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Proving Infinitary Normalization

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Abstract. We investigate the notion of 'infinitary strong normalization' (SN^∞), introduced in [6], the analogue of termination when rewriting infinite terms. A (possibly infinite) term is SN^∞ if along every rewrite sequence each fixed position is rewritten only finitely often. In [9], SN^∞ has been investigated as a system-wide property, i.e. SN^∞ for all terms of a given rewrite system. This global property frequently fails for trivial reasons. For example, in the presence of the collapsing rule $\mathsf{tail}(x:\sigma) \to \sigma$, the infinite term $t = \mathsf{tail}(0:t)$ rewrites to itself only. Moreover, in practice one usually is interested in SN^∞ of a certain set of initial terms. We give a complete characterization of this (more general) 'local version' of SN^∞ using interpretations into weakly monotone algebras (as employed in [9]). Actually, we strengthen this notion to *continuous* weakly monotone algebras (somewhat akin to [5]). We show that tree automata can be used as an automatable instance of our framework; an actual implementation is made available along with this paper.

1 Introduction

In first-order term rewriting a major concern is how to prove termination, or in another terminology, originating in the tradition of the λ-calculus, how to prove strong normalization (SN), i.e. the property that all rewrite sequences must end eventually in a normal form. Numerous advanced techniques and tools have been developed to prove SN, including interpretations of terms in monotone algebras [7,8] and in weakly monotone algebras [4].

Another development in term rewriting, in line with the increased attention for coalgebraic and coinductive notions and techniques, was concerned with the generalization of finitary to infinitary rewriting, where normal forms are infinite objects such as streams or infinite trees. Such trees need not be well-founded. At first sight, termination is then no longer an issue. But a notion analogous to strong normalization emerges, bearing in mind the same goal of reaching normal forms. This is infinitary normalization, SN^∞, stating that eventually always a normal form will be reached, although, depending on the chosen rewriting strategy, this may take an infinite or even a transfinitely infinite number of steps.

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The property SN^∞ has been investigated in Klop and de Vrijer [6], where it is shown that it can be rephrased as: all transfinite rewrite sequences converge, or, equivalently, along every transfinite rewrite sequence each fixed term position is rewritten only finitely often.

Zantema [9] initiated the development of proof methods for infinitary normalization by adapting the weakly monotone algebras to the infinitary setting. As a matter of fact, Zantema also studies a weaker notion than SN^∞, which he calls SN^ω, and which states that all rewrite sequences of length ω are convergent, in the sense that throughout the infinite reduction any position is rewritten at most finitely often.[1]

The properties SN^∞ and SN^ω can be viewed *locally*, as properties of individual terms or of sets of terms in a TRS, or *globally*: the entire TRS is SN^∞ (or SN^ω) if all its terms are. In [9] only the global versions are investigated, obtaining characterization theorems for the global properties SN^ω and SN^∞.

The first objective of this paper is to adapt the method of weakly monotone algebras for proving *local* versions of SN^∞ and SN^ω, which means that we can parametrize these properties to arbitrary sets S of finite or infinite terms. The gain is that the global system-wide version may fail, whereas the local version for a set S of intended terms may still succeed. Thus we are able to fine-tune the infinitary termination result for just the terms we want, removing the spoiling effect of unintended terms. Note that the global properties are special cases of the local ones. In that sense our results generalize those of [9].

The characterization theorems in [9] impose a certain continuity requirement on the algebras. However, we found that for the characterization of the stronger property SN^∞ that requirement does not suffice. In order to obtain a full characterization of SN^∞ we will strengthen the requirement to what we call below *continuous* weakly monotone algebras. They appear to be connected to an early study of continuous algebraic semantics by Goguen et al. [5].

The second contribution of this paper is the employment of tree automata to actually prove SN^∞ for a set S of infinite terms. Here the tree automaton $\mathcal{T}$ plays a double role: first, it specifies the set S of intended terms, namely as those infinite terms generated by $\mathcal{T}$, and second, it provides a 'termination certificate' for S. Moreover, and here is the bridge between this second part and the first part described above, the tree automaton $\mathcal{T}$ gives rise to a continuous weakly monotone algebra that guarantees the property SN^∞ for S. Thus the tree automata method is an 'instance' of the general set-up using continuous weakly monotone algebras.

An explicit goal of our study is finding automatable methods to establish infinitary normalization properties. Indeed, finding such a tree automaton can be automated, and we provide and discuss the actual implementation of the search process using SAT solvers. The implementation is available via the web page: `http://infinity.few.vu.nl/sni/`

[1] This property SN^ω does not imply that in ω many steps a normal form will always be reached (see Remark 2.5). Therefore "ω-convergence" would seem a more appropriate name. To keep consistency we stick here to the terminology used in [9].

2 Infinitary Rewriting

We will consider a finite or infinite term as a function on a prefix-closed subset of $\mathbb{N}^*$ taking values in a first-order signature. A *signature* Σ is a finite set of symbols each having a fixed *arity* $\sharp(f) \in \mathbb{N}$. We use $\Sigma_n := \{f \in \Sigma \mid \sharp(f) = n\}$ for the set of *n-ary function symbols*.

Let $\mathcal{X}$ be a set of symbols, called *variables*, such that $\mathcal{X} \cap \Sigma = \varnothing$. Then, a *term over* Σ is a partial map $t : \mathbb{N}^* \to \Sigma \cup \mathcal{X}$ such that the *root* is defined, $t(\epsilon) \in \Sigma \cup \mathcal{X}$, and for all $p \in \mathbb{N}^*$ and all $i \in \mathbb{N}$ we have $t(pi) \in \Sigma \cup \mathcal{X}$ if and only if $t(p) \in \Sigma_n$ for some $n \in \mathbb{N}$ and $1 \leq i \leq n$. The set of (not necessarily well-founded) terms over Σ and $\mathcal{X}$ is denoted by $Ter^\infty(\Sigma, \mathcal{X})$. Usually we will write $Ter^\infty(\Sigma)$ for the set of terms over Σ and a countably infinite set of variables, which is assumed to be fixed as underlying the definition of terms.

The set of *positions* $\mathcal{P}os(t)$ *of a term* $t \in Ter^\infty(\Sigma)$ is the domain of t, that is, the set of values $p \in \mathbb{N}^*$ such that $t(p)$ is defined: $\mathcal{P}os(t) := \{p \in \mathbb{N}^* \mid t(p) \in \Sigma \cup \mathcal{X}\}$. Note that, by the definition of terms, the set $\mathcal{P}os(t)$ is prefix closed. A term t is called *finite* if the set $\mathcal{P}os(t)$ is finite. We write $Ter(\Sigma)$ for the set of finite terms. For positions $p \in \mathcal{P}os(t)$ we use $t|_p$ to denote the *subterm of* t *at position* p, defined by $t|_p(q) := t(pq)$ for all $q \in \mathbb{N}^*$.

For $f \in \Sigma_n$ and terms $t_1, \ldots, t_n \in Ter^\infty(\Sigma)$ we write $f(t_1, \ldots, t_n)$ to denote the term t defined by $t(\epsilon) = f$, and $t(ip) = t_i(p)$ for all $1 \leq i \leq n$ and $p \in \mathbb{N}^*$. For *constants* $c \in \Sigma_0$ we simply write c instead of $c()$. We use $x, y, z, \ldots$ to range over variables. We write $s \equiv t$ for *syntactic equivalence* of terms s and t, that is, if $\forall p \in \mathbb{N}^*.\, s(p) = t(p)$ and $s \equiv_{\leq n} t$ for *syntactic equivalence up to depth* n, that is, if for all positions p with length $|p| \leq n$ we have $s(p) = t(p)$.

A *substitution* is a map $\sigma : \mathcal{X} \to Ter^\infty(\Sigma, \mathcal{X})$. For terms $t \in Ter^\infty(\Sigma, \mathcal{X})$ and substitutions σ we define $t\sigma$ as the result of replacing each $x \in \mathcal{X}$ in t by $\sigma(x)$. Formally, $t\sigma$ is defined, for all $p \in \mathbb{N}^*$, by: $t\sigma(p) = \sigma(t(p_0))(p_1)$ if there exist $p_0, p_1 \in \mathbb{N}^*$ such that $p = p_0 p_1$ and $t(p_0) \in \mathcal{X}$, and $t\sigma(p) = t(p)$, otherwise. Let $\Box$ be a fresh symbol, $\Box \notin \Sigma \cup \mathcal{X}$. A *context* C is a term from $Ter^\infty(\Sigma, \mathcal{X} \cup \{\Box\})$ containing precisely one occurrence of $\Box$. By $C[s]$ we denote the term $C\sigma$ where $\sigma(\Box) = s$ and $\sigma(x) = x$ for all $x \in \mathcal{X}$.

Dropping in the definition of terms the requirement that the number of subterms coincides with the arity of the symbols, we obtain the general notion of *labelled trees*. For trees we reuse the notation introduced above for terms.

Definition 2.1. An *infinitary term rewrite system (TRS)* is a set R of rewrite rules over a first-order signature Σ (and a set of variables $\mathcal{X}$): a *rewrite rule* is a pair $\langle \ell, r \rangle$ of terms $\ell, r \in Ter^\infty(\Sigma)$, usually written as $\ell \to r$, such that for *left-hand side* ℓ and *right-hand side* r we have $\ell(\epsilon) \notin \mathcal{X}$ and $Var(r) \subseteq Var(\ell)$.

Restriction. In this paper we restrict attention to TRSs R in which for all rules $\ell \to r \in R$ both ℓ and r are finite terms.

Definition 2.2. On the set of terms $Ter^\infty(\Sigma)$ we define a *metric* d by $d(s, t) = 0$ whenever $s \equiv t$, and $d(s, t) = 2^{-k}$ otherwise, where $k \in \mathbb{N}$ is the least length of all positions $p \in \mathbb{N}^*$ such that $s(p) \neq t(p)$.

Definition 2.3. Let R be a TRS over Σ. For terms $s, t \in Ter^{\infty}(\Sigma)$ and $p \in \mathbb{N}^*$ we write $s \to_{R,p} t$ if there exist $\ell \to r \in R$, a substitution σ and a context C with $C(p) = \Box$ such that $s \equiv C[\ell\sigma]$ and $t \equiv C[r\sigma]$. A step $s \to_{R,\epsilon} t$ is called a *root step*. We write $s \to_R t$ if there exists a position p such that $s \to_{R,p} t$.

