1: Overzicht einddoelen* [Specimen of a National Program for Primary Mathematics teaching. Part 1: Survey on Educational Aims]. Tilburg: Zwijsen.

Vedder, P. (2002). Realistisch rekenen en rekenzwakke, allochtone kinderen. Onderwijskansen op tafel [RME and low achievers from ethnic minorities. Educational chances on the table]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *20*(4), 15-20.

7 Theoretical reflection

Children's mathematisation processes form the most central issue in this study. We studied students' fraction learning as a typical example. We developed an experimental fraction programme to foster these mathematisation processes. We therefore opt for situational contexts and models for fractions that enable students to negotiate meanings in whole class discussions. In chapter 6 we found arguments that this approach disadvantages low achievers, while high performers in mathematics benefit from this approach. This finding will be elaborated upon here by reconsidering several issues concerning mathematisation processes and low achievers' learning in retrospect.

7.1 Abstraction, formalisation and generalisation – revisited

In chapter 2 we emphasised that this study, to a large extent, is about processes of formalisation and generalisation in learning fractions. When referring to formalising, we consider the process of constructing mathematical operations with abstract objects (for example abstract numbers). It is the process in which learners become conscious of relations (Van Hiele, 1986). It follows that the process of abstracting in some sense precedes formalisation.

Harel and Tall (1991) interrelate abstraction and generalisation. They distinguish three types of generalisation: expansive generalisation, where the applicability of a known formula or rule is extended, reconstructive generalisation, where a formula or rule is constructed and enriched to apply in a broader field, and disjunctive generalisation, where related formulas or rules are recognised as belonging to the same category (p. 38). In addition they sketch the relation between generalisation and abstraction:

> 'The formal abstraction process coupled with the construction of the formal concept, when achieved, leads to a mental object that is easier for the expert to manipulate mentally because the precise properties of the concept have been abstracted and can lead to precise general proofs based on these properties.' (p. 39-40)

Harel and Tall observed the difficulties students experience in formal abstracting. They propose 'generic examples' to diminish the leap from concrete experiences to formal abstraction. 'Generic examples' contain a prototype of the generalisable object. When students understand the abstraction, Harel and Tall speak of 'generic abstraction', abstraction from typical examples, wherein the action of abstraction is still included (cf. Terwel, 1984). Also Krutetskii (1976) and Semadeni (1984) addressed the issue of using examples to reach abstraction. They showed how well

chosen examples elicit children to develop generalisations and abstractions. Moreover, this notion of reaching abstraction via well chosen examples links up with the notion of 'horizontal mathematisation' – constructing mathematics from well chosen contexts, in which the abstract object is embedded (Treffers, 1987; Freudenthal, 1991; Nelissen, 1998; Gravemeijer & Terwel, 2000).

Another model in this sense is provided by Pirie and Kieren (1989), who sketch how mathematical understanding can be seen as a recursive and levelled phenomenon. Moss and Case (1999) show how this could work out for the subject of fractions:

> 'Children's understanding of whole and rational number develops in a formally similar way. In each case, children's numerical and global quantitative schemas develop separately at the outset. While they make the transition to a higher level of thought, children gradually co-ordinate these two schemas to yield a core understanding both of the way in which the simplest numbers in the field in question are structured and of the notation that is used for representing them. This core understanding is then extended to more complex numbers and forms of representation until the overall structure of the entire field is understood.' (Moss & Case, 1999, 124/5)

Underlying the work of Pirie and Kieren, Harel and Tall and that of Moss and Case is the assumption that students develop from one level to the next. Easley Jr. (1981) opposes this point of view. According to Easley a distinction between levels of abstraction is artificial to describe students' strategies, as at any moment several levels are under discussion. Dubinsky's (1991) scheme of 'reflective abstraction' in some sense addresses Easley's objections as it describes abstraction as a cyclic process with several forms of constructing schemas for understanding concepts, leaving room for different levels of strategy in one problem solution (cf. Van Hiele, 1986).

In the previous chapters we analysed how these different levels in solving fraction problems came forward during whole class and small group discussions. We saw how this classroom interaction in some cases led students to better fraction understanding. Moreover, we noticed that low achievers benefited less from these interactions than their gifted peers.

In this reflection we will therefore first examine the classroom as a learning environment in a constructivist setting. Next we will confront this setting with an RME (realistic mathematics education) setting, which is the background of the research reported here. In revisiting the problems low achievers experience, we will first consider the interweaving of learning strands in itself, to next use this analysis, among other arguments re-addressing the issue of opportunities for all students.

7.2   Teaching and learning strategies

Mathematics teaching and learning in a classroom environment is a social enterprise, where the class becomes a community of mathematical judgement, which develops

standards to judge (the results of) mathematical activity (Schoenfeld, 1994, 62). And, when number sense acquisition is a central aim in mathematics teaching – as it is in this study, this 'environment that fosters curiosity and exploration' should facilitate developing and negotiating meaning of terms and make sense of numbers and quantities in situations (Greeno, 1991, 173). And thus, those involved in developing mathematics education should consider '*what kinds* of complex, social activities to arrange, for which aspects of participation, and in what sequence to use them.' (Greeno, 1997, 10).

From the perspective of RME Elbers and Streefland (Elbers, 2001) elaborated this idea into classroom settings, where they invited the students to formulate research questions and alternated class discussions on these questions with work in small groups. In these settings the teacher takes the position of senior researcher to facilitate a role in the discussions for himself, but also to make it clear that the validation of the students' solutions comes from mathematical argument and not from the teacher's authority (cf. Van Oers & Wardekker, 1999). Moreover, the teacher here takes several measures to assure student-involvement in this learning environment, for example: by organising student presentations for the whole class, by stimulating variations in solutions and by helping students to see problems from another perspective.

Elbers and Streefland thus create an active role for the teacher in mathematics education in developing the learning environment and learning processes; a position we will follow here. In doing so, we take up position against very open learning environments such as for example suggested by Steffe and Wiegel (1994), where mathematical activity is mainly independent mathematical activity, which should not be directed (cf. Perkins & Unger, 1999; Van Lieshout, 2000; Menne, 2001).

Moreover, in our opinion, problem solving is best supported by a proactive teacher, who bases his or her teaching decisions on pre-set programme aims. We therefore take a different position than is brought forward by Freudenthal (1991), who argues that fractions can be introduced, when the teacher observes the utmost restraint in guiding the learner. We, furthermore, take a different position than is common in constructivist circles, as the teaching we intended is more structured (cf. Van Lieshout, 2000). Therefore for the experimental fraction programme, we developed situations that explicitly aimed at guiding students to curriculum goals like fraction language, fraction comparison strategies and equivalent fractions as points on a number line.

We thus created a situation where educational goals were set and clear and the classroom environment facilitated reaching these aims. In our view teachers should direct students' activities to promote mathematisation processes. This for example means encouraging the construction of mathematical language, for instance by transforming natural language into symbolic language (Forman & Fyfe, 1998). Moreover, this includes fostering the development of a mathematical point of view (Schoenfeld,

1992), so that students learn to value mathematics, become confident of their own ability, and learn to communicate and reason mathematically (Romberg, 1994).

7.3 Interweaving learning strands

Dividing an object is probably the most traditional starting point in teaching fractions, as it is anchored in the most obvious fraction appearance, the fraction as part-whole. The fraction as part-whole is still considered an important fraction appearance (cf. Streefland, 1991; Connell & Peck, 1993). The last decades have shown other promising teaching approaches for fractions, where other appearances were used to support the fraction learning process. Hunting (1986) introduced fractions as operators operating on quantities. He taught students to get an appropriate number of popsticks to take of the fraction, for example 30 popsticks could be taken to 'make' $\frac{4}{5}$ by taking four groups of six popsticks. Streefland (1991) presented fractions in fair-sharing situations, to then organise these situations into ratio-tables. And Carraher (1993) introduced fractions as ratio and as results of measurements. The experimental fraction programme presented in this book starts with various aspects of fractions, like measuring, the part-whole approach and the approach of fair-sharing. From a certain point on, measuring is emphasised. To establish formal fraction operations, measuring is elaborated upon to provide the bar and number line models, which next give rise to developing equivalent fractions which form the basis for fraction operations.

Moss and Case (1999) and Lembke and Reys (1994) in their studies show that many students are helped when they can relate fraction problems to calculations with decimals and percents. They thus interweave fraction learning with learning percents and decimals. In a similar manner Streefland relates the learning of fractions and ratio (see also Streefland (1993)). In the experimental programme as it is presented in this book, measurement situations are used to develop the number line as a model for fractions. As the number line is also a model for integers, the experimental programme interweaves learning strands for whole number arithmetic with that of fractions (cf. Treffers & De Moor, 1990; Beishuizen, 1997; Klein, Beishuizen & Treffers, 1998; Klein, 1998). Interweaving of learning strands is at hand, when mathematising is one of the objectives of a curriculum to be developed (Streefland, 1993). Moreover, interweaving of learning strands is a valuable perspective when formal and abstract concept formation is an aim in education, as students have to realise which elements of reality are similar in some respects and different in other respects (Trzcieniecka-Schneider, 1993). As a consequence in teaching mathematics there is a constant need to focus students on the relation between concrete experiences and formal mathematical concepts (Hart, 1987) and a need to consider student activities as aimed at learning how to think on one's own responsibility (Kohnstamm, 1948; Terwel, 1994; Terwel, 2002; cf. Van Dijk, 2002).

So far, the discussions on the experimental programme in this book did not concentrate on interweaving the fraction learning strand with other strands. However, as we have seen, this newly developed fraction programme explicitly relates learning integer operations with those of fractions, as the supporting model in both cases is the number line, with fractions effectively situated between integers. Furthermore, the number-line model provides a model for decimals, especially as the number line arose from measurements. And the bar-model, the forerunner of the number line in the programme, is also used in teaching percents, provides for interweaving learning and teaching fractions and percents (cf. Treffers, Streefland & De Moor, 1994; Van den Heuvel-Panhuizen & Streefland, 1993).

In chapter 2 we argued that the experimental programme aimed at progressive mathematising by the students. We analysed interweaving of learning strands as a direct consequence of this aim. Students construct their mathematical concepts starting from meaningful situations. They follow their own road through the landscape of learning, follow different tracks and will not always encounter similar experiences (Fosnot & Dolk, 2001). On this journey many students will pass the (somewhat artificial and often invisible) borders between fractions and other learning strands, especially as the experimental programme is designed for easy strolls from fraction learning to other topics and subjects.

7.4 High and low achievers: differential effects and possible solutions

In chapter 6 we argued that low achievers are disadvantaged in a setting where meaning are negotiated, among other things because they are unable to make use of the language presented in the class discussions. In addition, in chapter 5 we characterised low achievers in learning fractions as students who first of all are confronted with difficulties in acquiring a fraction language, a language that mediates between meaningful situations and the uttered or symbolic fraction. These students often experience a fraction as two single numbers instead of one whole (Carpenter, Coburn, Reys & Wilson, 1976; cf. Dubinsky, 1991). They, in general, follow numerical patterns without really understanding the fractions involved (Carraher & Schliemann, 1991) and have problems to oversee the problem as one entity (Sweller, 1994). So there is a need in teaching fractions to emphasise this fraction language (cf. Capps & Pickreign, 1993). Moreover, fraction learning for especially low achieving students can be improved by finding suitable situations to analyse fractions and to develop models for fractions, to relate fraction operations to those on integers, and to stimulate qualitative fraction reasoning in teaching (Behr, Harel, Post & Lesh, 1992).

In the previous chapters of this book we found that low achievers experience serious

problems in learning fractions. This also applies to our experimental programme. We found some support for qualitative differences between normal performing students and low achievers (cf. Van Lieshout, 1997; Van Lieshout & Meijers, in press). Combining these findings with the vast amount of literature on low achievers' difficulties in learning fractions made us conclude that we should consider a rather limited fraction programme for these students. Furthermore, we saw that learning meaningful (formal) fractions within a reasonable time is a feasibility for normal performing students, where the acquired fraction knowledge can be typified as number sense. We therefore concluded that these normal performing students can benefit from a fraction programme aimed at formal fraction acquisition. Looking at the differences in learning gains between lower and higher achieving students we conclude that there are 'differential effects'. These findings concern an important problem in fraction learning. We will revisit this main theme in this book.

Griffin, Case and Sandieson (1992), while considering students' cognitive growth, formulated arguments to postpone fraction learning until they reach a stage in which students' representations are developed appropriately. Coming from a neo-Piagetian tradition Griffin et al. (1992) suggest that students reach such a stage at more or less the same time. We noticed the opposite. While gifted students acquire formal fractions in a short time, their less achieving peers experience serious problems in even learning the fraction language.

Here we take the position that education should prepare students for living in a complex technological society (Greeno, Collins & Resnick, 1996; Greeno, 1997; Gravemeijer, 2001; cf. De Corte, Greer & Verschaffel, 1996; cf. Hopmann & Künzli, 1997). This leaves us with the question what mathematical concepts, skills and strategies are required to function in present-day society. Further, this issue forces us to consider to what extent these concepts, skills and strategies should be considered as mathematics for all.

Number sense acquisition is generally considered to partly answer these societal needs. If we thus limit ourselves to number sense acquisition, we could rephrase the above as follows: should we aim education at acquiring number sense, where each student develops his or her number sense to his or her possibilities and to his or her (personal) needs (cf. Ter Heege, 2000). This position can lead to limiting mathematics in primary school for the lowest achievers to a coherent collection of experiences to support essential daily life activities, like dealing with money and measurements (Keijzer, Baltussen, Ter Heege, Kaskens & Veldhuis, 2001). Fractions are included in such a programme, as long as they result from making sense of the world in a mathematical manner, to the extent that this is needed by the student. And while the lowest achievers stick to horizontal mathematising, their gifted peers are stimulated to model, symbolise and abstract the experiences; they are stimulated to extend the mathematising process in a vertical direction (Treffers, 1987; Freudenthal, 1991).

This differential model thus provides opportunities for all, with only horizontal mathematisation for some students and both horizontal and vertical mathematisation for others, to thus provide the same range of topics for all, but on different levels. Moreover, it does establish mathematics as an interwoven construction, without leaving out certain topics for specific student groups. It links up with students' right to be involved in mathematics as a meaningful human activity (Freudenthal, 1971).

7.5 Conclusion

The experimental fraction programme presented in this book offers a means to construct a curriculum that presents mathematics as interwoven construction as it is firmly related to other topics. Moreover, it can provide opportunities for all, when it is embedded in a curriculum where mathematics is explored in both horizontal and vertical directions. For example, the fraction programme starts with meaningful situations and it provides models that embed fraction operations in integer manipulation. Furthermore, the programme relates fractions, decimals and percents, and – in an indirect manner – ratio. The programme is easily embedded in a curriculum where decimals, as points on a number line, and percents, as part-wholes, are explored first, and where fractions are next introduced as alternative descriptions of decimals and percents (cf. Moss & Case, 1999). In this manner learning fractions comes from exploring daily life experiences and it offers students the chance to remain on the level of fraction language acquisition. It, however, leaves room for others to go beyond this point and acquire formal fractions.

In our point of view large differences between students do not necessarily implicate a form of ability grouping. In this respect we again follow Freudenthal's (1991) ideal not to break down or stream classes. On the contrary we seek heterogeneous classes and small co-operative groups to accommodate differences between students.

References

Behr, M.J., G. Harel, Th. Post & R. Lesh (1992). Rational number, ratio, and proportion. In: Douglas A. Grouws (ed.) *Handbook of research on mathematics teaching and learning.* New York: Macmillan Publishing Company, 296-332.

Beishuizen, M. (1997). Development of mathematical strategies and procedures up to 100. In: M. Beishuizen, K.P.E. Gravemeijer & E.C.D.M. van Lieshout (eds.). *The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures.* Utrecht: Freudenthal Institute,127-162.

Capps, L. R. & J. Pickreign (1993). Language connections in mathematics: A critical part of mathematics instruction. *Arithmetic Teacher*, *41*(1), 8-12.

Carpenter, Th.P., T.G. Coburn, R.E. Reys, & J.W. Wilson (1976). Notes from National Assessment: addition and multiplication with fractions. *The Arithmetic Teacher*, *23*(2), 137-142.

Carraher, D.W. & A.D. Schliemann (1991). Children's understanding of fractions as expressions of relative magnitude. In: Fulvia Furinghetti (ed.). *Proceedings of the fifteenth PME*

conference. Volume 1. Assisi, Italy: Dipartimento di Matematica dell'Università di Genova, 184-191.

Carraher, D.W. (1993). Lines of thought: A ratio and operator model of rational number. *Educational Studies in Mathematics*, 25, 281-305.

Connell, M.L. & D.M. Peck (1993). Report of a Conceptual Intervention in Elementary Mathematics. *Journal of Mathematical Behavior*, *12*, 329-350.

Corte, E. de, B. Greer & L. Verschaffel (1996). Mathematics Teaching and Learning. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology.* New York: Simon & Schuster Macmillan, 491-549.

Dijk, I.M.A.W. van (2002). *The learner as designer: Processes and effects of an experimental programme in modelling in primary mathematics education* (dissertation). Amsterdam: VU.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In: David Tall (ed.). *Advanced mathematical thinking.* Dordrecht/Boston/London: Kluwer Academic Publishers, 95-123.

Easley Jr., J.A. (1981). Levels of abstraction and intellectual development. *The Journal of Children's Mathematical Behavior*, *Volume 1*(2), 76-83.

Elbers, E. (2001). Social interaction as reflection: Leen Streefland as a teacher of primary school children. In: M. van den Heuvel-Panhuizen (ed.). *Proceedings of the 25th conference of the international group for the Psychology of Mathematics Education.* Utrecht: Freudenthal Institute, 226-232.

Forman, G. & B. Fyfe (1998). Negotiated Learning Through Design, Documentation, and Discourse. In: Carolyn Edwards, Lella Gandini & George Forman (eds.). *The hundred languages of children The Reggio Emilia Approach − Advanced Reflections (second edition).* Greenwich, Connecticut/London, England: Ablex Publishing Corporation, 239-260.

Fosnot, C.T. & M. Dolk (2001). *Young Mathematicians at Work. Constructing Number Sense, Addition and Substraction.* Portsmouth, NH: Heinemann.

Freudenthal, H. (1971) Geometry between the devil and the deep sea. *Educational Studies in Mathematics*, *3*(3/4), 413-435.

Freudenthal, H. (1991). *Revisiting Mathematics Education, China lectures.* Dordrecht: Kluwer Academic Publishers.

Gravemeijer, K.P.E. & J. Terwel (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of Curriculum Studies*, *32*(6), 777-796.

Gravemeijer, K.P.E. (2001). *Reken-wiskundeonderwijs voor de 21e eeuw* [Mathematics education for the 21th century]. Utrecht: Utrecht University.

Greeno, J.G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education*, *22*(3), 170-218.

Greeno, J.G., A.M. Collins & L. Resnick (1996). Cognition and Learning. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology.* New York: Simon & Schuster Macmillan, 15-46.

Greeno, J.G. (1997). On claims That Answer the Wrong Questions. *Educational Researcher*, *26*(1), 5-17.

Griffin, S., R. Case & R. Sandieson (1992). Synchrony and Asynchrony in the Acquisition of Children's Everyday Mathematical Knowledge. In: Robbie Case (ed.). *The Mind's Staircase. Exploring the Conceptual Underpinnings of Children's thought and Knowledge.* Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers, 75-97.

Harel, G. & D. Tall (1991). The General, the Abstract, and the Generic in Advanced Mathematics. *For the Learning of Mathematics*, *11*(1), 38-42.

Hart, K.M. (1987). Practical work and formalisation, too great a gap. In: Jacques C. Bergeron, Nicolas Herscovics & Carolyn Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Volume II*. Montreal: s.n., 408-415.

Heege, H. ter (2000). *Onderwijs in het meten. Een opmaat voor het voorkomen van ongecijferdheid*. [Education in measuring. A prelude in the prevention of innumeracy.] Enschede: SLO.

Heuvel-Panhuizen, M van den & L. Streefland (1993). Per Sense – een onderwijspakketje over procenten [Per Sense – an educational unit on percents]. In: M. Dolk & W. Uittenbogaard (eds.). *Procenten – op de grens van basisschool en basisvorming*. Utrecht: Panama/HMN/Freudenthal Institute, 25-48.

Hiele, P.M. van (1986). *Structure and Insight. A theory of Mathematics Education*. Orlando: Academic Press, Inc.

Hopmann, S. & R. Künzli (1997). Close our schools! Against current trends in policy making, educational theory and curriculum studies. *Journal of Curriculum Studies*, *29*(3), 259-266.

Hunting, R.P. (1986). Rachels schemes for constructing fraction knowledge. *Educational Studies in Mathematics*, *17*, 49-66.

Keijzer, R., M. Baltussen, H. ter Heege, J.M.M. Kaskens & E.C. Veldhuis (2001). Rekenen in het speciaal onderwijs [Arithmetic in special education]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *19*(3), 14-18.

Klein, A.S. (1998). *Flexibilization of mental arithmetic strategies on a different knowledge base* (dissertation). Utrecht: CD-ß Press.

Klein, A.S., M. Beishuizen & A. Treffers (1998). The Empty Number Line in Dutch Second Grades: Realistic Versus Gradual Program Design. *Journal for Research in Mathematics Education*, *29*(4)**,** 443-465.

Kohnstamm, P.H. (1948). *Keur uit het didactisch werk* [Selection from the didactic work]. Groningen: J.B. Wolters' Uitgeversmaatschappij.

Lembke, L.O. & B.J. Reys (1994). The development of, and interaction between, intuitive and school-taught ideas about percent. *Journal for Research in Mathematics Education*, *25*(3), 237-259.

Lieshout, E.C.D.M. van (1997). What can research on word and context problems tell about effective strategies to solve subtraction problems? In: M. Beishuizen, K.P.E. Gravemeijer & E.C.D.M van Lieshout (eds.). *The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures*. Utrecht: CD-ß Press, 79-111.

Lieshout, E.C.D.M. van (2000). *Ontwikkelingen in de Behandeling van Leerproblemen: Aandacht voor Aanpakken en Vasthouden* [Developments in Learning Problems Treatment: A Focus on Approaches and Holding]. Amsterdam: Vrije Universiteit.

Lieshout, E.C.D.M. van & F. Meijers (in press). Hoofdrekenstrategieën voor optellen en aftrekken tot 100 bij hoogbegaafde leerlingen en leerlingen met leerproblemen. [Strategies in mental arithmetic for addition and subtraction up to 100 of gifted students and pupils with learning problems.] *Pedagogische Studiën*.

Menne, J.J.M. (2001). *Met Sprongen Vooruit: Een productief oefenprogramma voor zwakke rekenaars in het getallengebied tot 100 – een onderwijsexperiment* [By leaps and bounds ahead: a productive training-programme in arithmetic up to 100 for low achievers – a teaching experiment] (dissertation). Utrecht: CD-ß Press.

Moss, J. & R. Case (1999). Developing Children's Understanding of the Rational Numbers: A New Model and an Experimental Curriculum. *Journal for Research in Mathematics Education*, *30*(2), 122-147.

Nelissen, J.M.C. (1998). Representaties in het reken-wiskundeonderwijs [Representations in mathematics education]. *Pedagogische Studiën*, *75*, 169-183.

Oers, B. van & W. Wardekker (1999). On becoming an authentic learner: semiotic activity in the early grades. *Journal of Curriculum Studies*, *31*(2), 229–249.

Pirie, S.E.B. & T.E. Kieren (1989). A Recursive Theory of Mathematical Understanding. *For the Learning of Mathematics*, *9*(3), 7-11.

Perkins, D.N. & Ch. Unger (1999). Teaching and Learning for Understanding. In: Charles M. Reigeluth (ed.). *Instructional-design theories and models Volume II. A new paradigm of instructional theory.* Mahwah, NJ: Lawrence Erlbaum Associates, Publishers, 91-114.

Romberg, Th.A. (1994). Classroom Instruction That Fosters Mathematical Thinking and Problem Solving: Connections Between Theory and Practice. In: Alan H. Schoenfeld (ed.). *Mathematical thinking and problem solving*. Hilldale, NJ: Lawrence Erlbaum Associates, Publishers, 287-304.

Schoenfeld, A.H. (1992). Learning to think mathematically: problem solving, metacognition, and sense making in mathematics. In: Douglas A. Grouws (ed.). *Handbook of research on mathematics teaching and learning*. New York: Macmillan Publishing Company, 334-370.

Schoenfeld, A.H. (1994). Reflections on Doing and Teaching Mathematics. In: Alan H. Schoenfeld (ed.). *Mathematical thinking and problem solving*. Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers, 53-70.

Semadeni, Z. (1984). A principle of concretization permanence for the formation of arithmetical concepts. *Educational Studies in Mathematics*, *15*, 379-395.

Steffe, L.P. & H.G. Wiegel (1994). Cognitive play and mathematical learning in computer microworlds. *Educational Studies in Mathematics*, *26*, 111-134.

