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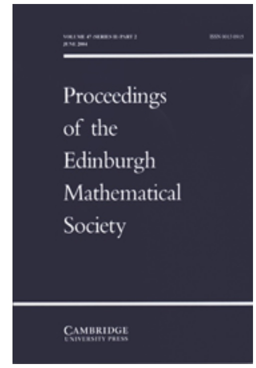

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A CRITERION FOR ERDŐS SPACES

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Abstract In 1940 Paul Erdős introduced the 'rational Hilbert space', which consists of all vectors in the real Hilbert space ℓ^2 that have only rational coordinates. He showed that this space has topological dimension one, yet it is totally disconnected and homeomorphic to its square. In this note we generalize the construction of this peculiar space and we consider all subspaces $\mathcal{E}$ of the Banach spaces ℓ^p that are constructed as 'products' of zero-dimensional subsets E_n of $\mathbb{R}$. We present an easily applied criterion for deciding whether a general space of this type is one dimensional. As an application we find that if such an $\mathcal{E}$ is closed in ℓ^p, then it is homeomorphic to complete Erdős space if and only if $\dim \mathcal{E} > 0$ and every E_n is zero dimensional.

Keywords: Erdős space; complete Erdős space; Lelek fan; cohesive; topological dimension; almost zero dimensional

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Let $p \geqslant 1$ and consider the Banach space ℓ^p. This space consists of all sequences $z = (z_1, z_2, \dots)$ of real numbers such that $\sum_{i=1}^{\infty} |z_i|^p < \infty$. The topology on ℓ^p is generated by the norm

$$\|z\| = \left(\sum_{i=1}^{\infty} |z_i|^p\right)^{1/p}.$$

Let $\hat{\mathbb{R}}$ be the compactification $[-\infty, \infty]$ of $\mathbb{R}$. We extend the p-norm over $\hat{\mathbb{R}}^{\mathbb{N}}$ by putting $\|z\| = \infty$ for each $z \in \hat{\mathbb{R}}^{\mathbb{N}} \setminus \ell^p$.

For the remainder of this note let $E_1, E_2, \dots$ be a fixed sequence of subsets of $\mathbb{R}$ and let

$$\mathcal{E} = \{z \in \ell^p : z_n \in E_n \text{ for every } n \in \mathbb{N}\}$$

be a corresponding subspace of some fixed ℓ^p. If we choose $p = 2$ and $E_n = \mathbb{Q}$ for every n, then $\mathcal{E}$ is called *Erdős space* $\mathfrak{E}$ and, if $E_n = \mathbb{R} \setminus \mathbb{Q}$, then we obtain *complete Erdős space* $\mathfrak{E}_c$ (cf. [**9**] and [**10**]). Erdős [**9**] proved that Erdős space and complete Erdős space have topological dimension one. We present an easily applied criterion for deciding whether a general space of the type $\mathcal{E}$ is one dimensional. As an application we find that if $\mathcal{E}$ is closed in ℓ^p, then it is homeomorphic to complete Erdős space if and only if $\dim \mathcal{E} > 0$ and every E_n is zero dimensional. Other applications are particularly simple models of complete Erdős space and the Lelek fan.

Every space under consideration is assumed to be separable metric. A space is called *cohesive* if every point has a neighbourhood that contains no non-empty clopen subsets of the space.

Theorem 1. *Assume that $\mathcal{E}$ is not empty and that every E_n is zero dimensional. For each $k \in \mathbb{N}$ we let $\eta(k) \in \mathbb{R}^{\mathbb{N}}$ be given by*

$$\eta(k)_n = \sup\{|a| : a \in E_n \cap [-1/k, 1/k]\},$$

where $\sup \emptyset = 0$. The following statements are equivalent:

(1) $\|\eta(k)\| = \infty$ *for each* $k \in \mathbb{N}$;

(2) *there exists an* $x \in \prod_{n=1}^{\infty} E_n$ *with* $\|x\| = \infty$ *and* $\lim_{n\to\infty} x_n = 0$;

(3) *every non-empty clopen subset of* $\mathcal{E}$ *is unbounded;*

(4) $\mathcal{E}$ *is cohesive; and*

(5) $\dim \mathcal{E} > 0$.

Proof. (1) $\Rightarrow$ (2). Assume (1). We shall construct sequences $n_0 < n_1 < \cdots$ in $\mathbb{N}$ and $y_1, y_2, \ldots$ in $\mathbb{R}$ such that, for each $k \in \mathbb{N}$,

(i) $y_m \in \{0\} \cup (E_m \cap [-1/k, 1/k])$ for $n_{k-1} \leqslant m < n_k$, and

(ii) $\sum_{m=1}^{n_k-1} |y_m|^p \geqslant k$.