A *transfinite rewrite sequence* (of length α) is a sequence of rewrite steps $(t_\beta \to_{R,p_\beta} t_{\beta+1})_{\beta<\alpha}$ such that for every limit ordinal $\lambda < \alpha$ we have that if β approaches λ from below (i) the distance $d(t_\beta, t_\lambda)$ tends to 0 and, moreover, (ii) the depth of the rewrite action, i.e. the length of the position p_β, tends to infinity. The sequence is called *strongly convergent* if the conditions (i) and (ii) are fulfilled for every limit ordinal $\lambda \leq \alpha$. In this case we write $t_0 \twoheadrightarrow\!\!\!\!\rightarrow_R t_\alpha$, or $t_0 \to^\alpha t_\alpha$ to explicitly indicate the length α of the sequence. Note that this ordinal will always be countable (see [6,7]). In the sequel we will use the familiar fact that countable limit ordinals have cofinality ω.

A transfinite rewrite sequence that is not strongly convergent will be called *divergent*. Note that all proper initial segments of a divergent reduction are yet strongly convergent.

Definition 2.4. A TRS R is *infinitary strongly normalizing on* $S \subseteq Ter^{\infty}(\Sigma)$, denoted $\mathsf{SN}^{\infty}_R(S)$, if every rewrite sequence starting from a term $t \in S$ is strongly convergent. We write $\mathsf{SN}^{\omega}_R(S)$ if all rewrite sequences of length $\leq \omega$ starting from a term $t \in S$ are strongly convergent. We write SN^{∞}_R shortly for $\mathsf{SN}^{\infty}_R(Ter^{\infty}(\Sigma))$, that is, infinitary normalization on all terms. Likewise SN^{ω}_R. Furthermore, the subscript R may be suppressed if it is clear from the context.

Remark 2.5. The notion SN^{ω} was introduced in [9]. Note that it does not imply that every reduction of length ω converges to a normal form, as examplified by a reduction $f(a, b) \to^{\omega} f(g^{\omega}, g^{\omega})$ in the TRS $\{a \to g(a),\ b \to g(b),\ f(x, x) \to c\}$. For the TRS R obtained by adding the extra rewrite rule $c \to c$ we will even have SN^{ω}_R without SN^{∞}_R. For this reason the terminology SN^{ω} seems a bit deceptive. We suggest to call it *ω-convergence*. For rewrite systems with rules that are left-linear and have finite left-hand sides the notions SN^{ω} and SN^{∞} coincide.

Infinitary strong normalization is related to root termination, as follows.

Definition 2.6. Let R be a TRS over Σ and $S \subseteq Ter^{\infty}(\Sigma)$. The *$\omega$-family $\mathcal{F}^{\omega}_R(S)$ of S* is the set of all subterms of $\twoheadrightarrow_R$-reducts of terms $t \in S$. Likewise the *∞-family $\mathcal{F}^{\infty}_R(S)$ of S* is the set of all subterms of $\twoheadrightarrow\!\!\!\!\rightarrow_R$-reducts of terms $t \in S$. We suppress the subscript R whenver R is clear from the context.

Definition 2.7. We call a term $t \in Ter^{\infty}(\Sigma)$ *root terminating* if t admits no rewrite sequence of length $\leq \omega$ which contains infinitely many root steps. Likewise, t is called *∞-root terminating* if t does not admit a transfinite reduction containing infinitely many root steps.

We obtain the following lemma, a refinement of Theorem 2 in [6].

Lemma 2.8. *A set of terms $S \subseteq Ter^{\infty}(\Sigma)$ is $\mathsf{SN}^{\infty}_R(S)$ if and only if all ∞-family members $t \in \mathcal{F}^{\infty}(S)$ are ∞-root terminating. Likewise we have $\mathsf{SN}^{\omega}_R(S)$ if and only if all ω-family members $t \in \mathcal{F}^{\omega}(S)$ are root terminating.*

Proof. For the 'only if'-direction, assume there exists a term $t \in \mathcal{F}^\infty(S)$ which admits a rewrite sequence $t \twoheadrightarrow$ containing infinitely many root steps. Then there exists a divergent rewrite sequence $s \twoheadrightarrow C[t] \twoheadrightarrow$ for some $s \in S$.

For the 'if'-direction, assume that $\mathsf{SN}^\infty_R(S)$ does not hold. Then there exists a rewrite sequence $\sigma : s \twoheadrightarrow$ for some $s \in S$ which is not strongly convergent. Then for some depth $d \in \mathbb{N}$ there are infinitely many rewrite steps at depth d in σ; let d be minimal with this property. There are only finitely many steps above depth d and therefore σ factors into $\sigma : s \twoheadrightarrow s' \twoheadrightarrow$ such that after s' there are no rewrite steps above depth d (but infinitely many steps at depth d). The term s' has only finitely many subterms at depth d, and by the Pigeonhole Principle one of these subterms admits a rewrite sequence containing infinitely many root steps. Hence there exists a term $t \in \mathcal{F}^\infty(S)$ which is not root terminating.

The proof for $\mathsf{SN}^\omega_R(S)$ proceeds analogously. □

3 Characterizations of Local SN^ω and Local SN^∞

We give a complete characterization of the local version of SN^∞, based on an extension of the monotone algebra approach of [9].

Definition 3.1. A Σ-algebra $\langle A, [\cdot]\rangle$ consists of a non-empty set A and for each n-ary $f \in \Sigma$ a function $[f] : A^n \to A$, the *interpretation of f*.

Let $\mathcal{A} = \langle A, [\cdot]\rangle$ be a Σ-algebra, and $\alpha : \mathcal{X} \to A$ be an assignment of variables. The interpretation of finite terms $t \in Ter(\Sigma)$ is inductively defined as follows:

$$[x]^\alpha := \alpha(x) \qquad\qquad [f(t_1, \ldots, t_n)]^\alpha := [f]([t_1]^\alpha, \ldots, [t_n]^\alpha)$$

For ground terms $t \in Ter(\Sigma, \varnothing)$ we write $[t]$ for short, since the interpretation does not depend on α. We define the interpretation $[t]$ of infinite terms t as the limit of the interpretations of finite terms converging towards t. In the sequel we assume (without loss of generality) that the signature Σ contains at least one constant symbol; in case it does not, we add one. This ensures that every infinite term is indeed the limit of a sequence of finite terms.

Let A_i, A be sets equipped with metrics. A function $f : A_1 \times \ldots \times A_n \to A$ is *continuous* if whenever for $i = 1, \ldots, n$ the sequence $a_{i,1}, a_{i,2}, \ldots$ in A_i converges with limit a_i, then $\lim_{j\to\infty} f(a_{1,j}, \ldots, a_{n,j})$ exists and is equal to $f(a_1, \ldots, a_n)$.

Definition 3.2. A Σ-algebra $\langle A, [\cdot], d\rangle$ equipped with a metric $d : A \times A \to \mathbb{R}^+_0$ is called *continuous* if:

(i) for every $f \in \Sigma$ the function $[f]$ is continuous, and
(ii) for every sequence $\{t_i\}_{i\in\mathbb{N}}$ of finite ground terms $t_i \in Ter(\Sigma, \varnothing)$ that is convergent in $Ter^\infty(\Sigma, \varnothing)$, the sequence $\{[t_i]\}_{i\in\mathbb{N}}$ is convergent.

Note that clause (ii) of Definition 3.2 is a necessary and sufficient condition for the existence of a unique continuous extension $[\cdot] : Ter^\infty(\Sigma) \to \mathcal{A}$ to (possibly) infinite terms of the interpretation $[\cdot] : Ter(\Sigma) \to \mathcal{A}$. As a matter of fact this observation motivates the definition.

Lemma 3.3. *Let $\mathcal{A} = \langle A, [\cdot] \rangle$ be a continuous Σ-algebra. Let $t \in Ter(\Sigma, \mathcal{X})$ be a finite term, and $\sigma : \mathcal{X} \to Ter^{\infty}(\Sigma, \varnothing)$ a ground substitution. We define the map $\alpha : \mathcal{X} \to A$ for all $x \in \mathcal{X}$ by $\alpha(x) = [\sigma(x)]$. Then we have $[t\sigma] = [t]^{\alpha}$.*

Proof. We use induction on the term structure of t. The case of t being a variable is trivial, hence assume $t = f(t_1, \ldots, t_n)$. For $i = 1, \ldots, n$ let $\{t_{i,j}\}_{j \in \mathbb{N}}$ be a sequence of finite terms converging towards $t_i\sigma$. Then we have:

$$\begin{aligned}
[t\sigma] &= \lim_{j\to\infty}[f(t_{1,j}, \ldots, t_{n,j})] && \text{by continuity of } [\cdot] \\
&= [f](\lim_{j\to\infty}[t_{1,j}], \ldots, \lim_{j\to\infty}[t_{n,j}]) && \text{by continuity of } f \\
&= [f]([t_1\sigma], \ldots, [t_n\sigma]) = [f]([t_1]^{\alpha}, \ldots, [t_n]^{\alpha}) = [t]^{\alpha} && \text{by IH} \qquad \square
\end{aligned}$$

Let R be a binary relation on A. A function $f : A^n \to A$ is *monotone* with respect to R if $a \mathrel{R} b$ implies $f(\ldots, a, \ldots) \mathrel{R} f(\ldots, b, \ldots)$ for every $a, b \in A$.

Definition 3.4. A *weakly monotone Σ-algebra* $\mathcal{A} = \langle A, [\cdot], \succ, \sqsupseteq \rangle$ is a Σ-algebra $\langle A, [\cdot] \rangle$ where $\succ$ is a strict partial order, and $\sqsupseteq$ a quasi-order, on A such that:

(i) $\succ$ is well-founded,
(ii) $\forall xyz. (x \succ y \sqsupseteq z \Rightarrow x \succ z)$ and $\forall xy. (x \succ y \Rightarrow x \sqsupseteq y)$ (compatibility), and
(iii) for every symbol $f \in \Sigma$ the function $[f]$ is monotone with respect to $\sqsupseteq$.

A *weakly monotone Σ-algebra with undefined elements* is a weakly monotone Σ-algebra $\mathcal{A} = \langle A, [\cdot], \succ, \sqsupseteq \rangle$ with a set $\Omega \subseteq A$ of *undefined elements* for which:

(iv) for every $b \in \Omega$ and $a \in A \setminus \Omega$ we have $b \succ a$ (maximality), and
(v) for every $f \in \Sigma$ and $b \in \Omega$ we have $[f](\ldots, b, \ldots) \in \Omega$ (strictness).