Streefland, L. (1991). *Fractions in Realistic Mathematics Education, a Paradigm of Developmental Research*. Dordrecht: Kluwer Academic Publishers.

Streefland, L. (1993). The design of a mathematics course: a theoretical reflection. *Educational Studies in Mathematics*, *25*, 109-135.

Sweller, J. (1994). Cognitive load theory, learning difficulty, and instructional design. *Learning and Instruction*, *4*, 295-312.

Terwel, J. (1984). *Onderwijs maken* [The making of education] (dissertation). Harlingen: Flevodruk.

Terwel, J. (1994). *Samen onderwijs maken. Over het ontwerpen van adaptief onderwijs* [Making education together. On developing adaptive education] (inaugural lecture University of Amsterdam). Groningen: Wolters-Noordhoff.

Terwel, J. (2002). Curriculumdifferentiatie en leren denken: een onderwijspedagogisch perspectief. *Pedagogische Studiën*, *79*(3), 192-211

Treffers, A. (1987). *Three dimensions. A model of Goal and Theory Description in Mathematics Instruction – the Wiskobas Project*. Dordrecht: Reidel Publishing Company.

Treffers, A. & E. de Moor (1990). *Proeve van een nationaal programma voor het reken-wiskundeonderwijs op de basisschool. Deel 2: Basisvaardigheden en cijferen* [Specimen of a National Program for Primary Mathematics teaching. Part 2: Basic Mental Strategies and Written Computation]. Tilburg: Zwijsen.

Treffers, A., L. Streefland & E. de Moor (1994). *Proeve van een nationaal programma voor het reken-wiskundeonderwijs op de basisschool. Deel 3: Breuken* [Specimen of a National

Program for Primary Mathematics teaching. Part 3: Fractions]. Tilburg: Zwijsen.

Trzcieniecka-Schneider, I. (1993). Some remarks on creating mathematical concepts. *Educational Studies in Mathematics*, *24*, 257-264.

8 Conclusion and discussion

This chapter presents the results and conclusions of the study described in this book. We will formulate answers to research questions posed in the previous chapters. Findings from previous chapters show that when a fraction programme aims at number sense acquisition, where knowledge and strategies are (also) acquired by negotiation of meaning, many students – but not all – can learn formal fractions in a meaningful manner. We will discuss this finding here. Furthermore, from theoretical positions taken in chapter 2 and 7, we will take a step ahead and formulate issues for future research. From these analyses we will sketch this study's implications for teaching fractions and educational practice in general.

8.1 Study's conclusions

In chapter 1 we stated that the study presented here researches the feasibility and effectiveness of a newly developed fraction programme in primary school as compared to a widely used programme. Moreover, we stated that we consider fraction learning as a mere example of a learning process in mathematising, i.e. 'making it more mathematical' (Freudenthal, 1968). This brought us to formulate the following research questions:

1 How are mathematisation processes facilitated for 10-11 year old children, especially in the case of fraction acquisition?
2 What main obstacles can be distinguished regarding the processes of vertical mathematisation for 10-11 year olds, especially in the case of fraction acquisition?

We hypothesised that a majority of 10-11 year old students can acquire abstract and formal mathematical concepts and strategies. We specified that students who learned fractions in a programme in which the number line is used as a model for fractions, and meanings are established by negotiation, will outperform students who learned fractions in a more traditional programme, where the circle is the central model and where learning is to a large extent a solitary activity.

Twenty students from two grade 6 groups (aged 10-11) participated as subjects in this study. These students attended a school near Amsterdam. We matched the twenty students in ten pairs, with one student in the experimental group and one in the control group. In this way ten students followed the experimental curriculum, in which the number line is used as a model for fractions and where meanings are established by negotiation. Their ten counterparts in the control group followed the more traditional fraction programme of the school, where the circle is the central model

and where learning, to a large extent, is a solitary activity. The researcher was the teacher in the experimental group during the lessons in fractions. The experimental group's regular teacher took care of all other mathematics topics in this group. The control curriculum was taught by the group's regular teacher, as were other mathematics topics in this group. The teaching styles of the two regular teachers were highly comparable, and apart from the fraction programmes, all mathematics topics were dealt with in a similar manner. We compared the outcomes of the two groups using a non-equivalent pre-test post-test control group design (Cook & Campbell, 1979). We measured students' proficiency in general mathematical skills at three points. Furthermore, we established students' fraction strategies from three standardised interviews.

8.1.1 Effects on general mathematical strategies

Firstly, we analysed the effects of the experimental programme by testing students several times on general mathematical strategies. On only one occasion did we find significant differences when comparing the student results in a *paired t-test*. Normal performing students who followed the experimental programme significantly outperformed their matched counterparts in the control group on the post-tests 'numbers and operations' and 'measurement and geometry'. This finding links up with the differential effects of the experimental programme which we described in chapter 3 and elaborated on in chapter 6.

We did not find significant results when comparing general mathematical strategies for all the paired students in paired t-tests and in regression analyses. However, we also searched for interaction between the condition and post-tests, and established differential effects of the experimental programme as compared to the control condition. Namely, the experimental programme benefits the better performing students more than the control programme, a 'Matthew-effect' (Kerckhoff & Glennie, 1999). We were able to distinguish intervals of significance; intervals of pre-test scores where one condition does significantly better than the other. We found that for the post-test 'numbers and operations' low achievers benefited more from the control condition than from the experimental condition. For the post-test 'measuring and geometry' we also did find significant differential effects. Here we were able to distinguish two intervals of significance. These intervals in pre-test scores indicated that the highest achieving students benefited more from the experimental condition as compared to the control condition, whereas the lowest achieving students benefited more from the control condition as compared to the experimental condition. We thoroughly analysed these effects in the previous chapters.

8.1.2 Effects on fraction strategies

Secondly, we analysed the effects of the programme on students' fraction strategies. We established these fraction strategies in three interviews (see appendix B). We

constructed the interviews so that students who were unable to clarify the problem situation themselves received standardised help (see chapter 3 and 6). In a paired t-test we found that students in the experimental group did significantly better in these fraction interviews than their peers in the control group ($p \leq 0.05$). However, a regression analysis, that corrected the scores for small initial differences, revealed that two of the three fraction interviews turned out to be clear, but non-significant, trends. Moreover, for the fraction interviews we established no interaction between the condition and pre-test scores. We therefore concluded that high and low achieving students do not benefit differently from the experimental condition as compared to the control condition (chapter 6).

We did not find significant results in the interviews in all cases when we limited ourselves to the condition that no help was offered. We reckon that the small number of subjects plays a role here. However, in chapter 3 we argued that the students in the experimental condition did better when standardised help was offered, because this enabled students to use strategies that were accentuated in the experimental programme, like making estimations, checking answers by making a sketch, et cetera. In chapter 6 we elaborated on this finding. We found that the students with possibilities to learn and use mathematical language more easily, are the high performers in mathematics, and they therefore gain in a situation where common understandings are discussed. Moreover, we argued that low achievers are possibly hindered by the formal and abstract nature of the fractions involved or by the difference in teaching style between researcher-teacher in fraction lessons and the students' regular teacher. Furthermore, a qualitative analysis of the students' work revealed that students in the experimental condition in their answers more frequent reflected number sense than students in the control group. We interpreted that these results indicated that the experimental programme leaded to better results.

The results in this paragraph therefore are what they are: mostly non significant but serious trends, which suggest that students in the experimental programme gain better understanding of fractions than their matched peers in the control condition. Experiments on a larger scale in the future should provide additional evidence on these effects of the experimental programme.

8.1.3 Fraction learning processes

We presented this study as one where quantitative and qualitative analyses are used to provide in-depth insight in students' learning processes. We expressed this choice in the study's research questions. This study is performed to uncover *how* mathematisation processes are facilitated for 10-11 year old children, especially in the case of fraction acquisition. Moreover, this study looks for obstacles that can be distinguished regarding the processes of vertical mathematisation for 10-11 year olds. Therefore qualitative analyses are needed to answering our research questions.

We addressed the research questions in this sense in chapter 3, 4 and 5, where we analysed learning processes in the experimental fraction curriculum as compared to the effects of the programme in a control group. An analysis of the data provided indications for the following:

– Students in the experimental condition show more proficiency in fractions than students in the control condition (chapter 3).
– Students who perform average or above average in general mathematical skills can learn formal fractions in a meaningful manner and within reasonable time (chapter 4).
– Low achievers in mathematics experience considerable difficulties in learning formal fractions (chapter 5).

These findings partly confirm the study's specific hypothesis. We found strong indications that students in the experimental curriculum do outperform control students in the more traditional programme in fraction proficiency; however, if we focus on general proficiency in mathematics, only average and above average performers in the experimental group benefit more from the experimental condition than from the control condition. These results, moreover, provide strong indications for this study's general hypothesis that states that a majority of 10-11 year old students can acquire abstract and formal mathematical concepts and strategies, when these concepts and strategies are learned in an educational setting where meanings are negotiated in meaningful and recognisable situations and where mathematical activities on different levels are closely connected, as the experimental condition can be considered to be such a setting.

8.2 Discussion

We found differential effects for high and low achieving students ('Matthew-effect') in the acquisition of general mathematical proficiency. This set the tone for reflections on this study's results. We analysed that an educational setting where formal mathematical concepts are an educational aim and where meanings are negotiated in meaningful and recognisable situations, could well be disadvantageous for low achievers in mathematics. We furthermore argued that mathematics as product of the learners mathematising processes, makes the discussion of mathematical ideas in order to extend meanings in processes of modelling, symbolising, schematising, formalising and abstracting, inevitable.

In this manner the described differential issue leads more or less to the following problematic situation. In order to accomplish the construction of mathematics, meanings are to be negotiated in meaningful and recognisable situations. However, low achievers are hardly in a position to sufficiently benefit from these discussions (cf. Keijzer, Baltussen, Ter Heege, Kaskens & Veldhuis, 2001; Milo & Ruijssenaars,

2002), and this is where this study leaves questions for future research.

Many propositions have been made regarding advancing low achievers in the learning of mathematics. These propositions include:

a   specific ideas on mathematics teaching and learning;
b   general ideas on learning as a social enterprise;
c   ideas on classroom organisation;
d   ideas from the political arena.

Keijzer c.s. (2001) argued that, as vertical mathematisation forms a major problem for low achievers, we better aim mathematics learning for these students at horizontal mathematisation, where mathematics learning is largely limited to making sense of those meaningful situations that are useful for functioning in present-day society. This point of view implies a different approach in curriculum sequence, where mathematics teaching will no longer be anchored in seperate learning strands (like fractions, ratio, etc.), but will move from one meaningful and useful situation to another, to form a relation network of meaningful situations with the potential to form the basis for vertical mathematisation, rather than forming more or less linear strands of activities mainly aimed at vertical mathematisation.

Starting from learning as a social enterprise, the focus is mainly on facilitating processes that support mathematics learning: discussing sociomathematical norms (Yackel & Cobb, 1996; Yackel, 2001), stimulating students to elaborate on what they perceive and construct explanations for themselves (Greeno, Collins & Resnick, 1996) and specific training in small group discussions, where students learn to ask for help and to provide adequate help for others (Terwel, Gillies, Van den Eeden & Hoek, 2001). All these initiatives opt for beneficial learning situations for both low and high achievers, where low achievers learn efficient strategies from their better performing peers, while the normal and high achieving students get the chance to bring their approaches on a higher level by explicating their reasoning. Learning sociomathematical norms in this situation provides students with ideas of what counts as a valid argumentation, to facilitate participation in group discussions for them. Moreover, discussing sociomathematical norms learn students to appreciate mathematics as constructive activity, where one has to construct explanations from (mathematical) experiences and situations. And (small group) discussions are more effective in terms of mathematics construction – especially for low achievers, if students are trained in (social) discussion skills (Terwel c.s., 2001).

However, considerable differences between students are difficult to manage in a teaching setting where learning processes largely depend on interpreting others' contributions. In order to have a fruitful conversation, it is necessary that the participants in the conversation understand and value each other's arguments. This is especially true if the discourse is embedded in an educational setting, where understanding and valuing arguments of others is supposed to support learning processes.

One solution for this problem, subdividing the class in (small) level groups, makes that students are no longer hindered by the arguments of those who function at another level. But, this also excludes low achievers from arguments on the near higher level, which potentially offers them a pathway to functioning at that higher level. Huitema (2002) therefore proposed a model that provides both: starting the lesson with a whole class discussion, followed by a discussion in smaller groups, where the teacher devotes his/her attention to those students that need extra support (see also the AGO-model, Terwel, Herfs, Mertens & Perrenet (1994)). During this second discussion the students who do not experience problems, work individually or in small groups. Huitema claims that his approach makes that low achievers are not isolated from the rest of the group, as they get the chance to catch up and to bring forward their problems in the safe small group of low achievers.

The aforementioned perspectives provide possible directions for future research. This research, however, should not focus solely on one of the perspectives mentioned. It should rather take all three perspectives – specific ideas on mathematics teaching and learning, general ideas on learning as a social enterprise, and ideas on classroom organisation – into account. In continuation of the present study on fraction learning this could mean that fraction learning would be explicitly embedded in learning percent, decimals and ratio. Moreover, Huitema's ideas on class organisation offer a possibility to, on the one hand, focus low achievers' learning processes on fraction language acquisition and very elementary operations with fractions, while other students are offered the chance to take the path to vertical mathematisation to finally acquire the concept of equivalent fractions. Explicit attention for effective co-operation and sociomathematical norms subsequently provides material for valuable discussions aimed at mathematics construction by all students.
Such a multiple perspective approach in future research on mathematics learning and teaching potentially offers adequate effective opposition against recent ideas from the political arena on learning standards in primary schools (Commissie kerndoelen basisonderwijs [Committee Curriculum Standards in Primary Education], 2002). These plans reduce mathematics in primary schools to a set of seemingly disconnected formal goals, that seem to focus mainly on testing set standards and do no do justice to mathematics as the product of students' mathematising processes.

8.3 Methods revisited

We tried to arrange the study described here so that its results are both of practical use for mathematics in primary education, and contribute to theories on learning mathematics, especially the theory of realistic mathematics education (RME). We therefore considered developmental research as an overarching research design to meet these intentions, as it consists of experiencing of and reflection on the devel-

opmental process (Terwel, 1984; Freudenthal, 1991; Gravemeijer, 1994) and therefore can, as a continuous process of reflecting and revising, result in suitable or renewed teaching programmes (Streefland, 1993). Moreover, as developmental research theory offers a research design to systematically work out relations between data in the course of the research, it potentially generates, modifies and reformulates theory (Glaser & Strauss, 1977).

We elaborated this developmental research design in a quasi-experimental design and two case studies. In the quasi-experimental design we compared the outcomes of the experimental programme, in terms of general mathematical skills and fraction reasoning skills, with outcomes of a control programme. In addition, the case studies provided for explanations of patterns observed in the quantitative comparison of the fraction programmes involved.

Developmental research seeks 'experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience' (Freudenthal, 1991, 161). This implies that there is a strong need to combine the researcher's role with that of developer and teacher. In his role of developer the researcher uses his previous experiences and others' findings to construct (parts of) a teaching programme – which we consider here to consist of both teaching materials and the way these are used by the teacher. The teacher's role provides the researcher with information on the validity of his ideas, when constructing the first version of the programme or parts thereof, to thus indicate what changes in the programme are required (Cobb & Steffe, 1983). This teacher-role is even more eminent when the first version of a programme is more or less in draft and mainly consists of a bulk of ideas (Cobb, Yackel & Wood, 1989), when teaching is supported by continuous analysing of its outcomes (Cobb & Whitenack, 1996) or when there is a need to frequently shift from teaching experiences to individual learning processes (Streefland, 1987). From a different perspective Lee (2001) also pleaded for and practiced the double role of researcher and teacher, to understand the process of implementing the programme and to develop peer-like relationship with 'colleagues' in the school to promote mutual understanding and participation. In other words, the teacher role implies that you do your utmost best to support the learning of students you have to care for and – as a consequence – acquire understanding of the learning processes involved (Simon, 1995; Roth, 2001; Lee, 2001). The outcomes of this teaching process are research data for the researcher, who in this unique situation is in the position to take the teacher's considerations directly into consideration as a means to explain the study's outcomes.

Terwel (1984), however, warns against this combination of researcher roles, especially if the research situation makes it difficult for the researcher to combine participation and distance. He here distinguishes between research where change is the

objective of the research, for example action research, and research that aims at reconstructing developers' theoretical notions and presuppositions, as is the case with developmental research. Terwel rejects the first type of research designs, as it makes the researcher's role unclear, and welcomes the second type if certain conditions concerning the needed researcher's distance are met.

In the research presented here the researcher is placed in a situation Terwel warns against. The researcher is one of the developers of the experimental programme, he is the teacher in the experimental condition, he developed the first and second interview and conducted the interviews, where the students in the experimental group talked with a familiar person, whereas the students in the control condition were confronted with a researcher they knew only vaguely. However, we constructed the study so that the research results and outcomes are independent of the researcher's preferences (cf. Streefland, 1993). We chose (researcher independent) quantitative analyses of programme outcomes to precede qualitative analyses. Moreover, these quantitative analyses determined what themes should be to focussed on in qualitative analyses. This way, on the one hand we realised a basis to combine the roles of teacher, developer and researcher as a necessary element of developmental research, while on the other hand we warranted the researcher's distance. We found a research design that facilitated us to observe, analyse and reflect upon the mathematisation processes that form the heart of this study.

In this way, the study's small scale facilitates us to formulate an explanatory model for the observed effects of the experimental programme. However, this small scale character of the study also forms one of its limitations. Although regression analyses provided indications on effects of the experimental and control condition, the small number of subjects made it impossible to perform a time series analysis, where *all* outcomes are combined within one statistical model. The results in the previous paragraphs therefore are what they are: mostly non significant but serious trends, which suggest that students in the experimental programme gain better understanding of fractions than their matched peers in the control condition. Experiments on a larger scale in the future should provide additional evidence on these effects of the experimental programme.

8.4 Fraction learning in primary school discussed

The twenty-first century is the century of computers, calculators, decimals and percents. Why, then, should we trouble students with difficult fraction learning? The discussion to what extent we should teach fractions is more than a century old. Dondorff (1879) already argued that multiplying and dividing fractions can be simplified, when fractions are transformed into decimals, as decimals behave – to a large extent – like natural numbers. The difficulty in learning fractions provides a second argument not to introduce them in primary school. It is suggested that if fraction

learning is not prudently embedded in meaningful situations, it soon degenerates to meaningless formal fraction manipulations (Erlwanger, 1973; Hart, 1981; Connell & Peck, 1993) or requires students to engage in activities which assume a processing capacity that is greater than their limits (Sweller, 1994; Van Lieshout, 2000). Van Hiele (1986), in analysing his level theory, finds that operations with fractions mainly take place at the third (formal) level. He sighs (cf. Goddijn, 1992):

> 'Why should we teach young children to add such fractions? They will never need it!'
> (Van Hiele, 1986, 87)

Many researchers disagree with Van Hiele and Goddijn. They for instance argue that fractions can indeed be useful in secondary education when learning algebra (Freudenthal, 1991; Lee & Wheeler, 1989). Or they provide more general arguments why fractions should be a part of primary school curriculum, for example by arguing that mathematics is of both formative and practical value and that there is a positive transfer from learning (formal) mathematics, like fractions, to other domains (De Moor, 2000). This argument links up with arguments that mathematics – considered as a 'building' constructed by students – is unsteady when an important basis like fraction knowledge is missing.

These arguments, combined with the experience that many students face difficulties in learning fractions, challenged many developers and researchers to bring forward ideas to improve fraction teaching (e.g. Weckesser, 1970; Carpenter, Coburn, Reys & Wilson, 1976; Kieren, 1976; Noelting, 1980; Behr, Lesh, Post & Silver, 1983; Hunting, 1983 & 1986; Streefland, 1987 & 1991; Mack, 1990 & 2000; Behr, Harel, Post & Lesh, 1992; Lek, 1992; Bezuk & Bieck, 1993; Carraher, 1993; Connell & Peck, 1993; Davis, Hunting & Pearn, 1993; Graeber & Tanenhaus, 1993; Armstrong & Novillis Larson, 1995; Kamii & Clark, 1995; Bokhove, Buys, Keijzer, Noteboom & Treffers, 1996; Brinker, 1997; Moss & Case, 1999; Tzur, 1999). They do so by bringing forward the suggestion to embed fraction operations in recognisable experiences, for instance those with natural numbers, to obtain better understanding of these operations (e.g. Greaber & Tanenhaus), choosing an appropriate level as educational aim (e.g. Streefland, 1990) or changing classroom environments into 'places where interesting problems are explored using important mathematical ideas' (Romberg, 1994, 302).

By introducing a number line as central model, the fraction programme discussed in this book, we develop fraction operations from recognisable experiences. The (empty) number line model embeds fractions in the set of natural numbers and facilitates the application of natural number knowledge in the domain of fractions (cf. Menne, 2001). Moreover, well-chosen contexts generate the number line as a mental model and bring forward equivalent fractions as being positioned at the same location on the line. Furthermore, we follow Streefland by promoting problem solutions

on the context or model level at any stage in the fraction programme.

On a more general level, the developed programme aims at promoting number sense and the acquisition of problem solving skills. More precisely, as both number sense and problem solving skills are (only) learned in well-chosen contexts and appropriate educational settings (Schoenfeld, 1992; Greeno, 1991) the programme promotes number sense and problem solving skills starting from fraction learning processes. Fraction operation meanings are negotiated with the students in whole class discussions (Greeno, 1991). In line with the Dutch RME tradition there is an eminent role for the teacher, who introduces problems and 'aims' discussions at pre-set educational goals (cf. Sfard, 1991; Menne, 2001). As such the presented teaching differs considerably from more 'traditional' constructivist teaching (Bakker & Van Galen, 2000; Cobb, Yackel & Wood, 1989; Steffe & Cobb, 1988), where teaching is less programme-directed.

We thus made a fraction programme in the RME tradition, that aims at essential strategies in mathematising. It presents fractions as embedded in other topics and therefore does not aim at isolated fraction knowledge. Moreover, if we limit fraction learning to context exploration for low achievers and offer other students the opportunity to reach for vertical mathematisation, the programme offers chances for all students in learning mathematics in primary school.

8.5 Implications for educational practice

Although the study reported here is a relatively small scale study, not all findings of which can be generalised, this study has some remarkable implications for educational practice. The study's consequences of course concern teaching fractions. In the experimental programme we emphasised the use of the bar and the number line as models for fractions. We here determine to what extent these educational choices should have implications for new fraction programmes. Consequences of the study also concern differential effects of the experimental fraction programme. We observed how normal and high achievers benefited on all aspects from an educational setting where fractions were learned through discussions on constructing fraction language and formal relations between fractions. We saw that low achievers benefit less from these exchanges of views, especially when the perspective changes too rapidly to number relations that lost their obvious bound to recognisable contexts. We followed Keijzer c.s. (2001), who proposed to aim mathematics learning for low achievers at horizontal mathematisation, where mathematics learning is largely limited to making sense of those meaningful situations that are useful for functioning in present day society.

Because we consider the experimental fraction programme a typical example of a mathematics programme that aims at mathematising processes by discussing and sharing point of views with students to negotiate meanings (Greeno, 1991; Forman

& Fyfe, 1998), we reckon our arguments hold for many RME and constructivist programmes. This means that this study suggests that mathematics teaching no longer should be anchored in seperate learning strands (like fractions, ratio, etc.), but should move from one meaningful and useful situation to another, to form a relation network of meaningful situations with the potential to form the basis for vertical mathematisation, rather than forming more or less linear strands of activities mainly aimed at vertical mathematisation.

From this perspective this study strongly indicates that in general we should rethink teaching mathematics in heterogeneous groups. New mathematics programmes should be developed from the viewpoint of interweavement. These programmes, that preferably are not limited to one learning strand, should offer opportunities for all students, by discussing contexts that are recognisable for all. These context are chosen so that context bound mathematical knowledge which is constructed by a process of horizontal mathematisation is connected and coherent. Moreover, this knowledge is useful in society and offers possibilities for processes of vertical mathematisation. Most of the students, but not all, should then get the chance to take this road to vertical mathematisation to discover formal mathematics as exiting and challenging subject.

This point of view does not only affect the community of mathematics educators. It implies that we should rewrite education goals for mathematics in terms of more general aims focused at fostering students' mathematising processes to their abilities. It therefore presupposes a radical change in setting educational goals in this direction.

In some sense, the experimental programme at the heart of this book provides several suggestions to construct mathematics education as described above. The number line model and bar as models for fractions can be embedded in recognisable contexts, that offer the opportunity to use fractions in daily life. Furthermore, these models provide an effective pathway to formalisation and number sense acquisition. By targeting fraction learning on bar and number line model, fractions are interweaved with percents, ratio, and natural numbers. And, although we assume other fraction programmes share similar bonds with other learning strands and can be used to stimulate students' number sense development, we reckon that the experimental programme offers many characteristics that are worthwhile in considering renewing fraction programmes, namely the use of measuring as key context, the development of the bar and number line as models and the characterisation of equivalent fractions as fractions sharing their position on the line.