Put $n_0 = 1$ and assume that $n_0, \ldots, n_{k-1}$ and $y_1, \ldots, y_{n_{k-1}-1}$ have been found. Select for each $m \in \mathbb{N}$ an $a_m \in \{0\} \cup (E_m \cap [-1/k, 1/k])$ such that $|a_m| \geqslant \frac{1}{2}\eta(k)_m$. Since $\|a\| \geqslant \frac{1}{2}\|\eta(k)\| = \infty$, we can select an $n_k > n_{k-1}$ such that

$$\sum_{m=n_{k-1}}^{n_k-1} |a_m|^p \geqslant 1.$$

If we define $y_m = a_m$ for $n_{k-1} \leqslant m < n_k$, then the hypotheses are satisfied. Clearly, we have $\|y\| = \infty$ and $\lim_{m\to\infty} y_m = 0$. Select a $z \in \mathcal{E}$ and define $x \in \prod_{m=1}^{\infty} E_m$ by $x_m = y_m$ if $y_m \neq 0$ and $x_m = z_m$ if $y_m = 0$. Condition (2) is proved.

(2) $\Rightarrow$ (3). Erdős [**9**] proved statement (3) for the case $E_n = \{1/i : i \in \mathbb{N}\}$ for all n. We adapt his method to suit the general situation. Assume that $x \in \prod_{n=1}^{\infty} E_n$ is such that $\|x\| = \infty$ and $\lim_{n\to\infty} x_n = 0$. Let A be a bounded and non-empty subset of $\mathcal{E}$. Select an $M \in \mathbb{N}$ such that $\|z\| \leqslant M$ for every $z \in A$. For $i \in \mathbb{N}$ let $\xi_i : \mathbb{R}^{\mathbb{N}} \to \ell^p$ be the projection $\xi_i(z) = (z_1, \ldots, z_i, 0, 0, \ldots)$. We construct inductively a sequence of points $a^0, a^1, \ldots$ in A and natural numbers $n_0 < n_1 < \cdots$ such that, for $i \geqslant 1$,

(a) $\xi_{n_i}(a^i) = \xi_{n_i}(a^{i-1})$,

(b) $\|a^{i-1} - \xi_{n_i}(a^{i-1})\| < 2^{-i}$, and

(c) the distance between a^i and $\mathcal{E} \setminus A$ is less than 2^{-i}.

We put $n_0 = 1$ and choose $a_0 \in A$. Assume that a^{i-1} and n_{i-1} have been found. Select an n_i such that $n_i > n_{i-1}$, $|x_j| < 2^{-i-1}$ for all $j > n_i$, and $\|a^{i-1} - \xi_{n_i}(a^{i-1})\| < 2^{-i-1}$, satisfying hypothesis (b). For $j \in \mathbb{N}$ we define $b^j \in \mathcal{E}$ by

$$b^j_m = \begin{cases} x_m, & \text{if } n_i < m \leqslant n_i + j, \\ a^{i-1}_m, & \text{otherwise.} \end{cases}$$

Observe that $b^0 = a^{i-1} \in A$ and that, since $\|x\| = \infty$,

$$\lim_{j\to\infty} \|b^j\| \geqslant \left(\sum_{m=n_i+1}^{\infty} |x_m|^p \right)^{1/p} = \infty.$$

Since A is bounded we can find a j such that $b^j \in A$ and $b^{j+1} \notin A$. We put $a^i = b^j$ and note that hypothesis (a) is satisfied. For the third hypothesis we note that, for $j \in \mathbb{N}$,

$$\begin{aligned} \|b^{j+1} - b^j\| &= |x_{n_i+j+1} - a^{i-1}_{n_i+j+1}| \\ &\leqslant |x_{n_i+j+1}| + |a^{i-1}_{n_i+j+1}| \\ &< 2^{-i-1} + \|a^{i-1} - \xi_{n_i}(a^{i-1})\| \\ &< 2^{-i}. \end{aligned}$$

This completes the induction.

By hypothesis (a) there is a $c \in \prod_{i=1}^{\infty} E_i$ such that $\xi_{n_{i+1}}(c) = \xi_{n_{i+1}}(a^i)$ for every $i \geqslant 0$. We then have

$$\|c\| = \lim_{i\to\infty} \|\xi_{n_{i+1}}(a^i)\| \leqslant \lim_{i\to\infty} \|a^i\| \leqslant M,$$

thus $c \in \mathcal{E}$. We find that

$$\lim_{i\to\infty} \|c - a^i\| \leqslant \lim_{i\to\infty} (\|c - \xi_{n_{i+1}}(c)\| + \|\xi_{n_{i+1}}(a^i) - a^i\|) \leqslant 0 + \lim_{i\to\infty} 2^{-i-1} = 0$$

and thus $\lim_{i\to\infty} a^i = c$. This means that c is in the closure of A and, by hypothesis (c), it also means that c is in the closure of $\mathcal{E} \setminus A$. So c is a boundary point of A and the proof is complete.

The implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are trivial.

(5) $\Rightarrow$ (1). Assume that $n \in \mathbb{N}$ is such that $\|\eta(n)\| < \infty$. Let $z \in \mathcal{E}$ and let $\varepsilon \in (0, 1/n)$. Select a $k \in \mathbb{N}$ such that

$$\sum_{i=k}^{\infty} (\eta(n)_i)^p < \tfrac{1}{2}\varepsilon^p \quad \text{and} \quad \sum_{i=k}^{\infty} |z_i|^p < \varepsilon^p.$$

Define $z' \in \ell^p$ by $z'_i = z_i$ if $i < k$ and $z'_i = 0$ if $i \geqslant k$, thus $\|z - z'\| < \varepsilon$. Put $\delta = \varepsilon/(2k)^{1/p}$ and let $i < k$. Since E_i is zero dimensional we may select a_i and b_i in $\mathbb{R} \setminus E_i$ such that

$z_i - \delta < a_i < z_i = z'_i < b_i < z_i + \delta$. We define the clearly clopen neighbourhood C of z in $\mathcal{E}$ by

$$C = \{x \in \mathcal{E} : a_i < x_i < b_i \text{ for each } i < k\}.$$

Define $U = \{x \in C : \|x - z'\| \leqslant \varepsilon\}$ and note that U is a closed neighbourhood of z in $\mathcal{E}$ with diameter at most 2ε. Let x be a point in U. If $i \geqslant k$, then $|x_i| = |x_i - z'_i| \leqslant \|x - z'\| \leqslant \varepsilon < 1/n$. This means that $|x_i| \leqslant \eta(n)_i$ and hence that

$$\sum_{i=k}^{\infty} |x_i - z'_i|^p = \sum_{i=k}^{\infty} |x_i|^p < \tfrac{1}{2}\varepsilon^p.$$

On the other hand, since $x \in C$ we have $\sum_{i=1}^{k-1} |x_i - z'_i|^p < k\delta^p = \frac{1}{2}\varepsilon^p$. Thus $\|x - z'\| < \varepsilon$ and x is an interior point of U because C is clopen. We have that U is a clopen neighbourhood of z with small diameter and we may conclude that $\dim \mathcal{E} = 0$. □

Note that in Theorem 1 the conditions (1)–(3) are metric, whereas conditions (4) and (5) are topological. Let us compare (4) and (5). Cohesion is a weakening of connectedness and plays a crucial role in characterizing Erdős space and complete Erdős space (see [**4**–**6**]). Clearly, a cohesive space is at least one dimensional at every point but the converse is not valid. An extreme example can be found in [**3**], where a one-dimensional homogeneous space that is not cohesive is constructed. However, if X is either a topological group or a complete and homogeneous space, then X is cohesive if and only if $\dim X \neq 0$ (see [**5**, Proposition 6.3], respectively [**3**]). In addition, it follows from Theorem 3.1 in [**7**] that a closed subspace of complete Erdős space is cohesive if and only if it is one dimensional at every point. Theorem 1 extends this list of positive results.

Recall that if $A_1, A_2, \dots$ is a sequence of subsets of a space X, then

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k=n}^{\infty} A_k}.$$

The following sufficient condition for $\dim \mathcal{E} \neq 0$ is a useful one because it is easily tested.

Corollary 2. *If* 0 *is a cluster point of* $\limsup_{n\to\infty} E_n$*, then every non-empty clopen subset of* $\mathcal{E}$ *is unbounded (and hence* $\dim \mathcal{E} \neq 0$*).*

Proof. If $\mathcal{E}$ is empty, then the conclusion is void. Let $\mathcal{E} \neq \emptyset$ and let $n \in \mathbb{N}$. Select a $t \in \limsup_{k\to\infty} E_k$ such that $0 < |t| < 1/n$. Choose a sequence $k_0 < k_1 < k_2 < \cdots$ in $\mathbb{N}$ such that there is, for each $j \in \mathbb{N}$, a $t_j \in E_{k_j}$ with $\lim_{j\to\infty} t_j = t$. We may assume that for every j, $\frac{1}{2}|t| < |t_j| < 1/n$. Thus $\eta(n)_{k_j} \geqslant |t_j| > \frac{1}{2}|t|$ for each j and hence $\|\eta(n)\| = \infty$, proving statement (1) of Theorem 1. The desired conclusion follows when we note that the zero dimensionality of the E_n was used only for the implication (5) $\Rightarrow$ (1) in the proof of Theorem 1. □