All of the results in this paper remain valid if instead of requiring $\succ$ to be a strict partial order and $\sqsupseteq$ a quasi-order we allow arbitrary binary relations fulfilling conditions (i)–(v) of Definition 3.4.

Remark 3.5. The reason to consider weakly monotone algebras with more than just one undefined element is the following. For every TRS R, we want to be able to build a continuous weakly monotone algebra from the term algebra with carrier-set $Ter^{\infty}(\Sigma)$ by interpreting the terms t with $\mathsf{SN}^{\infty}_{R}(\{t\})$ by themselves, and the other terms by suitably chosen undefined objects. However, by just dropping the terms t that are not SN^{∞}_{R}, and replacing them by a single undefined element usually a continuous algebra is not obtained.

For example, let $\Sigma = \{\mathsf{I}, \mathsf{J}, \mathsf{c}\}$, where I, J are unary function symbols and c a constant. Let R be the (orthogonal) TRS over Σ with the rules $\mathsf{I}(x) \to x$ and $\mathsf{J}(x) \to x$. Here the terms $t \in Ter^{\infty}(\Sigma)$ with $\mathsf{SN}^{\infty}_{R}(\{t\})$ are precisely the finite terms, the terms $t \in Ter(\Sigma)$. Now suppose that $\mathcal{A} = \langle A, [\cdot], d_A, \succ, \sqsupseteq \rangle$ is a continuous, weakly monotone algebra with $A \supseteq Ter(\Sigma)$, an interpretation $[\cdot] : \Sigma \to A$ with the property that $[f]([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$ for all $f \in Ter(\Sigma)$, and d_A an extension of the metric in Definition 2.2. Then we find that $A \setminus Ter(\Sigma)$ contains more than one element (and in fact uncountably many elements). Note that for the induced interpretation function

$[\cdot] : Ter^\infty(\Sigma) \to A$ it holds that $[t] = t$ for all $t \in Ter(\Sigma)$. We find that $[\mathsf{I}^\omega] = [\mathsf{I}(\mathsf{I}(\mathsf{I}(\ldots)))] = [\lim \mathsf{I}^n(x)] = \lim[\mathsf{I}^n(x)] = \lim \mathsf{I}^n(x) \in A \setminus Ter(\Sigma)$, and similarly, $[\mathsf{J}^\omega] = \lim \mathsf{J}^n(x) \in A \setminus Ter(\Sigma)$. From this we conclude that the interpretations $[\mathsf{I}^\omega]$ and $[\mathsf{J}^\omega]$ of the infinite terms I^ω and J^ω are different elements in $A \setminus Ter^\infty(\Sigma)$: $[\mathsf{I}^\omega] \neq [\mathsf{J}^\omega]$ follows from $d_A([\mathsf{I}^\omega], [\mathsf{J}^\omega]) = d_A(\lim \mathsf{I}^n(x), \lim \mathsf{J}^n(x)) = \lim d_A(\mathsf{I}^n(x), \mathsf{J}^n(x)) = \lim d(\mathsf{I}^n(x), \mathsf{J}^n(x)) = 1$.

Definition 3.6. Let $\mathcal{A} = \langle A, [\cdot], \succ, \sqsupseteq \rangle$ be a weakly monotone Σ-algebra with undefined elements Ω.

(i) A set $S \subseteq Ter^\infty(\Sigma, \varnothing)$ is called *defined w.r.t.* Ω if, for all $s \in S$, $[s] \notin \Omega$.
(ii) A TRS R over Σ is called *(weakly) decreasing w.r.t.* Ω if for all $\ell \to r \in R$ and every assignment $\alpha : \mathcal{X} \to A$, $[\ell]^\alpha \notin \Omega$ implies $[\ell]^\alpha \succ [r]^\alpha$ ($[\ell]^\alpha \sqsupseteq [r]^\alpha$).

Theorem 3.7. *Let R be a TRS over Σ, and $S \subseteq Ter^\infty(\Sigma, \varnothing)$. Then the following statements are equivalent:*

(i) $\mathsf{SN}^\omega_R(S)$.
(ii) There exists a continuous weakly monotone Σ-algebra $\mathcal{A} = \langle A, [\cdot], d, \succ, \sqsupseteq \rangle$ with a set Ω of undefined elements such that S is defined w.r.t. Ω, and R is decreasing with respect to Ω.

Proof. For (i) $\Rightarrow$ (ii) assume that $\mathsf{SN}^\omega_R(S)$ holds. We define $\mathcal{A} := \langle A, [\cdot], d, \succ, \sqsupseteq \rangle$ with $A := Ter^\infty(\Sigma, \varnothing)$, equipped with the metric d on A from Definition 2.2, and let $\Omega := A \setminus \mathcal{F}^\omega(S)$ be the set of undefined elements. We define the relations $\succ := (\to_{R,\epsilon} \cdot \to^*) \cap (\mathcal{F}^\omega(S) \times \mathcal{F}^\omega(S))$ and $\sqsupseteq := \to^*$, extended by $s \succ t$ for all $s \in \Omega$, $t \in \mathcal{F}^\omega(S)$ and $s \sqsupseteq t$ for all $s \in \Omega$, $t \in \mathcal{A}$. The interpretation $[\cdot]$ is defined for all $f \in \Sigma$ by $[f](t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$.

Clearly $\mathcal{A}$ is a continuous Σ-algebra; we check that $\mathcal{A}$ is a weakly monotone Σ-algebra with undefined elements Ω. Assume that $\succ$ would not be well-founded. Then there exists a term $t \in \mathcal{F}^\omega(S)$ admitting an ω-rewrite sequence containing infinitely many root steps, contradicting $\mathsf{SN}^\omega_R(S)$. The compatibility $\succ \cdot \sqsupseteq \subseteq \succ$ and $\succ \subseteq \sqsupseteq$ holds by definition. For every $b \in \Omega$ and $a \in A \setminus \Omega$ we have $b \succ a$ by definition. Furthermore $b \in \Omega$ implies $[f](\ldots, b, \ldots) = f(\ldots, b, \ldots) \in \Omega$, since the family $\mathcal{F}^\omega(S)$ is closed under subterms. For monotonicity with respect to $\sqsupseteq$, we consider $f \in \Sigma$ and $s, t \in \mathcal{A}$ with $s \sqsupseteq t$. If $s \in \Omega$ then $[f](\ldots, s, \ldots) \in \Omega \sqsupseteq [f](\ldots, t, \ldots)$. If $s \in \mathcal{F}^\omega(S)$, then $[f](\ldots, s, \ldots) \sqsupseteq [f](\ldots, t, \ldots)$ as a consequence of the closure of rewriting $\to^*$ under contexts.

We check the remaining requirements of the theorem. For all $s \in S$ we have $[s] \notin \Omega$ by definition. Consider $\ell \to r \in R$ and $\alpha : \mathcal{X} \to A_I$ such that $[\ell]^\alpha \notin \Omega$. Then $[\ell]^\alpha \in \mathcal{F}^\omega(S)$ and hence $\alpha(x) \in \mathcal{F}^\omega(S)$ for all $x \in Var(\ell)$. Therefore we obtain $[\ell]^\alpha \equiv \ell\alpha \to_{R,\epsilon} r\alpha \equiv [r]^\alpha$ and $[r]^\alpha \in \mathcal{F}^\omega(S)$, hence $[\ell]^\alpha \succ [r]^\alpha$.

For (ii) $\Rightarrow$ (i) assume that $\mathcal{A} := \langle A, [\cdot], \succ, \sqsupseteq \rangle$ and Ω fulfilling the requirements of the theorem are given. We show the following auxiliary lemmas:

$$\forall s, t \in Ter^\infty(\Sigma).\ [s] \notin \Omega \wedge s \to t \Rightarrow [t] \notin \Omega \wedge [s] \sqsupseteq [t] \tag{$*$}$$

$$\forall s.\ [s] \notin \Omega \Rightarrow \forall t \in \mathcal{F}^\omega(s).\ [t] \notin \Omega \tag{$**$}$$

Let $s, t \in Ter^{\infty}(\Sigma)$ with $[s] \notin \Omega$ and $s \to t$. There exist a context C, a rule $\ell \to r \in R$ and a substitution σ such that $s \equiv C[\ell\sigma] \to C[r\sigma] \equiv t$. By Lemma 3.3 together with the assumptions we obtain $[\ell\sigma] = [\ell]^{\alpha} \succ [r]^{\alpha} = [r\sigma]$ where the map $\alpha : \mathcal{X} \to A$ is defined by $\alpha(x) = [\sigma(x)]$ for all $x \in \mathcal{X}$. Since $\succ \subseteq \sqsupseteq$ and $[f]$ is monotone with respect to $\sqsupseteq$ for $f \in \Sigma$, we obtain $[s] \sqsupseteq [t]$. Furthermore $[t] \notin \Omega$, otherwise $[t] \in \Omega \succ [s] \sqsupseteq [t]$ and hence $[t] \succ [t]$, contradicting well-foundedness of $\succ$. We obtain $(**)$ by induction together with 'monotonicity' of Ω.

Assume $\mathsf{SN}^{\omega}_{R}(S)$ would not hold. By Lemma 2.8 there exists a term $t_0 \in \mathcal{F}^{\omega}(S)$ which admits an ω-reduction $t_0 \to t_1 \to \dots$ containing infinitely many root steps. Then $t_0 \in \mathcal{F}^{\omega}(s)$ for some $s \in S$ and by assumption $[s] \notin \Omega$, hence by $(**)$ we obtain $t_i \notin \Omega$ for all $i \in \mathbb{N}$. Furthermore by $(*)$ if follows $[t_i] \sqsupseteq [t_{i+1}]$ for all $i \in \mathbb{N}$. Moreover for root steps $t_i \to_{R,\epsilon} t_{i+1}$ we get $[t_i] \succ [t_{i+1}]$ since then the context C in the proof of $(*)$ is empty. As a consequence we have infinitely often a strict decrease $\succ$ in the sequence $[t_0] \sqsupseteq [t_1] \dots$, and by applying $\succ \cdot \sqsupseteq \subseteq \succ$ we can remove all $\sqsupseteq$ between them; giving rise to an infinite decreasing $\succ$-sequence, contradicting well-foundedness of $\succ$. $\square$

Remark 3.8. A close inspection of the above proof yields that for Theorem 3.7 the requirement on the algebra to be continuous can be weakened. It suffices to require that for every infinite ground term t the sequence $[\mathsf{trunc}(t, n)]$ converges for $n \to \infty$. Here $\mathsf{trunc}(t, n)$ stands for the *truncation of t at depth n* defined for all $p \in \mathbb{N}^*$ by $\mathsf{trunc}(t, n)(p)$ is $t(p)$ if $|p| < n$, $\bot$ if $|p| = n$, and undefined, otherwise; where $\bot$ is an arbitrary, fixed constant symbol from the signature Σ.