We saw (chapter 4 and 5) that the process of vertical mathematisation, where the number line becomes an abstract model, can be stimulated by using computer games, where two students discuss work with the program and consider the peculiarities of the abstract computer world (cf. Van Galen & Buter, 1997). Our experiences, working with the experimental programme, suggest that limited software

learning environments – small micro-worlds where rules of the game coincide with formal mathematical relations – offer a worthy element in an educational programme. Moreover, such computer use can help in organising education so all students profit from mathematics education.

8.6 Final remarks

Developmental research implies that all developed teaching programmes are subjected to new reflections, to thus form the basis for new developments, where the newly developed programme soon becomes the old one, predestined to be replaced by something new. Fresh ideas, established during the developmental process, enter developers' repertoire to become tools for further developmental work (cf. Gravemeijer, 1994). In this respect, this book can be seen as a source of ideas for further development of mathematics education. These ideas include theoretical notions on learning and teaching mathematics and in particular on the learning and teaching of fractions. Moreover, these include arguments to consider all involved in mathematics education: the students – both low achievers and those who are proficient in mathematics – and their teachers.

To further develop RME, as it was developed over the past thirty years in the Netherlands, in such a way that it can face many of the present problems in mathematics education, there is a need to approach relevant issues from multiple research disciplines. Developmental research to some extent offers a presentation of the researchers' learning processes. The proposed co-operation between different disciplines on important issues in the teaching of mathematics, will present several views that will place others in perspective. But, more importantly, discussions with others than members of the own discipline will lead to negotiation of meanings to redevelop a language to talk about education. The learning processes that will follow can offer a broadened view for all those involved, to thus enlarge the repertoire of future researchers and developers in mathematics education.

Some researchers took a first step in this direction. The results of this efforts look promising. The study described here can also be seen as an attempt to consider the development of mathematics education from multiple perspectives; constructing new arguments while uncovering different territory and preparing for new roads to fresh approaches in teaching mathematics.

References

Armstrong, B.E. & C. Novillis Larson (1995). Students' use of part-whole and direct comparison strategies for comparing partitioned rectangles. *Journal for Research in Mathematics Education*, *26*(1), 2-19.

Bakker, A. & F. van Galen (2000). Cathy Fosnot over constructivisme [Cathy Fosnot on constructivism]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*,

19(1), 21-23.

Behr, M.J., R. Lesh, Th. Post & E.A. Silver (1983). Rational-Number Concepts. In: Richard Lesh & Marsha Landau (eds.). *Acquisition of Mathematical Concepts and Processes*. New York/London: Academic Press, 91-126.

Behr, M.J., G. Harel, Th. Post & R. Lesh (1992). Rational number, ratio, and proportion. In: Douglas A. Grouws (ed.). *Handbook of research on mathematics teaching and learning*. New York: Macmillan Publishing Company, 296-332.

Bezuk, N.S. & M. Bieck (1993). Current Research on Rational Numbers and Common Fractions: Summary and Implications for Teachers. In: Douglas T. Owens (ed.). *Research Ideas for the Classroom – Middle Grades Mathematics*. New York: Mcmillan, 118-136.

Bokhove, J., K. Buys (ed.), R. Keijzer, A. Lek, A. Noteboom & A. Treffers (1996). *De Breukenbode. Een leergang voor de basisschool* (werkbladen en handleiding) [The Fraction gazette]. Enschede/Utrecht: SLO/FI/Cito.

Brinker, L. (1997). *Using Structured Representations to Solve Fraction Problems: A Discussion of Seven Students' Strategies*. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL.

Carpenter, Th.P., T.G. Coburn, R.E. Reys, & J.W. Wilson (1976). Notes from National Assessment: addition and multiplication with fractions. *The Arithmetic Teacher*, *23*(2), 137-142.

Carraher, D.W. (1993). Lines of thought: A ratio and operator model of rational number. *Educational Studies in Mathematics*, *25*, 281-305.

Cobb, P. & L.P. Steffe (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education*, *14*(2), 83-94.

Cobb, P., E. Yackel & T. Wood (1989). Young Children's Emotional Acts While Engaged in Mathematical Problem Solving. In: D.B. McLeod & V.M. Adams (eds.). *Affect and Mathematical problem solving: A new perspective*. New York: Springer Verlag, 117-148.

Cobb, P. & J.W. Whitenack (1996). A Method for Conducting Longitudinal Analyses of Classroom Videorecordings and Transcripts. *Educational Studies in Mathematics*, *30*, 213-228.

Commissie kerndoelen basisonderwijs [Committee Curriculum Standards in Primary Education] (2002). Verantwoording delen. Herziening van de kerndoelen basisonderwijs met het oog op beleidsruimte voor scholen [Sharing responsibility. Revision of the teaching standards for primary education with a view to the scope for school policymaking]. 's Gravenhage: Ministerie van OC & W.

Connell, M.L. & D.M. Peck (1993). Report of a Conceptual Intervention in Elementary Mathematics. *Journal of Mathematical Behavior*, *12*, 329-350.

Cook, Th.D. & D.T. Campbell (1979). *Quasi-Experimentation. Design & Analysis Issues for Field Settings*. Chicago: Rand McNally College Publishing Company.

Davis, G., R.P. Hunting & C. Pearn (1993). What Might a Fraction Mean to a Child and How would a Teacher Know? *Journal of Mathematical Behavior*, *12*, 63-76.

Dondorff, L. (1879). Vermenigvuldiging en deeling der breuken [Multiplication and division of fractions]. *Weekblad voor het lager, middelbaar en gymnasiaal onderwijs*, *jrg. XVIII* (26/4/1879), 1.

Erlwanger, S.H. (1973). Benny's conception of Rules and Answers in IPI Mathematics. *Childern's Mathematical Behavior*, *1*(2), 7-26.

Freudenthal, H. (1968). Why to teach mathematics as to be useful? *Educational Studies in Mathematics*, *1*(1), 3-8.

Freudenthal, H. (1991). *Revisiting Mathematics Education, China lectures.* Dordrecht:

Kluwer Academic Publishers.

Forman, G. & B. Fyfe (1998). Negotiated Learning Through Design, Documentation, and Discourse. In: Carolyn Edwards, Lella Gandini & George Forman (eds.). *The hundred languages of children The Reggio Emilia Approach – Advanced Reflections (second edition)*. Greenwich, Connecticut/London, England: Ablex Publishing Corporation, 239-260.

Galen, F. van & A. Buter (1997). De rol van interactie bij leren rekenen met de computer [The role of interaction while learning arithmetic with the computer]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *16*(1), 11-18.

Glaser, B.G. & A.L. Strauss (1977). *The Discovery of Grounded Theory. Strategies for Qualitative Research*. Chicago: Aldine Publishing Company.

Goddijn, A.J. (1992). De oudste en de nieuwste breuken [The oldest and the newest fractions]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *11*(2), 18-31.

Graeber, A.O. & E. Tanenhaus (1993). Multiplication and Division: From Whole Numbers to Rational Numbers. In: Douglas T. Owens (ed.). *Research Ideas for the Classroom – Middle Grades Mathematics*. New York: Macmillan, 99-117.

Gravemeijer, K.P.E. (1994). *Developing Realistic Mathematics Education* (dissertation). Utrecht: CD-ß Press.

Greeno, J.G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education*, *22*(3), 170-218.

Greeno, J.G., A.M. Collins & L. Resnick (1996). Cognition and Learning. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology*. New York: Simon & Schuster Macmillan, 15-46.

Hart, K.M. (1981). Fractions. In: K.M. Hart (ed.). *Children's understanding of mathematics: 11-16*. London: John Murray, 66-81.

Hiele, P.M. van (1986). *Structure and Insight. A theory of Mathematics Education.* Orlando: Academic Press, Inc.

Huitema, S. (2002). Interactie en de dilemma's van de leerkracht. Differentiatie in interactie [Interaction and the teacher's dilemma's. Differentiation in interaction]. In: R. Keijzer & W. Uittenbogaard (eds.). *Interactie in het reken-wiskundeonderwijs*. Utrecht: Panama/ Freudenthal Institute, 41-48.

Hunting, R.P. (1983). Alan: A case study of knowledge of units and performance with fractions. *Journal for Research in Mathematics Education*, *14*(3), 182-197.

Hunting, R.P. (1986). Rachels schemes for constructing fraction knowledge. *Educational Studies in Mathematics*, *17*, 49-66.

Kamii, C. & F.B. Clark (1995). Equivalent Fractions: Their Difficulty and Educational Implications. *Journal of Mathematical Behavior*, *14*, 365-378.

Keijzer, R., M. Baltussen, H. ter Heege, J.M.M. Kaskens & E.C. Veldhuis (2001). Rekenen in het speciaal onderwijs [Arithmetic in special education]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *19*(3), 14-18.

Kerckhoff, A.C. & E. Glennie (1999). The Matthew effect in American education. In: Aaron M. Pallas (ed.). *Research in sociology of education and socialization, Volume 12*. Stamford, Connecticut: Jai Press Inc., 35-66.

Kieren, Th.E. (1976). On the Mathematical, Cognitive and Instructional Foundations of Rational Number. In: R. Lesh (ed.). *Number and Measurement*. Columbus, OH:ERIC/ SMEAC, 101-144.

Lee, L. & D. Wheeler (1989). The arithmetic connection. *Educational Studies in Mathematics,* 20, 41-54.

Lee, C.D. (2001). Is October Brown Chinese? A Cultural Modeling Activity System for Underachieving Students. *American Educational Research Journal*, *38*(1), 97-141.

Lek, A. (1992). Met repen begrepen [Understood with chocolate bars]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *11*(2), 9-17.

Lieshout, E.C.D.M. van (2000). *Ontwikkelingen in de Behandeling van Leerproblemen: Aandacht voor Aanpakken en Vasthouden* [Developments in Learning Problems Treatment: A Focus on Approaches and Holding]. Amsterdam: Vrije Universiteit.

Mack, N.K. (1990). Learning fractions with understanding: building on informal knowledge. *Journal for Research in Mathematics Education*, *21*(1), 16-32.

Mack, N.K. (2000). *Long-term effects of building on informal knowledge in a complex content domain: the case of multiplication of fractions*. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, April 2000.

Menne, J.J.M. (2001). *Met Sprongen Vooruit: Een productief oefenprogramma voor zwakke rekenaars in het getallengebied tot 100 – een onderwijsexperiment* [By leaps and bounds ahead: a productive training-programme in arithmetic up to 100 for low achievers – a teaching experiment] (dissertation). Utrecht: CD-ß Press.

Milo, B.F. & A.J.J.M. Ruijssenaars (2002). Interactie in het speciaal basisonderwijs [Interaction in special education]. In: R. Keijzer & W. Uittenbogaard (eds.). *Interactie in het reken-wiskundeonderwijs*. Utrecht: Panama/Freudenthal Institute, 77-88.

Moor, E. de (2000). Heeft wiskunde vormende waarde? [Does mathematics have an educational value?]. *Vernieuwing. Tijdschrift voor onderwijs en opvoeding*, *59*(1/2),11.

Moss, J. & R. Case (1999). Developing Children's Understanding of the Rational Numbers: A New Model and an Experimental Curriculum. *Journal for Research in Mathematics Education*, *30*(2), 122-147.

Noelting, G. (1980). The development of proportional reasoning and the ratio concept. Part I – differentiation of stages. *Educational Studies in Mathematics*, *11,* 217-253.

Romberg, Th.A. (1994). Classroom Instruction That Fosters Mathematical Thinking and Problem Solving: Connections Between Theory and Practice. In: Alan H. Schoenfeld (ed.). *Mathematical thinking and problem solving*. Hilldale, NJ: Lawrence Erlbaum Associates, Publishers, 287-304.

Roth, W.-M. (2001). Situating Cognition. *The Journal of the Learning Sciences*, *10*(1&2), 27-61.

Schoenfeld, A.H. (1992). Learning to think mathematically: problem solving, metacognition, and sense making in mathematics. In: Douglas A. Grouws (ed.). *Handbook of research on mathematics teaching and learning*. New York: Macmillan Publishing Company, 334-370.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different side of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

Simon, M.A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114-145.

Steffe, L.P. & P. Cobb (1988). *Construction of arithmetical meanings and strategies*. New York, NY: Springer Verlag.

Streefland, L. (1987). Free production of fraction monographs. In: Jacques C. Bergeron, Nicolas Herscovics & Carolyn Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Volume I*. Montreal: s.n., 405-410.

Streefland, L. (1991). *Fractions in Realistic Mathematics Education: a Paradigm of Developmental Research*. Dordrecht: Kluwer Academic Publishers.

Streefland, L. (1993). The design of a mathematics course: a theoretical reflection. *Educational Studies in Mathematics*, *25*, 109-135.

Sweller, J. (1994). Cognitive load theory, learning difficulty, and instructional design. *Learning and Instruction*, *4*, 295-312.

Terwel, J. (1984). *Onderwijs maken* [The making of education] (dissertation). Harlingen: Flevodruk.

Terwel, J., P.G.P. Herfs, M. Mertens & J.Chr. Perrenet (1994). Co-operative learning and adaptive instruction in a mathematics curriculum. *Journal of Curriculum Studies*, *26*(2), 217-233.

Terwel, J., R.M. Gillies, P. van den Eeden & D. Hoek (2001). Co-operative learning processes of students: A longitudinal multilevel perspective. *British Journal of Educational Psychology*, *71*, 619-645.

Treffers, A., E. de Moor & E. Feijs (1989). *Proeve van een nationaal programma voor het reken-wiskundeonderwijs op de basisschool. Deel 1: Overzicht einddoelen* [Specimen of a National Program for Primary Mathematics teaching. Part 1: Survey on Educational Aims]. Tilburg: Zwijsen.

Tzur, R. (1999). An Integrated Study of Children's Construction of Improper Fractions and the Teacher's Role in Promoting That Learning. *Journal for Research in Mathematics Education*, *30*(4), 390-416.

Weckesser, H. (1970). Die Einführung von Brüchen mit Hilfe von Modellen [The introduction of fractions with the help of models]. *Der Mathematikunterricht*, *16*, 30-77.

Yackel, E. & P. Cobb (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458-477.

Yackel, E. (2001). Explanation, justification and argumentation in mathematics classrooms. In: M. van den Heuvel-Panhuizen (ed.). *Proceedings of the 25th conference of the international group for the Psychology of Mathematics Education*, *Volume 1*. Utrecht: Freudenthal Institute, 9-24.

References

Ainley, J. (1996). Purposeful Contexts for Formal Notation in a Spreadsheet Environment. *Journal of Mathematical Behavior*, *15*(4), 405-422.

Alexander, P.A., S. Graham & K.R. Harris (1998). A Perspective on Strategy Research: Progress and Prospects. *Educational Psychology Review*, *10*(2), 129-151.

Alexander, P.A. (2000). Toward a Model of Academic Development: Schooling and the Acquisition of Knowledge. *Educational Researcher*, *29*(2), 28-34.

Anderson, J.R., J.G. Greeno, L.M. Reder & H.A. Simon (2000). Perspectives on Learning, Thinking, and Activity. *Educational Researcher*, *29*(4), 11-13.

Arcavi, A. & A.H. Schoenfeld (1992). Mathematics Tutoring Through a Constructivist Lens: The Challenges of Sense-Making. *Journal of Mathematical Behavior*, *11*, 321-335.

Arcavi, A. (1994). Symbol Sense: Informal Sense-making in Formal Mathematics. *For the Learning of Mathematics*, *14*(3), 24-35.

Armstrong, B.E. & C. Novillis Larson (1995). Students' use of part-whole and direct comparison strategies for comparing partitioned rectangles. *Journal for Research in Mathematics Education*, *26*(1), 2-19.

Bakker, A. & F. van Galen (2000). Cathy Fosnot over constructivisme [Cathy Fosnot on constructivism]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *19*(1), 21-23.

Bednarz, N. & B. Janvier (1988). A constructivist approach in primary school: results of a three year intervention with the same group of children. *Educational Studies in Mathematics*, *19*, 299-331.

Behr, M.J., R. Lesh, Th.R. Post & E.A. Silver (1983). Rational-Number Concepts. In: Richard Lesh & Marsha Landau (eds.). *Acquisition of Mathematical Concepts and Processes*. New York/London: Academic Press, 91-126.

Behr, M.J., G. Harel, Th. Post & R. Lesh (1992). Rational number, ratio, and proportion. In: Douglas A. Grouws (ed.). *Handbook of research on mathematics teaching and learning*. New York: Macmillan Publishing Company, 296-332.

Beishuizen, M. (1997). Development of mathematical strategies and procedures up to 100. In: M. Beishuizen, K.P.E. Gravemeijer & E.C.D.M. van Lieshout (eds.). *The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures*. Utrecht: Freudenthal Institute, 127-162.

Bergeron, A., N. Herscovics & J.C. Bergeron (1987). Kindergartners' knowledge of numbers: A longitudinal study. Part II: Abstraction and formalization. In: J.C. Bergeron, N. Herscovics & C. Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education, Volume II*. Montreal: s.n., 352-360.

Bezuk, N.S. & M. Bieck (1993). Current Research on Rational Numbers and Common Fractions: Summary and Implications for Teachers. In: Douglas T. Owens (ed.). *Research Ideas for the Classroom – Middle Grades Mathematics*. New York: Macmillan Publishing Company, 118-136.

Bokhove, J., K. Buys (ed.), R. Keijzer, A. Lek, A. Noteboom & A. Treffers (1996). *De Breukenbode. Een leergang voor de basisschool* (werkbladen en handleiding) [The Fraction gazette]. Enschede/Utrecht: SLO/FI/Cito.

Bokhove, J., F. van der Schoot & T. Eggen (1996). *Balans van het rekenonderwijs aan het einde van de basisschool 2* [Balance-sheet of mathematics education at the end of prima-

ryschool 2]. Arnhem: Cito.

Booth, R.D.L. & M.O.J. Thomas (2000). Visualization in Mathematics Learning: Arithmetic Problem-solving and Student Difficulties. *Journal of Mathematical Behavior*, *18*(2), 169-190.

Brandt, B. (1998). Forms of participation in elementary mathematics classroom interaction. In: P. Abrantes, J. Porfírio & B. Baía (eds.). *The interactions in the mathematics classroom.* Proceedings of the CIEAEM 49. Setúbal, Portugal (July, 1997): s.n., 157-163.

Brenner, M.E., R.E. Mayer, B. Moseley, T. Brar, R. Durán, B. Smith Reed & D. Webb (1997). Learning by Understanding: The Role of Multiple Representations in Learning Algebra. *American Educational Research Journal*, *34*(4), 663-689.

Brinker, L. (1997). *Using Structured Representations to Solve Fraction Problems: A Discussion of Seven Students' Strategies.* Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL.

Brown, J.S., A. Collins & P. Duguid (1989). Situated cognition and the culture of learning. *Educational Researcher*, *18*(1), 32-42.

Bruner, Jerome S. (1966). *Toward a Theory of Instruction*. Cambridge, MA: The Belknap Press of Harvard University Press.

Bruner, J.S. (1966). On Cognitive Growth II. In: Jerome S. Bruner, Rose R. Olver & Patricia M. Greenfield (eds.). *Studies in Cognitive Growth. A Collaboration at the Center for Cognitive Studies*. New York/London/Sydney: John Wiley & Sons, Inc., 30-67.

Burton, L. (with the help of Mark Burton) (1980a). Problems and Puzzles. *For the Learning of Mathematics*, *1*(2), 20-23.

Burton, L. (1980b). The teaching of mathematics to young children using a problem solving approach. *Educational Studies in Mathematics*, *11*, 43-58.

Buys, K. & A. Noteboom (1995). Breuken: begrip en taalontwikkeling [Fractions: comprehension and language development]. In: C. van den Boer & M. Dolk (eds.). *Rekenen in de bovenbouw van de basisschool*. Utrecht: Panama/HvU/Freudenthal Institute, 39-52.

Capps, L.R. & J. Pickreign (1993). Language connections in mathematics: A critical part of mathematics instruction. *Arithmetic Teacher*, *41*(1), 8-12.

Carpenter, Th.P., T.G. Coburn, R.E. Reys & J.W. Wilson (1976). Notes from National Assessment: addition and multiplication with fractions. *The Arithmetic Teacher*, *23*(2), 137-142.

Carraher, D.W. & A.D. Schliemann (1991). Children's understanding of fractions as expressions of relative magnitude. In: Fulvia Furinghetti (ed.). *Proceedings of the fifteenth PME conference. Volume 1*. Assisi, Italy: Dipartimento di Matematica dell'Università di Genova, 184-191.

Carraher, D.W. (1993). Lines of thought: A ratio and operator model of rational number. *Educational Studies in Mathematics*, *25*, 281-305.

Clements, M.A. (1980). Analyzing children's errors on written mathematical tasks. *Educational Studies in Mathematics*, *11*, 1-21.

Cobb, P. & L.P. Steffe (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education*, *14*(2), 83-94.

Cobb, P., E. Yackel & T. Wood (1989). Young Children's Emotional Acts While Engaged in Mathematical Problem Solving. In: D.B. McLeod & V.M. Adams (eds.). *Affect and Mathematical problem solving: A new perspective*. New York: Springer Verlag, 117-148.

Cobb, P. & J.W. Whitenack (1996). A Method for Conducting Longitudinal Analyses of Classroom Videorecordings and Transcripts. *Educational Studies in Mathematics*, *30*, 213-228.

Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Curriculum Standards in Primary Education] (1994). *Doelbewust leren; kerndoelen in maatschappelijk perspectief* [Purposeful learning; curriculum standards in social perspective]. Den Haag: SDU.

Commissie kerndoelen basisonderwijs [Committee Curriculum Standards in Primary Education] (2002). Verantwoording delen. Herziening van de kerndoelen basisonderwijs met het oog op beleidsruimte voor scholen [Sharing responsibility. Revision of the teaching standards for primary education with a view to the scope for school policymaking]. 's Gravenhage: Ministerie van OC & W.

Connell, M.L. & D.M. Peck (1993). Report of a Conceptual Intervention in Elementary Mathematics. *Journal of Mathematical Behavior*, *12*, 329-350.

Cook, Th.D. & D.T. Campbell (1979). *Quasi-Experimentation. Design & Analysis Issues for Field Settings*. Chicago: Rand McNally College Publishing Company.

Corte, E. de, B. Greer & L. Verschaffel (1996). Mathematics Teaching and Learning. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology*. New York: Simon & Schuster Macmillan, 491-549.

Corwin, R.B. (1989). Multiplication as Original Sin. *Journal of Mathematical Behavior*, *8*, 223-225.

Dar, Y. & N. Resh (1994). Separating and mixing students for learning: Concepts and research. *Pedagogisch Tijdschrift*, *19*, 109-126.

Davis, G., R.P. Hunting & C. Pearn (1993). What Might a Fraction Mean to a Child and How would a Teacher Know? *Journal of Mathematical Behavior*, *12*, 63-76.

Davis, P.J. & R. Hersh (1984). *The Mathematical Experience*. London: Penguin Books.

Davis, R.B. (1992). Understanding 'Understanding'. *Journal of Mathematical Behavior*, *11*, 225-241.

Davis, R.B. (1994). Comments on Thomas Romberg's Chapter. In: Alan H. Schoenfeld (ed.). *Mathematical thinking and problem solving*. Hilldale, NJ: Lawrence Erlbaum Associates, Publishers, 312-321.

Deal, D., M.M. Buehl, M.G.O. Tron, L. Harris, S. Clements & S. Hardy (2000). *Portraying Mathematical and Analogical Reasoning in the Young: Three Case Studies.* Paper presented at the annual meeting of the American Educational Research Association, New Orleans, April 2000.

Dekker, R. (1991). *Wiskunde leren in kleine heterogene groepen* [Learning mathematics in small heterogeneous groups] (dissertation). De Lier: Academisch Boeken Centrum.

Dijk, I.M.A.W. van, B. van Oers & J. Terwel (2000). Aanreiken of ontwerpen? Leren modelleren als strategie voor het werken met problemen in contexten in het reken-wiskundeonderwijs [Handing or developing? Learning to model as a strategy in working with problems in context in mathematics education]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *18*(2), 38-51.

Dijk, I.M.A.W. van (2002). *The learner as designer: Processes and effects of an experimental programme in modelling in primary mathematics education* (dissertation). Amsterdam: VU.

Dondorff, L. (1879). Vermenigvuldiging en deeling der breuken [Multiplication and division of fractions]. *Weekblad voor het lager, middelbaar en gymnasiaal onderwijs*, jrg. XVIII (26/4/1879), 1.

Doornbos, K. (1997). *De pedagogische balans* [The pedagogical balance]. Groningen: Wolters-Noordhoff.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In: David

Tall (ed.). *Advanced mathematical thinking*. Dordrecht/Boston/London: Kluwer Academic Publishers, 95-123.

