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Faculty of Economics and Business Administration

A Taylor series expansion approach to the functional approximation of finite queues

Research Memorandum 2011-49

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A Taylor Series Expansion Approach to the Functional Approximation of Finite Queues

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Abstract

This paper presents a new approach to the functional approximation of the M/G/1/N built on a Taylor series approach. Specifically, we establish an approximative expression for the remainder term of the Taylor series that can be computed in an efficient manner. As we will illustrate with numerical examples, the resulting Taylor series approximation turns out to be of practical value.

keywords: Series expansion approach; Taylor series; $M/G/1/N$ queue; Performance Measures; Deviation matrix

1 Introduction

Queueing models are a well-established tool for the analysis of stochastic systems from areas as divers as manufacturing, telecommunication, transport and the service industry. Typically, a queueing model is a simplified representation of the real-world system under consideration. In addition, often there is not sufficient statistical data to determine the service and interarrival time distribution, or, in case the type of distribution is known, there is statical uncertainty on the exact values of the parameters of the distribution. For these reasons, perturbation analysis of queueing systems (PAQS) has been developed. PAQS studies the dependence of the performance of a given queueing system on the underlying distributional assumptions. Consider, for example, an M/M/1 model with interarrival rate λ and service rate μ and suppose that one is interested in the variance of the stationary queue length. A first question addressed by PAQS is that on what the effect of small change in, say, λ on the performance would be. This type of perturbation analysis is called *parameter sensitivity.* In addition, PAQS addresses the issue what would happen if, say, the service time distribution is uniform on a certain interval rather than exponential. This

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is called *distributional sensitivity.* With the advent of sample path techniques, such as infinitesimal perturbation analysis [10, 5, 2], the score function method [18, 20], and measure-valued differentiation [17], parameter sensitivity is well understood.

The distributional sensitivity is less developed. Importance sampling, see e.g. [19], is a sample path method applicable to this problem. Unfortunately, importance sampling suffers from large variance if the nominal and the perturbed distribution are not 'sufficiently close' and it requires that the perturbed distribution is absolutely continuous with respect the nominal one. For these reasons, most authors use numerical analysis to come up with bounds on the effect of a distributional perturbation. See, for example, the strong stability method [12, 16].

In this paper we present a new approach to PAQS. Firstly, we comprise parameter and distributional sensitivity in one framework. This is achieved by interpreting the distributional sensitivity as a parametric problem: we assume that the distribution under consideration is a convex combination of the nominal and the perturbed distribution, where the weight factor of the convex combination introduces an artificial parameter into the model. Secondly, we develop the performance of the system under consideration into a Taylor series with respect to the parameter of interest, where we make use of a fundamental result on Taylor series for Markov chains, see [7].

For explicatory purposes, we will consider in the following the single server queue with Poisson arrival process and generally distributed service times. Service following the first-come first-served (FCFS) discipline, and there is a limited capacity of in total N customers that can be present at the queue (the one in service included). Customers that do not find an empty place at the queue upon their arrival are lost. Let π^* denote the stationary distribution of the continuous-time queue-length process of the M/G/1N queue. It is well known that π^* can be expressed via the stationary distribution of the Markov chain embedded at departure points of customers, denoted by π. Specifically, let ρ denote the traffic rate, then it holds that

$$\pi^*(i) = \frac{\pi(i)}{\pi(0) + \rho}, i = 0, \dots N-1, \tag{1}$$

and

$$\pi^*(N) = \frac{1}{\rho}\left(\rho - 1 + \frac{\pi(0)}{\pi(0)+\rho}\right) = \frac{\pi(0)+\rho-1}{\pi(0)+\rho}, \tag{2}$$

see [6] for details.

For the PAQS presented in this paper we consider π^* as a mapping of some real-valued parameter θ, in notation π^*_θ. For example, θ may denote the mean service time of the queue. We are interested in obtaining the *functional* dependence of $\pi^*(\theta)$ on θ in a simplified form. For our approach we will compute π^*_θ for some parameter value θ numerically. However, then we will approximate the function $\pi^*(\theta + \Delta)$ on some Δ-interval. More specifically, we will approximate

$\pi^*(\theta + \Delta)$ by a polynomial in Δ. To achieve this we will use the Taylor series expansion approach established in [7]. More specifically, let π_θ denote the stationary distribution of the queue-length process embedded at departure epoches in the M/G/1/N queue, where $\theta \in \mathbb{R}$ denotes a control parameter. Under quite general conditions it holds that $\pi_{\theta+\Delta}$ can be developed into a Taylor series of the following from

$$\pi_{\theta+\Delta} = \sum_{n=0}^{k} \frac{\Delta^n}{n!} \pi_\theta^{(n)},$$

where $\pi_\theta^{(n)}$ denotes the n-th order derivative of π_θ with respect to θ. We call

$$H_\theta(k, \Delta) = \sum_{n=0}^{k} \frac{\Delta^n}{n!} \pi_\theta^{(n)}$$

the k-th order Taylor approximation of $\pi_{\theta+\Delta}$ at θ, and

$$r_\theta(k, \Delta) = \pi_{\theta+\Delta} - H_\theta(k, \Delta)$$

the k-th order remainder term at θ. Let η denote the width of the interval $(\theta - \eta, \theta + \eta)$ on which the stationary distribution has to be approximated.

The usefulness of any Taylor series based approach relies on two factors:

1. Fast convergence of the series (a Taylor polynomial of small order yields already a satisfying approximation), i.e., if for small k it holds that

$$\sup_{|\Delta| \le \eta} |r_\theta(k, \Delta)| \tag{3}$$

is sufficiently small.

2. The ability of computing the remainder term of the Taylor series in an efficient way so that order of the Taylor polynomial that is sufficient to achieve a desired precision of the approximation can be decided *a priori*.

The contribution of the paper is as follows. We investigate the applicability use of the Taylor polynomial $H_\theta(k, \Delta)$ for numerical purposes for M/G/1/N queue. Specifically,

- we identify a recursive form of the derivatives which simplifies coding the algorithm;
- our numerical studies show that already a Taylor series of small order yields good approximations (this addresses topic (1) above);
- a simplified and easily computable expression bounding the remainder of the Taylor series, see (3), is established (this addresses topic (2) above).

The paper is organized as follows. The embedded Markov chain of the M/G/1/N model is presented in Section 2. Our series expansions approach is detailed in Section 3. Numerical examples are provided in Section 4.

We conclude the introduction with a brief discussion of implications of our approach to the numerical approximation of the M/G/1/∞ queue. The stationary distribution π of the embedded jump chain can be obtained in a closed-from only for special cases. This has lead to a rich literature on approximations of the stationary distribution π^*. A very efficient numerical approach is to approximate the M/G/1/N queue by the M/G/1/∞ queue, see [24, 23]. However, this approach is limited to the cases where the traffic load is less than one (so that the corresponding M/G/1/∞ model is stable). For a detailed overview on numerical approaches to the M/G/1/N queue we refer to the excellent survey in [22]. The approach presented in this paper yields a new numerical approach to the M/G/1/N queue that is also feasible for traffic loads larger than one. In addition, our approach yields an approximation of a performance functional on an entire interval and allows for an error bound of the approximation that holds uniformly on an interval.

2 The M/G/1/N Queue

Consider the $M/G/1/N$ queue, where customers arrive according to a Poisson process with rate λ and demand an independent and identically distributed service time with common distribution function $B(t)$ with mean $1/\mu$. There can at most be N customers be present at the queue (including the one in service), and customers attempting to enter the queue when there are already N customers present are lost. The service discipline is FCFS.

Let $X(t)$ denote the number of customers in the $M/G/1/N$ queue at time t, for $t \geq 0$. Note that the queue-length processes $\{X(t) : t \geq 0\}$ of the $M/G/1/N$ system fails to be a Markov process because the service time distribution does not have the memoryless property. Since we have assumed that customers that do not find an empty buffer place upon their arrival are lost, the stationary distribution of $\{X(t) : t \geq 0\}$, denoted by π, exists (independent of the traffic rate). Let $\{X_n : n \in \mathbb{N}\}$ denote the queue–length process embedded right after the departure of the nth customer, see [6]. Note that X_n has state-space $\{0, \ldots, N-1\}$ as after the departure of a customer the system cannot be full.