However, we emphasise that for the characterization of $\mathsf{SN}^{\infty}_{R}(S)$ this weaker condition is *not* sufficient. Continuity of $[\cdot] : Ter^{\infty}(\Sigma) \to A$ is essential for the correctness of Theorem 3.10. It guarantees that for the limit steps in transfinite rewrite sequences, the limit of the interpretations coincides with the interpretation of the limit term.

We note that the weaker continuity condition used in [9, Theorem 3] does not suffice; see Example 3.9. Strengthening the condition to full continuity of the interpretation mapping would validate the theorem.

Example 3.9. We consider a TRS R which is SN^{ω} but not SN^{∞}. Interestingly, although the TRS is SN^{ω}, we display a term of which a normal form cannot be reached in ω many steps. Let R be the TRS consisting of the following rules:

$$f(x, x) \to f(A, B) \qquad\qquad A \to s(A) \qquad\qquad B \to s(B)\ .$$

It is not difficult to verify that R is indeed SN^{ω}, but SN^{∞} does not hold:

$$f(A, B) \to f(s(A), B) \to f(s(A), s(B)) \twoheadrightarrow f(s^{\omega}, s^{\omega}) \to f(A, B) \to \dots\ .$$

Note that the TRS R forms a counterexample to [9, Theorem 3], as the following Σ-algebra $\mathcal{A}$ fulfills all requirements of the theorem, but SN^{∞} does not hold. We choose the Σ-algebra $\mathcal{A} = \{A, B, F, a, b, f\}$ with $A \succ a$, $B \succ b$, $F \succ f$ and $\sqsupseteq := \succ \cup =$. The interpretation $[\cdot]$ is defined as follows:

$$[A] = A$$
$$[B] = B$$
$$[s](A \mid a) = a \qquad [s](B \mid b) = b \qquad [s](F \mid f) = f$$
$$[f](A \mid a, B \mid b) = f \qquad [f](\text{otherwise}) = F$$

where $\mid$ denotes ‘ or’ and as truncation symbol c we chose $c := A$. Furthermore, for the metric we choose $d(x, y) = 0$ if $x = y$ and 1 otherwise. Then for all variable interpretations $\alpha : \mathcal{X} \to \mathcal{A}$ we have:

$$[f(x, x)]^\alpha = F > f = [f(A, B)]^\alpha$$
$$[A]^\alpha = A > a = [s(A)]^\alpha$$
$$[B]^\alpha = B > b = [s(B)]^\alpha \,.$$

Thus all rules are strictly decreasing. It is straightforward to verify that all functions $[g]$ are continuous, for every infinite ground term t the sequence $[\mathsf{trunc}(t, n)]$ converges (with limit in $\mathcal{A}$) for $n \to \infty$, and for every descending sequence $a_1 \sqsupseteq a_2 \sqsupseteq \cdots$ for which $\lim_{n\to\infty} a_i$ exists we have $a_1 \sqsupseteq \lim_{n\to\infty} a_i$.

Let A be a set equipped with a metric d and let $\sqsupseteq$ be a binary relation on A. We call the relation $\sqsupseteq$ *compatible with limits* if for every converging sequence $\{a_i\}_{i\in\mathbb{N}}$ with $a_0 \sqsupseteq a_1 \sqsupseteq \ldots$ we have $a_0 \sqsupseteq \lim_{i\to\infty} a_i$.

Theorem 3.10. *Let R be a TRS over Σ and $S \subseteq Ter^\infty(\Sigma, \varnothing)$. Then the following statements are equivalent:*

(i) $\mathsf{SN}^\infty_R(S)$.

(ii) There exists a continuous weakly monotone Σ-algebra $\mathcal{A} = \langle A, [\cdot], d, \succ, \sqsupseteq\rangle$ with a set Ω of undefined elements such that S is defined w.r.t. Ω, R is decreasing with respect to Ω, and $\sqsupseteq$ is compatible with limits.

Proof. We give the crucial steps for both directions. The remainder of the proof proceeds analogously to the proof of Theorem 3.7.

For (i) $\Rightarrow$ (ii) assume that $\mathsf{SN}^\infty_R(S)$ holds. We define $\mathcal{A} := \langle A, [\cdot], d, \succ, \sqsupseteq\rangle$ with $A := Ter^\infty(\Sigma, \varnothing)$, d the metric from Definition 2.2, and $\Omega := A \setminus \mathcal{F}^\infty(S)$; we define the relations $\succ := (\to_{R,\epsilon} \cdot \twoheadrightarrow) \cap (\mathcal{F}^\infty(S) \times \mathcal{F}^\infty(S))$, $\sqsupseteq := \twoheadrightarrow \cap (\mathcal{F}^\infty(S) \times \mathcal{F}^\infty(S))$, extended by $s \succ t$ for all $s \in \Omega$, $t \in \mathcal{F}^\infty(S)$ and $s \sqsupseteq t$ for all $s \in \Omega$, $t \in \mathcal{A}$. The interpretation $[\cdot]$ is defined for all $f \in \Sigma$ by $[f](t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$. Consider a sequence $a_0 \sqsupseteq a_1 \sqsupseteq \ldots$ with $a_0 \in \mathcal{F}^\infty(S)$. Then $a_0 \twoheadrightarrow a_1 \twoheadrightarrow \ldots$ by definition and by $\mathsf{SN}^\infty_R(S)$ we obtain that $a := \lim_{i\to\infty} a_i$ exists, $a_0 \twoheadrightarrow a$ and $a_0 \sqsupseteq a$. Hence $\sqsupseteq$ is compatible with limits.

For the implication (ii) $\Rightarrow$ (i), the crucial step is to show that $s \twoheadrightarrow t$ implies $s \sqsupseteq t$. We use induction on the length of the rewrite sequence $s \to^\alpha t$. Note that the length α of a reduction is a countable ordinal, c.f. [6]. For $\alpha = \beta + 1$ we obtain $s \sqsupseteq t$ by induction hypothesis together with $(*)$ from the proof of Theorem 3.7. Assume that α is a (countable) limit ordinal. Then there exists a non-decreasing sequence $\{\beta_i\}_{i\in\mathbb{N}}$ of ordinals $\beta_i < \alpha$ such that $\alpha = \lim_{i\to\infty} \beta_i$. Let s_γ denote the term before the γ-th rewrite step in $s \to^\alpha t$. Then $s \twoheadrightarrow s_{\beta_1} \twoheadrightarrow s_{\beta_2} \ldots$

and $t = \lim_{i\to\infty} s_{\beta_i}$. Hence by induction hypothesis $s \sqsupseteq s_{\beta_1} \sqsupseteq s_{\beta_2} \ldots$; and by compatibility of $\sqsupseteq$ with limits we obtain $s \sqsupseteq t$. This gives us a handle for limit steps; the rest of the proof is analogous to the proof of Theorem 3.7. □

Finally, we generalize the Theorems 3.7 and 3.10 together with the concept of 'root termination' allowing for simpler, stepwise proofs of $\mathsf{SN}^{\infty}_{R}(S)$. This facility is incorporated in our tool. The following definition and theorem allow for modular proofs of SN^{∞} and root termination of infinite terms. This is reminiscent to modular proofs of finitary root termination [1] (the dependency pairs method).

Definition 3.11. Let R_1 and R_2 be TRS over Σ, and $S \subseteq \mathit{Ter}^{\infty}(\Sigma)$. We say that R_1 *is ∞-root terminating relative to R_2 on S*, denoted $\mathsf{RT}^{\infty}_{R_1/R_2}(S)$, if no $s \in S$ admits a $\rightarrow_{R_1,\epsilon} \cup \rightarrow_{R_2}$-reduction containing infinitely many $\rightarrow_{R_1,\epsilon}$-steps.

We say R_1 *is root terminating relative to R_2 on S*, denoted $\mathsf{RT}^{\omega}_{R_1/R_2}(S)$, if the condition holds for rewrite sequences of length $\leq \omega$.

The following lemma is a direct consequence of Lemma 2.8 and Definition 3.11.

Lemma 3.12. *(i)* $\mathsf{SN}^{\infty}_{R}(S) \Leftrightarrow \mathsf{RT}^{\infty}_{R/R}(\mathcal{F}^{\infty}(S))$*; (ii)* $\mathsf{SN}^{\omega}_{R}(S) \Leftrightarrow \mathsf{RT}^{\omega}_{R/R}(\mathcal{F}^{\omega}(S))$.

For proving $\mathsf{SN}^{\infty}_{R}(S)$ using Theorem 3.10 we have to make all rules in R decreasing at once. For practical purposes it is often desirable to prove $\mathsf{SN}^{\infty}_{R}(S)$ stepwise, by repeatedly removing rules until no top-rules remain, that is, $\mathsf{RT}^{\infty}_{\varnothing/R}(\mathcal{F}^{\infty}(S))$ trivially holds. The following theorem enables us to do this, we can remove all decreasing rules, as long as the remaining rules are weakly decreasing.

Theorem 3.13. *Let $R_1 \subseteq R_2$, $R_1' \subseteq R_2$ be TRS over Σ, and $S \subseteq \mathit{Ter}^{\infty}(\Sigma, \varnothing)$. Let $\mathcal{A} = \langle A, [\cdot], d, \succ, \sqsupseteq\rangle$ be a continuous weakly monotone Σ-algebra with a set Ω of undefined elements such that S is defined w.r.t. Ω and it holds:*

(i) $R_1 \cup R_2$ *is weakly decreasing with respect to Ω, and*
(ii) R_1' *is decreasing with respect to Ω.*

Then $\mathsf{RT}^{\omega}_{R_1/R_2}(\mathcal{F}^{\omega}_{R_2}(S))$ *implies* $\mathsf{RT}^{\omega}_{(R_1\cup R_1')/R_2}(\mathcal{F}^{\omega}_{R_2}(S))$. *If additionally $\sqsupseteq$ is compatible with limits, then* $\mathsf{RT}^{\infty}_{R_1/R_2}(\mathcal{F}^{\infty}_{R_2}(S))$ *implies* $\mathsf{RT}^{\infty}_{(R_1\cup R_1')/R_2}(\mathcal{F}^{\infty}_{R_2}(S))$.