Easley Jr., J.A. (1981). Levels of abstraction and intellectual development. *The Journal of Children's Mathematical Behavior*, *1*(2), 76-83.

Eeden, P. van den & J. Terwel (1994). Evaluation of a mathematics curriculum: differential effects. *Studies in Educational Evaluation*, *20*, 457-475.

Elbers, E. (2001). Social interaction as reflection: Leen Streefland as a teacher of primary school children. In: M. van den Heuvel-Panhuizen (ed.). *Proceedings of the 25th conference of the international group for the Psychology of Mathematics Education*. Utrecht: Freudenthal Institute, 226-232.

Erlwanger, S.H. (1973). Benny's conception of Rules and Answers in IPI Mathematics. *Childern's Mathematical Behavior*, *1*(2), 7-26.

Featherstone, H. (2000). '− Pat + Pat = 0': Intellectual Play in Elementary Mathematics. *For the Learning of Mathematics*, *20*(2), 14-23.

Fischbein, E. (1982). Intuition and Proof. *For the Learning of Mathematics, 3*(2), 9-18 and 24.

Forman, G. & B. Fyfe (1998). Negotiated Learning Through Design, Documentation, and Discourse. In: Carolyn Edwards, Lella Gandini, George Forman (eds.). *The hundred languages of children The Reggio Emilia Approach − Advanced Reflections* (second edition). Greenwich, Connecticut/London, England: Ablex Publishing Corporation, 239-260.

Fosnot, C.T. & M. Dolk (2001). *Young Mathematicians at Work: Constructing the Number System, Addition and Subtraction*. Portsmouth, NH: Heinemann Press.

Freudenthal, H. (1953). Zur Geschichte der vollständigen Induktion. *Archives Internationales d'histoire des sciences*, *22*, 17-37.

Freudenthal, H. (1968). Why to teach mathematics as to be useful? *Educational Studies in Mathematics*, *1*(1), 3-8.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics*, *3*(3/4), 413-435.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: Reidel.

Freudenthal, H. (1980). *Weeding and Sowing: Preface to a Science of Mathematics Education*. Dordrecht: Reidel.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: Reidel-Kluwer.

Freudenthal, H. (1989). Einführung der negativen Zahlen nach dem geometrisch-algebraischen Permanenzprinzip. *Mathematiklehren, Heft 35*, 26-37.

Freudenthal, H. (1991). *Revisiting Mathematics Education, China lectures*. Dordrecht: Kluwer Academic Publishers.

Galen, F. van & A. Buter (1997). De rol van interactie bij leren rekenen met de computer [The role of interaction while learning arithmetic with the computer]. *Tijdschrift voor naschling en onderzoek van het reken-wiskundeonderwijs*, *16*(1), 11-18.

Glaser, B.G. & A.L. Strauss (1977). *The Discovery of Grounded Theory. Strategies for Qualitative Research*. Chicago: Aldine Publishing Company.

Glasersfeld, E. von (1987). Learning as a Constructive Activity. In: C. Janvier (ed.). *Problems of representation in the teaching and learning of mathematics*. Hillsdale, New Jersey/London: Lawrence Erlbaum Associates, Publishers, 3-17.

Goddijn, A.J. (1992). De oudste en de nieuwste breuken [The oldest and the newest fractions]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *11*(2), 18-31.

Goldin, G.A. (1998). Representational Systems, Learning, and Problem Solving in Mathematics. *Journal of Mathematical Behavior*, *17*(2), 137-165.

Graeber, A.O. & E. Tanenhaus (1993). Multiplication and Division: From Whole Numbers to Rational Numbers. In: Douglas T. Owens (ed.). *Research Ideas for the Classroom − Middle Grades Mathematics*. New York: Macmillan Publishing Company, 99-117.

Gravemeijer, K.P.E. (1994). *Developing Realistic Mathematics Education* (dissertation). Utrecht: CD-ß Press.

Gravemeijer, K. (1998). Symboliseren en modelleren als wiskundige activiteit [Symbolising and modelling as mathematical activity]. In: N. Boswinkel & M. Dolk (eds.). *Over rekenen gesproken − taal in/en rekenen −*. Utrecht: Panama/Freudenthal Institute, 35-52.

Gravemeijer, K.P.E. & J. Terwel (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of Curriculum Studies*, *32*(6), 777-796.

Gravemeijer, K.P.E. (2001). *Reken-wiskundeonderwijs voor de 21^e eeuw* [Mathematics education for the 21th century]. Utrecht: Utrecht University.

Greeno, J.G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education*, *22*(3), 170-218.

Greeno, J.G., A.M. Collins & L. Resnick (1996). Cognition and Learning. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology*. New York: Simon & Schuster Macmillan, 15-46.

Greeno, J.G. (1997). On claims That Answer the Wrong Questions. *Educational Researcher*, *26*(1), 5-17.

Greeno, J.G. & S.V. Goldman (1998). *Thinking Practices in Mathematics and Science Learning*. Mahwah, New Jersey/London: Lawrence Erlbaum Associates, Publishers.

Greer, B. (1987). Didactical obstacles in the development of the concepts of multiplication and division. In: Jacques C. Bergeron, Nicolas Herscovics & Carolyn Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Volume II*. Montreal: s.n., 402-407.

Greer, B. & G. Harel (1998). The Role of Isomorphisms in Mathematical Cognition. *Journal of Mathematical Behavior*, *17*(1), 5-24.

Griffin, S., R. Case & R. Sandieson (1992). Synchrony and Asynchrony in the Acquisition of Children's Everyday Mathematical Knowledge. In: Robbie Case (ed.). *The Mind's Staircase. Exploring the Conceptual Underpinnings of Children's thought and Knowledge*. Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers, 75-97.

Griffin, S. & R. Case (1997). Re-thinking the primary school math curriculum: An approach based on cognitive science. *Issues in Education*, *3*(1), 1-49.

Gudmundsdottir, S. (1995). The narrative nature of pedagogical content knowledge. In: H. McEwan & K. Egan (eds.). *Narrative in teaching, learning and research*. New York: Teachers College, Columbia University, 24-38.

Hamers, J.H.M. & A.J.J.M. Ruijssenaars (1984). *Leergeschiktheid en Leertests. Een leertestonderzoek bij eersteklassers in het gewoon lager onderwijs* [The concept of learning ability and learning ability tests. A research on learning ablility tests in first graders in primary school] (dissertation). Harlingen: Flevodruk.

Hanna, G. (1989). More than Formal Proof. *For the Learning of Mathematics*, *9*(1), 20-23.

Harel, G. & D. Tall (1991). The General, the Abstract, and the Generic in Advanced Mathematics. *For the Learning of Mathematics*, *11*(1), 38-42.

Hart, K.M. (1981). Fractions. In: K.M. Hart (ed.). *Children's understanding of mathematics: 11-16*. London: John Murray, 66-81.

Hart, K.M. (1987). Practical work and formalisation, too great a gap. In: Jacques C. Bergeron, Nicolas Herscovics & Carolyn Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Volume II*. Montreal: s.n.,

408-415.

Hasemann, K. (1981). On difficulties with fractions. *Educational Studies in Mathematics*, *12*, 71-87.

Have, P. ten (1977). *Sociologisch veldonderzoek* [Sociological field-research]. Meppel: Boom.

Haylock, D.W. (1987). A framework for assessing mathematical creativity in schoolchildren. *Educational Studies in Mathematics*, *18*, 59-74.

Hedegaard, M. (1990). The zone of proximal development as basis for instruction. In: L.C. Moll (ed.). *Vygotsky and education*. Cambridge: Cambridge University Press, 349-371.

Heege, H. ter (2000). *Onderwijs in het meten. Een opmaat voor het voorkomen van ongecijferdheid* [Education in measuring. A prelude in the prevention of innumeracy]. Enschede: SLO.

Hefendehl-Hebeker, L. (1991). Negative Numbers: Obstacles in their Evolution from Intuitive to Intellectual Constructs. *For the Learning of Mathematics*, *11*(1), 26-32.

Herfs, P.G.P., E.H.M. Mertens, J.Chr. Perrenet & J. Terwel (1991). *Leren door samenwerken. Adaptief groepsonderwijs (Ago) als een curriculum-innovatie in het voortgezet onderwijs* [Learning by co-operation. Adaptive co-operative education as a curriculum-innovation in secondary education]. Amsterdam/Lisse: Swets & Zeitlinger BV.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, *24*, 389-399.

Heuvel-Panhuizen, M. van den & L. Streefland (1993). Per Sense − een onderwijspakketje over procenten [Per Sense − an educational unit on percents]. In: M. Dolk & W. Uittenbogaard (eds.). *Procenten − op de grens van basisschool en basisvorming*. Utrecht: Panama/HMN/Freudenthal Institute, 25-48.

Heuvel-Panhuizen, M. van den (1996). *Assessment and realistic mathematics education* (dissertation). Utrecht: CD-ß Press.

Hiebert, J. (1988). A theory of developing competence with written mathematical symbols. *Educational Studies in Mathematics*, *19*, 333-355.

Hiebert, J., Th.P. Carpenter, E. Fennema, K. Fuson, P. Human, H. Murray, A. Olivier & D. Wearne (1996). Problem Solving as a Basis for Reform in Curriculum and Instruction: The Case of Mathematics. *Educational Researcher*, *25*(4), 12-21.

Hiele, P.M. van (1986). *Structure and insight, a theory of mathematics education*. Orlando: Academic Press, Inc.

Hoek, D., J. Terwel & P. van den Eeden (1997). Effects of Training in the Use of Social and cognitive Strategies: An Intervention Study in Secondary Mathematics in Co-operative Groups. *Educational Research and Evaluation*, *3*(4), 364-389.

Hoek, D.J. (1998). *Social and cognitive strategies in co-operative groups. Effects of strategy instruction in secondary mathematics* (dissertation). Amsterdam: University of Amsterdam.

Hoek, D., P. van den Eeden & J. Terwel (1999). The effects of integrated social and cognitive strategy instruction on the mathematics achievement in secondary education. *Learning & Instruction*, *9*(5), 427-448.

Holt, J. (1964). *How Children Fail*. New York: Dell Publishing Co.

Hopmann, S. & K. Riquarts (1995). Starting a dialogue: issues in a beginning conversation between didactic and the curriculum traditions. *Journal of Curriculum Studies*, *27*(1), 3-12.

Hopmann, S. & R. Künzli (1997). Close our schools! Against current trends in policy making, educational theory and curriculum studies. *Journal of Curriculum Studies*, *29*(3), 259-

266.

Huhse, K. (1968). *Studien und Berichte. Theorie und Praxis der Curriculum Entwicklung*. Berlin: Institut für Bildungsforschung in der Max-Planck-Gesellschaft.

Huitema, S. (2002). Interactie en de dilemma's van de leerkracht. Differentiatie in interactie [Interaction and the teacher's dilemma's. Differentiation in interaction]. In: R. Keijzer & W. Uittenbogaard (eds.). *Interactie in het reken-wiskundeonderwijs*. Utrecht: Panama/ Freudenthal Institute, 41-48.

Huitema, S., A. van der Klis, F. van de Molengraaf, M. Timmermans & L. Erich (n.y.). *De wereld in getallen 6. Handleiding a & b* [The world in numbers 6. Teacherguide a & b]. Den Bosch: Malmberg.

Hunting, R.P. (1983). Alan: A case study of knowledge of units and performance with fractions. *Journal for Research in Mathematics Education*, *14*(3), 182-197.

Hunting, R.P. (1984). Understanding equivalent fractions. *Journal of Science and Mathematics Education in S.E. Asia, Volume VII*(1), 26-33.

Hunting, R.P. (1986). Rachels schemes for constructing fraction knowledge. *Educational Studies in Mathematics*, *17*, 49-66.

Janssen, J., J.-M. Kraemer & A. Noteboom (1995). *Leerling Volg Systeem. Rekenen-Wiskunde 2* [Student Registration System. Mathematics 2]. Arnhem: Cito.

Janssen, J., F. van der Schoot, B. Hemker & N. Verhelst (1999). *Balans van het reken-wiskundeonderwijs in de basisschool 3* [Balance-sheet of mathematics education in primary school 3]. Arnhem: Cito.

Kambartel, F. (1961). Logische Stellung und konstruktive Bedeutung mathematischer Permanenzprinzipien. *Der Mathematikunterricht*, *7*(1) (Logische Probleme im Mathematikunterricht II), 57-78.

Kamii, C. & F.B. Clark (1995). Equivalent Fractions: Their Difficulty and Educational Implications. *Journal of Mathematical Behavior*, *14*, 365-378.

Keijzer, R. (1994). De ontwikkeling van een breukenleergang – een voorbeeld van ontwikkelingsonderzoek [The development of a curriculum on fractions – an example of developmental research]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *13*(1), 24-31.

Keijzer, R. & A. Lek (1995). Breuken: kerndoelen en formaliseren [Fractions: curriculum standards and formalising]. In: C. van den Boer & M. Dolk (eds.). *Rekenen in de bovenbouw van de basisschool*. Utrecht: Panama/HvU/Freudenthal Institute, 53-64.

Keijzer, R. & K. Buys (1996a). *Making sense of mathematics in the newspaper*. Paper presented at ICME 8. Seville, Spain (July 1996).

Keijzer, R. & K. Buys (1996b). Groter of kleiner. Een doorkijkje door een nieuwe leergang breuken [Bigger or smaller. A close look at a new curriculum on fractions]. *Willem Bartjens*, *15*(3), 10-17.

Keijzer, R. (1997). Formeel rekenen met breuken [Formal arithmetic with fractions]. In: C. van den Boer & M. Dolk (eds.). *Naar een balans in de reken-wiskundeles – interactie, oefenen, uitleggen en zelfstandig werken –*. Utrecht: Panama/Freudenthal Instituut, 101-116.

Keijzer, R. (1999). Een klein verschil? [A small difference?]. In: W. Faes & W. Oonk (eds.). *Van Rekenend Nederland voor Fred Goffree. 65/2 Onderwijsverhalen voor pabostudenten*. Groningen: Wolters-Noordhoff, 37-40.

Keijzer, R. & J. Terwel (2000). *Learning for mathematical insight: a longitudinal comparison of two Dutch curricula on fractions*. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, April, 2000.

Keijzer, R. & J. Terwel (2001). Audrey's acquisition of fractions: a case study into the learning of formal mathematics. *Educational Studies in Mathematics*, *47*(1), 53-73.

Keijzer, R., M. Baltussen, H. ter Heege, J.M.M. Kaskens & E.C. Veldhuis (2001). Rekenen in het speciaal onderwijs [Arithmetic in special education]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *19*(3), 14-18.

Keijzer, R. & J. Terwel (in press). Learning for mathematical insight: a longitudinal comparative study on modelling. *Learning & Instruction*.

Kerckhoff, A.C. & E. Glennie (1999). The Matthew effect in American education. In: Aaron M. Pallas (ed.). *Research in sociology of education and socialization, Volume 12*. Stamford, Connecticut: Jai Press Inc., 35-66.

Kieren, Th.E. (1976). On the Mathematical, Cognitive and Instructional Foundations of Rational Number. In: R. Lesh (ed.). *Number and Measurement*. Columbus, OH: ERIC/ SMEAC, 101-144.

Kieren, T., D. Nelson & G. Smith (1985). Graphical Algorithms in Partitioning Tasks. *Journal of Mathematical Behavior*, *4*(1), 25-36.

Klein, A.S. (1998). *Flexibization of mental arithmetic strategies on a different knowledge base* (dissertation). Utrecht: CD-ß Press.

Klein, A.S., M. Beishuizen & A. Treffers (1998). The Empty Number Line in Dutch Second Grades: Realistic Versus Gradual Program Design. *Journal for Research in Mathematics Education*, *29*(4), 443-465.

Koch, L.C. & X. Li (1996). Students' Understanding of Computation-Related Rational Number Skills. *Journal of Mathematical Behavior, 15*, 193-205.

Kohnstamm, P.H. (1948). *Keur uit het didactisch werk* [Selection from the didactic work]. Groningen: J.B. Wolters Uitgeversmaatschappij.

Kraemer, J.-M. (2000). Met handen en voeten rekenen. Naar een natuurlijke benadering van rekenzwakke kinderen [Arithmetic with hands and feet. Towards a natural approach for low arithmetic achievers] *Vernieuwing. Tijdschrift voor onderwijs en opvoeding*, *59*(1/2), 29-32.

Kraemer, J.-M. & J. Janssen (2000). Druppels op een gloeiende plaat – aandachtspunten voor een grondige discussie over de continue ontwikkeling van rekenzwakke leerlingen tussen vier en zestien jaar [Drops in the ocean – points of attention for a thorough discussion on the continuous development of low-attainers in arithmetic between four and sixteen years of age]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *18*(3), 3-14.

Kraemer, J.-M., F. van der Schoot & R. Engelen (2000). *Balans van het reken- wiskunde-onderwijs op LOM- en MLK-scholen 2: uitkomsten van de tweede peiling in 1997* [Balance-sheet of mathematics education in special schools 2: results of the second gauging in 1997]. Arnhem: Cito.

Krutetskii, V.A. (1976 (American edition)). *The psychology of Mathematical Abilities in Schoolchildren*. (Translated by Joan Teller, edited by Jeremy Kilpatrick and Izaak Wirszup). Chicago/London: The University of Chicago Press.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer. *American Educational Research Journal*, *27*(1), 29-63.

Lawler, R.W. (1990). Constructing Knowledge From Interactions. *Journal of Mathematical Behavior*, *9*, 177-192.

Lee, C.D. (2001). Is October Brown Chinese? A Cultural Modeling Activity System for Underachieving Students. *American Educational Research Journal*, *38*(1), 97-141.

Lee, L. & D. Wheeler (1989). The arithmetic connection. *Educational Studies in Mathemat-*

ics, *20*, 41-54.

Lek, A. (1992a). Met repen begrepen [Understood with chocolate bars]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *11*(2), 9-17.

Lek, A.T.E. (1992b). *Met repen begrepen* [Understood with chocolate bars]. Unpublished doctoral thesis. Utrecht: Utrecht University.

Lembke, L.O. & B.J. Reys (1994). The development of, and interaction between, intuitive and school-taught ideas about percent. *Journal for Research in Mathematics Education*, *25*(3), 237-259.

Lemoyne, G. & C. Tremblay (1986). Addition and multiplication: problem-solving and inter-pretation of relevant data. *Educational Studies in Mathematics*, *17*, 97-123.

Leron, U. (1985). A direct approach to indirect proofs. *Educational Studies in Mathematics*, *16*, 321-325.

Lieshout, E.C.D.M. van (1997). What can research on word and context problems tell about effective strategies to solve subtraction problems? In: M. Beishuizen, K.P.E. Gravemeijer & E.C.D.M van Lieshout (eds.). *The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures*. Utrecht: CD-ß Press, 79-111.

Lieshout, E.C.D.M. van (2000). *Ontwikkelingen in de Behandeling van Leerproblemen: Aan-dacht voor Aanpakken en Vasthouden* [Developments in Learning Problems Treatment: A Focus on Approaches and Holding]. Amsterdam: Vrije Universiteit.

Lieshout, E.C.D.M. van & F. Meijers (in press). Hoofdrekenstrategieën voor optellen en aftrekken tot 100 bij hoogbegaafde leerlingen en leerlingen met leerproblemen [Strategies in mental arithmetic for addition and subtraction up to 100 of gifted students and pupils with learning problems]. *Pedagogische Studiën*.

Linchevski, L. (1995). Algebra With Numbers and Arithmetic With Letters: A Definition of Pre-Algebra. *Journal of Mathematical Behavior*, *14*, 113-120.

Luit, J.E.H. van & B.A.M. van de Rijt (1997). Stimulation of early mathematical competence. In: M. Beishuizen, K.P.E. Gravemeijer & E.C.D.M. van Lieshout (eds.). *The Role of Con-texts and Models in the Development of Mathematical Strategies and Procedures*. Utrecht: CD-ß Press, 215-237.

Lyle, S. (2000). Narrative understanding: developing a theoretical context for understanding how children make meaning in classroom settings. *Journal of Curriculum Studies*, *32*(1), 45-63,

Mack, N.K. (1990). Learning fractions with understanding: building on informal knowledge. *Journal for Research in Mathematics Education*, *21*(1), 16-32.

Mack, N.K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education*, *26*(5), 422-441.

Mack, N.K. (2000). *Long-term effects of building on informal knowledge in a complex content domain: the case of multiplication of fractions*. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, April 2000.

Mason, J. (1989). Mathematical Abstraction as the Result of a Delicate Shift of Attention. *For the Learning of Mathematics*, *9*(2), 2-8.

Mayer, R.E. (1989). Models for Understanding. *Review of Educational Research*, *59*(1), 43-64.

Mayer, R.E. (1996). Learners as information processors: Legacies and limitations of educa-tional psychology's second metaphor. *Educational Psychologist*, *31*(3/4), 151-161.

Mayer, R.E. & M.C. Wittrock (1996). Problem-Solving Transfer. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology*. New York: Simon & Schuster Macmillan, 47-62.

Mayer, R.E. (1999). Designing Instruction for Constructivist Learning. In: Charles M. Reigeluth (ed.). *Instructional-design theories and models. A new paradigm of instructional theory. Volume II.* Mahwah, NJ: Lawrence Erlbaum Associates, Publishers, 141-160.

Mayer, R.E. (2001). What Good is Educational Psychology? The Case of Cognition and Instruction. *Educational Psychologist, 36*(2), 83-88.

Mcintosh, A., B.J. Reys & R.E. Reys (1992). A Proposed Framework for Examining Basic Number Sense. *For the Learning of Mathematics, 12*(3), 2-8.

McNeill, R. (1988). A Reflection on When I Loved Math and How I Stopped. *The Journal of Mathematical Behavior, 7*(1), 45-50.

Mead, M. (1959). *Coming of age in Samoa: a psychological study of primitive youth for Western civilisation.* New York: New American Library.

Menne, J.J.M. (2001). *Met Sprongen Vooruit: Een productief oefenprogramma voor zwakke rekenaars in het getallengebied tot 100 – een onderwijsexperiment* [By leaps and bounds ahead: a productive training-programme in arithmetic up to 100 for low achievers – a teaching experiment] (dissertation). Utrecht: CD-ß Press.

Milo, B.F. & A.J.J.M. Ruijssenaars (2002). Interactie in het speciaal basisonderwijs [Interaction in special education]. In: R. Keijzer & W. Uittenbogaard (eds.). *Interactie in het reken-wiskundeonderwijs.* Utrecht: Panama/Freudenthal Institute, 77-88.

Moor, E. de (2000). Heeft wiskunde vormende waarde? [Does mathematics have an educational value?]. *Vernieuwing. Tijdschrift voor onderwijs en opvoeding, 59*(1/2), 11.

Moss, J. & R. Case (1999). Developing Children's Understanding of the Rational Numbers: A New Model and an Experimental Curriculum. *Journal for Research in Mathematics Education, 30*(2), 122-147.

Movshovits-Hadar, N. (1988). School Mathematics Theorems – an Endless Source of Surprise. *For the Learning of Mathematics, 8*(3), 34-40.

Naujok, N. (1998). How Helpful Is Help among Students? Theoretical Basis of and Methodological Considerations for an Interaction Theory of Learning in School. In: P. Abrantes, J. Porfírio & B. Baía (eds.). *The interactions in the mathematics classroom.* Proceedings of the CIEAEM 49. Setúbal, Portugal (July, 1997): s.n., 52-62.

Nelissen, J.M.C. (1998a). Representaties in het reken-wiskundeonderwijs [Representations in mathematics education]. *Pedagogische Studiën, 75,* 169-183.

Nelissen, J.M.C. (1998b). Taal en betekenis in het realistisch reken-wiskundeonderwijs [Language and meaning in realistic mathematics education]. In: N. Boswinkel & M. Dolk (eds.). *Over rekenen gesproken – taal in/en rekenen –.* Utrecht: Panama/Freudenthal Institute, 11-34.

Nesher, P. (1986). Are Mathematical Understanding and Algorithmic Performance Related? *For the Learning of Mathematics, 6*(3), 2-9.

Newell, A. & H.A. Simon (1972). *Human problem solving.* Englewood Cliffs, NJ: Prentice-Hall.

Noelting, G. (1980). The development of proportional reasoning and the ratio concept. Part I – differentiation of stages. *Educational Studies in Mathematics, 11,* 217-253.

Noteboom, A. (1994). De Kop van Jut. Een breukenactiviteit in groep 6. [The try-your-strength machine. A fraction-activity in grade 6]. *Willem Bartjens, 14*(2), 9-13.

Novillis Larson, C. (1980). Locating Proper Fractions On Number Lines: Effect of Length and Equivalence. *School Science and Mathematics, Volume LXXX*(5), 423-428.