Then $\{X_n : n \in \mathbb{N}\}$ is a Markov chain with transition matrix

$$P = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-2} & 1 - \sum_{k=0}^{N-2} a_k \\ a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-2} & 1 - \sum_{k=0}^{N-2} a_k \\ 0 & a_0 & a_1 & a_2 & \cdots & a_{N-3} & 1 - \sum_{k=0}^{N-3} a_k \\ 0 & 0 & a_0 & a_1 & \cdots & a_{N-4} & 1 - \sum_{k=0}^{N-4} a_k \\ & & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_0 & 1 - a_0 \end{pmatrix}, \tag{4}$$

where

$$a_k \quad = \quad \int_0^\infty \frac{(\lambda t)^k}{k!} e^{-\lambda t} dB(t), \quad k = 0, \ldots, N-2. \tag{5}$$

In words, a_k is the probability of k Poisson arrivals during an $B(\cdot)$ distributed service time.

3 The Taylor Series Expansion Approach

In this section, we present the Taylor series approximation for the M/G/1/N queue.

Let $B(\cdot)$ have density mapping $b(\cdot)$. Let $\Theta = (a, b) \subset \mathbb{R}$, for $0 < a < b < \infty$.

(A) For $0 \leq k \leq N-2$ it holds that a_k is n-times differentiable with respect to θ on Θ.

Under **(A)** it holds that the first n derivatives of P exists. Let $P^{(k)}$ denote the kth order derivative of P with respect to θ, then it holds that

$$P^{(k)}(i,j) = \frac{d^{(k)}}{d\theta^{(k)}} P(i,j), \quad 0 \leq i, j \leq N-1,$$

or, more specifically,

$$P^{(k)} = \begin{pmatrix} a_0(k) & a_1(k) & a_2(k) & a_3(k) & \cdots & a_{N-2}(k) & -\sum_{j=0}^{N-2} a_j(k) \\ a_0(k) & a_1(k) & a_2(k) & a_3(k) & \cdots & a_{N-2}(k) & -\sum_{j=0}^{N-2} a_j(k) \\ 0 & a_0(k) & a_1(k) & a_2(k) & \cdots & a_{N-3}(k) & -\sum_{j=0}^{N-3} a_j(k) \\ 0 & 0 & a_0(k) & a_1(k) & \cdots & a_{N-4}(k) & -\sum_{j=0}^{N-4} a_j(k) \\ & & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_0(k) & -a_0(k) \end{pmatrix} \tag{6}$$

where

$$a_j(k) = \frac{d^k}{d\theta^k} a_j, \quad 0 \le j \le N-2.$$

Example 1 *Consider the M/D/1 queue with arrival rate λ and deterministic service rate c. Then, a_k is given by the probability to see k arrivals in an time interval of length c:*

$$a_k = \frac{(\lambda c)^k}{k!} e^{-\lambda c}.$$

Example 2 *Consider the M/G/1 queue with arrival rate λ and Weibull-(μ,γ)-distributed service times, where the density function is given by*

$$b(t) = \mu\gamma(\mu t)^{\gamma-1} e^{-(\mu t)^\gamma}, \quad t \ge 0.$$

Note that the mean of the Weibull-(μ,γ)-distribution is $1/\mu$ and that γ defines the shape of the distribution. In particular, $\gamma = 1$ yields the exponential distribution with rate μ. The influence of γ on the shape of the service time distribution is illustrated in more detail in Section 4.2. Then,

$$a_k = \frac{\gamma\mu^\gamma\lambda^k}{k!} \int_0^\infty t^{k+\gamma-1} e^{-\lambda t - (\mu t)^\gamma} \, dt.$$

Let π_θ denote the stationary distribution of the embedded chain, where θ denotes the parameter of interest, and denote the deviation matrix by D_θ defined by

$$D_\theta = \sum_{n=0}^\infty (P_\theta^n - \Pi_\theta),$$

where Π_θ is a square matrix with rows equal to $\pi_\theta^\top$, with $x^\top$ denoting the transposed of vector x. As shown in [9], for any finite-state aperiodic Markov chain the deviation matrix exists.

Theorem 1 *Let $\theta \in \Theta$ and let $\Theta_0 \subset \Theta$ a closed interval with θ be an interior point such that the queue is stable on Θ_0. Provided that the entries of P are n-times differentiable with respect to θ, let*

$$K_\theta(n) = \sum_{\substack{1 \le m \le n \\ 1 \le l_k \le n \\ l_1 + \cdots + l_m = n}} \frac{n!}{l_1! \cdots l_m!} \prod_{k=1}^m \left(P_\theta^{(l_k)} D_\theta \right).$$

Then it holds that

$$\pi_\theta^{(n)} = \pi_\theta \, K_\theta(n) \, .$$

Proof: We prove the theorem by induction. For $n = 1$, we have to show that $\pi'_\theta = \pi_\theta P'_\theta D_\theta$. By simple algebra, it holds that for Δ such that $\theta + \Delta \in \Theta$ that

$$\pi_{\theta+\Delta} - \pi_\theta = \pi_{\theta+\Delta}(P_{\theta+\Delta} - P_\theta) D_\theta,$$

see, e.g., [9] for a proof, which yields

$$\frac{1}{\Delta}(\pi_{\theta+\Delta} - \pi_\theta) = \pi_\theta \frac{1}{\Delta}(P_{\theta+\Delta} - P_\theta)D_\theta + (\pi_{\theta+\Delta} - \pi_\theta)\frac{1}{\Delta}(P_{\theta+\Delta} - P_\theta)D_\theta. \quad (7)$$

Element-wise differentiability of P implies that

$$\lim_{\Delta \to 0} \pi_\theta \frac{1}{\Delta}(P_{\theta+\Delta} - P_\theta)D_\theta = \pi_\theta P'_\theta D_\theta. \quad (8)$$

Since, Θ_0 is a compact neighborhood of θ, and π_θ is finite for any $\theta \in \Theta_0$, it holds that

$$\sup_{\theta \in \Theta_0} |\pi_\theta|$$

is finite. Moreover,

$$0 \leq |(\pi_{\theta+\Delta} - \pi_\theta)| \left| \frac{1}{\Delta}(P_{\theta+\Delta} - P_\theta)D_\theta \right| \leq \sup_{\theta \in \Theta_0} |\pi_\theta| \left| \frac{1}{\Delta}(P_{\theta+\Delta} - P_\theta)D_\theta \right|.$$

Element-wise differentiability of P then yields

$$\lim_{\Delta \to 0} \left| \frac{1}{\Delta}(P_{\theta+\Delta} - P_\theta)D_\theta \right| = 0,$$

which implies that the term on the righthand side of (7) tends to zero as Δ tends to zero. Hence, taking the limit for Δ to zero in (7) reduces to (8), which proves the claim for $n = 1$.

The proof for the general case follows by induction with respect to n like in conventional analysis. □

Remark 1 *The result put forward in Theorem 1 appears to be a special case of Theorem 4 in [7]. However, the latter result was established for Markov chains with general state-space and in the general case the analysis of the differentiability of π_θ requires elaborate conditions concerning the geometric ergodicity of the Markov chain with respect to a particular type of norm corresponding to the Lyapunov function of the system. This results in conditions that are often hard to check, even for simple systems. For this reason, we provide in Theorem 1 a new, simple and self- contained proof that is tailored to the class of problems studied in this paper.*

The derivatives in Theorem 1 enjoy a recursive structure in the sense that a $(k + 1)$-st order derivative is mainly constituted out of information already provided by the k-th order derivative. The following lemma provides the exact statement.