Proof. Minor modification of the proofs of Theorem 3.7 and 3.10, respectively. □

4 Tree Automata

We now come to the second contribution of our note, consisting of an application of tree automata to prove infinitary strong normalization, SN^{∞}, and a connection of tree automata with the algebraic framework treated above. For the notion of tree automata the reader is referred to [2]. We repeat the main definitions, for the sake of completeness, and to fix notations.

Definition 4.1. A *(finite nondeterministic top-down) tree automaton $\mathcal{T}$ over a signature Σ* is a tuple $\mathcal{T} = \langle Q, \Sigma, I, \Delta\rangle$ where Q is a finite set of *states*, disjoint from Σ; $I \subseteq Q$ is a set of *initial states*, and $\Delta \subseteq \mathit{Ter}(\Sigma \cup Q, \varnothing)^2$ is a ground term rewriting system over $\Sigma \cup Q$ with rules, or *transitions*, of the form:

$$q \to f(q_1, \ldots, q_n)$$

for n-ary $f \in \Sigma$, $n \geq 0$, and $q, q_1, \ldots, q_n \in Q$.

We define the notion of 'run' of an automaton on a term. For terms containing variables, we assume that a map $\alpha : \mathcal{X} \to \mathbf{2}^Q$ is given, so that each variable $x \in \mathcal{X}$ can be generated by any state from $\alpha(x)$.

Definition 4.2. Let $\mathcal{T} = \langle Q, \Sigma, I, \Delta \rangle$ be a tree automaton. Let $t \in \mathit{Ter}^\infty(\Sigma, \mathcal{X})$ be a term, $\alpha : \mathit{Var}(t) \to \mathbf{2}^Q$ a map from variables to sets of states, and $q \in Q$. Then a *q-run of $\mathcal{T}$ on t with respect to α* is a tree $\rho : \mathcal{P}\mathit{os}(t) \to Q$ such that:

(i) $\rho(\epsilon) = q$, and
(ii) $\rho(p) \to t(p)(\rho(p1), \ldots, \rho(pn)) \in \Delta$ for all $p \in \mathcal{P}\mathit{os}(t)$ with $t(p) \in \Sigma_n$, and
(iii) $\rho(p) \in \alpha(t(p))$ for all $p \in \mathcal{P}\mathit{os}(t)$ with $t(p) \in \mathcal{X}$.

We define $Q_\alpha(t) := \{q \in Q \mid \text{there exists a } q\text{-run of } \mathcal{T} \text{ on } t \text{ with respect to } \alpha\}$.

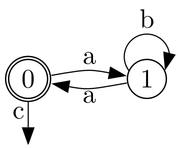
For ground terms t the above notions are independent of α. Then we say $\mathcal{T}$ *has a q-run on a term t* and write $Q(t)$ in place of $Q_\alpha(t)$. Moreover, we say that an automaton $\mathcal{T}$ *generates* a ground term t if $\mathcal{T}$ has a q-run on t such that $q \in I$. The language of an automaton is the set of ground terms it generates.

Definition 4.3. The *language* $\mathcal{L}(\mathcal{T})$ of a tree automaton $\mathcal{T}$ is defined by:

$$\mathcal{L}(\mathcal{T}) := \{t \in \mathit{Ter}^\infty(\Sigma, \varnothing) \mid Q(t) \cap I \neq \varnothing\} .$$

$\mathcal{T}$ is called *complete* if it generates all ground terms, i.e. if $\mathcal{L}(\mathcal{T}) = \mathit{Ter}^\infty(\Sigma, \varnothing)$.

Example 4.4. Consider the tree automaton $\mathcal{T} = \langle Q, \Sigma, I, \Delta \rangle$ with $Q := \{0, 1\}$, $I := \{0\}$, and with Δ consisting of the rules:



$$0 \to \mathsf{a}(1) \mid \mathsf{c} \qquad\qquad 1 \to \mathsf{a}(0) \mid \mathsf{b}(1)$$

where $\ell \to r_1 \mid \ldots \mid r_n$ is shorthand for rules $(\ell \to r_i)_{1 \leq i \leq n}$. The language of $\mathcal{T}$ is $\mathcal{L}(\mathcal{T}) = (\mathsf{a}\,\mathsf{b}^*\mathsf{a})^*\mathsf{c} \mid (\mathsf{a}\,\mathsf{b}^*\mathsf{a})^\omega \mid (\mathsf{a}\,\mathsf{b}^*\mathsf{a})^*\mathsf{a}\,\mathsf{b}^\omega$.

The following lemma states a continuity property of tree automata.

Lemma 4.5. *Let $\mathcal{T} = \langle Q, \Sigma, I, \Delta \rangle$ be a tree automaton, $q \in Q$, and $t \in \mathit{Ter}^\infty(\Sigma)$. Then $q \in Q(t)$ if and only if for all $n \in \mathbb{N}$ exists t_n with $q \in Q(t_n)$ and $t \equiv_{\leq n} t_n$.*

Proof. The 'only if'-direction is trivial, take $t_n := t$ for all $n \in \mathbb{N}$.

For the 'if'-direction, we prove $q \in Q(t)$ by constructing a q-run $\rho : \mathcal{P}\mathit{os}(t) \to Q$ of $\mathcal{T}$ on t. For ever $i \in \mathbb{N}$ there exists a q-run ρ_{t_i} of $\mathcal{T}$ on t_i by assumption. Define $T_0 := \{t_i \mid i \in \mathbb{N}\}$. In case T_0 is finite, then it follows that $t \in T_0$ and $q \in Q(t)$. Hence assume that T_0 is infinite.

First we define a decreasing sequence $T_0 \supseteq T_1 \supseteq T_2 \supseteq \ldots$ of infinite subsets of T_0 by induction as follows. Assume that T_i has already been obtained. By the Pigeonhole Principle there exists an infinite subset $T_{i+1} \subseteq T_i$ such that for all $v_1, v_2 \in T_{i+1}$ we have $v_1 \equiv_{\leq i} v_2$ and $\rho_{v_1} \equiv_{\leq i} \rho_{v_2}$.

We define the q-run ρ on t as follows. For each $i \in \mathbb{N}$ we pick a term $s_i \in T_{i+1}$ and define $\rho(p) := \rho_{s_i}(p)$ for all $p \in \mathcal{P}os(t)$ with $|p| = i$. Note that the definition of ρ does not depend no the choice of s_i. Furthermore note that for every $i \in \mathbb{N}$ the term s_i coincides with the term s_{i+1} on all positions $p \in \mathcal{P}os(t)$ with $|p| = i + 1$. Therefore the condition $\rho(p) \to t(p)(\rho(p1), \ldots, \rho(pn)) \in \Delta$ for every $p \in \mathcal{P}os(t)$ follows from $s_{|p|}$ fulfilling this condition. Hence ρ is a q-run on t and $q \in Q(t)$. ☐

Lemma 4.6. *Each of the following properties imply completeness of a tree automaton $\mathcal{T} = \langle Q, \Sigma, I, \Delta \rangle$:*

(i) there exists a single core state $q_c \in I$ *such that:*

$$\forall n \in \mathbb{N}. \forall f \in \Sigma_n. \, q_c \to f(q_c, \ldots, q_c) \in \Delta \, ;$$

(ii) there exists a set of core states $Q_c \cap I \neq \varnothing$ *such that for all* core inputs $q \in Q_c$ *there exist a tuple of* core outputs $q_1, \ldots, q_n \in Q_c$*:*

$$\forall n \in \mathbb{N}. \forall f \in \Sigma_n. \forall q \in Q_c. \exists q_1, \ldots, q_n \in Q_c. \, q \to f(q_1, \ldots, q_n) \in \Delta \, ;$$

(iii) there exists a set of core states $Q_c \subseteq I$ *such that for all tuples of* core outputs $q_1, \ldots, q_n \in Q_c$ *there exists a* core input $q \in Q_c$*:*

$$\forall n \in \mathbb{N}. \forall f \in \Sigma_n. \forall q_1, \ldots, q_n \in Q_c. \exists q \in Q_c. \, q \to f(q_1, \ldots, q_n) \in \Delta \, .$$

Proof. Note that (i) is an instance of (ii). For (ii) let $\Delta' \subseteq \Delta$ be such that the set Δ' contains for every $q \in Q$ exactly one transition of the form $\langle q, f(q_1, \ldots, q_n) \rangle$. We define $\rho(t, q)$ coinductively: $\rho(f(t_1, \ldots, t_n), q) := q(\rho(t_1, q_1), \ldots, \rho(t_n, q_n))$ where $\langle q, f(q_1, \ldots, q_n) \rangle \in \Delta'$. By construction $\rho(t, q)$ is a q-run on t. For (iii) it follows by induction that for every finite term $t \in Ter(\Sigma, \varnothing)$ has a q-run for some $q \in Q_c$. For infinite terms t take a sequence $\{t_i\}_{i \in \mathbb{N}}$ of finite terms converging towards t. By the Pigeonhole Principle there exists $q \in Q_c$ and a subsequence $\{s_i\}_{i \in \mathbb{N}}$ of $\{t_i\}_{i \in \mathbb{N}}$ such that every s_i has a q-run. Then by Lemma 4.5 we conclude that t has a q-run. ☐

5 Tree Automata as Certificates for SN^∞

We are now ready to use tree automata as 'certificates' for SN^∞.

Definition 5.1. Let R be a TRS over Σ, and let $S \subseteq Ter^\infty(\Sigma)$. A *certificate for* $\mathsf{SN}^\infty_R(S)$ is a tree automaton $\mathcal{T} = \langle Q, \Sigma, I, \Delta \rangle$ such that:

(i) $\mathcal{T}$ generates S, i.e. $S \subseteq \mathcal{L}(\mathcal{T})$, and
(ii) $Q_\alpha(\ell) \subsetneq Q_\alpha(r)$ if $Q_\alpha(\ell) \neq \varnothing$, for all $\ell \to r \in R$, and $\alpha : Var(\ell) \to \mathbf{2}^Q$.