Oers, B. van (1996). Learning mathematics as a meaningful activity. In: L.P. Steffe, P. Nesher, P. Cobb, G.A. Goldin & B. Greer (eds.). *Theories of mathematical learning.* Mahwah, NJ: Lawrence Erlbaum Associates, Publishers, 91-113.

Oers, B. van & W. Wardekker (1999). On becoming an authentic learner: semiotic activity in the early grades. *Journal of Curriculum Studies*, *31*(2), 229-249.

Oers, B. van (2001). Educational forms of initiation in mathematical culture. *Educational Studies in Mathematics*, *46*, 59-85.

Padberg, F. (1989). *Didaktik der Bruchrechnung: Gemeine Brüche – Dezimalbrüche. Lehrbücher und Monographien zur Didaktik der Mathematik. Band 11*. Mannheim/Wien/ Zürich: Wissenschaftsverlag.

Parreren, C.F. van (1975). De relatie onderwijs – cognitieve ontwikkeling in de Russische psychologie [The relation education – cognitive development in Russian psychology]. In: C.F. van Parreren & W.A. van Loon-Vervoorn (eds.). *Teksten en Analyses Sovjetpsychologie 1: Denken*. Groningen: H.D. Tjeenk Willink BV, 10-31.

Paulos, J.A. (1988). *Innumeracy, mathematical illiteracy and its consequences*. New York: Hill and Wang.

Pedhazur, E.J. (1982). *Multiple regression in behavioral research – explanation and prediction*. New York: Holt, Rinehart and Winston.

Perkins, D.N. & Ch. Unger (1999). Teaching and Learning for Understanding. In: Charles M. Reigeluth (ed.). *Instructional-design theories and models. A new paradigm of instructional theory. Volume II*. Mahwah, NJ: Lawrence Erlbaum Associates, Publishers, 91-114.

Perrenet, J.Ch. (1995). *Leren probleemoplossen in het wiskunde-onderwijs: samen of alleen. Onderzoek van wiskunde leren bij 12- tot 16-jarigen* [Learning problem solving in mathematics education: together or alone. Research on learning mathematics of 12- to 16-year olds] (dissertation). Amsterdam: Universiteit van Amsterdam/Deventer: Sneldruk.

Piaget, J. (1973). Comments on mathematical education. In: A.G. Howson (ed.). *Developments in Mathematical Education*. Proceedings of the Second International Congress on Mathematical Education. Cambridge: Cambridge University Press, 79-87.

Pinar, W.M., W.M. Reynolds, P. Slattery & P.M. Taubman (1995). *Understanding Curriculum*. New York: Peter Lang.

Pirie, S.E.B. (1988). Understanding: Instrumental, Relational, Intuitive, Constructed, Formalized...? How Can We Know? *For the Learning of Mathematics*, *8*(3), 2-6.

Pirie, S.E.B. & T.E. Kieren (1989). A Recursive Theory of Mathematical Understanding. *For the Learning of Mathematics*, *9*(3), 7-11.

Pirie, S.E.B. & Th.E. Kieren (1994). Beyond Metaphor: Formalising in Mathematical Understanding within Constructivist Environments. *For the Learning of Mathematics*, *14*(1), 39-43.

Principles and Standards for School Mathematics (2000). Reston, VI: NCTM.

Rasmussen, J. (2001). The importance of communication in teaching: a system-theory approach to the scaffolding metaphor. *Journal of Curriculum Studies*, *33*(5), 569-582.

Reigeluth, Ch.M. (1999a). *Instructional-design theories and models. Volume II. A new paradigm of instructional theory*. Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Reigeluth, Ch.M. (1999b). What Is Instructional-Design Theory and How Is It Changing. In: Charles M. Reigeluth (ed.). *Instructional-design theories and models. A new paradigm of instructional theory. Volume II*. Mahwah, NJ: Lawrence Erlbaum Associates, Publishers, 5-30.

Reeuwijk, M. van (1991). Benoemde breuken: herinneringen van vroeger [Named fractions: memories of the past]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskunde-onderwijs*, *10*(1), 39-41.

Romberg, Th.A. (1994). Classroom Instruction That Fosters Mathematical Thinking and

Problem Solving: Connections Between Theory and Practice. In: Alan H. Schoenfeld (ed.). *Mathematical thinking and problem solving*. Hilldale, NJ: Lawrence Erlbaum Associates, Publishers, 287-304.

Roth, W.-M. (2001). Situating Cognition. *The Journal of the Learning Sciences*, *10*(1&2), 27-61.

Schoenfeld, A.H. (1985). *Mathematical problem solving*. San Diego, CA: Academic Press.

Schoenfeld, A.H. (1992). Learning to think mathematically: problem solving, metacognition, and sense making in mathematics. In: Douglas A. Grouws (ed.). *Handbook of research on mathematics teaching and learning*. New York: Macmilian Publishing Company, 334-370.

Schoenfeld, A.H. (1994). Reflections on Doing and Teaching Mathematics. In: Alan H. Schoenfeld (ed.). *Mathematical thinking and problem solving*. Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers, 53-70.

Semadeni, Z. (1984). A principle of concretization permanence for the formation of arithmetical concepts. *Educational Studies in Mathematics*, *15*, 379-395.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different side of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics*, *14*(1), 44-56.

Shulman, L.S. (1995). Wisdom for Practice and Wisdom of Practice: Two Aspects of a Didactics Substance. In: Stefan Hopmann & Kurt Riquarts (eds.). *Didaktik and/or Curriculum*. Kiel: Institut für die Pädagogik der Naturwissenschaften, 201-204.

Shulman, L.S. & K.M. Quinlan (1996). The Comparative Psychology of School Subjects. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology*. New York: Simon & Schuster Macmillan, 399-423.

Siegler, R.S. (1991). Strategy choice and strategy discovery. *Learning and Instruction*, *1*(1), 89-102.

Sierpinska, A. (1995). Mathematics: 'in Context', 'Pure', or 'with Applications'? A contribution to the question of transfer in the learning of mathematics. *For the Learning of Mathematics*, *15*(1), 2-15.

Sierpinska, A. & S. Lerman (1996). Epistemologies of Mathematics and of Mathematics Education. In: Alan J. Bishop, Ken Clements, Christine Keitel, Jeremy Kilpatrick & Colette Laborde (eds.). *International Handbook of Mathematics Education (part 2)*. Dordrecht, Boston, London: Kluwer Academic Publishers, 826-876.

Simon, M.A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, *26*(2), 114-145.

Steffe, L.P. & P. Cobb (1988). *Construction of arithmetical meanings and strategies*. New York, NY: Springer Verlag.

Steffe, L.P. & H.G. Wiegel (1994). Cognitive play and mathematical learning in computer microworlds. *Educational Studies in Mathematics*, *26*, 111-134.

Steinbring, H. (1989). Routine and Meaning in the Mathematics Classroom. *For the Learning of Mathematics*, *9*(1), 24-33.

Streefland, L. (1982). Subtracting fractions with different denominators. *Educational Studies in Mathematics*, *13*, 233-255.

Streefland, L. (1983). *Aanzet tot een nieuwe breukendidactiek volgens wiskobas* [Impuls for a new pedagogy on fractions according to Wiskobas]. Utrecht: OW&OC.

Streefland, L. (1987). Free production of fraction monographs. In: Jacques C. Bergeron, Nico-

las Herscovics & Carolyn Kieran (eds.). *Proceedings of the eleventh international conference Psychology of Mathematics Education (PME-XI) Volume I.* Montreal: s.n., 405-410.

Streefland, L. (1991a). *Fractions in Realistic Mathematics Education, a Paradigm of Developmental Research.* Dordrecht: Kluwer Academic Publishers.

Streefland, L. (1991b). *Realistic Mathematics Education in Primary School.* Utrecht: CD-ß Press/Freudenthal Institute.

Streefland, L. (1993). The design of a mathematics course: a theoretical reflection. *Educational Studies in Mathematics*, *25*, 109-135.

Streefland, L. & E. Elbers (1995). Interactief realistisch reken-wiskundeonderwijs werkt [Interactive realistic mathematics education works]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *14*(1), 12-21.

Streefland, L. (1996). Negative Numbers: Reflections of a Learning Researcher. *Journal of Mathematical Behavior*, *15*, 57-77.

Streefland, L. (1997). Een geval van reflectief denken in ontwikkeling met verhoudingen als paradigma (slot) [A case of developing reflective thinking with ratio as paradigm]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *15*(3), 22-31.

Streefland, L. & E. Elbers (1997). De klas als onderzoeksgemeenschap [Class as research community]. In: C. van den Boer & M. Dolk (eds.). *Naar een balans in de reken-wiskundeles – interactie, oefenen, uitleggen en zelfstandig werken –.* Utrecht: Freudenthal Institute, 11-24.

Streun, A. van (1989). *Heuristisch wiskunde-onderwijs – verslag van een onderwijsexperiment* [Heuristic Mathematics Education] (dissertation). Groningen: Rijksuniversiteit Groningen.

Struik, D.J. (1987). *A Concise History of Mathematics* (fourth revised edition). New York: Dover Publications, Inc.

Sweller, J. (1994). Cognitive load theory, learning difficulty, and instructional design. *Learning and Instruction 4*, 295-312.

Terwel, J. (1984). Onderwijs maken [The making of education] (dissertation). *Stichting voor Onderzoek van het Onderwijs.* Harlingen: Flevodruk.

Terwel, J. (1990). Real Maths in Cooperative Groups in Secondary Education. In: N. Davidson (ed.). *Co-operative Learning in Mathematics.* Menlo Park: Addison-Wesley, 228-264.

Terwel, J. (1994). *Samen onderwijs maken. Over het ontwerpen van adaptief onderwijs* [Making education together. On developing adaptive education] (inaugural lecture University of Amsterdam). Groningen: Wolters-Noordhoff.

Terwel, J., P.G.P. Herfs, M. Mertens & J.Chr. Perrenet (1994). Co-operative learning and adaptive instruction in a mathematics curriculum. *Journal of Curriculum Studies*, *26*(2), 217-233.

Terwel, J. (1999). Constructivism and its implications for curriculum theory and practice. *Journal of Curriculum Studies*, *31*(2), 195-199.

Terwel, J., M. Volman, W. Wardekker & U. Hameyer (2000). *Substantive trends in curriculum design and implementation. An analysis of innovations in the Netherlands.* Paper presented at the annual meeting of the American Educational Research Association, New Orleans, April 22-29, 2000.

Terwel, J., R.M. Gillies, P. van den Eeden & D. Hoek (2001). Co-operative learning processes of students: A longitudinal multilevel perspective. *British Journal of Educational Psychology*, *71*, 619-645.

Terwel, J. (2002). Curriculumdifferentiatie en leren denken: een onderwijspedagogisch per-

spectief. *Pedagogische Studiën, 79*(3), 192-211

Tharp, R. & R. Gallimore (1998). The theory of teaching as assisted performance. In D. Faulkner, K. Littleton & M. Woodhead (eds.). *Learning relationships in the classroom.* London: Routledge, 93-111.

Treffers, A. (1987). *Three dimensions. A model of Goal and Theory Description in Mathematics Instruction – the Wiskobas Project*. Dordrecht: Reidel Publishing Company.

Treffers, A., E. de Moor & E. Feijs (1989). *Proeve van een nationaal programma voor het reken-wiskundeonderwijs op de basisschool. Deel 1: Overzicht einddoelen* [Specimen of a National Program for Primary Mathematics teaching. Part 1: Survey on Educational Aims]. Tilburg: Zwijsen.

Treffers, A. & E. de Moor (1990). *Proeve van een nationaal programma voor het reken-wiskundeonderwijs op de basisschool. Deel 2: Basisvaardigheden en cijferen* [Specimen of a National Program for Primary Mathematics teaching. Part 2: Basic Mental Strategies and Written Computation]. Tilburg: Zwijsen.

Treffers, A., L. Streefland & E. de Moor (1994). *Proeve van een nationaal programma voor het reken-wiskundeonderwijs op de basisschool. Deel 3: Breuken* [Specimen of a National Program for Primary Mathematics teaching. Part 3: Fractions]. Tilburg: Zwijsen.

Trzcieniecka-Schneider, I. (1993). Some remarks on creating mathematical concepts. *Educational Studies in Mathematics*, *24*, 257-264.

Tzur, R. (1999). An Integrated Study of Children's Construction of Improper Fractions and the Teacher's Role in Promoting That Learning. *Journal for Research in Mathematics Education, 30*(4), 390-416.

Vedder, P. (2002). Realistisch rekenen en rekenzwakke, allochtone kinderen. Onderwijskansen op tafel [RME and low achievers from ethnic minorities. Educational chances on the table]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs*, *20*(4), 15-20.

Verschaffel, L. (1995). Beïnvloeden van leerprocessen [Influencing of learning processes]. In: J. Lowyck & N. Verloop (eds.). *Onderwijskunde. Een kennisbasis voor professionals*. Groningen: Wolters-Noordhoff, 153-187.

Walker, D. (1990). *Fundamentals of Curriculum*. San Diego: Harcourt, Brace, Jovanovich.

Watson, J.M., K.J. Cambell & K.F. Collis (1993). Multimodal Functioning in Understanding Fractions. *Journal of Mathematical Behavior, 12*, 45-62.

Weckesser, H. (1970). Die Einführung von Brüchen mit Hilfe von Modellen [The introduction of fractions with the help of models]. *Der Mathematikunterricht, 16*, 30-77.

Westbury, I. (1995). Didaktik and Curriculum Theory: Are They the Two Sides of the Same Coin. In: Stefan Hopmann & Kurt Riquarts (eds.). *Didaktik and/or Curriculum*. Kiel: Institut für die Pädagogik der Naturwissenschaften, 233-263.

Wijnstra, J.M. (ed.) (1988). *Balans van het rekenonderwijs in de basisschool* [Balance-sheet of arithmetics education in primary school]. Arnhem: Cito.

Wittmann, E. (1981). The complementary roles of intuitive and reflective thinking in mathematics teaching. *Educational Studies in Mathematics, 12*, 389-397.

Wood, T. (1998). Creating classroom interactions for mathematical reasoning: beyond 'natural teaching'. In: P. Abrantes, J. Porfírio & B. Baía (eds.). *The interactions in the mathematics classroom*. Proceedings of the CIEAEM 49 (July, 1997). Setúbal, Portugal: s.n., 34-43.

Yackel, E. & P. Cobb (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458-477.

Yackel, E. (2001). Explanation, justification and argumentation in mathematics classrooms.

In: Marja van den Heuvel-Panhuizen (ed.). *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education. Volume 1*. Utrecht: Freudenthal Institute, 9-24.

Yin, R.K. (1984). *Case study research. Designs and Methods*. Beverly Hills, CA: SAGE Publications.

Summary

Mathematising in Realistic Mathematics Education

A demand of our present day technological society is that human beings need to learn to deal with abstract concepts and formal relations. From these societal needs, educational psychology has strongly aimed at cognitive processes and problem solving strategies (Greeno, Collins and Resnick, 1996). In order to enable students to participate as competent members of their community, it is important to guide children in the required strategies of symbolisation, modelling, abstraction, formalisation and generalisation; to cope with key elements in mathematising one's world (Sierpinska & Lerman, 1996; Streefland, 1997; Schoenfeld, 1994).

More than thirty years ago constructing mathematics as human (mental) activity formed the basis of Dutch Realistic Mathematics Education (RME) (Freudenthal, 1973; Gravemeijer & Terwel, 2000) and mathematising was established as a major learners' activity (Gravemeijer, 1994; De Corte, Greer & Verschaffel, 1996; Gravemeijer, 2001). A phenomenological analysis of mathematical structures and its learning lead to programmes based on the principles of RME (cf. Freudenthal, 1983). RME teaching and learning starts with recognisable contexts. These meaningful situations, in time, are mathematised to form more formal relations and abstract structures (Van den Heuvel-Panhuizen, 1996). Treffers (1987), here, made a distinction between horizontal and vertical mathematisation. The former involves converting a contextual problem into a mathematical problem, the latter involves taking mathematical matter onto a higher plane. Vertical mathematisation can be induced by setting problems which admit solutions on different mathematical levels (Freudenthal, 1991; Gravemeijer & Terwel, 2000). Moreover, embedding vertical mathematisation in RME makes that RME is not limited to applying mathematics in recognisable contexts, but has the potential to develop formal mathematics in students.

This study focuses on learning mathematics in an RME-context. Moreover, it considers the developmental process leading to an RME fraction programme. This study therefore is about constructing and teaching mathematics. With this we follow in particular Freudenthal (1991), and take a phenomenological point of view, where mathematical strategy acquisition is considered as a process of *'guided reinvention'*. Freudenthal underlined the necessity of student guidance while learning mathematics; he pleaded for guided *reconstruction*. Moreover, he recognised that mankind developed mathematics to solve all sorts of practical problems and that students should be guided to re-experience this lengthy process in only a few years, similar to the way in which mathematics evolved to the science of structures and became formal mathematics, which in some cases lost many of the obvious links to daily life (Struik, 1987; De Corte, Greer & Verschaffel, 1996).

Fraction learning

Streefland (1991) constructed a fraction programme in a RME setting. By using situations of fair sharing he stimulated students to develop a fraction language. Streefland argued how the sharing context lead to the fraction notation and provided a means to model fractions both as circle parts and as numbers in a ratio table. He elaborated on this last fraction representation to constitute equivalent fractions and formal fraction subtraction and addition. The experimental programme described here is also constructed in an RME setting. This programme, in a similar manner, evokes fraction language, but uses measuring situations instead of situations of fair sharing. These measuring situations next prepare for positioning fractions on a number line, where equivalent fractions come forward as fractions in the same position, thus forming a base for fraction operations.

Fraction learning in an RME context, thus considered, is a typical example of a mathematising process, as it can be seen as a process of modelling, symbolising, generalisation, abstraction and formalisation (Treffers, 1987; Freudenthal, 1991; Nelissen, 1998; Gravemeijer & Terwel, 2000). Moreover, fraction learning aims at vertical mathematisation, as one rarely experiences fractions in daily life; the fraction concept is about abstract and formal relations. However, this limited bound to reality probably makes fractions the most difficult subject in primary school (Behr, Harel, Post & Lesh, 1992; Hasemann, 1981). This furthermore makes that some question the need for (formal) fractions in primary school (cf. Goddijn, 1992).

Research questions and hypothesis

This study researches the feasibility and effectiveness of a newly developed fraction programme in primary school, as compared to a widely used programme in a somewhat traditional educational setting. The newly developed programme is reflected upon, to further improve it to the needs of present day education (Gravemeijer, 1994; cf. Gravemeijer, 2001). Moreover, the study is conducted to provide arguments on to what extent students can and should participate in a fraction programme aimed at formal fraction acquisition.

From this point of view we formulated the following general research questions:

1. How are mathematisation processes facilitated for 10-11 year old children, especially in the case of fraction acquisition?
2. What main obstacles can be distinguished concerning the processes of vertical mathematisation for 10-11 year olds, especially in the case of fraction acquisition?

We specified these questions in the following specific research question:

How do student learning processes develop in an experimental curriculum in

which the number line is used as a model for fractions, and meanings are established by negotiation, and what are the learning outcomes of the developed programme as compared to a control group, which learns fractions in a more traditional programme, where the circle is the central model and where learning, to a large extent, is a solitary activity?

In elaborating the outcomes of the programme comparison, we will focus especially on possible differential effects (cf. Hoek, Terwel & Van den Eeden, 1997; Hoek, 1998; Hoek, Van den Eeden & Terwel, 1999).

From this perspective, our research questions lead us to the study's hypotheses:

A majority of 10-11 year old students can acquire abstract and formal mathematical concepts and strategies, when these concepts and strategies are learned in an educational setting, where meanings are negotiated in meaningful and recognisable situations (Greeno, 1991) and where mathematical activities on different levels are closely connected (Van Hiele, 1986; Pirie & Kieren, 1989; Pirie & Kieren, 1994; Gravemeijer, 1994).
We specified that students who learned fractions in a programme in which the number line is used as a model for fractions, and meanings are established by negotiation, will outperform students who learned fractions in a more traditional programme, where the circle is the central model and where learning is to a large extent a solitary activity.

Methods

The research described here was conducted at a school in a small town north of Amsterdam. The students visiting the school in general have a middle class background. From two parallel grade 6 groups (9 to 10 years) we selected an experimental group which followed a newly developed programme, and a control group which followed the more traditional fraction programme of the school.We thoroughly observed and analysed the fraction learning processes of students in both the experimental and the control group during a whole school year.
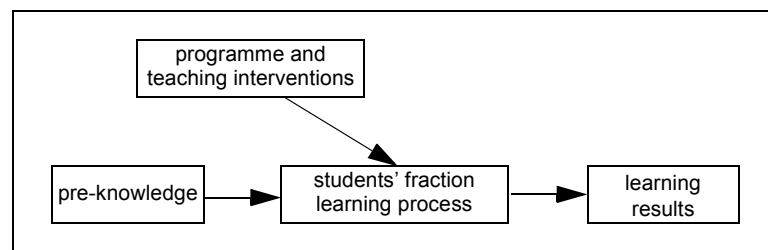

figure 1: conceptual model guiding the study

We ordered our analyses to further identify and clarify key elements in fraction learning and to generalise findings to learning processes that are involved in formal mathematics acquisition. The conceptual model of the study schematises key elements in the research described here (figure 1). Teaching in the experimental group initially focuses on fraction language acquisition through student exploration of measuring situations. Here, the bar and the number line are subsequently introduced as (mental) models for fractions. In the control group the subject is introduced by fair sharing and the circle-model. In the experimental group students are invited to discuss, in the control group students mostly work individually.
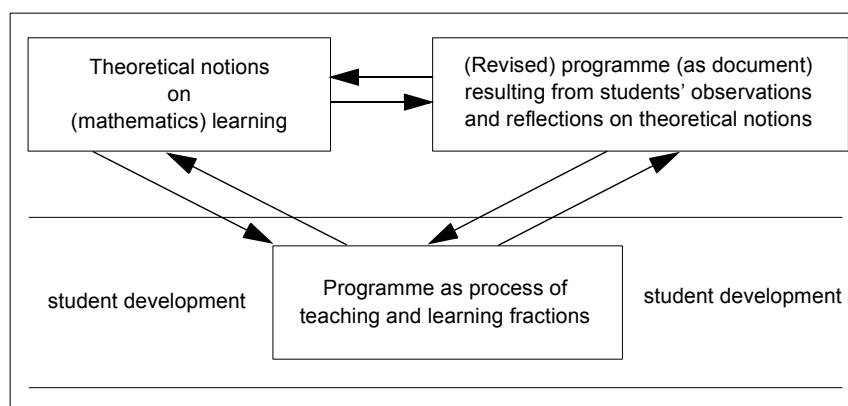
figure 2: framework for developing mathematics education

In line with what Anderson, Greeno, Reder & Simon (2000) suggest on attuning appropriate research approaches to relevant research issues, we used several methodological approaches to analyse the aforementioned fraction programmes and their students outcomes with various coherent instruments. This is based on considering developmental research to be an overarching research design, that is to say, the developmental research design could be considered to contain any concise form of active search or reflection leading to arguments to (a continuous) improvement of mathematics programmes (cf. Gravemeijer, 1994; Terwel, 1984) and this study is conducted to provide those arguments. We visualise this overarching character of developmental research in figure 2, where we relate considerations in teaching and learning fractions. In the conceptual framework of the study (figure 1), which could be considered as derivation from the developmental research scheme, we schematise key-elements in the research: the two programmes, teaching experiments, student development and pre- and post-tests. Research interventions are embedded in a quasi-experimental research design (Cook & Campbell, 1979), more precisely a non-equivalent pre-test post-test control group design, where students in the exper-

imental group follow the experimental curriculum and students in the control group follow the control programme. In table 1 we provide a specification of the research design. In this design O_1, O_2 and O_3 are standardised tests to ascertain students' general mathematical strategies at the start, halfway through and at the end of the research year. Three standardised interviews, I_1, I_2 and I_3 were held to determine students' fraction knowledge at the start, halfway through and at the end of the research year, and thorough observations during fraction instruction in both experimental group (Xe_1, Xe_2, Xe_3, Xe_4) and control group (Xc_1, Xc_2, Xc_3, Xc_4) were conducted to provide for additional information on student development.

O_1	Xe_1	I_1	Xe_2	O_2	Xe_3	I_2	Xe_4	O_3	I_3
O_1	Xc_1	I_1	Xc_2	O_2	Xc_3	I_2	Xc_4	O_3	I_3

table 1: specification of the research design

Student observations provided us with a means to perform two case studies. In each of these case studies we describe the development in fraction learning of one of the students in the developed programme. To do so, we analysed all lessons in the experimental curriculum, to select the moments that clearly exhibit the students' development. Next the theory of RME helped us to interpret and – in some sense – generalise the findings (cf. Yin, 1984).