Lemma 1 *Under the conditions put forward in Theorem 1 it holds for $k < n$ that*

$$\pi_\theta^{(k+1)} = \sum_{m=0}^{k} \binom{k+1}{m} \pi_\theta^{(m)} P_\theta^{(k+1-m)} D_\theta \, .$$

Example 3 *For ease of reference we will provide in the following an explicit representation of the first derivatives of π_θ:*

$$\pi_\theta^{(1)} = \pi_\theta P_\theta^{(1)} D_\theta$$

and

$$\pi_\theta^{(2)} = \pi_\theta P_\theta^{(2)} D_\theta + 2\pi_\theta (P_\theta^{(1)} D_\theta)^2.$$

Elaborating on the recursive formula for higher order derivatives in Lemma 1, the second order derivative can be written as

$$\pi_\theta^{(2)} = \pi_\theta P_\theta^{(2)} D_\theta + 2\pi_\theta^{(1)} P_\theta^{(1)} D_\theta.$$

In the same vein, we obtain for the third order derivative

$$\pi_\theta^{(3)} \quad = \quad \pi_\theta P_\theta^{(3)} D_\theta + 3\pi_\theta^{(2)} P_\theta^{(1)} D_\theta + 3\pi_\theta^{(1)} P_\theta^{(2)} D_\theta,$$

and

$$\pi_\theta^{(4)} \quad = \quad \pi_\theta P_\theta^{(4)} D_\theta + 4\pi_\theta^{(3)} P_\theta^{(1)} D_\theta + 6\pi_\theta^{(2)} P_\theta^{(2)} D_\theta + 4\pi_\theta^{(1)} P_\theta^{(3)} D_\theta.$$

A Taylor polynomial yields an approximation and the error introduced by this approximation can be expressed by the Lagrange form of the remainder as follows

$$r_\theta(k, \Delta) = \int_0^\Delta \frac{x^k}{k!} \pi_{\theta+x}^{(k+1)} \, dx. \tag{9}$$

From a numerical point of view the above expression is rather pointless as, by Theorem 1, it holds that

$$\pi_{\theta+x}^{(k+1)} = \pi_{\theta+\Delta} K_{\theta+\Delta}(k+1)$$

which implies that for computing the remainder we already have to know the very entity we want to approximate, namely, π_η for $\eta \in [\theta, \theta + \eta]$. To overcome this drawback we will present an alternative form for the remainder term.

The basic idea is that analyticity of π_θ implies that of $\pi_\theta^{(k)}$ for all k and we can again use a Taylor series to approximate $\pi_{\theta+x}^{(k+1)}$ in (9). By doing so we initiate the Taylor series in the tail of original Taylor series, and we expect that the error of this second Taylor approximation step is negligibly small. We explain this approach in the following.

Let

$$G_\theta(k, m, \delta) = \sum_{n=0}^{m} \frac{\delta^n}{n!} \pi_\theta^{(k+1+n)}$$

denote the Taylor polynomial of order m for $\pi_\theta^{(k+1)}$, i.e.,

$$\pi_{\theta+\delta}^{(k+1)} \approx G_\theta(k, m, \delta)$$

for m sufficiently large. Inserting the above approximation into (9) yields

$$\begin{aligned} r_\theta(k,\Delta) &\approx \int_0^\Delta \frac{x^k}{k!} G_\theta(k,m,x)\,dx \\ &= \int_0^\Delta \frac{x^k}{k!} \sum_{n=0}^{m} \frac{x^n}{n!} \pi_\theta^{(k+1+n)}\,dx \\ &= \sum_{n=0}^{m} \frac{1}{k!\,n!} \pi_\theta^{(k+1+n)} \int_0^\Delta x^{k+n}\,dx \\ &= \sum_{n=0}^{m} \frac{\Delta^{k+n+1}}{k!\,n!\,(k+n+1)} \pi_\theta^{(k+1+n)}\,. \end{aligned} \tag{10}$$

We denote by

$$g_\theta(k,m,\Delta) = \sum_{n=0}^{m} \frac{\Delta^{k+n+1}}{k!\,n!\,(k+n+1)} \pi_\theta^{(k+1+n)}$$

the expression for the approximation of the remainder term obtained from (10). Provided that $|g_\theta(k,m,\Delta) - r_\theta(k,\Delta)|$ is small for m small, we will use $g_\theta(k,m,\Delta)$ in our Taylor series approach to determine the order of the polynomial that is sufficient for achieving the desired precision of the approximation. As the following theorem shows, $g_\theta(k,m,\Delta)$ is of order $c\Delta^m r_\theta(k,\Delta)$ for some small constant c. In other words, letting, for example, $\Delta = 0.1$ and choosing $m = 3$, the error introduced by our approximation of the remainder term at k is typically smaller than $10^{-(5+k)}$.

In order to state the precise statement, we introduce the norm

$$||x|| = \sum_{i=1}^{n} |x_i|$$

on $\mathbb{R}^n$.

Theorem 2 *Let $\theta \in \Theta$ be an interior point of Θ and let $\Delta > 0$ be such that $\theta + \Delta \in \Theta$. Assume that the entries of P are $(k+m+2)$-times continuously differentiable with respect to θ on Θ. Suppose that a finite constant d exists such that*

$$d \geq \sup_{x \in [\theta, \theta+\Delta]} ||\pi_x^{(k+2+m)}||,$$

then

$$|g_\theta(k,m,\Delta) - r_\theta(k,\Delta)| \leq d \frac{\Delta^{(m+k+2}}{m!\,k!\,(m+k+2)}$$

Proof: Note that the Lagrange form of the remainder for $G_\theta(k,m,\delta)$ reads

$$\int_0^\delta \frac{u^m}{m!} \pi_{\theta+u}^{(k+m+2)}\,du.$$

Applying the norm $||\cdot||$ and using the bound d, yields

$$\begin{aligned} |g_\theta(k,m,\Delta) - r_\theta(k,\Delta)| &\leq \left| \int_0^\Delta \frac{x^k}{k!} \left(\int_0^x \frac{u^m}{m!} \pi_{\theta+u}^{(k+m+2)} \, du \right) dx \right| \\ &\leq d \int_0^\Delta \frac{x^k}{k!} \left(\int_0^x \frac{u^m}{m!} du \right) dx \\ &= d \frac{\Delta^{m+k+2}}{m!\, k!\, (m+k+2)}, \end{aligned}$$

which proves the claim. □

In the numerical examples presented in the following sections, we will show that choosing $m = 2$ already yields a sufficient precision for approximating the remainder term.

Remark 2 *Taylor series like approaches for performance approximation have been studied in the literature before, see, e.g. [3, 4]. However, no* a priori *knowledge on the quality of the approximation of these approach could be established.*

Remark 3 *The Taylor series approximation developed above applies to differentiable Markov kernels. This extends the case of linear θ dependence that has been studied in the literature so far; see, for example, [1, 14, 21]. More specifically, for linear perturbations the standard assumption is that P_θ, the Markov kernel of the embedded jump chain, is of form*

$$P_\theta = \theta P + (1-\theta)\hat{P}, \quad \theta \in [0,1]. \tag{11}$$

In this case, P_θ has derivative $P'_\theta = P - \hat{P}$ and elements of Taylor series are of simple algebraical form

$$\pi_\theta^{(n)} = \pi_\theta((P-\hat{P})D_\theta)^n, \tag{12}$$

where D_θ is the deviation matrix associated with P_θ. See [1] for the denumerable state-space case and [8] for the general state-space case. Moreover, it has been shown in [9] that the remainder term can be bounded in terms of π_θ, $P, \hat{P}$ and D_θ. This algorithm has earned its merits as the bound of the remainder term is numerically efficient and due to its simple algebraical form, the Taylor series can be easily computed. Extensions of this basic algorithm to continuous time processes are provided in [7, 15].

In contrast to the model in (11), the Taylor series expansions established in this section applies to non-linear perturbations. The case of a non-linear perturbation hasn't been studied in the literature so far. This case is significantly more difficult than the linear case as in this case all higher-order derivatives of P_θ maybe different from zero. Moreover, the elements of the Taylor series in the linear case are of rather simple form, compare the expression for $\pi_\theta^{(n)}$ in Theorem 1 with (12). This stems from the fact that in the linear case all but the first derivative of P with respect to θ are zero.