Theorem 5.2. *Let R be a TRS over Σ, and $S \subseteq Ter^\infty(\Sigma)$. Then $\mathsf{SN}^\infty_R(S)$ holds if there exists a certificate for $\mathsf{SN}^\infty_R(S)$.*

The proof will be based on Theorem 3.10, the characterization of SN^∞ in terms of interpretability in a continuous algebra. For this purpose we establish a bridge between tree automata certificates and continuous algebras. This bridge may need some intuitive explanation first. This concerns our use of tree automata states q decorated with a real numbers $r \in [0,1] = \{r \in \mathbb{R} \mid 0 \leq r \leq 1\}$, to be perceived as the degree of accuracy with which q can generate a certain term. Here 'accuracy' refers to the distance d in Definition 2.2. An example may be helpful.

Example 5.3. Consider the tree automaton $\mathcal{T}$ with the transitions

$$0 \to \mathsf{a} \qquad 1 \to \mathsf{b} \qquad 0 \to \mathsf{c}(0) \qquad 1 \to \mathsf{c}(1)$$

First we consider the 'run'-semantics $Q(\cdot)$ from Definition 4.1. Then for all $n \in \mathbb{N}$ we have $Q(\mathsf{c}^n(\mathsf{a})) = \{0\}$, meaning that $\mathsf{c}^n(\mathsf{a})$ can be generated by state 0, and likewise $Q(\mathsf{c}^n(\mathsf{b})) = \{1\}$. However, $Q(\mathsf{c}^\omega) = \{0,1\}$, and since c^ω is both the limit of $\mathsf{c}^n(\mathsf{a})$ and $\mathsf{c}^n(\mathsf{b})$, we face a problem if we aim at a continuous interpretation.

We redo this example, now with the accuracies r mentioned as superscripts of states 0, 1. More precisely, we use the continuous Σ-algebra $\mathcal{A}_\mathcal{T}$ defined below. Then $[\mathsf{c}^n(\mathsf{a})] = \{0^1, 1^{1-2^{-n}}\}$, meaning that $\mathsf{c}^n(\mathsf{a})$ can be generated from state 1 with accuracy 1, and also from state 0 but only with accuracy $1 - 2^{-n}$. Likewise, $[\mathsf{c}^n(\mathsf{b})] = \{0^{1-2^{-n}}, 1^1\}$. Furthermore $[\mathsf{c}^\omega] = \{0^1, 1^1\}$, which is indeed the limit of both $\{0^1, 1^{1-2^{-n}}\}$ and $\{0^{1-2^{-n}}, 1^1\}$, thereby resolving the clash with the continuity requirement.

Definition 5.4. Let $\mathcal{T} = \langle Q, \Sigma, I, \Delta\rangle$ be a tree automaton. We define a continuous weakly monotone Σ-algebra $\mathcal{A}_\mathcal{T} = \langle A, [\cdot], d, \succ, \sqsupseteq\rangle$ as follows. We let $A := \{\gamma \mid \gamma : Q \to [0,1]\}$ with undefined elements $\Omega_\mathcal{T} := \{\gamma \in A \mid \forall q \in Q.\ \gamma(q) < 1\}$.

For every $f \in \Sigma$ with arity n we define the interpretation $[f]$ by:

$$[f](\gamma_1, \ldots, \gamma_n) := \lambda q. \sup\big\{0.5 + 0.5 \cdot \min(\gamma_1(q_1), \ldots, \gamma_n(q_n)) \mid \\ q \to f(q_1, \ldots, q_n) \in \Delta\big\}$$

where $\sup \varnothing := 0$.

For $\gamma \in A$ define $Q(\gamma) := \{q \in Q \mid \gamma(q) = 1\}$. Then $\succ$ and $\sqsupseteq$ on A are defined by: $\gamma_1 \succ \gamma_2 := Q(\gamma_1) \subsetneq Q(\gamma_2)$ and $\gamma_1 \sqsupseteq \gamma_2 := Q(\gamma_1) \subseteq Q(\gamma_2)$. As the metric d on A we choose $d(\gamma_1, \gamma_2) := \max\{|\gamma_1(q) - \gamma_2(q)| \mid q \in Q\}$.

The definition gives rise to a natural, continuous semantics associated with tree automata.

Lemma 5.5. *The algebra $\mathcal{A}_\mathcal{T}$ from Definition 5.4 is a continuous weakly monotone Σ-algebra with undefined elements Ω.*

Proof. We have $\succ \cdot \sqsupseteq \subseteq \succ$, and $\succ$ is well-founded since Q is finite. Consider a state $q \in Q$ for which $[f](\gamma_1, \ldots, \gamma_n)(q) = 1$, then there is $q \to f(q_1, \ldots, q_n) \in \Delta$

such that $\gamma_1(q_1) = 1,\ldots,\gamma_n(q_n) = 1$. Whenever additionally $\gamma_j \sqsupseteq \gamma'_j$ for some $1 \le j \le n$, then $\gamma'_j(q_j) = 1$ and therefore $[f](\ldots,\gamma'_j,\ldots)(q) = 1$. Hence $[f]$ is monotone with respect to $\sqsupseteq$ for all $f \in \Sigma$. Using the same reasoning it follows that Ω fulfills both requirements imposed on undefined elements. Hence $\mathcal{A}_\mathcal{T}$ is a weakly monotone Σ-algebra with undefine elements Ω.

For every $f \in \Sigma$ with arity n and every $\gamma_1, \gamma'_1, \ldots, \gamma_n, \gamma'_n \in A$ we have

$$d([f](\gamma_1,\ldots,\gamma_n),[f](\gamma'_1,\ldots,\gamma'_n)) \le 0.5 \cdot \max\{d(\gamma_i,\gamma'_i) \mid 1 \le i \le n\}\,.$$

As a consequence, for the interpretation $[\cdot] : \mathit{Ter}(\Sigma,\varnothing) \to A$ of finite terms we have $d([s],[t]) \le d(s,t)$ for all $s,t \in \mathit{Ter}(\Sigma,\varnothing)$. As a uniformly continuous map on the metric space $\langle \mathit{Ter}(\Sigma,\varnothing), d\rangle$, this interpretation can be extended to a continuous function $[\cdot] : \mathit{Ter}^\infty(\Sigma,\varnothing) \to A$ on the completion space $\langle \mathit{Ter}^\infty(\Sigma,\varnothing), d\rangle$. Hence $\mathcal{A}_\mathcal{T}$ is a continuous Σ-algebra. □

The following lemma connects the standard semantics of tree automata with the continuous algebra $\mathcal{A}_\mathcal{T}$. Roughly, in the continuous algebra the automaton can be found back, when considering only states with 'accuracy' 1 ($\gamma(q) = 1$).

Lemma 5.6. *Let $\mathcal{A}_\mathcal{T} = \langle A, [\cdot], d, \succ, \sqsupseteq\rangle$ be the Σ-algebra as in Definition 5.4. Then for all $t \in \mathit{Ter}^\infty(\Sigma,\varnothing)$, and $\alpha : \mathit{Var}(t) \to \mathbf{2}^Q$, $\beta : \mathit{Var}(t) \to A$ such that $\forall x \in \mathit{Var}(t).\ \alpha(x) = Q(\beta(x))$, it holds $Q_\alpha(t) = Q([t]^\beta)$.*

Proof. For the case $t \in \mathcal{X}$, there is nothing to be shown. Thus let $t \equiv f(t_1,\ldots,t_n)$. For '$\supseteq$', assume $q \in Q([t]^\beta)$. Then there exists $q \to f(q_1,\ldots,q_n) \in \Delta$ such that for $i = 1,\ldots,n$ we have $q_i \in Q([t_i]^\beta)$. Applying this argument (coinductively) to the subterms t_i we obtain a q-run $\rho := q(\rho_1,\ldots,\rho_n)$ of $\mathcal{T}$ on t (with respect to α) where ρ_i is a q_i-run of $\mathcal{T}$ on t_i for $i = 1,\ldots,n$. For '$\subseteq$', we show that $[t]^\beta(q) \ge 1 - 0.5^d$ for all $t \in \mathit{Ter}^\infty(\Sigma)$, $d \in \mathbb{N}$ and $q \in Q$ with $q \in Q_\alpha(t)$. Assume contrary this claim would not hold. Consider a counterexample with minimal $d \in \mathbb{N}$. Since $q \in Q_\alpha(t)$ there exists $q \to f(q_1,\ldots,q_n) \in \Delta$ such that $q_i \in Q_\alpha(t_i)$ for $i = 1,\ldots,n$. This implies $d \ge 1$ and from minimality of d we obtain $\forall i.\ [t_i]^\beta(q_i) \ge 1 - 0.5^{d-1}$. But then $[t]^\beta(q) \ge 0.5 + 0.5 \cdot \min([t_i]^\beta(q_i)) \ge 1 - 0.5^d$, contradicting the assumption. Hence $[t]^\beta(q) = 1$, and $q \in Q([t]^\beta)$. □

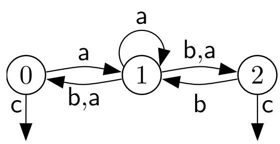
Using $\mathcal{A}_\mathcal{T}$ we now give the proof of Theorem 5.2.

Proof (Theorem 5.2). Let $\mathcal{T} = \langle Q, \Sigma, I, \Delta\rangle$ be a certificate for $\mathsf{SN}^\infty_R(S)$. Let $\mathcal{A}_\mathcal{T} = \langle A, [\cdot], d, \succ, \sqsupseteq\rangle$ and Ω as defined in Definition 5.4. According to Lemma 5.5 $\mathcal{A}_\mathcal{T}$ is a continuous weakly monotone Σ-algebra with undefined elements Ω. We prove that $\mathcal{A}_\mathcal{T}$ fulfills the requirements of Theorem 3.10.

As a consequence of Lemma 5.6 we obtain that $[s] \notin \Omega$ for all $s \in S$, since by assumption $S \subseteq \mathcal{L}(\mathcal{T})$; and $[\ell]^\alpha \notin \Omega$ implies $[\ell]^\alpha \succ [r]^\alpha$, for all rules $\ell \to r \in R$ and every $\alpha : \mathcal{X} \to A$. Finally, we check compatibility of $\sqsupseteq$ with limits. Let $\{\gamma_i\}_{i\in\mathbb{N}}$ be a converging sequence with $\gamma_0 \sqsupseteq \gamma_1 \sqsupseteq \ldots$, and define $\gamma := \lim_{i\to\infty} \gamma_i$. Note that $Q(\gamma_i) \subseteq Q(\gamma_{i+1})$ for all $i \in \mathbb{N}$. For every $q \in Q$ with $\gamma_0(q) = 1$ we have $\gamma_i(q) = 1$ for all $i \in \mathbb{N}$ and therefore $\gamma(q) = 1$. Hence $\gamma_0 \sqsupseteq \gamma$.