Research setting

Twenty students from two grade 6 groups (aged 10/11) participated as subjects in this study. We matched the twenty students in ten pairs, with one student in the experimental group and one in the control group. In this way ten students followed the experimental curriculum, in which the number line is used as a model for fractions and where meanings are established by negotiation. Their counterparts in the control group followed the more traditional fraction programme of the school, where the circle is the central model and where learning, to a large extent, is a solitary activity. The researcher was the teacher in the experimental group during the lessons in fractions. The experimental group's regular teacher took care of all other mathematics topics in this group. The control curriculum was taught by the regular teacher of the group, as were other mathematics topics in this group. The teaching styles of the two teachers were highly comparable and apart from the fraction programmes, all mathematics topics were dealt with in a similar manner. At three points we measured students' proficiency in general mathematical strategies. Additionally, we established students' fraction strategies from three standardised interviews.

Results

The study presented here researches the feasibility and effectiveness of a newly

developed fraction programme in primary school, as compared to a widely used programme. In this programme fraction learning is just an example of a learning process in mathematising, i.e. 'making it more mathematical' (Freudenthal, 1968).

Effects

We analysed the effects of the experimental programme by testing students several times on general mathematical strategies. On only one occasion did we find significant differences when comparing the student results in a paired t-test. Normal performing students who followed the experimental programme significantly outperformed their matched counterparts in the control group on the post-tests 'numbers and operations' and 'measurement and geometry'. This finding indicates the differential effects.

We did not find significant results when comparing general mathematical strategies for all the paired students in paired t-tests and in regression analyses. However, clear interaction effects between the condition and post-tests were found, meaning that we established differential effects of the experimental programme as compared to the control condition. Namely, the experimental programme benefits the better performing students more than the control programme, a 'Matthew-effect' (Kerckhoff & Glennie, 1999). We were able to distinguish intervals of significance; intervals of pre-test scores where one condition does significantly better than the other. We found that for the post-test 'numbers and operations' low achievers benefited more from the control condition than from the experimental condition. For the post-test 'measuring and geometry' we also did find significant differential effects. Here we were able to distinguish two intervals of significance. These intervals in pre-test scores indicated that the highest achieving students benefited more from the experimental condition as compared to the control condition, whereas the lowest achieving students benefited more from the control condition as compared to the experimental condition.

We analysed the effects of the programme on students' fraction strategies. We established these fraction strategies in three interviews. We constructed the interviews so that students who were unable to clarify the problem situation themselves received standardised help. In a paired t-test we found that students in the experimental group did significantly better in these fraction interviews than their peers in the control group ($p \leq 0.05$). However, a regression analysis, that corrected the scores for small initial differences, revealed that two of the three fraction interviews turned out to be clear, but non-significant, trends. Moreover, for the fraction interviews we established no interaction between the condition and pre-test scores. We therefore concluded that high and low achieving students do not benefit differently from the experimental condition as compared to the control condition.

We did not find significant results in the interviews in all cases when we limited our-

selves to the condition that no help was offered. We reckon that the small number of subjects plays a role here. However, students in the experimental condition possibly did better when standardised help was offered, because this enabled students to use strategies that were accentuated in the experimental programme, like making estimations, checking answers by making a sketch, et cetera. Students with possibilities to learn and use mathematical language more easily, are the high performers in mathematics, and they therefore gain in a situation where common understandings are discussed. Moreover, we argued that low achievers are possibly hindered by the formal and abstract nature of the fractions involved or by the difference in teaching style between researcher-teacher in fraction lessons and the students' regular teacher. Furthermore, a qualitative analysis of the students' work revealed that students in the experimental condition in their answers more frequent reflected number sense than students in the control group. We interpreted that these results indicated that the experimental programme leaded to better results.

Fraction learning processes

We presented this study as one where quantitative and qualitative analyses are used to provide in-depth insight in students' learning processes. We expressed this choice in the study's research questions. This study is performed to uncover *how* mathematisation processes are facilitated for 10-11 year old children, especially in the case of fraction acquisition. Moreover, this study looks for obstacles that can be distinguished regarding the processes of vertical mathematisation for 10-11 year olds. Therefore qualitative analyses are needed to answering our research questions.

We analysed learning processes in the experimental fraction curriculum as compared to the effects of the programme in a control group. An analysis of the data provided indications for the following:

– Students in the experimental condition show more proficiency in fractions than students in the control condition.
– Students who perform average or above average in general mathematical skills can learn formal fractions in a meaningful manner and within reasonable time.
– Low achievers in mathematics experience considerable difficulties in learning formal fractions.

These findings partly confirm the study's specific hypothesis. We found strong indications that students in the experimental curriculum do outperform control students in the more traditional programme in fraction proficiency; however, if we focus on general proficiency in mathematics, only above average performers in the experimental group benefit more from the experimental condition than from the control condition. These results, moreover, provide strong indications for this study's general hypothesis that states that a majority of 10-11 year old students can acquire abstract and formal mathematical concepts and strategies, when these concepts and

strategies are learned in an educational setting where meanings are negotiated in meaningful and recognisable situations and where mathematical activities on different levels are closely connected, as the experimental condition can be considered to be such a setting.

Conclusion

The main conclusion based on the quantitative part of this study, using the outcomes of the regression analysis while controlling for small initial differences and searching for interaction effects, can thus be stated as follows. Although clear trends were found in the expected direction, strictly speaking we cannot substantiate a significant general effect of the programme on the learning gains of the students. This conclusion holds true for both the effects on general mathematical strategies as on the outcomes of fraction learning. Therefore, in general the hypothesis has to be rejected. However, thorough qualitative analyses of students' learning strategies indicated that the experimental programme offers many students chances to learn fractions in a meaningful way. Students that followed the control programme showed less flexibility in operating and manipulating with fractions then their matched peers in the experimental condition.

Moreover, although no general effects for all students were found, the study showed significant differential effects. As far as the effects on general mathematics strategies is concerned, high achieving students clearly benefitted from the programme while low achieving students showed significantly less gains as compared to their counterparts in the control programme. Thus there is an effect of the programme which is not only statistical significant but also relevant for both educational theory and practice.

Overview

Chapter 1 – Introduction

National standards for teaching mathematics in primary schools in the Netherlands leave little room for formal fractions. However, a newly developed programme in fractions aims at learning formal fractions. The first chapter provides an introduction of the research project on mathematics education, the main issue of which concerns the effects of a newly developed fraction programme on the learning processes and outcomes of primary school students. The two guiding research questions for the study are set here: 'How are processes of learning formal fractions facilitated for 10-11 year old children?' and 'What main obstacles can be distinguished concerning the processes of learning formal fractions for 10-11 year olds?' In this chapter we formulate the study's hypotheses that a majority of 10-11 year old students can acquire

abstract and formal mathematical concepts and strategies, when these concepts and strategies are learned in an educational setting, where meanings are negotiated in meaningful and recognisable situations and where mathematical activities on different levels are closely connected.

Chapter 2 – Theoretical background

This chapter analyses research paradigms in psychology that in the 20th century provided a basis for research in mathematics education. Starting from this analysis we sketch the development of pedagogical content knowledge – a theory of learning generated by key aspects of the subject matter. From this point of view this chapter focuses on the activity of 'mathematising' as an aim in mathematics education. Mathematising embeds processes of modelling, symbolising, generalising, formalising and abstracting. In this chapter we argue that these aspects of the mathematising process are not meant to be regarded in isolation, as only the combination of all these aspects does justice to the mathematising process. We will analyse mathematising in the context of RME, which notion next provides us with instruments to embed the research issues in the book.

Chapter 3 – Learning for mathematical insight: a longitudinal comparative study on modelling

The third chapter reports on a longitudinal study of teaching and learning the subject of fractions in two matched groups of ten 9–10-year-old students. In the experimental group, fractions are introduced using the bar and the number line as (conceptual) models, in the control group the subject is introduced by fair sharing and the circle-model. In the experimental group students are invited to discuss, in the control group students work individually. The groups are compared on several occasions during one year. After this year, the experimental students show more proficiency in fractions than those in the control group. However data analysis also reveal a so-called 'Matthew effect'. We see how those students that are proficient in mathematics profit more from the experimental curriculum than low achievers.

Chapter 4 – Audrey's acquisition of fractions: a case study into the learning of formal mathematics

Chapter four consists of a case study describing the growth in reasoning ability with fractions of one student in this newly developed programme of 30 lessons during one whole school year. The study indicates that the programme and its teaching stimulated the progress of an average performer in mathematics. Moreover, arguments were found for as to what extent formal operations with fractions suits as an educational goal.

Chapter 5 – The fraction learning process in low-achieving students: Shirley's choice and use of strategies in primary mathematics

Research in mathematics education offers a considerable body of evidence that both high and low-achievers can benefit from learning mathematics in meaningful contexts. The case study in this chapter offers an in-depth analysis of the learning process of a low-achieving student in a RME context. The focus is on the use of productive and counterproductive strategies in the solution of problems with fractions. Arguments were found to support the idea that low-achievers do benefit from RME, but experience difficulties in the formalisation process with regard to fractions. We have seized upon the observed difficulties by discussing the implications of uniform standards in mathematics education.

Chapter 6 – Effects of an experimental fraction programme in primary mathematics: a longitudinal analysis

This chapter offers the study's quantitative results and discusses the effects of the two fraction programmes. Clear trends will show that students have better chances when they learn fractions in the experimental programme. Moreover, a focus on general mathematical strategies indicates that the experimental programme enlarges differences between high and low achieving students as compared to the control programme. We provide a model to explain these programme effects from the viewpoint of mathematical language and cognitive load theory.

Chapter 7 – Theoretical reflection

Constructivism offers valuable notions to RME for designing learning environments, as it shows how classroom discussions could be focused on uncovering mathematical structures, for example by explicating socio-mathematical norms; what is valued as mathematical argument and what is not. When, in this way, mathematics is constructed by negotiating meanings and understandings, teaching aims at students' mathematising. And if mathematising is one of the objectives of a curriculum to be developed, interweaving of learning strands is at hand. Moreover, interweaving of learning strands is a valuable perspective when formal and abstract concept formation is an aim in education, as students then have to realise what elements of reality are similar in some respects and different in others. Furthermore, interweaving learning strands can provide a means to make all students, especially low achievers, benefit from mathematics education.

In this chapter we elaborate on these arguments to reconsider this study's embeddedness in notions from constructivism and RME. Moreover, these considerations are elaborated on to reconsider low achievers' learning processes in mathematics education.

190

Chapter 8 – Conclusion and discussion

This chapter presents the results and conclusions of the study described in this book. We will formulate answers to research questions posed in the previous chapters. Findings from chapter 3-5 will show that when a fraction programme aims at number sense acquisition, where knowledge and strategies are (also) acquired by negotiation of meaning, many students – but not all – can learn formal fractions in a meaningful manner. We will discuss this finding in this chapter. In addition, from theoretical positions taken in chapter 2 and 6 we will take a step forward and formulate issues for future research. In doing this we establish that the experimental programme offers perspectives for all students in learning mathematics.

Reflection

Developmental research implies that all developed teaching programmes are subjected to new reflections, to thus form the basis for new developments, where the newly developed programme soon becomes the old one predestined to be replaced by something new. Fresh ideas, established during the developmental process, enter developers' repertoire to become tools for further developmental work (cf. Gravemeijer, 1994). In this respect, this book can be seen as a source of ideas for further developing mathematics education.

References

Anderson, J.R., J.G. Greeno, L.M. Reder & H.A. Simon (2000). Perspectives on Learning, Thinking, and Activity. *Educational Researcher, 29*(4), 11–13

Behr, M.J., G. Harel, Th. Post & R. Lesh (1992). Rational number, ratio, and proportion. In: Douglas A. Grouws (ed.) *Handbook of research on mathematics teaching and learning*. (296-332). New York: Macmilian Publishing Company

Cook, Th.D. & D.T. Campbell (1979). *Quasi-Experimentation. Design & Analysis Issues for Field Settings*. Chicago: Rand McNally College Publishing Company.

De Corte, E., B. Greer & L. Verschaffel (1996). Mathematics Teaching and Learning. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology.* New York: Simon & Schuster Macmillan, 491-549.

Freudenthal, H. (1968) Why to teach mathematics as to be useful? *Educational Studies in Mathematic*s, *1*(1), 3-8.

Freudenthal, H. (1973). *Mathematics as an Educational Task.* Dordrecht: Riedel.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures.* Dordrecht: Riedel-Kluwer.

Freudenthal, H. (1991). *Revisiting Mathematics Education, China lectures.* Dordrecht: Kluwer Academic Publishers.

Goddijn, A.J. (1992). De oudste en de nieuwste breuken [The oldest and the newest fractions]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs, 11*(2), 18-31.

Gravemeijer, K.P.E. (1994). *Developing realistic mathematics education* (dissertation). Utrecht: CD-ß Press.

Gravemeijer, K.P.E. & J. Terwel (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of Curriculum Studies, 32*(6), 777-796.

Gravemeijer, K.P.E. (2001). *Reken-wiskundeonderwijs voor de 21^e eeuw* [Mathematics education for the 21th century]. Utrecht: Utrecht University.

Greeno, J.G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22*(3), 170-218.

Greeno, J.G., A.M. Collins & L. Resnick (1996). Cognition and Learning. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology.* New York: Simon & Schuster Macmillan, 15-46.

Hasemann, K. (1981). On difficulties with fractions. *Educational Studies in Mathematics, 12*, 71-87.

Heuvel-Panhuizen, M. van den (1996). *Assessment and realistic mathematics education* (dissertation)*.* Utrecht: CD-ß Press.

Hiele, P.M. van (1986). *Structure and Insight. A theory of Mathematics Education.* Orlando: Academic Press, Inc.

Hoek, D., J. Terwel & P. van den Eeden (1997). Effects of Training in the Use of Social and cognitive Strategies: An Intervention Study in Secondary Mathematics in Co-Operative Groups. *Educational Research and Evaluation, 3*(4), 364-389.

Hoek, D.J. (1998). *Social and cognitive strategies in co-operative groups. Effects of strategy instruction in secondary mathematics* (dissertation). Amsterdam: University of Amsterdam.

Hoek D., P. van den Eeden, J. Terwel (1999). The effects of integrated social and cognitive strategy instruction on the mathematics achievement in secondary education. *Learning & Instruction, 9*, 427–448.

Kerckhoff, A.C. and E. Glennie (1999). The Matthew effect in American education. In: Aaron M. Pallas (ed.). *Research in sociology of education and socialization (Vol. 12).* Stamford, Connecticut: Jai Press Inc, 35-66.

Nelissen, J.M.C. (1998). Representaties in het reken-wiskundeonderwijs [Representations in mathematics education]. *Pedagogische Studiën, 75*, 169-183.

Pirie, S.E.B. & T.E. Kieren (1989). A Recursive Theory of Mathematical Understanding. *For the Learning of Mathematics, 9*(3), 7-11.

Pirie, S.E.B. & T.E. Kieren (1994). Beyond Metaphor: Formalising in Mathematical Understanding within Constructivist Environments. *For the Learning of Mathematics, 14*(1), 39-43.

Schoenfeld, A.H. (1994). Reflections on Doing and Teaching Mathematics. In: Alan H. Schoenfeld (ed.). *Mathematical thinking and problem solving.* Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers, 53-70.

Sierpinska, A. & S. Lerman (1996). Epistemologies of Mathematics and of Mathematics Education. In: Alan J. Bishop, Ken Clements, Christine Keitel, Jeremy Kilpatrick & Colette Laborde (eds.). *International Handbook of Mathematics Education (part 2).* Dordrecht, Boston, London: Kluwer Academic Publishers, 826-876.

Streefland, L. (1990). *Fractions in Realistic Mathematics Education, a Paradigm of Developmental Research.* Dordrecht: Kluwer Academic Publishers.

Streefland, L. (1997). Een geval van reflectief denken in ontwikkeling met verhoudingen als paradigma (slot) [A case of developing reflective thinking with ratio as paradigm]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs, 15*(3), 22-31.

Struik, D.J. (1987). *A Concise History of Mathematics (fourth revised edition).* New York: Dover Publications, Inc.

Terwel, J. (1984). *Onderwijs maken* [Making education] (dissertation). Harlingen: Flevodruk.

Treffers, A. (1987). *Three dimensions. A model of Goal and Theory Description in Mathematics Instruction – the Wiskobas Project*. Dordrecht: Reidel Publishing Company.

Yin, R.K. (1984). *Case study research. Designs and Methods.* Beverly Hills, CA: SAGE Publications.

Samenvatting

Onderwijzen van formele wiskunde in het basisonderwijs
- het leren van breuken als proces van mathematiseren -

Mathematiseren in realistisch reken-wiskundeonderwijs

Onze hedendaagse technologische maatschappij vereist dat mensen leren omgaan met abstracte concepten en formele relaties. Dit gegeven vormde voor de onderwijspsychologie een reden zich sterk te richten op cognitieve processen en het leren probleemoplossen (Greeno, Collins & Resnick, 1996). Om leerlingen de kans te bieden om op passende wijze maatschappelijk te functioneren, is het van belang ze te begeleiden in het leren symboliseren, modelleren, abstraheren en generaliseren; leren om om te gaan met centrale elementen in het mathematiseren van de eigen wereld (Sierpinska & Lerman, 1996; Streefland, 1997; Schoenfeld, 1994).

Meer dan dertig jaar geleden vormde het construeren van wiskunde als menselijke (mentale) activiteit de basis voor het huidige realistisch reken-wiskundeonderwijs (Freudenthal, 1973; Gravemeijer & Terwel, 2000) en binnen dit onderwijs werd het mathematiseren de belangrijkste activiteit van de lerende (Gravemeijer, 1994; De Corte, Greer & Verschaffel, 1996; Gravemeijer, 2001). Een didactisch-fenomenologische analyse van wiskundige structuren leidde tot leergangen gebaseerd op de uitgangspunten van het realistisch reken-wiskundeonderwijs (vgl. Freudenthal, 1983). Het onderwijs en het leren in het realistisch reken-wiskundeonderwijs start met voor leerlingen herkenbare contexten. Een proces van mathematiseren leidt er vervolgens toe dat deze van betekenisvolle situaties verworden tot formele en abstracte relaties (Van den Heuvel-Panhuizen, 1996). Hierbij maakt Treffers (1987) overigens een onderscheid tussen horizontaal en verticaal mathematiseren. Bij het eerste, horizontaal mathematiseren, gaat het om het omzetten van een in een context gesteld probleem naar een wiskundig probleem. Bij het tweede, verticaal mathematiseren, gaat het erom de ontstane wiskunde op een hoger plan te brengen en dit kan worden bereikt door leerlingen problemen voor te leggen die op verschillende niveaus kunnen worden opgelost (Freudenthal, 1991; Gravemeijer & Terwel, 2000). Verder maakt het inbedden van verticaal mathematiseren in het realistisch reken-wiskundeonderwijs dat dit onderwijs niet beperkt blijft tot het toepassen van wiskunde in herkenbare situaties, maar dat het de potentie heeft bij leerlingen formele wiskunde te ontwikkelen.

Deze studie beschouwt het leren van wiskunde in de context van het realistisch reken-wiskundeonderwijs. Het neemt daarnaast het proces in ogenschouw dat leidde tot een realistische breukenleergang. Deze studie gaat daarom over het construeren en onderwijzen van rekenen-wiskunde. Op deze manier volgen we in het bijzonder Freudenthal (1991) en kiezen voor een fenomenologisch standpunt, waarbij het

leren van wiskunde wordt beschouwd als 'geleide heruitvinding'. Freudenthal bena-drukte de noodzaak van het leiden van leerlingen bij het leren van wiskunde; hij pleitte met nadruk voor geleide *reconstructie*. Bovendien onderkende hij dat de mensheid wiskunde ontwikkelde om tal van praktische problemen het hoofd te bieden en dat leerlingen begeleid moeten worden om dit langdurige proces te herbe-leven in slechts enkele jaren; herbeleven hoe wiskunde de wetenschap van structu-ren en formele wiskunde werd, die vaak zijn voor de hand liggende binding met het dagelijkse leven heeft verloren (Struik, 1987; De Corte, Greer & Verschaffel, 1996).

Het leren van breuken

Streefland (1991) ontwierp een realistische leergang breuken. Hij gebruikte daarin eerlijk-verdeelsituaties om leerlingen breukentaal te laten ontwikkelen. Hij beschreef verder hoe de eerlijk-verdeel-context kon leiden tot een notatie voor breu-ken en hij introduceerde de cirkel en de verhoudingstabel als modellen voor breu-ken. Hij bewerkte deze laatste representatie voor breuken om gelijkwaardige breu-ken te genereren en aldus te komen tot het formele optellen en aftrekken van breuken. De experimentele leergang die hier wordt beschreven is ook van realis-tische snit. In dit programma is ook aandacht voor de breukentaal, maar om deze taal te genereren worden meetsituaties gepresenteerd in plaats van situaties van eerlijk verdelen. Deze meetsituaties vormen vervolgens een voorbereiding op het plaatsen van breuken op de getallenlijn, waarbij gelijkwaardige breuken naar voren komen als breuken op eenzelfde positie, om op die manier een mogelijkheid te bieden voor het rekenen met breuken.

Zo beschouwd is het leren van breuken in genoemde realistische leergangen een typisch voorbeeld van het leren mathematiseren dat kan worden gezien als proces van modelleren, symboliseren, abstraheren en formaliseren (Treffers, 1987; Freu-denthal, 1991; Nelissen, 1998; Gravemeijer & Terwel, 2000). Bovendien, omdat breuken nauwelijks voorkomen in het dagelijkse leven, richt het leren van breuken zich vooral op het verticale mathematiseren; bij breuken gaat het immers vooral om abstracte en formele relaties. Echter, juist het gegeven dat breuken zo weinig te maken hebben met de realiteit maakt ze wellicht tot het moeilijkste onderwerp van de basisschool (Behr, Harel, Post & Lesh, 1992; Hasemann, 1981). En daarom zijn er ook mensen die hun vraagtekens plaatsen bij de zin van breukenonderwijs in het primair onderwijs (vgl. Goddijn, 1992).

Onderzoeksvraag en hypothese

Deze studie onderzoekt de uitvoerbaarheid en de effectiviteit van een nieuw ontwik-kelde leergang breuken voor de basisschool, in vergelijking met een algemeen gebruikte leergang in een ietwat traditionele onderwijssetting. Overwegingen rond deze nieuwe leergang en de opbrengst ervan zijn gebruikt om de leergang verder te

verbeteren, met het oog op de behoeften van het huidige onderwijs (Gravemeijer, 1994; vgl. Gravemeijer, 2001). Daarnaast wordt binnen het hier gepresenteerde onderzoek nagegaan in welke mate leerlingen moeten deelnemen aan een leergang breuken die gericht is op het verwerven van formele breuken.

Aldus komen we tot de volgende onderzoeksvragen:

1 Hoe kunnen processen van mathematiseren worden ondersteund voor kinderen van 10 tot 11 jaar, in het bijzonder in het geval van het leren van breuken?
2 Welke obstakels kunnen onderscheiden worden in verband met het proces van verticale mathematisering voor kinderen van 10 tot 11 jaar, in het bijzonder in het geval van het leren van breuken?

We spitsen deze vragen toe in de volgende specifieke onderzoeksvraag:

Hoe ontwikkelen leerprocessen van leerlingen zich in een experimentele leergang, waarin de getallenlijn wordt gebruikt als model voor breuken en waarin betekenissen worden gevonden door onderhandelen, en wat is de leeropbrengst van het ontwikkelde programma in vergelijking met een controlegroep die breuken leert in een meer traditionele setting, waar de cirkel een centraal model is en waar leren, in het algemeen, een solitaire activiteit is?

Bij het bewerken van de opbrengst van de vergelijking van de twee leergangen, zullen we met name kijken naar mogelijke differentiële effecten (vgl. Hoek, Terwel & Van den Eeden, 1997; Hoek, 1998; Hoek, Van den Eeden & Terwel, 1999).

Aldus leiden onze onderzoeksvragen ons tot de hypothesen van deze studie:

Een meerderheid van de 10- tot 11-jarige leerlingen is in staat om formele en abstracte wiskundeconcepten en -aanpakken te verwerven, wanneer deze worden geleerd in een onderwijssituatie waarin betekenissen worden bediscussieerd in herkenbare en betekenisvolle situaties (Greeno, 1991), en waar wiskundige activiteiten op verschillende niveaus nauw met elkaar verbonden zijn (Van Hiele, 1986; Pirie & Kieren, 1989; Pirie & Kieren, 1994; Gravemeijer, 1994).
Toegespitst: leerlingen die breuken leren in een leergang, waarin de getallenlijn centraal staat en waarin betekenissen tot stand komen via onderhandelen, zullen betere resultaten behalen dan leerlingen die breuken leren in een meer traditionele setting, waar de cirkel het centrale breukenmodel is en waar leren vooral een solitaire activiteit is.