4 Applications to the M/G/1/N Queue

In this section we present the two numerical examples. In the first one, we let the distribution of the service times be deterministic, with θ denoting the deterministic service time, and in the second one we assume Weibull distributed service times, where θ is the shape parameter of the distribution. As performance measure we focus on the blocking probability $\pi^*_\theta(N)$, which is due the fact that customers arrive according to a Poisson arrival stream, equal to the probability that an arriving customer is lost due to no available free waiting space. Let $E[B(\theta)]$ denote the mean service time depending on θ and assume that the arrival rate λ is independent of θ. Then, the traffic rate is given by

$$\rho(\theta) = \lambda \, E[B(\theta)].$$

Recall, that by (2), it holds that

$$\pi^*_{\theta+\Delta}(N) = \frac{\pi_{\theta+\Delta}(0) + \rho(\theta+\Delta) - 1}{\pi_{\theta+\Delta}(0) + \rho(\theta+\Delta)}. \tag{13}$$

Inserting our Taylor series expansion for $\pi_{\theta+\Delta}(0)$ into the above expression yields a functional representation of $\pi^*_{\theta+\Delta}(N)$ as function in Δ. Elaborating on (2), a similar procedure leads the a functional representation of the mean queue length and via Little's law to one of the stationary waiting time.

4.1 Parameter Sensitivity: The M/D/1/N Queue

Consider the M/D/1 queue with arrival rate λ and deterministic service time $c = \theta$. The elements of P are provided in Example 4.1.

Lemma 2 *For the M/D/1/N queue, P is infinitely often differentiable with respect to c.*

Proof: By (6) differentiability properties of P can be deduced from that of the α_j entries. By Example 1, all higher-order derivatives exist for a_j, which proves the claim. □

The following numerical examples illustrate our approach. In all numerical examples we have set $\lambda = 1$ and $N = 5$ We first illustrate the numerical behavior of $\pi^{(n)}_\theta(i)$ for $0 \leq i \leq N-1$ and $n = 1, 2, 3$, for the stationary distribution of the embedded jump chain of the M/D/1/N queue.

As a first example, we apply the Taylor series of order 1, i.e,

$$\begin{aligned} H_\theta(1,\Delta) &= \pi_\theta + \Delta \pi'_\theta \\ &= \pi_\theta + \Delta \pi_\theta P^{(1)}_\theta D_\theta, \end{aligned}$$

see Example 3. and we plot in Figure 1 the relative error given by

$$\frac{H_\theta(1,\Delta)(i) - \pi_{\theta+\Delta}(i)}{\pi_{\theta+\Delta}(i)}$$

for each element $i = 0, \ldots, 4$. In words, the relative error times the factor 100 yields the error percentage of the Taylor series approximation. For Figure 1, we let $\theta = 1$ and vary Δ by at most 10 percent of θ, i.e., $0 \leq \Delta \leq \delta = 0.1$. Note that in this situation the traffic load of the system is one but, due to the fact that we consider a loss system, stability is still guaranteed.

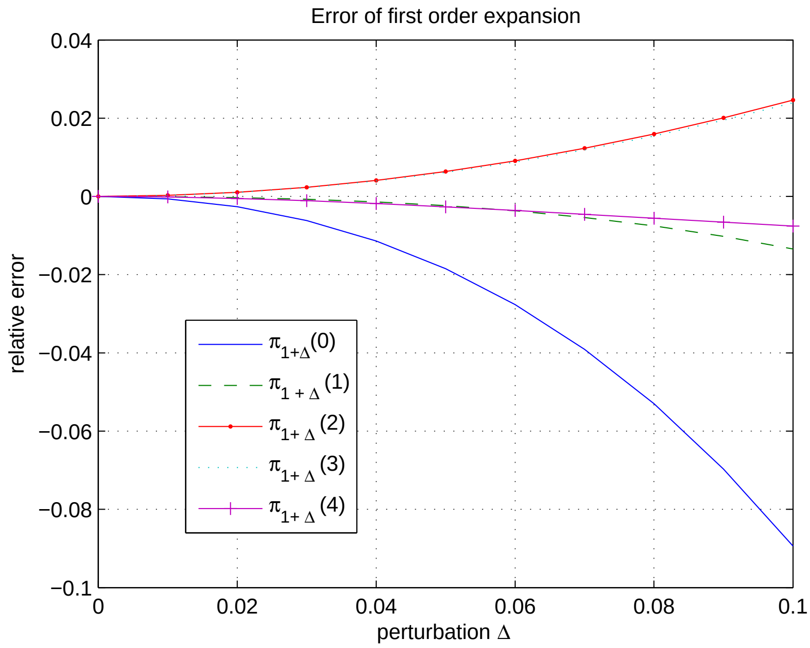

Figure 1: The relative error in predicting $\pi_{1+\Delta}$ by $H_1(1, \Delta)$ for $\rho = 1$.

We now repeat this example, but we apply a Taylor series of order 2, i.e,

$$\begin{aligned} H_\theta(2, \Delta) &= \pi_\theta + \Delta \pi_\theta^{(1)} + \frac{\Delta^2}{2} \pi_\theta^{(2)} \\ &= \pi_\theta + \Delta \pi_\theta P_\theta^{(1)} D_\theta + \frac{\Delta^2}{2} \pi_\theta P_\theta^{(2)} D_\theta + \Delta^2 \pi_\theta ((P_\theta^{(1)} D_\theta)^{(2)}, \end{aligned}$$

see Example 3, and we plot in Figure 2 the relative error given by

$$\frac{H_\theta(2, \Delta)(i) - \pi_{\theta+\Delta}(i)}{\pi_{\theta+\Delta}(i)}$$

for each element $i = 0, \ldots, 4$. As for Figure 1, we let $\theta = 1$ and vary Δ by at most 10 percent of θ, i.e., $0 \leq \Delta \leq 0.1$.

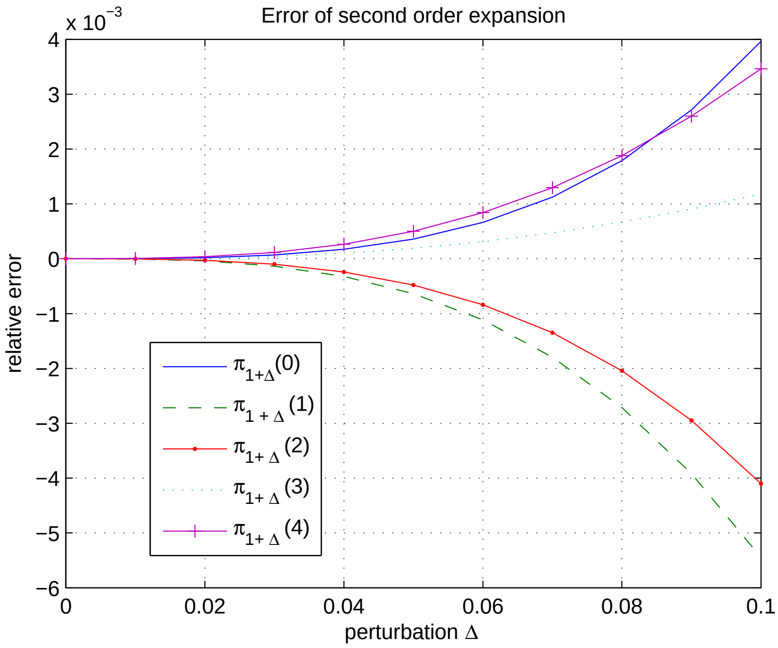

Figure 2: The relative error in predicting $\pi_{1+\Delta}$ by $H_1(2, \Delta)$ for $\rho = 1$.

We conclude this series of examples by plotting the relative error of the Taylor series expansion for order 3 in Figure 3, where

$$\begin{aligned} H_\theta(3, \Delta) &= \pi_\theta + \Delta \pi_\theta^{(1)} + \frac{\Delta^2}{2} \pi_\theta^{(2)} + \frac{\Delta^3}{6} \pi_\theta^{(3)} \\ &= \pi_\theta + \Delta \pi_\theta P_\theta^{(1)} D_\theta + \frac{\Delta^2}{2} \pi_\theta P_\theta^{(2)} D_\theta + \Delta^2 \pi_\theta (P_\theta^{(1)} D_\theta)^2 \\ &\quad + \frac{\Delta^3}{6} \left(\pi_\theta P_\theta^{(3)} D_\theta + 3\pi_\theta^{(2)} P_\theta^{(1)} D_\theta + 3\pi_\theta^{(1)} P_\theta^{(2)} D_\theta \right), \end{aligned}$$

see Example 3.