The algebra $\mathcal{A}_\mathcal{T}$ fulfills all requirements of Theorem 3.10, hence $\mathsf{SN}^\infty_R(S)$ holds. □

Example 5.7. Let $\Sigma := \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$ and $R := \{\mathsf{a}(\mathsf{c}) \to \mathsf{a}(\mathsf{b}(\mathsf{c})),\ \mathsf{b}(\mathsf{b}(\mathsf{c})) \to \mathsf{c}\}$ where a and b are unary symbols, and c is a constant. We are interested in SN^{∞}_{R}, that is, in infinitary normalization of R on the set of all (possibly infinite) terms. Consider the tree automaton $\mathcal{T} = \langle Q, \Sigma, I, \Delta\rangle$ depicted below:



where $Q := \{0, 1, 2\}$, $I := Q$ and Δ consists of the following rules:

$$0 \to \mathsf{a}(1) \mid \mathsf{c} \qquad 1 \to \mathsf{a}(0) \mid \mathsf{a}(1) \mid \mathsf{a}(2) \mid \mathsf{b}(0) \mid \mathsf{b}(2) \qquad 2 \to \mathsf{b}(1) \mid \mathsf{c}$$

We show that $\mathcal{T}$ is a certificate for SN^{∞}_{R}, by checking the conditions of Definition 5.1. Completeness of $\mathcal{T}$ follows from Lemma 4.6 (iii), take $Q_{\mathrm{c}} = Q$. Second, as both rules of R have no variables, we do not have to consider assignments α. We verify that $Q(\ell) \subsetneq Q(r)$ for both rules. For the rule $\mathsf{a}(\mathsf{c}) \to \mathsf{a}(\mathsf{b}(\mathsf{c}))$ we compute $Q(\mathsf{a}(\mathsf{c})) = \{1\}$, for only from state 1 we can generate $\mathsf{a}(\mathsf{c})$: $1 \to \mathsf{a}(2) \to \mathsf{a}(\mathsf{c})$ (or $1 \to \mathsf{a}(0) \to \mathsf{a}(\mathsf{c})$). From state 2 there is no 'a-transition', and from state 0 we get stuck at $\mathsf{a}(1)$, for there is no rule $1 \to \mathsf{c}$. Similarly we find $Q(\mathsf{a}(\mathsf{b}(\mathsf{c}))) = \{0, 1\}$, hence $Q(\mathsf{a}(\mathsf{c})) \subsetneq Q(\mathsf{a}(\mathsf{b}(\mathsf{c})))$. For the second rule of R we find $Q(\mathsf{b}(\mathsf{b}(\mathsf{c}))) = \{2\} \subsetneq \{0, 2\} = Q(\mathsf{c})$. Thus we have shown $\mathcal{T}$ to be a certificate, and by Theorem 5.2 we may conclude SN^{∞}_{R}.

6 Improving Efficiency: Strict Certificates

The second requirement for an automaton to be a certificate for SN^{∞} (item (ii) of Definition 5.1) is computationally expensive to check, since there are $2^{|Q| \cdot |\mathit{Var}(\ell)|}$ different maps $\alpha : \mathit{Var}(\ell) \to \mathbf{2}^{Q}$, leading to an exponential explosion in the number of states when searching for such an automaton.

Remark 6.1. For Theorem 5.2 it is not sufficient to check that the second condition holds for maps from variables to single states, that is, maps $\alpha : \mathit{Var}(\ell) \to \mathbf{2}^{Q}$ with $|\alpha(x)| = 1$ for all $x \in \mathcal{X}$.

To see this, consider the TRS $R := \{\mathsf{f}(x) \to \mathsf{f}(\mathsf{a}(x))\}$ with the tree automaton $\mathcal{T} = \langle Q, \Sigma, I, \Delta\rangle$ where $Q := I := \{0, 1\}$ and Δ consists of $0 \to \mathsf{f}(0)$, $1 \to \mathsf{f}(1)$, $0 \to \mathsf{a}(0)$, $0 \to \mathsf{a}(1)$, $1 \to \mathsf{a}(0)$, and $1 \to \mathsf{a}(1)$. Then $\mathcal{L}(\mathcal{T}) = \mathit{Ter}^{\infty}(\Sigma)$ and for every map $\alpha := x \mapsto \{q\}$ with $q \in Q$ we get $Q_{\alpha}(\ell) = \{q\} \subsetneq Q = Q_{\alpha}(r)$. Both conditions seem to be fulfilled, however SN^{∞}_{R} does not hold, since R admits an infinite root rewrite sequence $\mathsf{f}(\mathsf{a}^{\omega}) \to_{R,\epsilon} \mathsf{f}(\mathsf{a}^{\omega}) \to_{R,\epsilon} \ldots$.

For the purpose of efficient implementations and the envisaged SAT encoding, we define the notion of 'strict certificates', and show that they have the same theoretical strength while being easier to check.

Definition 6.2. Let R be a TRS over Σ, and $S \subseteq \mathit{Ter}^{\infty}(\Sigma)$. A *strict certificate for* $\mathsf{SN}^{\infty}_{R}(S)$ is a tree automaton $\mathcal{T} = \langle Q, \Sigma, I, \Delta\rangle$ with a strict total order $< \subseteq Q \times Q$ such that:

(i) $S \subseteq \mathcal{L}(\mathcal{T})$, and
(ii) for every $\ell \to r \in R$ and $\alpha : \mathit{Var}(\ell) \to \mathbf{2}^Q$ with $1 \leq |\alpha(x)| \leq \#_x(\ell)$, for all $x \in \mathit{Var}(\ell)$, where $\#_x(\ell) \in \mathbb{N}$ the number of occurrences of x in ℓ, it holds:

$$Q_\alpha(\ell) \neq \varnothing \implies Q_\alpha(\ell) \subseteq Q_\alpha(r) \text{ and } \\ \forall q \in Q_\alpha(\ell).\ \exists q' \in Q_\alpha(r).\ q' < q\,.$$

That strict certificates are certificates, the next theorem, will be proved below.

Theorem 6.3. *Let R be a TRS over Σ, and $S \subseteq \mathit{Ter}^\infty(\Sigma)$. Then every strict certificate for $\mathsf{SN}^\infty_R(S)$ is a certificate for $\mathsf{SN}^\infty_R(S)$.*

In the search for certificates, the computational complexity is improved when restricting the search to strict certicates, because the number of maps α which have to be considered is reduced to:

$$\textstyle\prod_{x \in \mathit{Var}(\ell)} (\sum_{i=1}^{\#_x(\ell)} \binom{|Q|}{i})$$

which is polynomial in the number of states $|Q|$. In particular if ℓ is linear then we need to consider $|Q|^{|\mathit{Var}(\ell)|}$ maps α.

Remark 6.4. Note that, in the definition of strict certificates, we cannot replace the condition $1 \leq |\alpha(x)| \leq \#_x(\ell)$ by $|\alpha(x)| = 1$. To see this, we consider the non-left-linear TRS $R := \{\mathsf{f}(x, x) \to \mathsf{f}(\mathsf{a}(x), \mathsf{a}(x))\}$ together with the tree automaton $\mathcal{T} = \langle Q, \Sigma, I, \Delta\rangle$ where $Q := I := \{0, 1\}$ and Δ consists of $1 \to \mathsf{f}(q, q)$, $0 \to \mathsf{f}(q, \overline{q})$ and $q \to \mathsf{a}(q')$ for all $q, q' \in Q$ where $\overline{q} = 1 - q$. Then $\mathcal{L}(\mathcal{T}) = \mathit{Ter}^\infty(\Sigma)$ and for every map $\alpha := x \mapsto \{q\}$ with $q \in Q$ we get $Q_\alpha(\ell) = \{1\}$ and $Q_\alpha(r) = \{0, 1\}$; thus $Q_\alpha(\ell) \subseteq Q_\alpha(r)$ and $0 < 1$ with $0 \in Q_\alpha(r)$. However R admits an infinite root rewrite sequence $\mathsf{f}(\mathsf{a}^\omega, \mathsf{a}^\omega) \to_{R,\epsilon} \mathsf{f}(\mathsf{a}^\omega, \mathsf{a}^\omega) \to_{R,\epsilon} \ldots$.

Note that the theorem holds even if one allows a partial order $<$ in the definition of strict certificates. However, that would not make the notion of strict certificates more general, because such a partial order can always be extended to a total order. The advantage of the definition as it stands is that we get the order for free. For every strict certificate with n states there exists an isomorphic automaton with states $Q := \{1, \ldots, n\}$ and $<$ being the natural order on integers. Thus, we can narrow the search for certificates to such automata.

Lemma 6.5. *Let $\mathcal{T} = \langle Q, \Sigma, I, \Delta\rangle$ be a tree automaton, $s \in \mathit{Ter}^\infty(\Sigma)$ and $\alpha : \mathit{Var}(s) \to \mathbf{2}^Q$. Let $\mathcal{B}$ consist of all maps $\beta : \mathit{Var}(s) \to \mathbf{2}^Q$ with $\beta(x) \subseteq \alpha(x)$ and $1 \leq |\beta(x)| \leq \#_x(s)$ for all $x \in \mathit{Var}(s)$. Then $Q_\alpha(s) = \bigcup_{\beta \in \mathcal{B}} Q_\beta(s)$.*

Proof. The part '$\supseteq$' is trivial, all maps $\beta \in \mathcal{B}$ are a restriction of α. For '$\subseteq$' let ρ be a q-run with respect to α on s. Let $\beta := \lambda x.\{\rho(p) \mid p \in \mathcal{P}os(s) \text{ with } s(p) = x\}$, then ρ is also a q-run with respect to β and $\forall x \in \mathit{Var}(s).\, 1 \leq |\beta(x)| \leq \#_x(s)$. $\square$

Now we prove Theorem 6.3.