Methoden

Het beschreven onderzoek vond plaats op een school even ten noorden van Amsterdam. De leerlingen die de school bezoeken komen in het algemeen niet uit achterstandsgezinnen. Uit twee parallelle groepen 6 selecteerden we een experimentele groep, die de nieuw ontwikkelde leergang volgde, en een controlegroep, die het meer traditionele onderwijs in breuken van de school volgde. We observeerden en analyseerden het leren van breuken in beide groepen gedurende een heel schooljaar grondig. Wij ordenden onze analyses vervolgens om belangrijke elementen van het leren van breuken te identificeren en om onze bevindingen te kunnen generaliseren naar processen die het leren van formele wiskunde beïnvloeden. Het conceptuele model voor de studie brengt deze sleutelelementen in beeld (figuur 1).
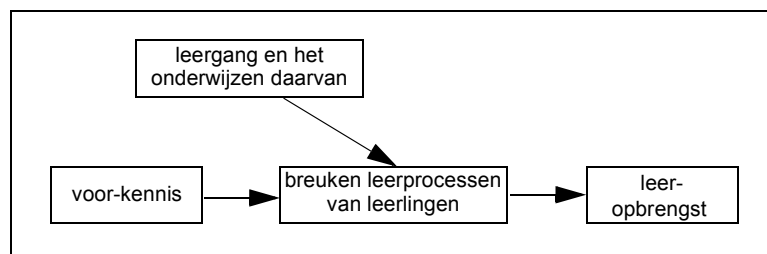
figuur 1: conceptueel model als leidraad voor de studie

Het onderwijs in de experimentele groep richt zich aanvankelijk op het verwerven van breukentaal door het verkennen van meetsituaties. Op deze manier worden achtereenvolgens de strook en de getallenlijn geïntroduceerd als modellen voor breuken. In de controlegroep wordt het onderwerp breuken geïntroduceerd via het eerlijk verdelen en het cirkelmodel.
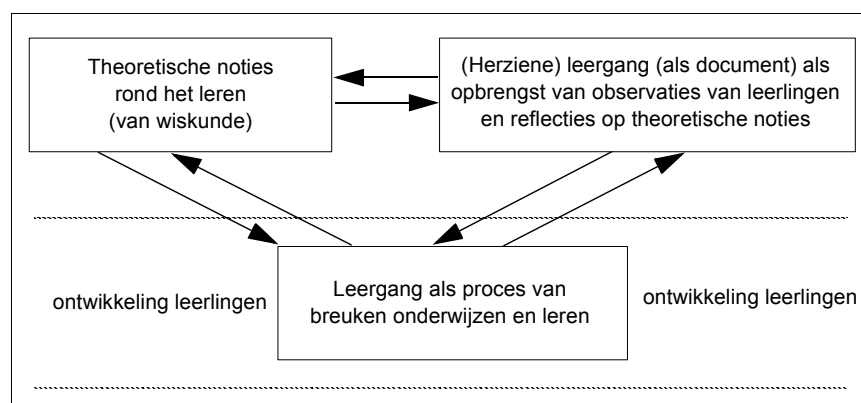
figuur 2: kader voor het ontwikkelen van reken-wiskundeonderwijs

In de experimentele groep worden leerlingen uitgenodigd resultaten te bediscussiëren; in de controlegroep werken de leerlingen vooral individueel.Anderson, Greeno, Reder en Simon (2000) geven aan dat onderzoeksaanpakken moeten passen bij gekozen onderzoeksthema's. In lijn met dit idee gebruiken we in de hier beschreven studie verschillende methodologische aanpakken om de genoemde leergangen breuken te analyseren en de leeropbrengsten te meten met passende instrumenten. Daarbij beschouwen we ontwikkelingsonderzoek als overkoepelend onderzoeksdesign, in de zin dat ontwikkelingsonderzoek wordt beschouwd als een samenhangend geheel van onderzoekingen en reflecties leidend tot argumenten die leiden tot (continue) verbetering van het reken-wiskundeonderwijs (vgl. Gravemeijer, 1994; Terwel, 1984). Deze studie is zo opgebouwd dat het voorziet in genoemde argumenten. We visualiseren dit overkoepelende karakter van ontwikkelingsonderzoek in figuur 2, waar overwegingen rond het onderwijzen en leren met elkaar in verband worden gebracht. In het conceptuele kader voor de studie (figuur 1), dat beschouwd kan worden als afgeleide van het schema voor ontwikkelingsonderzoek, schematiseren we sleutelelementen in het onderzoek: de twee leergangen, onderwijsexperimenten, de ontwikkeling van leerlingen en voor- en nametingen. Interventies in het onderzoek zijn ingebed in een quasi-experimenteel onderzoeksdesign (Cook & Campbell, 1979), meer precies een niet-equivalent pre-test post-test controlegroep design, waar leerlingen in de experimentele groep de experimentele leergang volgen en leerlingen in de controlegroep het controleprogramma. In tabel 1 is dit onderzoeksdesign verder gespecificeerd. In het design zijn O_1, O_2 en O_3 gestandaardiseerde toetsen, om de algemeen wiskundige vaardigheden van de leerlingen vast te stellen bij het begin, halverwege en aan het eind van het onderzoeksjaar. Drie gestandaardiseerde interviews, I_1, I_2 en I_3 werden gehouden om de breukenkennis van de leerlingen vast te stellen bij het begin, halverwege en aan het eind van het onderzoeksjaar, en grondige observaties tijdens breukenlessen in de experimentele groep (Xe_1, Xe_2, Xe_3, Xe_4) en controlegroep (Xc_1, Xc_2, Xc_3, Xc_4) werden gedaan om te voorzien in aanvullende informatie over de ontwikkeling van de leerlingen

O_1	Xe_1	I_1	Xe_2	O_2	Xe_3	I_2	Xe_4	O_3	I_3
O_1	Xc_1	I_1	Xc_2	O_2	Xc_3	I_2	Xc_4	O_3	I_3

tabel 1: specificatie van het onderzoeksdesign

Via deze observaties van leerlingen beschikten we over middelen om twee casestudies uit te voeren. In deze gevalsstudies beschrijven we telkens de ontwikkeling van een van de leerlingen in de ontwikkelde leergang. Om dit te bewerkstelligen analyseerden we alle lessen in het experimentele programma om zo momenten te selecteren die de ontwikkeling van de leerlingen duidelijk tonen. Voorts bood de theorie van realistisch reken-wiskundeonderwijs ons mogelijkheden om onze bevindingen te interpreteren en – in bepaald opzicht – te generaliseren (vgl. Yin, 1984).

Situatieschets

Twintig leerlingen uit twee groepen 6 deden mee aan het onderzoek. We maakten tien tweetallen leerlingen, waarbij een leerling uit de experimentele groep gekoppeld werd aan een leerling uit de controlegroep. Op deze manier volgenden tien leerlingen de experimentele leergang, waarin de getallenlijn als model gebruikt werd en waar betekenissen tot stand kwamen via discussie. De tien lotgenoten in de controlegroep volgden de meer traditionele leergang van de school, waarin de cirkel het centrale model is en waar in het algemeen individueel gewerkt wordt. De onderzoeker was de leerkracht in de experimentele groep tijdens de breukenlessen. De reguliere leerkracht van de experimentele groep nam de andere reken-wiskundeonderwerpen voor haar rekening. De leergang in de controlegroep werd onderwezen door de reguliere leerkracht van de groep, die ook alle andere reken-wiskundeonderwerpen voor haar rekening nam. De stijl van lesgeven van de twee reguliere leerkrachten kwam in grote mate overeen en, afgezien van het breukenonderwijs, kwamen alle onderwerpen rekenen-wiskunde op een gelijke manier aan de orde.

Op drie momenten toetsten we de algemeen reken-wiskundige vaardigheden van de leerlingen. Verder stelden we de breukenvaardigheden van de leerlingen vast in drie gestandaardiseerde interviews.

Resultaten

De hier gepresenteerde studie onderzoekt de haalbaarheid en effectiviteit van een nieuw ontwikkelde leergang breuken voor de basisschool, in vergelijking tot een algemeen gebruikte leergang. Daarbij beschouwen we het leren van breuken als voorbeeld van het leren mathematiseren, dat wil zeggen 'het meer wiskundig maken' (Freudenthal, 1968).

Effecten

We analyseerden de effecten van de experimentele leergang door leerlingen verschillende keren te toetsen op algemeen reken-wiskundige vaardigheden. Daarbij vonden we in slechts één geval een significant verschil bij het vergelijken van leeropbrengsten in een gepaarde t-toets. Leerlingen in de experimentele groep die in staat zijn het reken-wiskundeprogramma op een normale manier te doorlopen, doen het significant beter dan de gekoppelde lotgenoten in de controlegroep op de posttests 'getallen en bewerkingen' en 'meten en meetkunde'. Deze vaststelling geeft verder aan dat er sprake is differentiële effecten.

We konden namelijk geen significante verschillen vaststellen bij het vergelijken van algemeen reken-wiskundige vaardigheden voor alle leerlingen, in zowel een gepaarde t-toets als in een regressieanalyse. We vonden wel duidelijke interactie-effecten tussen de conditie en de post-test, waaruit we mogen afleiden dat er inderdaad differentiële effecten zijn van de experimentele leergang in vergelijking met het controle programma. Namelijk in de experimentele conditie profiteren de beter

presterende leerlingen meer dan in het controleprogramma, een zogenoemd 'Mattheüs-effect' (Kerckhoff & Glennie, 1999). We waren verder in staat om significantie-intervallen te berekenen; dit zijn intervallen van pre-test scores waarbij een van de condities het significant beter doet dan de andere. We vonden dat zwakke rekenaars op de post-test 'getallen en operaties' meer baat hadden van de controleconditie dan van de experimentele conditie. Een dergelijk effect vonden we ook voor de post-test 'meten en meetkunde'. Hier konden we twee significantie-intervallen bepalen. We zagen dat de sterkste rekenaars meer profiteerden van de experimentele conditie in vergelijking met de controleconditie, terwijl de zwakste rekenaars meer baat hadden van de controleconditie.

We analyseerden ook de effecten van de leergangen op het verwerven van breuken. In drie breukeninterviews achterhaalden we de kennis en vaardigheden van de leerlingen. We maakten de interviews zodanig, dat de leerlingen die moeite hadden de situatie te verhelderen gestandaardiseerde hulp kregen. In een gepaarde t-toets vonden we dat leerlingen in de experimentele groep het significant beter deden dan hun lotgenoten in de controlegroep ($p \leq 0.05$). Echter, een regressieanalyse die corrigeert voor kleine verschillen bij aanvang van het onderzoek, toonde dat bij twee van de drie breukeninterviews de significante verschillen veranderden in duidelijke, maar niet-significante trends. Verder konden we voor de breukeninterviews geen interactie vaststellen tussen de conditie en de scores op de pre-test. We stellen daarom vast dat goede en zwakke rekenaars in gelijke mate profiteren van de experimentele conditie, in vergelijking met de controleconditie.

Wanneer we ons beperkten tot de conditie dat er geen hulp gegeven werd, vonden we niet in alle gevallen significante opbrengsten in de interviews. We vermoeden dat het kleine aantal leerlingen hierbij een rol speelt. Overigens waren leerlingen in de experimentele conditie wellicht juist bevoordeeld wanneer hen gestandaardiseerde hulp werd geboden, omdat deze hulp ze in staat stelden de aanpakken die in de experimentele leergang centraal stonden toe te passen, zoals het nagaan van gegeven antwoorden, het maken van een schetsje, enzovoort. Juist de sterke rekenaars zijn de leerlingen die gemakkelijk wiskundige taal leren en gebruiken. Zij zijn daarom in het voordeel in een situatie waar wiskunde tot stand komt via discussie. Daarnaast, zo stelden we vast, worden zwakke rekenaars mogelijk gehinderd door het formele karakter van de breuken of door het verschil in onderwijsstijl tussen leerkracht-onderzoeker en reguliere leerkracht. Verder maakt een kwalitatieve analyse van het werk van de leerlingen duidelijk, dat leerlingen in de experimentele conditie vaak blijk geven van enige gecijferdheid, in tegenstelling tot de leerlingen in de controlegroep. We interpreteren dit daarom als dat de experimentele leergang tot betere resultaten leidt dan die in de controlesetting.

Breuken leren

We presenteerden deze studie als een studie waarbij kwalitatieve en kwantitatieve analyses worden gebruikt om goed inzicht te krijgen in leerprocessen van leerlingen. We lieten dit nadrukkelijk naar voren komen in de geformuleerde onderzoeksvragen. Deze studie richt zich op het blootleggen van *hoe* mathematiseerprocessen kunnen worden ondersteund voor 10-11-jarige kinderen, met name voor wat betreft het leren van breuken. Verder richt deze studie zich op het achterhalen van obstakels in het proces van verticale mathematisering voor 10-11-jarigen. Daarom zijn naast kwantitatieve ook kwalitatieve analyses nodig voor het beantwoorden van de gestelde onderzoeksvragen.

We analyseerden leerprocessen in de experimentele leergang in vergelijking met effecten van het programma in de controlegroep. Dit bracht aanwijzingen voor het volgende naar voren:

– Leerlingen in de experimentele conditie tonen zich sterker in breuken dan leerlingen in de controleconditie.
– Leerlingen die gemiddeld of bovengemiddeld presteren in rekenen-wiskunde zijn in staat om formele breuken te leren op een betekenisvolle manier en binnen een redelijke tijd.
– Zwakke rekenaars ervaren aanzienlijke moeilijkheden bij het leren van formele breuken.

Deze bevindingen bevestigen gedeeltelijk de gestelde specifieke hypothese in deze studie. Wij vonden nadrukkelijke indicaties dat leerlingen in de experimentele conditie betere prestaties behalen ten aanzien van breuken dan hun gekoppelde lotgenoten in de controlegroep; echter, wanneer we kijken naar algemeen reken-wiskundige vaardigheden zien we dat alleen leerlingen, die bovengemiddeld scoren voor rekenen-wiskunde meer baat hebben bij de experimentele conditie dan bij de controleconditie. Deze resultaten bieden verder sterke aanwijzingen in de richting van de algemene hypothese van deze studie, die stelt dat een meerderheid van de 10-11-jarige leerlingen abstracte en formele wiskundige concepten kunnen leren in een onderwijskundige setting, waar betekenissen onderwerp zijn van discussie in voor de leerlingen herkenbare situaties, en waar wiskundige activiteiten op verschillende niveaus nauw met elkaar verbonden zijn, omdat we de experimentele conditie kunnen aanmerken als een dergelijke setting.

Conclusie

De algemene conclusie, gebaseerd op het kwantitatieve deel van deze studie, beschouwd vanuit de uitkomsten van de regressieanalyse, die controleert voor kleine verschillen bij aanvang van de studie, kan op de volgende wijze geformuleerd worden. Hoewel er duidelijke trends gevonden zijn in de verwachte richting, zijn er

strikt genomen geen algemeen significante effecten van het programma gevonden op het leren van de leerlingen. Dit geldt zowel voor algemeen reken-wiskundige vaardigheden als voor het leren van breuken. Daarom moeten we vaststellen dat we onvoldoende bevestiging vonden voor de algemene hypothese.

Echter, grondige kwalitatieve analyses van het leren door de leerlingen laten zien dat de experimentele leergang veel leerlingen kansen biedt om op een betekenisvolle manier breuken te leren. Leerlingen die breuken leerden in de controleleergang toonden minder flexibiliteit in het opereren en manipuleren met breuken dan hun gekoppelde lotgenoten in het experimentele programma.

En hoewel geen algemene effecten voor alle leerlingen konden worden gevonden, toont deze studie significante differentiële effecten. Sterke rekenaars hebben baat bij de experimentele conditie voor wat betreft de algemeen reken-wiskundige vaardigheden, terwijl zwakke rekenaars profiteren van het controleprogramma. Dit betekent dat het programma niet alleen een statistisch significant effect heeft, maar dat dit een effect is dat zowel relevant is voor de onderwijspraktijk als voor de onderwijskunde als wetenschappelijke discipline.

Overzicht

Hoofdstuk 1 – Inleiding
De kerndoelen voor het basisonderwijs laten weinig ruimte voor formele breuken. Echter, een recent ontwikkelde leergang breuken richt zich hier wel op. Het eerste hoofdstuk voorziet in een introductie in het onderzoeksproject in het reken-wiskundeonderwijs, dat zich richt op de effecten van deze nieuw ontwikkelde leergang en de opbrengsten bij leerlingen in het basisonderwijs. Twee leidende onderzoeksvragen voor de studie worden gepresenteerd: 'Hoe kan het leren van formele breuken worden ondersteund voor leerlingen van 10-11 jaar oud?' en 'Welke belangrijke obstakels in het leren van breuken door leerlingen van 10 tot 11 jaar oud kunnen worden onderscheiden?' In dit hoofdstuk worden de hypothesen van de studie geformuleerd, namelijk dat het overgrote deel van de 10- en 11-jarige leerlingen abstracte en formele wiskundige concepten en aanpakken kunnen verwerven, wanneer deze worden geleerd in een onderwijskundige setting waar betekenissen worden bediscussieerd in herkenbare en betekenisvolle situaties en waar wiskundige activiteiten op verschillende niveaus nauw zijn verbonden.

Hoofdstuk 2 – Theoretische achtergrond
Dit hoofdstuk analyseert paradigma's in de psychologie die in de twintigste eeuw een basis vormde voor onderzoek van het reken-wiskundeonderwijs. Vanuit deze analyse schetsen we de ontwikkeling van vakdidactische kennis – een leertheorie die voortkomt uit de karakteristiek van het vak. Vanuit dit perspectief wordt het 'mathe-

matiseren' als doel in het reken-wiskundeonderwijs beschouwd. Mathematiseren omvat bijvoorbeeld het modelleren, symboliseren, generaliseren, formaliseren en abstraheren. In dit hoofdstuk wordt gesteld dat deze elementen van het proces van mathematiseren niet geïsoleerd kunnen worden beschouwd, omdat alleen een combinatie van deze aspecten recht kunnen doen aan het proces van mathematiseren. We zullen het 'mathematiseren' analyseren in de context van het realistisch reken-wiskundeonderwijs, wat ons vervolgens zal voorzien van instrumenten om onderzoeksthema's in dit boek integraal uit te werken.

Hoofdstuk 3 – Werken aan wiskundig inzicht: een longitudinale vergelijkende studie over modelleren

Het derde hoofstuk beschrijft een longitudinale studie van het onderwijzen en leren van breuken in twee gepaarde groepen van 9- tot 10-jarige leerlingen. In de experimentele groep worden breuken geïntroduceerd aan de hand van de strook en getallenlijn als (conceptuele) modellen, in de controlegroep wordt dit onderwerp geïntroduceerd via het eerlijk verdelen en het cirkelmodel. Leerlingen in de experimentele groep worden uitgenodigd om bevindingen te bediscussiëren, terwijl leerlingen in de controlegroep individueel werken. De groepen worden op verschillende momenten in het experimentele jaar vergeleken. Na een jaar blijkt dat leerlingen in de experimentele groep sterker zijn in breuken dan hun lotgenoten in de controlegroep. Echter, analyses van de gegevens tonen ook een zogenoemd 'Mattheüs-effect'. We zien dat sterke rekenaars meer baat hebben bij het experimentele programma dan zwakke rekenaars.

Hoofdstuk 4 – Het leren van breuken van Audrey: een gevalsstudie rond het leren van formele wiskunde

Hoofdstuk vier bestaat uit een gevalsstudie, die de ontwikkeling van werken met breuken van een leerling beschrijft in de nieuw ontwikkelde leergang van dertig lessen gedurende een heel schooljaar. Deze studie laat zien dat de leergang en het onderwijzen daarvan bijdraagt aan de groei van een gemiddelde rekenaar. Verder worden argumenten gepresenteerd in welke mate het formele rekenen met breuken gepast is als doel in het basisonderwijs.

Hoofdstuk 5 – Het leren van breuken door zwakke rekenaars: Shirleys keuzen in aanpakken in het reken-wiskundeonderwijs

Onderzoek aan het reken-wiskundeonderwijs toont dat zowel sterke als zwakke rekenaars profiteren van het leren van wiskunde in betekenisvolle contexten. De gevalsstudie in dit hoofdstuk biedt een uitgebreide analyse van het leerproces van een zwakke rekenaar in het realistisch reken-wiskundeonderwijs. Daarbij wordt met name gekeken naar productieve en contra-productieve aanpakken bij het oplossen

van problemen met breuken. We vinden argumenten voor het idee dat zwakke rekenaars zeker profiteren van het realistisch reken-wiskundeonderwijs, maar problemen ondervinden bij het verwerven van formele breuken. Deze problemen zijn aanleiding om uniforme doelen in het reken-wiskundeonderwijs ter discussie te stellen.

Hoofdstuk 6 – Effecten van een experimentele leergang breuken in het basisonderwijs: een longitudinale analyse

Dit hoofdstuk beschrijft de kwantitatieve opbrengsten van de onderhavige studie en bespreekt de effecten van de twee breukenleergangen. Heldere trends laten zien dat leerlingen voor het leren van breuken beter af zijn in de experimentele leergang. Verder laat het beschouwen van algemeen reken-wiskundige vaardigheden zien dat de experimentele leergang vergeleken met de controleleergang verschillen tussen leerlingen vergroot. We schetsen een verklarend model vanuit het gezichtspunt van wiskundetaal en de theorie rond 'cognitieve lading' om deze programmaeffecten te verhelderen.

Hoofdstuk 7 – Theoretische reflectie

Noties vanuit het constructivisme zijn waardevol voor het ontwikkelen van leeromgevingen voor het realistisch reken-wiskundeonderwijs, omdat het laat zien hoe discussies in de klas kunnen worden gericht op het blootleggen van wiskundige structuren, bijvoorbeeld door socio-wiskundige normen te bespreken; wat is geldig als wiskundig argument en wat niet. Wanneer wiskunde op deze manier wordt geconstrueerd door te onderhandelen over betekenissen en over hoe er begrepen wordt, richt het onderwijs zich op het mathematiseren door leerlingen. En wanneer het mathematiseren een van de doelen in het te ontwikkelen onderwijs is, ligt het voor de hand om leerlijnen te verstrengelen. Verder is juist het verstrengelen van leergangen een waardevol perspectief wanneer het ontwikkelen van formele en abstracte concepten een onderwijsdoel is, omdat leerlingen zich daarbij moeten realiseren in welk opzicht elementen van de werkelijkheid hetzelfde of verschillend zijn. Daarnaast kan het verstrengelen van leergangen een middel zijn om alle leerlingen te laten profiteren van het reken-wiskundeonderwijs, met name de zwakke rekenaars. In dit hoofdstuk worden deze argumenten bewerkt om aldus de inbedding van deze studie in het constructivisme en het realistisch reken-wiskundeonderwijs te heroverwegen. Daarnaast worden deze overwegingen aangegrepen om leerprocessen van zwakke rekenaars te plaatsen.

Hoofdstuk 8 – Conclusie en discussie

Dit hoofdstuk presenteert de resultaten en conclusies van de studie die is beschreven in dit boek. We zullen antwoorden formuleren op onderzoeksvragen die in voorafgaande hoofdstukken werden gesteld. Bevindingen uit de hoofdstukken 3 tot en met

5 tonen, dat wanneer een breukenleergang gericht wordt op het verwerven van gecijferdheid, waarbij kennis en vaardigheden (mede) worden verworven via het onderhandelen over betekenissen, veel leerlingen – maar niet alle leerlingen – breuken kunnen leren op een betekenisvolle manier. We zullen deze bevinding bespreken in dit hoofdstuk. Vervolgens zullen we, vanuit de theoretische beschouwingen in hoofdstuk 2 en 6, een volgende stap nemen op weg naar het formuleren van onderwerpen voor vervolgonderzoek. Terwijl we dit doen stellen we nogmaals vast dat het experimentele programma perspectief biedt op het leren van rekenen-wiskunde voor alle leerlingen.

Reflectie

Ontwikkelingsonderzoek impliceert dat ontwikkelde leergangen onderworpen zullen worden aan nieuwe overwegingen, om aldus de basis te vormen voor nieuwe ontwikkelingen, waarbij het nieuwe programma snel het oude wordt; voorbestemd om vervangen te worden door iets nieuws. Nieuwe ideeën, ontstaan tijdens het ontwikkelingsproces, worden toegevoegd aan het repertoire van de ontwikkelaar als gereedschap voor toekomstige ontwikkelingen (vgl. Gravemeijer, 1994). In deze zin kan dit boek beschouwd worden als ideeënbron voor het verder ontwikkelen van reken-wiskundeonderwijs.

Literatuur

Anderson, J.R., J.G. Greeno, L.M. Reder & H.A. Simon (2000). Perspectives on Learning, Thinking, and Activity. *Educational Researcher, 29*(4), 11–13.

Behr, M.J., G. Harel, Th. Post & R. Lesh (1992). Rational number, ratio, and proportion. In: Douglas A. Grouws (ed.) *Handbook of research on mathematics teaching and learning*. (296-332). New York: Macmilian Publishing Company.

Cook, Th.D. & D.T. Campbell (1979). *Quasi-Experimentation. Design & Analysis Issues for Field Settings*. Chicago: Rand McNally College Publishing Company.