Comparing Figure 1 to Figure 3 one notes that the state i which yields the dominant relative error changes with the order of the derivative. More specifically, while for the first order derivative, $\pi'_\theta(0)$ yields the dominant error in Figure 1, the dominant error for the second order derivative stems from state $i = 4$, see Figure 2. As can be seen from the figures, a third order Taylor series yields a remarkably good approximation of the stationary probabilities through the range $\theta \pm 0.1$. It is also worth noting that the quality of approximation increases with increasing traffic load. This is of particular interest as standard numerical methods for approximation the M/G/1/N queue are restricted to the case of $\rho < 1$.

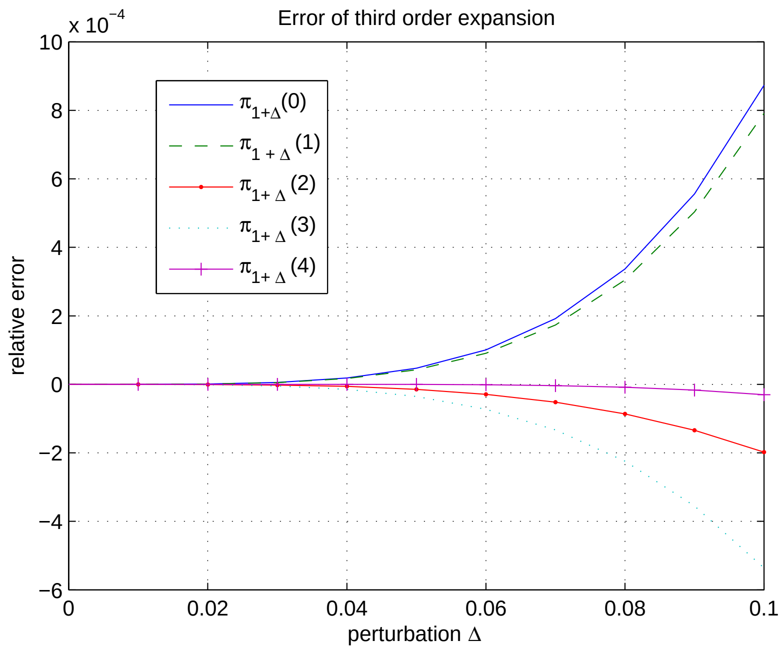

Figure 3: The relative error in predicting $\pi_{1+\Delta}$ by $H_1(3,\Delta)$ for $\rho = 1$.

We now turn to the blocking probability. Starting point is the expression for the loss probability in (13). The traffic rate is given by $\rho(\theta+\Delta) = \lambda(\theta+\Delta)$ and $\pi_{\theta+\Delta}(0)$ is approximated via a Taylor series polynomial of degree k, i.e., we replace $\pi_{\theta+\Delta}(0)$ by $H_\theta(k,\Delta)(0)$. Figure 4 shows the relative absolute error in medium traffic, i.e., $\theta = 0.5$, for $k = 2, 3$. The approximation in case of saturation, i.e., $\rho(\theta) = 1$, is illustrated in Figure 5. Finally, in Figure 6 we show the behavior of our approximation for the case of over saturation, i.e., $\rho = 1.2$.

As can be seen from the figures, the approximation yields a satisfying precision in predicting the loss probability $\pi^*_{\theta+\Delta}(N)$ as a mapping of Δ in a range of Δ being 10 percent of θ.

We conclude the discussion of the M/D/1/N queue by providing a bound on the error of the Taylor series approximation for $\pi^*_{\theta+\Delta}(N)$. Suppose that for given order k of the Taylor approximation, the error is bounded by R for any Δ such that $|\Delta| \leq \delta$, i.e., assume that

$$|\pi_{\theta+\Delta}(0) - H_\theta(k,\Delta)| \leq R,$$

see Theorem 2. Replacing $\pi_{\theta+\Delta}(0)$ in (2) or (13) by $H_\theta(k,\Delta)$ and noting that $\rho(\theta+\Delta) = \lambda\theta + \Delta\lambda$, implies thus that the true value for $\pi^*_{\theta+\Delta}(N)$ is bounded by

$$\frac{H_\theta(k,\Delta) - R + \lambda\theta + \Delta\lambda - 1}{H_\theta(k,\Delta) - R + \lambda\theta + \Delta\lambda} \leq \pi^*_{\theta+\Delta}(N) \leq \frac{H_\theta(k,\Delta) + R + \lambda\theta + \Delta\lambda - 1}{H_\theta(k,\Delta) + R + \lambda\theta + \Delta\lambda}$$

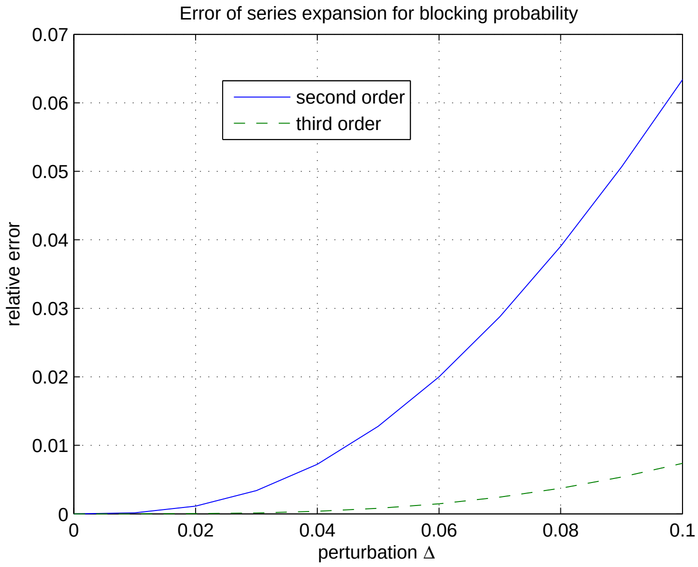

Figure 4: The relative error in predicting the loss probability in medium traffic ($\rho = 0.5, \theta = 2$).

The numerical error can thus be bounded by

$$\left| \frac{H_\theta(k,\Delta) + R + \lambda\theta + \Delta\lambda - 1}{H_\theta(k,\Delta) + R + \lambda\theta + \Delta\lambda} - \frac{H_\theta(k,\Delta) - R + \lambda\theta + \Delta\lambda - 1}{H_\theta(k,\Delta) - R + \lambda\theta + \Delta\lambda} \right|$$

$$= \frac{2R}{(H_\theta(k,\Delta) + R + \lambda\theta + \Delta\lambda)(H_\theta(k,\Delta) - R + \lambda\theta + \Delta\lambda)}$$

noting that $H_\theta(k,\Delta) \geq 0$ for all Δ, yields

$$\leq \frac{2R}{(R + \lambda\theta + \Delta\lambda)(-R + \lambda\theta + \Delta\lambda)} = \frac{2R}{\lambda^2(\theta - \Delta)^2 - R^2}$$

and from $|\Delta| \leq \delta$ it follows

$$\leq \frac{2R}{\lambda^2(\theta - \delta)^2 - R^2}.$$

We summarize our analysis in the following lemma.