Proof (Theorem 6.3). Let R be a TRS over Σ, $S \subseteq \mathit{Ter}^\infty(\Sigma)$ a set of terms, and $\mathcal{T} = \langle Q, \Sigma, I, \Delta \rangle$ a strict certificate for $\mathsf{SN}_R^\infty(S)$ with a strict total order $<$ on the states. We show that $\mathcal{T}$ satisfies the conditions of Definition 5.1. Let $\ell \to r \in R$ and $\alpha : \mathit{Var}(\ell) \to \mathbf{2}^Q$ with $Q_\alpha(\ell) \neq \varnothing$. Let $\mathcal{B}$ consist of all maps $\beta : \mathit{Var}(\ell) \to \mathbf{2}^Q$ with $\beta(x) \subseteq \alpha(x)$ and $1 \leq |\beta(x)| \leq \#_x(\ell)$ for all $x \in \mathit{Var}(\ell)$. Then $Q_\alpha(\ell) = \bigcup_{\beta \in \mathcal{B}} Q_\beta(\ell)$ and $Q_\alpha(r) = \bigcup_{\beta \in \mathcal{B}} Q_\beta(r)$ by Lemma 6.5. Note that we have $Q_\beta(\ell) \subseteq Q_\beta(r)$ for all $\beta \in \mathcal{B}$ by assumption, hence $Q_\alpha(\ell) \subseteq Q_\alpha(r)$. Take the least $q \in Q_\alpha(\ell)$ with respect to $<$. Then there exists $\beta \in \mathcal{B}$ with $q \in Q_\beta(\ell)$ and by assumption $\exists q' \in Q_\beta(r).\ q' < q$. Hence $q' \in Q_\alpha(r)$ and $Q_\alpha(\ell) \subsetneq Q_\alpha(r)$. □

The additional requirement of an ordering $<$ on the states is not a weakening. Indeed, we can show that any certificate can be transformed into a strict one.

Lemma 6.6. *Let R be a TRS over Σ, and $S \subseteq \mathit{Ter}^\infty(\Sigma)$. If there is a certificate for $\mathsf{SN}_R^\infty(S)$ then there is a strict certificate for $\mathsf{SN}_R^\infty(S)$.*

Proof. Let R be a TRS over Σ, $S \subseteq \mathit{Ter}^\infty(\Sigma)$, and $\mathcal{T} = \langle Q, \Sigma, I, \Delta \rangle$ a certificate for $\mathsf{SN}_R^\infty(S)$. We construct a tree automaton $\mathcal{T}' = \langle Q', \Sigma, I', \Delta' \rangle$ and show that it meets the requirements of Definition 6.2. Let $Q' := \mathbf{2}^Q$, and $I' := \{Q_I \subseteq Q \mid Q_I \cap I \neq \varnothing\}$. We define Δ' to consist of all transitions of the form $Q_0 \to f(Q_1, \ldots, Q_n)$ with $f \in \Sigma$, $Q_0, \ldots, Q_n \subseteq Q$ such that $\varnothing \neq Q_0 \subseteq Q_0'$ where

$$Q_0' := \{q \in Q \mid \text{exists } q \to f(q_1, \ldots, q_n) \in \Delta \text{ such that } \forall i.\ q_i \in Q_i\}\ .$$

Note that the construction is similar to the construction for making tree automata deterministic [2]. The main difference concerns the set Q_0, which is not uniquely defined as $Q_0 := Q_0'$ in our setting (we allow subsets $Q_0 \subseteq Q_0'$). Therefore the automaton $\mathcal{T}'$ will in general not be deterministic. For all terms $s \in \mathit{Ter}(\Sigma)$ and maps $\alpha' : \mathit{Var}(s) \to \mathbf{2}^{Q'}$ we have:

$$Q'_{\alpha'}(s) = \{Q_0' \subseteq Q_\alpha(s) \mid \alpha : \mathit{Var}(s) \to \mathbf{2}^Q \text{ with } \forall x.\ \alpha(x) \in \alpha'(x)\} \qquad (*)$$

This follows from the above-mentioned analogy; we refer to [2] for a proof. From $(*)$ it immediately follows that $\mathcal{L}(\mathcal{T}) = \mathcal{L}(\mathcal{T}')$.

We define the strict order $>$ on Q' as $\subsetneq$, arbitrarily extended to a total order. Let $\ell \to r \in R$ and $\alpha' : \mathit{Var}(\ell) \to \mathbf{2}^{Q'}$ such that $Q'_{\alpha'}(\ell) \neq \varnothing$. We know that for every $\alpha : \mathit{Var}(s) \to \mathbf{2}^Q$ it holds $Q_\alpha(\ell) \subsetneq Q_\alpha(r)$ by assumption. Then together with $(*)$ it follows that $Q'_{\alpha'}(\ell) \subseteq Q'_{\alpha'}(r)$. Finally let Q_0' be the least element with respect to $>$ from $Q'_{\alpha'}(\ell)$. Then there exists a map $\alpha : \mathit{Var}(s) \to \mathbf{2}^Q$ such that $\forall x. \alpha(x) \in \alpha'(x)$ and $Q_0' \subseteq Q_\alpha(\ell)$, even $Q_0' = Q_\alpha(\ell)$, since otherwise $Q_0' > Q_\alpha(\ell)$ would contradict minimality of Q_0'. Then we have $Q_\alpha(\ell) \subsetneq Q_\alpha(r)$ and therefore $Q_\alpha(r) \in Q'_{\alpha'}(r)$ with $\forall q' \in Q'_{\alpha'}(\ell).\ Q_\alpha(r) < q'$. □

7 Examples and Tool

Here we consider a few illustrating examples. We have implemented our method into a tool that aims at proving $\mathsf{SN}_R^\infty(S)$ automatically. Actually, all certificates in this section have been found fully automatically by our tool. The program is available via `http://infinity.few.vu.nl/sni/`, it may be used to try examples online. The tool shows the interpretation of all symbols and rules (with respect to all variable assignments) in the form of transition tables such that decreasingness can be recognized easily. The start language S can be specified by providing a tree automaton $\mathcal{T}$ that generates S; the program then searches an extension of $\mathcal{T}$ which fulfills the requirements of Theorem 6.3.

Example 7.1. Consider the following TRS R defining the sequence morse:

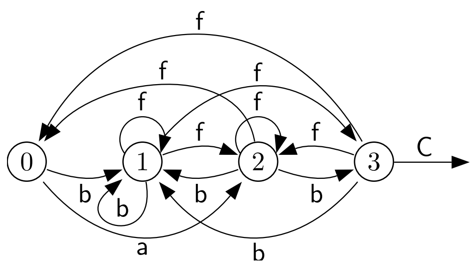
$$\begin{aligned}
\mathsf{morse} &\to \mathsf{cons}(0, \mathsf{zip}(\mathsf{inv}(\mathsf{morse}), \mathsf{tail}(\mathsf{morse})))\\
\mathsf{zip}(\mathsf{cons}(x, y), z) &\to \mathsf{cons}(x, \mathsf{zip}(z, y))\\
\mathsf{inv}(\mathsf{cons}(0, x)) &\to \mathsf{cons}(1, \mathsf{inv}(x))\\
\mathsf{inv}(\mathsf{cons}(1, x)) &\to \mathsf{cons}(0, \mathsf{inv}(x))\\
\mathsf{tail}(\mathsf{cons}(x, y)) &\to y
\end{aligned}$$

Our tool proves $\mathsf{SN}^{\infty}_{R}(\{\mathsf{morse}\})$ fully automatically. First it instantiates y in the rule $\mathsf{tail}(\mathsf{cons}(x, y)) \to y$ with non-variable terms covering all ground instances, and then it finds the tree automaton $\mathcal{T} = \langle Q, \Sigma, I, \Delta\rangle$ with $I = Q = \{0, 1, 2\}$ where the set Δ consists of: $2 \to \mathsf{morse}$, $1 \mid 2 \to \mathsf{0}$, $1 \mid 2 \to \mathsf{1}$, $2 \to \mathsf{tail}(0 \mid 2)$, $1 \mid 2 \to \mathsf{inv}(1 \mid 2)$, $0 \mid 1 \mid 2 \to \mathsf{cons}(1, 1)$, $1 \mid 2 \to \mathsf{zip}(1 \mid 2, 1)$, and $1 \mid 2 \to \mathsf{zip}(1, 2)$. Note that with the productivity tool of [3] we could already prove productivity of this specification fully automatically.

Example 7.2. Consider the term rewriting system R consisting of the rules:

$$\mathsf{c} \to \mathsf{f}(\mathsf{a}(\mathsf{b}(\mathsf{c}))) \qquad \mathsf{f}(\mathsf{a}(x)) \to \mathsf{f}(x) \qquad \mathsf{f}(\mathsf{b}(x)) \to \mathsf{b}(\mathsf{f}(x))$$

and the tree automaton $\mathcal{T} = \langle Q, \Sigma, I, \Delta\rangle$ with (initial) states $I = Q = \{0, 1, 2, 3\}$ over the signature $\Sigma = \{\mathsf{c}, \mathsf{a}, \mathsf{b}, \mathsf{f}\}$ where the set Δ of transition rules is given by:



We show that $\mathcal{T}$ is a strict certificate for $\mathsf{SN}^{\infty}_{R}(\{\mathsf{c}\})$. Clearly, we have $\{\mathsf{c}\} \subseteq \mathcal{L}(\mathcal{T})$. To verify condition (ii) of Definition 6.2 for the first rule of R, observe that $Q(\mathsf{c}) = \{3\} \subsetneq \{2, 3\} = Q(\mathsf{f}(\mathsf{a}(\mathsf{b}(\mathsf{c}))))$, and $2 < 3$. For the second rule, we only have to consider the map α given by $\alpha(x) = \{2\}$, for only then $Q_\alpha(\mathsf{f}(\mathsf{a}(x))) \neq \varnothing$. We observe $Q_\alpha(\mathsf{f}(\mathsf{a}(x))) = \{2, 3\} \subsetneq \{1, 2, 3\} = Q_\alpha(\mathsf{f}(x))$. For the third rule of R we have to consider two assignments: α_1 that maps x to $\{1\}$, and α_3 that maps x to $\{3\}$. We get that $Q_{\alpha_1}(\mathsf{f}(\mathsf{b}(x))) = \{1, 2, 3\} \subsetneq Q = Q_{\alpha_1}(\mathsf{b}(\mathsf{f}(x)))$ (and $0 < q$ for all $q \in \{1, 2, 3\}$), and $Q_{\alpha_3}(\mathsf{f}(\mathsf{b}(x))) = \{1, 2\} \subsetneq Q = Q_{\alpha_3}(\mathsf{b}(\mathsf{f}(x)))$ (and $0 < 1, 2$).

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