De Corte, E., B. Greer & L. Verschaffel (1996). Mathematics Teaching and Learning. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology*. New York: Simon & Schuster Macmillan, 491-549.

Freudenthal, H. (1968) Why to teach mathematics as to be useful? *Educational Studies in Mathematic*s, *1*(1), 3-8.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: Riedel.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: Riedel-Kluwer.

Freudenthal, H. (1991). *Revisiting Mathematics Education, China lectures*. Dordrecht: Kluwer Academic Publishers.

Goddijn, A.J. (1992). De oudste en de nieuwste breuken. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs, 11*(2), 18-31.

Gravemeijer, K.P.E. (1994). *Developing realistic mathematics education* (dissertation). Utrecht: CD-ß Press.

Gravemeijer, K.P.E. & J. Terwel (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of Curriculum Studies, 32*(6), 777-796.

Gravemeijer, K.P.E. (2001). *Reken-wiskundeonderwijs voor de 21^e eeuw*. Utrecht: Utrecht University.

Greeno, J.G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22*(3), 170-218.

Greeno, J.G., A.M. Collins & L. Resnick (1996). Cognition and Learning. In: David C. Berliner & Robert C. Calfee (eds.). *Handbook of Educational Psychology.* New York: Simon & Schuster Macmillan, 15-46.

Hasemann, K. (1981). On difficulties with fractions. *Educational Studies in Mathematics, 12*, 71-87.

Heuvel-Panhuizen, M. van den (1996). *Assessment and realistic mathematics education* (dissertation)*.* Utrecht: CD-ß Press.

Hiele, P.M. van (1986). *Structure and Insight. A theory of Mathematics Education.* Orlando: Academic Press, Inc.

Hoek, D., J. Terwel & P. van den Eeden (1997). Effects of Training in the Use of Social and cognitive Strategies: An Intervention Study in Secondary Mathematics in Co-Operative Groups. *Educational Research and Evaluation, 3*(4), 364-389.

Hoek, D.J. (1998). *Social and cognitive strategies in co-operative groups. Effects of strategy instruction in secondary mathematics* (dissertation). Amsterdam: University of Amsterdam.

Hoek D., P. van den Eeden & J. Terwel (1999). The effects of integrated social and cognitive strategy instruction on the mathematics achievement in secondary education. *Learning & Instruction, 9*, 427-448.

Kerckhoff, A.C. & E. Glennie (1999). The Matthew effect in American education. In: Aaron M. Pallas (ed.). *Research in sociology of education and socialization (Vol. 12)*. Stamford, Connecticut: Jai Press Inc, 35-66.

Nelissen, J.M.C. (1998). Representaties in het reken-wiskundeonderwijs. *Pedagogische Studiën, 75*, 169-183.

Pirie, S.E.B. & T.E. Kieren (1989). A Recursive Theory of Mathematical Understanding. *For the Learning of Mathematics, 9*(3), 7-11.

Pirie, S.E.B. & T.E. Kieren (1994). Beyond Metaphor: Formalising in Mathematical Understanding within Constructivist Environments. *For the Learning of Mathematics, 14*(1), 39-43.

Schoenfeld, A.H. (1994). Reflections on Doing and Teaching Mathematics. In: Alan H. Schoenfeld. *Mathematical thinking and problem solving*. Hillsdale, NJ: Lawrence Erlbaum Associates, Publishers, 53-70.

Sierpinska, A. & S. Lerman (1996). Epistemologies of Mathematics and of Mathematics Education. In: Alan J. Bishop, Ken Clements, Christine Keitel, Jeremy Kilpatrick & Colette Laborde (eds.). *International Handbook of Mathematics Education (part 2)*. Dordrecht, Boston, London: Kluwer Academic Publishers, 826-876.

Streefland, L. (1990). *Fractions in Realistic Mathematics Education, a Paradigm of Developmental Research.* Dordrecht: Kluwer Academic Publishers.

Streefland, L. (1997). Een geval van reflectief denken in ontwikkeling met verhoudingen als paradigma (slot). *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs, 15*(3), 22-31.

Struik, D.J. (1987). *A Concise History of Mathematics (fourth revised edition).* New York: Dover Publications, Inc.

Terwel, J. (1984). *Onderwijs maken* (dissertation). Harlingen: Flevodruk.

Treffers, A. (1987). *Three dimensions. A model of Goal and Theory Description in Mathematics Instruction – the Wiskobas Project*. Dordrecht: Reidel Publishing Company.

Yin, R.K. (1984). *Case study research. Designs and Methods.* Beverly Hills, CA: SAGE Publications.

Appendix A: The experimental fraction programme and its development

The fraction programme discussed here, the experimental programme, is an extension of 'The Fraction gazette'-programme [De Breukenbode] (Bokhove, et al., 1996). The experimental programme, as compared to 'The Fraction gazette'-programme, focuses more on the development of formal fraction reasoning. Where 'The Fraction gazette' only provides ideas for teaching formal fraction reasoning, in the experimental programme these ideas to reach formalisation were elaborated and expanded.

Because of this close connection between these programmes, both were grounded on the same general ideas on fraction learning. These ideas – at the start of the developmental process – included:

– educational time spent on fraction learning should be limited;
– learning fractions should contribute to number-sense acquisition and should therefore not primary be aimed at learning how the operations '+', '−', '×' and ':' work out in the case of fractions (cf. Moss & Case, 1999);
– learning fractions should be embedded in learning (to operate with) whole numbers (Treffers & De Moor, 1990), which (in the Dutch situation) implies that fractions need to be considered as points on a number line;
– as a consequence: equivalent fractions should emerge as numbers at the same position on the number line.

Positioning fractions on a number line, however, is somewhat problematic (Novillis Larson, 1980). Lek (1992) analysed that students have problems with positioning non-unit-fractions on a number line – in the context of a road, when their fraction language is not firmly grounded. Lek therefore developed several activities where bars were divided and where the division result was named as a fraction. She elaborated one of these contexts, the chocolate bar context, further to provide for equivalent fractions and fraction addition and subtraction. In this context fractions were named in terms of pieces of the bar and students could switch from fractions to pieces at will. For example, to add $\frac{1}{4}$ and $\frac{1}{3}$, these fractions could be taken from a chocolate bar with 12 pieces. $\frac{1}{4}$ of the bar is 3 pieces and $\frac{1}{3}$ of the bar consists of 4 pieces, 7 in total, $\frac{7}{12}$ of the bar.

Two major objections were made regarding this approach in formalising fraction addition:

1 The use of 'named fractions', where fractions are replaced by whole numbers (Van Reeuwijk, 1991) – parts of the bar replaced by number of pieces – does not

stimulate students to think in terms of fractions and therefore will not develop a 'sense of fractions', as fractions can be 'avoided' by replacing them by whole numbers (cf. Goddijn, 1992).

2   Chocolate bars as context provide for a part-whole notion of fractions. Moreover, as fractions are connected to a number of pieces, this context emphasises on the discrete character of fractions. This makes that this context does not link up with representing fractions on a number line. It also does not prepare for fractions larger than 1.

To resist the first objection, the chocolate bar was wrapped up so that the pieces became invisible, to prevent the students from counting the pieces. Now the students needed to figure out how many pieces the bar could have. This made that in communicating findings fractions came forward, as the number of pieces in the bar should be communicated together with the number of pieces; 'seven pieces' needed to be replaced by 'seven pieces of a bar with twelve pieces', which could be shortened to $\frac{7}{12}$. However, in this way the second objection was not dealt with. To be able to project operations on a bar to the number line in a proper way, the bar should represent a length and in the chocolate bar context the bar clearly does not.

These considerations made that one of the central fraction generating activities in the first lessons in 'The Fraction gazette' [De Breukenbode] was a measuring activity. Students are invited to measure all kinds of objects in the classroom with a folded bar, named a Amsterdam foot (or av) (figure 1). Naming the measurements stimulates the formation of a fraction language, as the number of pieces in the folded bar should be taken into consideration.
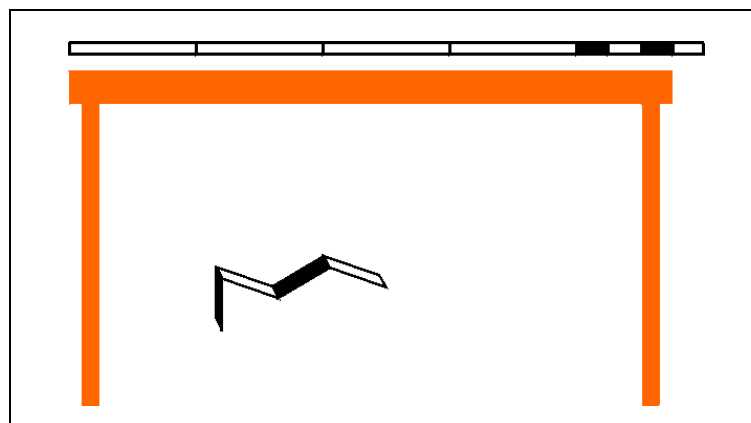
figure 1: the table is $4\frac{3}{4}$ av at large

Measuring activities, like the one presented, facilitate fractions to be represented on

a number line, if students are asked to project several measurements on a line in order to compare these. Moreover, as the number of parts in the measuring-bar is crucial, discussions on how to make the bars by folding, provide for the first fraction relations; by folding twice the bar is divided in four pieces, so one half has the same length as two fourths and – a little later – $\frac{5}{6}$ can be formed by dividing in three and next dividing one of the parts in two.

In the next stage in the programme we focus on comparison strategies for fractions. One of the central contexts here is the 'try-your-strength machine' (Noteboom, 1994). In this context, we discuss with the students, two children, Sarah and René, hit the machine (figure 2). After Sarah's hit the water runs to $\frac{2}{5}$ of the pipe. When René hits the machine the water runs to $\frac{5}{8}$. When we further discuss this situation with our students, one of them, Charlie, argues why the water runs higher at René's strike: '$\frac{4}{8}$ is halfway the machine and $\frac{2}{5}$ is below half.'
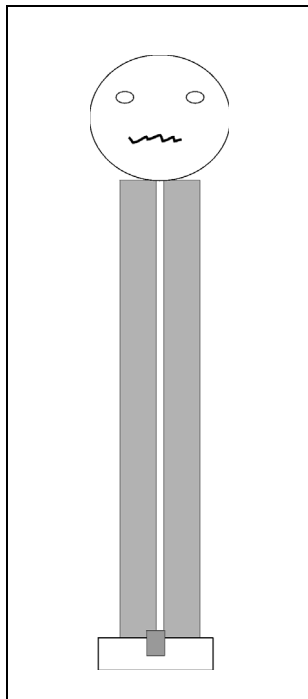
figure 2: try-your-strength machine

Charlie thus in this context compares the fractions $\frac{5}{8}$ and $\frac{2}{5}$ by comparing both fractions with a half. Using the try-your-strength machine-context and similar contexts, this strategy and other comparison strategies are discussed and – at a certain point –

become the students' repertoire. Moreover, these strategies are more or less bound to the number line-model for fractions. Halfway the experimental year the acquired comparison strategies include:

– comparing fractions by the look of it, when fractions are clearly far apart;
– comparing fractions by only considering numerator or denominator, for example when comparing $\frac{3}{5}$ and $\frac{4}{5}$ – by the number of parts – or when comparing $\frac{3}{8}$ and $\frac{3}{5}$ – by the size of the parts;
– comparing fractions via 1, for example when comparing $\frac{5}{6}$ and $\frac{7}{8}$;
– comparing fractions via $\frac{1}{2}$ (see above);
– comparing using (simple) equivalent fractions.

As we aimed at the acquisition of formal fractions, we were especially interested in the use of equivalent fractions as comparison strategy. We noticed that the choice of fractions to compare is crucial in evoking this strategy. Moreover, we developed several situations that focused students at considering equivalent fractions. For example we introduced the term 'roommates' to name fractions at the same position on the number line. We also introduced 'neighbours' for fractions close to the ones already placed on the line.

We ask the students to figure out what fractions could be 'neighbours' of $\frac{1}{6}$ and $\frac{1}{3}$, living between these fractions. To support students' arguments we first ask them to construct a number line with $\frac{1}{6}$ and $\frac{1}{3}$ and find 'roommates' of these fractions (figure 3).
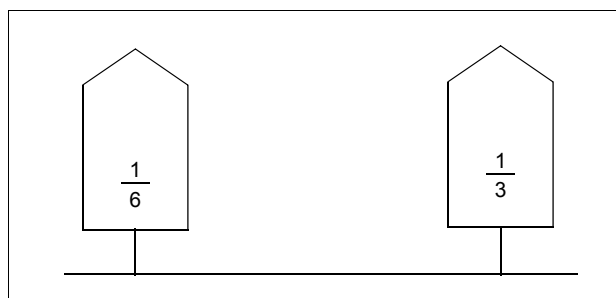
figure 3: 'roommates' and 'neighbours'

From there experiences in folding bars, students soon find $\frac{2}{12}$, $\frac{3}{18}$, $\frac{4}{24}$ and $\frac{10}{60}$ as roommates of $\frac{1}{6}$ and $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$, $\frac{8}{24}$ and $\frac{16}{48}$ as $\frac{1}{3}$'s roommates and soon discover how these roommates can be used to determine fractions between $\frac{1}{6}$ and $\frac{1}{3}$, the neighbours.

The 'roommates'-context provides for a basis for formal fraction addition and subtraction. Namely, when for example $\frac{4}{24}$ is found as roommate of $\frac{1}{6}$ and $\frac{8}{24}$ is found as a roommate of $\frac{1}{3}$, it is not difficult to understand that $\frac{1}{3} - \frac{1}{6}$, the distance from $\frac{1}{6}$

to $\frac{1}{3}$, is 4 leaps of $\frac{1}{24}$ or $\frac{4}{24}$. However, to fully grasp formal fraction operations a deeper understanding of equivalence relations is required. To reach this understanding students need to develop a language to consider the role of both numerator and denominator separately.

We used the 'Fraction-lift'-context to provide for such a situation.[1] In this context elevators are numbered, where these numbers indicate the number of steps needed to move from the ground floor to the top of the fraction-building. For example the 3-lift moves in three steps to the top, from 0 to $\frac{1}{3}$, from $\frac{1}{3}$ to $\frac{2}{3}$ and from $\frac{2}{3}$ to the top of the building.

One of the first lifts that the students investigate when exploring the fraction lifts is the 4-lift. They soon discover that the second stop of 4-lift is located at $\frac{1}{2}$, since two steps of $\frac{1}{4}$ brings the lift halfway, which makes them wonder what other lifts stop halfway the fraction building. The students soon find out: the 8-lift and the 16-lift. Then Edith notices that there are many lift that stop halfway: 'They all do…'

The teacher takes this argument to refocus the search: 'Here you have the 3-lift. Does this one stop at $\frac{1}{2}$?' The students soon agree. This one certainly does not stop there. The students easily name more lifts that do not call at $\frac{1}{2}$: the 5-lift, the 7-lift, the 9-lift. The students notice the pattern in this row of numbers: the odd-numbered lifts don't stop at $\frac{1}{2}$.
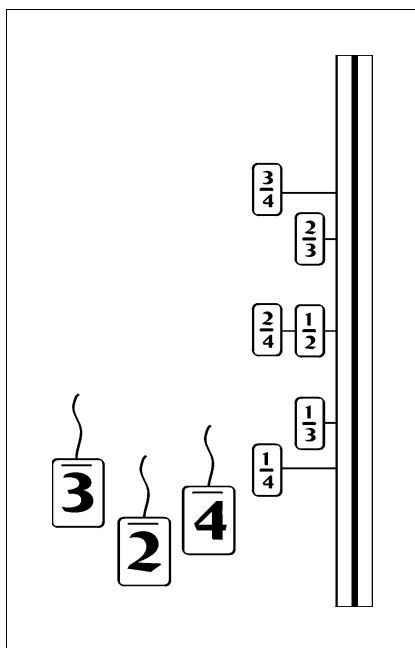

figure 4: fraction-lift

Discussing how fractions could be transported to the appropriate floors in one of the next lessons elicits reflection on fraction equivalence. Students construct series of lifts that stop at $\frac{3}{5}$, for example the 10-lift (6 stops) and the 15-lift (9 stops). Moreover, they acquire a method to determine which lifts stop at, for example, $\frac{3}{5}$ and $\frac{1}{3}$. As the 15 can be divided by 3 and 5, the 15-lift stops at $\frac{3}{5}$ (after 9 stops) and at $\frac{1}{3}$ (after 5 stops).

Students next learn how to tackle expressions like $\frac{3}{4}$: 5, by replacing $\frac{3}{4}$ by $\frac{15}{20}$ − 15 leaps of $\frac{1}{20}$ can be divided in five portions of 3 leaps of $\frac{1}{20}$, so $\frac{3}{4}$: 5 = $\frac{3}{20}$.[2]

One of the central points of discussion at the start of the developmental process that leaded to the programme described was the continuous number line vs. the discrete fraction appearance issue. Two solutions for this problem were imbedded in the described programme: the folding of measuring bars and the lift-stops in the 'Fraction-lift'-context. While the folded measuring bars provided a key-activity to stimulate fraction-language learning, the 'Fraction-lift' gives reason to reflect on fraction equivalence, to enable students to finally gain fraction operations; not as a direct aim of the programme, but as a by-product of a thorough analysis of fraction positions on the number line.

Notes

1 The 'Fraction-lift' was an idea of Adrian Treffers.
2 For a more thorough discussion of the 'Fraction-lift'-context, see chapter 3.

References

Bokhove, J., K. Buys (ed.), R. Keijzer, A. Lek, A. Noteboom & A. Treffers (1996). *De Breukenbode. Een leergang voor de basisschool* (werkbladen en handleiding) [The Fraction gazette]. Enschede/Utrecht: SLO/FI/Cito.

Goddijn, A.J. (1992). De oudste en de nieuwste breuken [The oldest and the newest fractions.]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs ,11*(2). 18-31.

Lek, A.T.E. (1992). *Met repen begrepen* [Understood with chocolate bars]. , Utrecht: University Utrecht (unpublished doctoral thesis).

Moss, J. & R. Case (1999). Developing Children's Understanding of the Rational Numbers: A New Model and an Experimental Curriculum. *Journal for Research in Mathematics Education, 30*(2), 122-147.

Noteboom, A. (1994). De Kop van Jut. Een breukenactiviteit in groep 6 [The try-your-strength machine. A activity on fractions in grade 6]. *Willem Bartjens, 14*(2), 9-13.

Novillis Larson, C. (1980). Locating Proper Fractions On Number Lines: Effect of Length and Equivalence. *School Science and Mathematics, Vol. LXXX*(5), 423-428.

Reeuwijk, M. van (1991). Benoemde breuken: herinneringen van vroeger [Named fractions: memories of the past]. *Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs, 10*(1), 39-41.

Treffers, A. & E. de Moor (1990). *Proeve van een nationaal programma voor het reken-wiskundeonderwijs op de basisschool. Deel 2: Basisvaardigheden en cijferen* [Specimen of a National Program for Primary Mathematics teaching. Part 2: Basic Mental Strategies and Written Computation]. Tilburg: Zwijsen.

Appendix B: Interview 1 and 2

Problems posed in interview 1

1 **Schilderen**
   Welk deel van de muur is geschilderd? [Painting. What part of the wall is painted?]
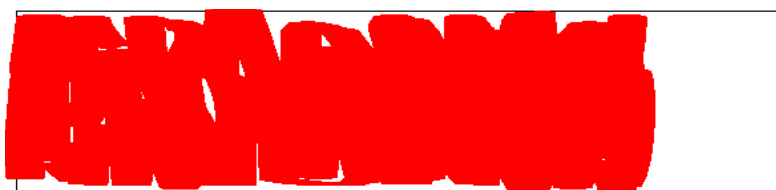
2 **Reep eten**
   Irene heeft $\frac{3}{5}$ deel van haar reep opgegeten.
   Dit heeft ze over. [Eating chocolate. Irene ate $\frac{3}{5}$ of her chocolate bar. Here you see what is left.]
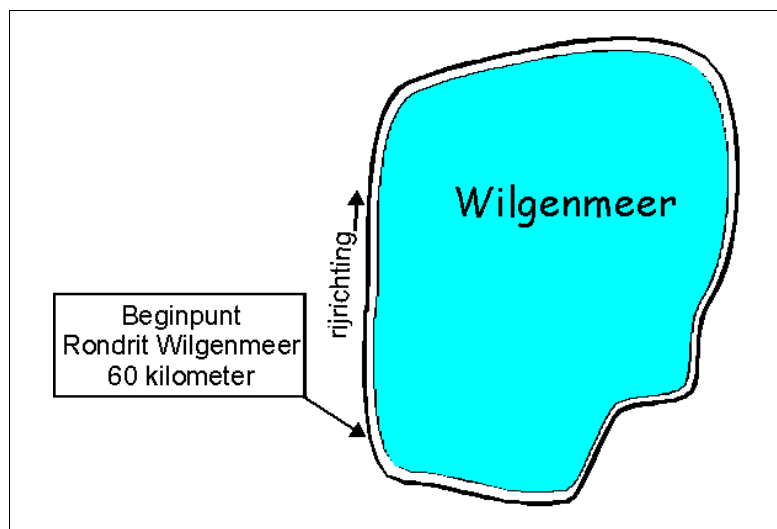

3 **Brood eten**
   Ik heb $\frac{3}{4}$ van het brood opgegeten. [Eating bread. I ate $\frac{3}{4}$ from this loaf of bread.]

Problems posed in interview 2

1   *Fietstocht*

Mieke, Janneke en Jelle maken een fietstocht om het Wilgenmeer.

Om 12 uur is Mieke tot $\frac{2}{3}$ gekomen en Janneke tot $\frac{5}{6}$ gekomen.
Jelle is dan nog verder gekomen.
[1. Bicycle tour. Mieke, Janneke and Jelle make a bicycle tour around Willow-lake. 12 O'clock Mieke came to $\frac{2}{3}$ of the route and Janneke to $\frac{5}{6}$. Jelle came even further.]

2   *Vergelijken met een half*
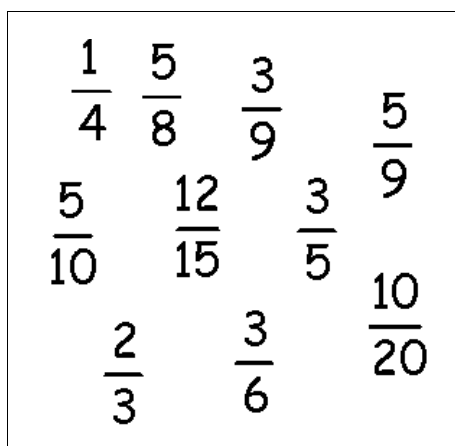
Kleiner dan half	Precies een half	Groter dan een half

[2. Comparing with a half. Choose from smaller than a half, exactly a half and bigger than a half.]

3 *Dropstaven*

De bovenste staaf is een hele dropstaaf

Daaronder staan:

$\frac{9}{10}$ dropstaaf

$\frac{1}{10}$ dropstaaf

$\frac{1}{4}$ dropstaaf

$\frac{5}{6}$ dropstaaf

$\frac{2}{3}$ dropstaaf

[3. Licorice sticks. The top stick is a whole licorice stick. Below are: $\frac{9}{10}$ stick, $\frac{1}{10}$ stick, $\frac{1}{4}$ stick, $\frac{5}{6}$ stick, $\frac{2}{3}$ stick.]

4 *Heel dicht in de buurt van $\frac{3}{4}$.*
Maak zo'n breuk.
[4. Very close to $\frac{3}{4}$. Make such a fraction.]

Curriculum Vitae

Ronald Keijzer was born on November 21, 1959, in Amsterdam. Between 1979 and 1985 he studied Mathematics and Psychology at Amsterdam University, where he majored in Algebraic Number Theory in 1985. After his study he worked for a year and a half as a Mathematics teacher trainer for secondary education. He was occasionally involved in teacher training for primary education, in which field he then worked as a mathematics educator from 1988.

In the early 1990s he published many short articles on mathematics in teacher education and participated in the PUIK-project, a SLO project. This project aimed at formulating standards for mathematics in teacher education for primary schools. Later he became a member of the subgroup Mathematics and Technology of the Process management Teacher Education (PmL). This subgroup's task was to write a document naming targets for mathematics in teacher education for primary schools.

Ronald Keijzer worked as a developer for mathematics in primary education in the Mathematics in Context-project – a joint project of the University of Wisconsin, Madison and the Freudenthal Institute – and in the 'fraction'-project – a joint project of the SLO, Cito and the Freudenthal Institute.

At the moment he is co-ordinator of the Panama-project, a project of the Freudenthal Institute. This project forms the network for school councillors, teacher educators, developers and researchers in the field of mathematics education for primary schools. Furthermore, he is editor-in-chief of the 'Tijdschrift voor nascholing en onderzoek van het reken-wiskundeonderwijs' ['Journal for in-service training and research in mathematics education'].