Lemma 3 *Consider the M/D/1/N queue with arrival rate λ and deterministic service time θ. Suppose that for k it holds for $|\Delta| \leq \delta$ that*

$$|\pi_{\theta+\Delta}(0) - H_\theta(k,\Delta)| \leq R_\theta(k,\delta),$$

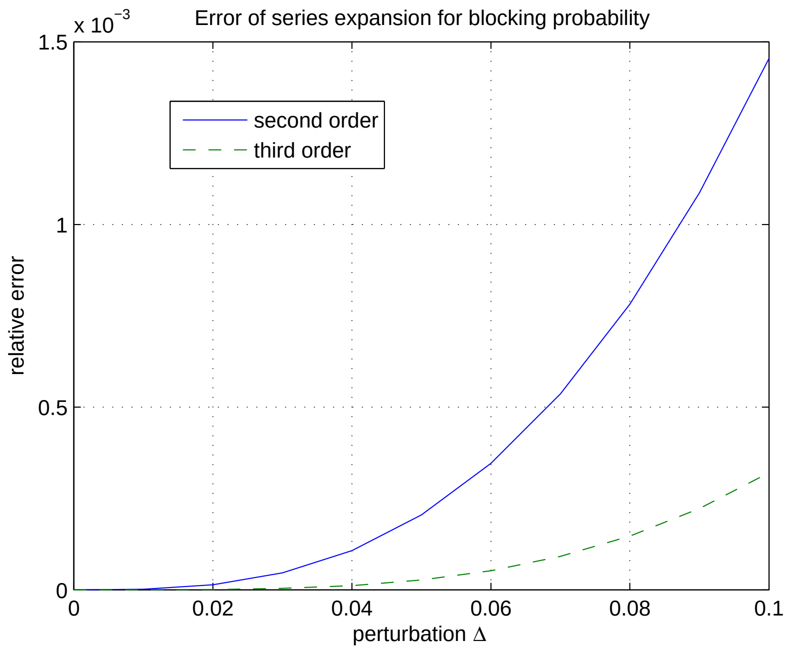

Figure 5: The relative error in predicting the loss probability in saturated traffic ($\rho = 1, \theta = 1$).

then

$$\sup_{|\Delta| \leq \delta} \left| \pi^*_{\theta+\Delta}(N) - \frac{H_\theta(k,\Delta) + \lambda(\theta+\Delta) - 1}{H_\theta(k,\Delta) + \lambda(\theta+\Delta)} \right| \leq \frac{2R_\theta(k,\delta)}{\lambda^2(\theta-\delta)^2 - (R_\theta(k,\delta))^2}.$$

We conclude this section with a discussion of the numerical bound on the error provided in Lemma 3 and Theorem 2. To this end we consider the Taylor series approximation of degree $k = 2$ for the blocking probability. Figure 7 plots the true remainder term against the approximation of the remainder term obtained from inserting the expression in Theorem 2 into the bound provided in Lemma 3, where we have chosen $m = 2$. As can be seen from Figure 7, the approximation of the remainder term yields good results for small values of Δ. For example, the approximative remainder term indicates that a Taylor series of degree 2 yields a maximal error of 5×10^{-4} in predicting blocking probability, whereas the true error is no greater than 3×10^{-4}.

4.2 Distributional Sensitivity: The M/Weibull/1/N Queue

In this section we illustrate the use of the Taylor series for the M/G/1/N queue where the service time distribution is given by the Weibull distribution, see Example 2. We let $\theta = \gamma$, which yields that for $\theta = 1$ the service distribution

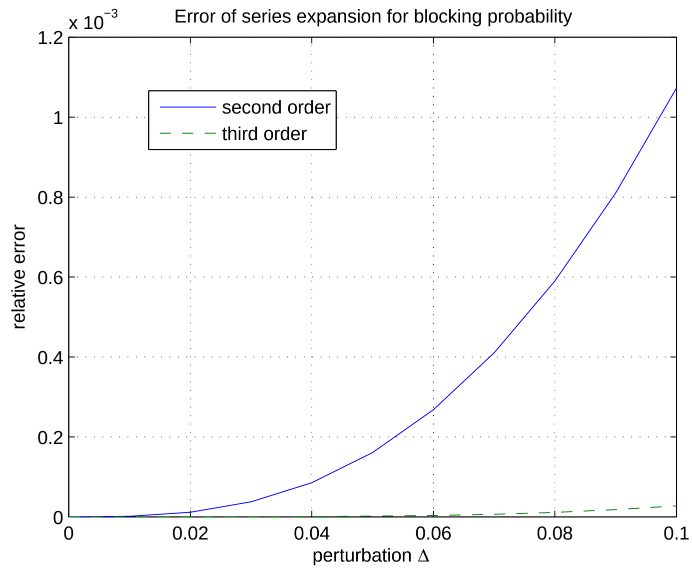

Figure 6: The relative error in predicting the loss probability in over-saturated traffic ($\rho = 1.2, \theta = 0.833$).

$B(t)$ yields the exponential distribution with mean $1/\mu$. Note that for $\theta > 1$, $B(t)$ is a heavy tailed distribution, whereas for $\theta < 1$ it becomes light tailed. Varying θ around one, provides a sensitivity analysis with respect to the shape of the distribution.

Lemma 4 *For the M/Weibull/1/N queue, P is infinitely often differentiable with respect to γ.*

Proof: By (6) differentiability properties of P can be deduced from that of the α_j entries. By Example 2, all higher-order derivatives exist for a_j, which proves the claim. □

In the following we will let $\lambda = \mu = 1$, which implies $\rho = 1$ and independent of θ. We let $|\Delta| \leq \delta = 0.1$ and we illustrate the effect a change of θ by $\pm\Delta$ has on the density in Figure 8.

Like for the M/D/1/N case, we present in the following the relative errors for predicting $\pi^*_{\theta+\Delta}$ for various order of the Taylor polynomial and traffic rate one, where we restrict Δ to positive values, i.e., $0 \leq \Delta \leq \delta$. Figure 9 illustrates the relative error of the Taylor series approximation of degree 1. The relative error of the Taylor series approximation of degree 2 is shown in Figure 10. Eventually, we show relative error of the Taylor series approximation of degree 3 is shown in Figure 11.

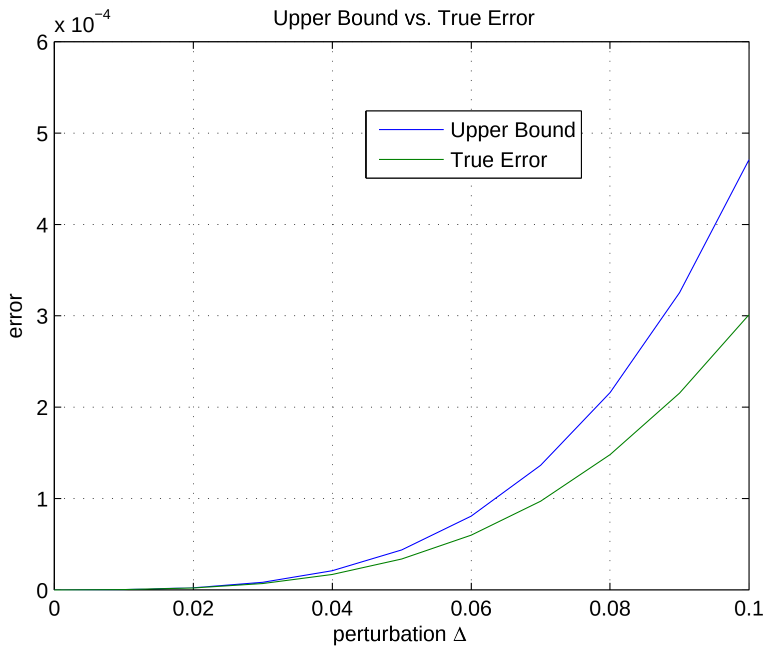

Figure 7: The remainder term vs. the bound for the remainder at $k = 2$ and $m = 2$.

Comparing the numerical results with those for the M/D/1/N case, one notes that the qualitative behavior of the relative error is similar but that the effect of perturbing the value of the mean service time in the M/G/1/N queue is better predictable than that of perturbing the shape of the service time distribution.

We conclude the example with presenting the absolute relative error for predicting the blocking probability in the M/G/1/N queue for various traffic rates, where for $\rho = 1.2$ we plot only the range $|\Delta| \leq 0.05$ for better visibility of the curve.

We now turn to the bound on the remainder term. Since for the M/W/1/N model the traffic rate $\rho = \lambda/\mu$ is independent of θ, we obtain the following adaptation of Lemma 3 to the M/W/1/N queue.

Lemma 5 *Consider the M/W/1/N queue with arrival rate λ, service rate μ, and shape parameter θ. Denote the traffic rate by $\rho = \lambda/\mu$. Suppose that for k it holds for $|\Delta| \leq \delta$ that*

$$|\pi_{\theta+\Delta}(0) - H_\theta(k, \Delta)| \leq R_\theta(k, \delta),$$

then

$$\sup_{|\Delta|\leq\delta} \left| \pi^*_{\theta+\Delta}(N) - \frac{H_\theta(k, \Delta) + \rho - 1}{H_\theta(k, \Delta) + \rho} \right| \leq \frac{2\, R_\theta(k, \delta)}{\rho^2 - (R_\theta(k, \delta))^2}.$$

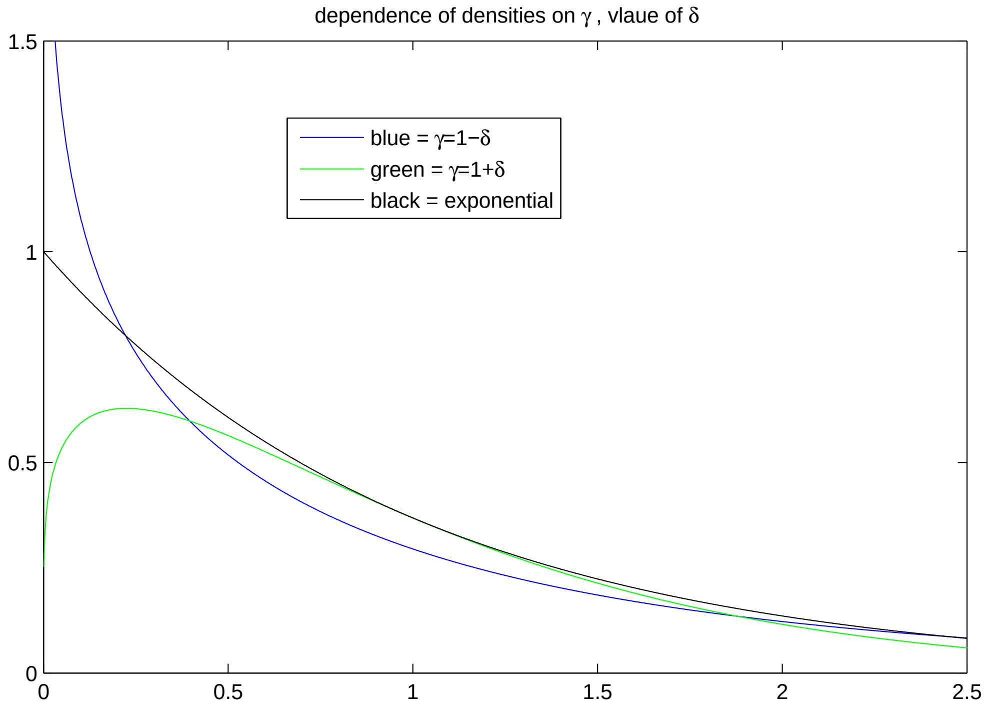

Figure 8: The extreme values of the density

We conclude this section with a discussion of the numerical bound on the error provided in Lemma 3 and Theorem 2. To this end we consider the Taylor series approximation of degree $k = 2$ for the blocking probability. Figure 15 plots the true remainder term against the approximation of the remainder term obtained from inserting the expression in Theorem 2 into the bound provided in Lemma 5, where we have chosen $m = 2$. As can be seen from Figure 15, the approximation of the remainder term yields good results for small values of Δ. For example, the approximative remainder term indicates that a Taylor series of degree 2 yields a maximal error of 4.5×10^{-5} in predicting blocking probability, whereas the true error is no greater than 3.7×10^{-5}.

5 Conclusion

We have presented a new approach to the functional approximation of finite queues. As illustrated by the numerical examples for the M/G/1/N queue, the convergence rate of the Taylor series is such that already a Taylor polynomial of

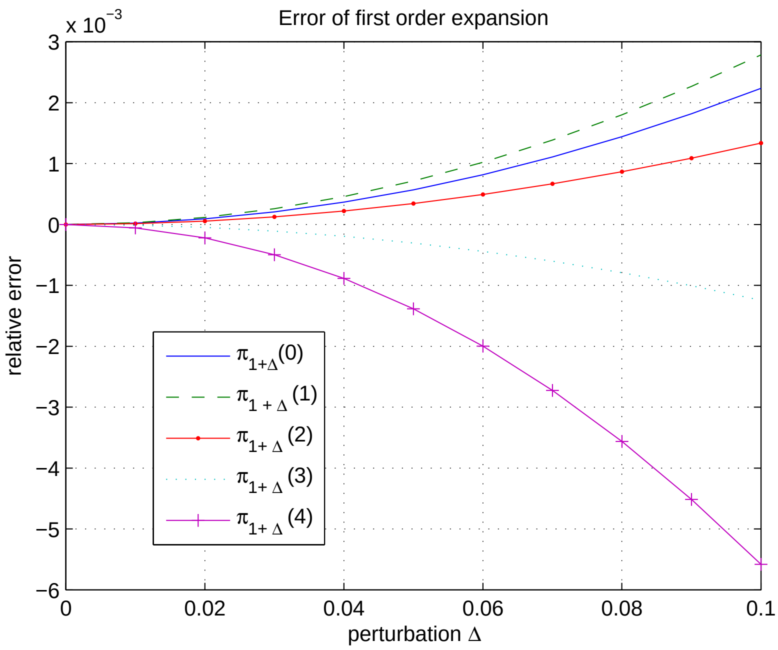

Figure 9: The relative error in predicting $\pi_{1+\Delta}$ by $H_1(1, \Delta)$ for $\rho = 1$.

degree 2 or 3 yields good numerical results. We established an approximation for the remainder term of the Taylor series that provides an efficient way of computing (approximately) the remainder term and thereby provides an algorithmic way of deciding which order of the Taylor polynomial is sufficient to achieve a desired precision of the approximation. This implies the proposed Taylor series approximation can be of practical value. Future research will be on investigating the behavior of the series expansion for multi-server queues.

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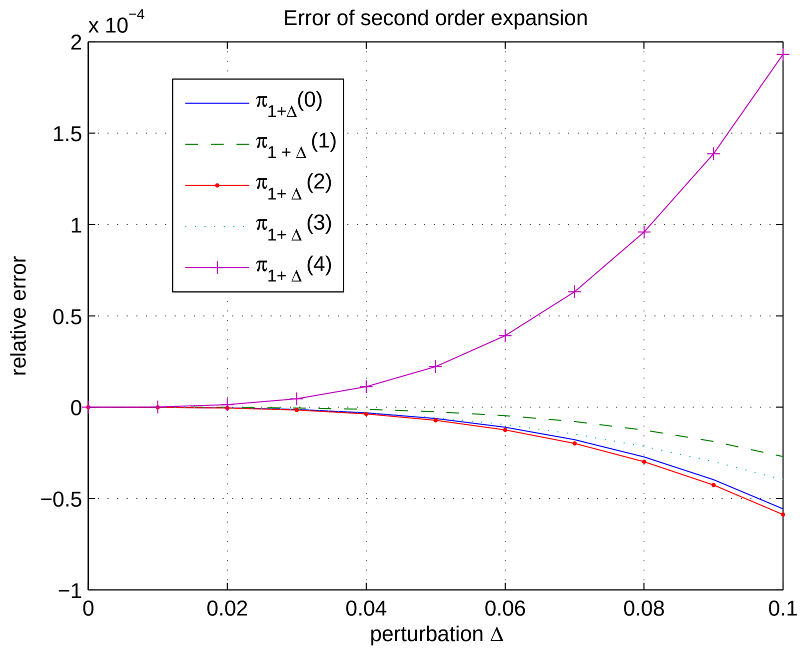

Figure 10: The relative error in predicting $\pi_{1+\Delta}$ by $H_1(2, \Delta)$ for $\rho = 1$.

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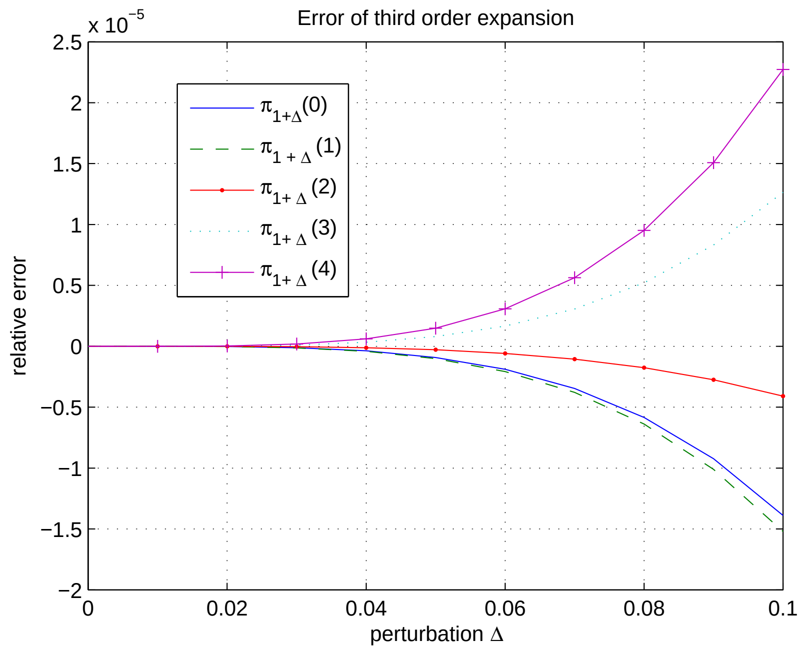

Figure 11: The relative error in predicting $\pi_{1+\Delta}$ by $H_1(3, \Delta)$ for $\rho = 1$.

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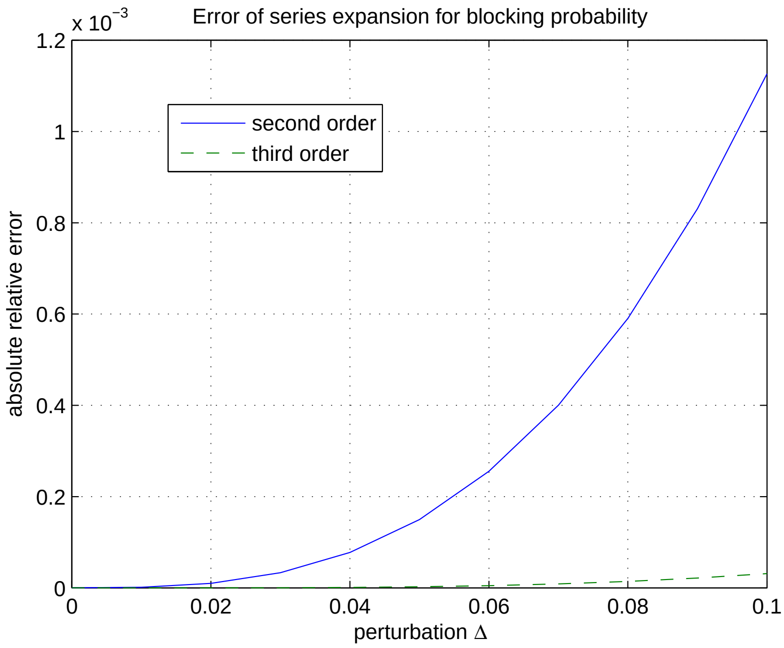

Figure 12: The relative error in predicting the loss probability in dense traffic ($\rho = 0.7$)

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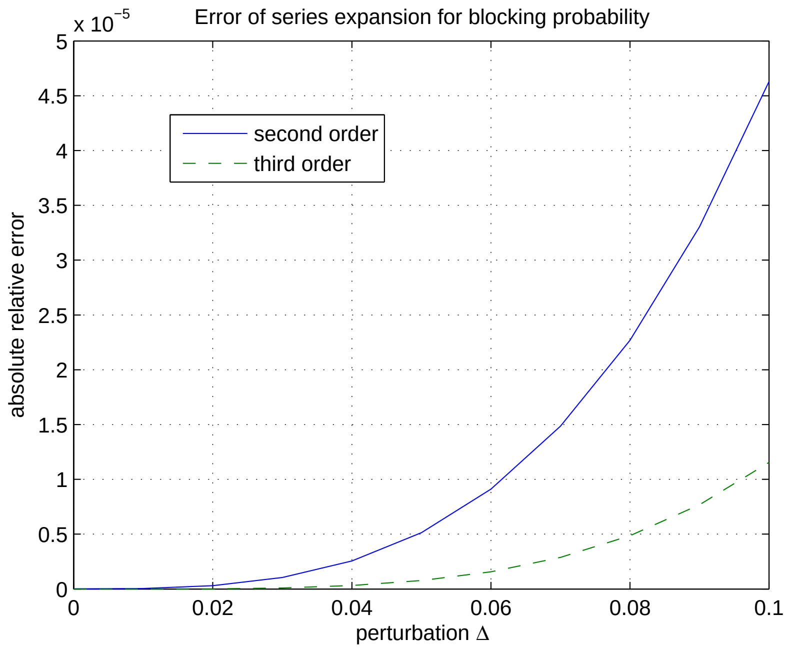

Figure 13: The relative error in predicting the loss probability in saturated traffic ($\rho = 1$)

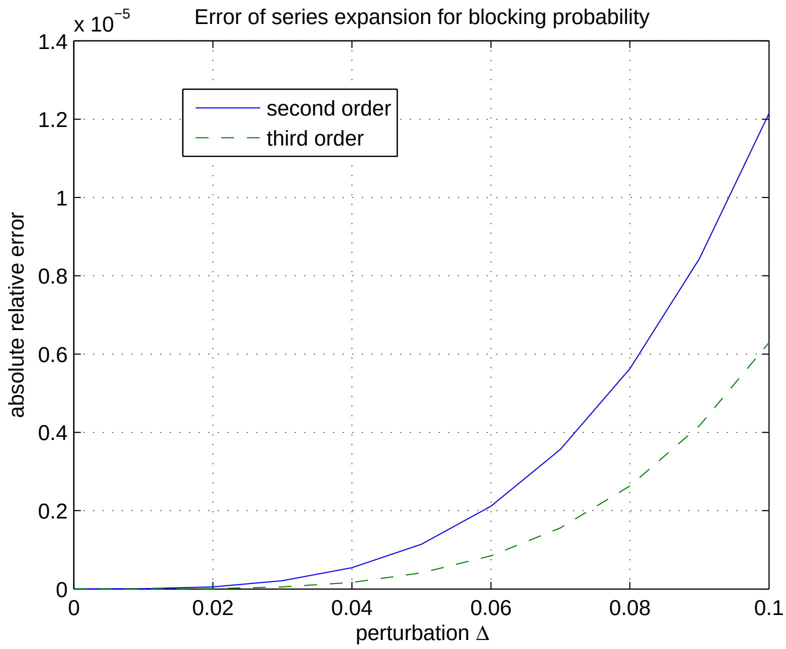

Figure 14: The relative error in predicting the loss probability in over saturated traffic ($\rho = 1.2$)

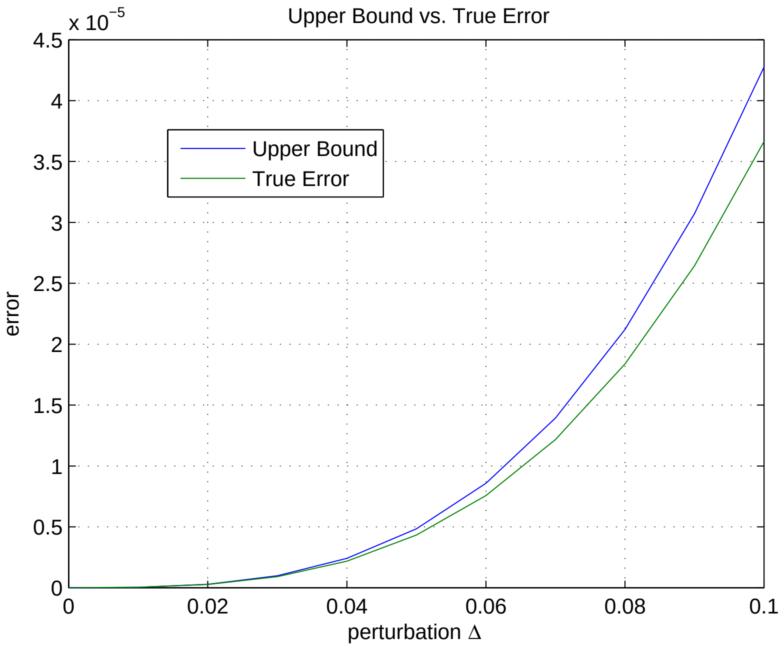

Figure 15: The remainder term vs. the bound for the remainder at $k = 2$ and $m = 2$.

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