Let $\varphi : X \to \hat{\mathbb{R}}$ be a function. We define the following subspaces of the product $X \times \hat{\mathbb{R}}$:

$$L_\varphi = \{(x, t) \in X \times \hat{\mathbb{R}} : \varphi(x) \leqslant t\}$$

and

$$G_\varphi = \{(x, \varphi(x)) : x \in X \text{ and } \varphi(x) < \infty\}.$$

Let C be a non-empty zero-dimensional compact space and let $\varphi : C \to \hat{\mathbb{R}}$ be a *lower semi-continuous* (LSC) function, which means that $\varphi^{-1}((t, \infty])$ is open in C for every $t \in \hat{\mathbb{R}}$. We call φ a *Lelek function* if G_φ is dense in L_φ. If φ is a Lelek function, then the quotient space L_φ/∞ we obtain when we identify the set $C \times \{\infty\}$ to a point in L_φ is called a *Lelek fan* (see [**11**]). According to Bula and Oversteegen [**1**] and Charatonik [**2**], the Lelek fans, and consequently also their endpoint sets G_φ, are topologically unique. Kawamura, Oversteegen and Tymchatyn [**10**] have shown that complete Erdős space is homeomorphic to G_φ.

A well-known and useful property of the norm topology on ℓ^p is that this topology is the weakest topology that makes all the coordinate projections $z \mapsto z_i$ and the norm function continuous. This fact can also be formulated as follows: the graph of the norm function when seen as a function from ℓ^p with the product topology to $\mathbb{R}$ is homeomorphic to the Banach space ℓ^p. Note that this fact means that the norm topology on spheres $S_\varepsilon(a) = \{x \in \ell^p : \|x - a\|\}$ coincides with the product topology and thus the spheres in $\mathcal{E}$ are zero dimensional if the E_n are zero dimensional. Consequently, we have $\dim \mathcal{E} \leqslant 1$ in that case.

The last result can also be obtained from the following more abstract analysis. A space is called *almost zero dimensional* if every point has a neighbourhood basis consisting of sets that are intersections of clopen sets of the space. If every E_i is zero dimensional, then the resulting space $\mathcal{E}$ is almost zero dimensional. The reason lies in the fact that closed balls in ℓ^p are also closed subsets of $\hat{\mathbb{R}}^{\mathbb{N}}$ with the product topology, or, in other words, that the norm function is LSC when seen as a function from $\hat{\mathbb{R}}^{\mathbb{N}}$ to $[0, \infty]$. Thus, closed balls in $\mathcal{E}$ are also closed subsets of the zero-dimensional space $\prod_{n=1}^{\infty} E_n$, making them intersections of clopen sets. Oversteegen and Tymchatyn [**12**] proved that every almost zero-dimensional space is at most one dimensional.

Theorem 3. *If every E_i is closed in $\mathbb{R}$, then $\mathcal{E}$ is homeomorphic to complete Erdős space if and only if $\dim \mathcal{E} > 0$ and every E_n is zero dimensional.*

Proof. According to Erdős [**9**], complete Erdős space is one dimensional and totally disconnected. If $\mathcal{E} \neq \emptyset$, then every E_n is clearly imbeddable in $\mathcal{E}$. Thus, if $\mathcal{E}$ is homeomorphic to complete Erdős space, then every E_n is also totally disconnected and hence zero dimensional as a subset of $\mathbb{R}$.

We now turn to the 'if' part. We follow the method in [**10**] and will represent $\mathcal{E}$ as an endpoint set G_φ of a Lelek fan. Let $\bar{E}_n$ be the closure of E_n in $\hat{\mathbb{R}}$ and consider the zero-dimensional compactum $C = \prod_{n=1}^{\infty} \bar{E}_n$ in $\hat{\mathbb{R}}^{\mathbb{N}}$. We let $\varphi : C \to [0, \infty]$ be the restriction of the p-norm $\|\cdot\|$. Since the p-norm together with the product topology on ℓ^p generates the norm topology on ℓ^p and $\mathcal{E}$ corresponds to $\{x \in C : \varphi(x) < \infty\}$ we have that $\mathcal{E}$ is homeomorphic to G_φ. It now suffices to show that G_φ is dense in L_φ. Let $x \in C$ and let

$$U = U_1 \times \cdots \times U_k \times \bar{E}_{k_1} \times \bar{E}_{k+2} \times \cdots$$

be a standard neighbourhood of x in C. Since every E_n and hence every $\bar{E}_n$ is zero dimensional, we may assume that the U_i are clopen. Consider $\mathcal{E} \cap U$ and note that it is clopen subspace of $\mathcal{E}$. Select an $a \in \mathcal{E}$ and select, for each $i \leqslant k$, a $b_i \in E_i \cap U_i$. If we put $b_i = a_i$ for $i > k$, then $b = (b_1, b_2, \dots) \in \mathcal{E} \cap U$, thus $\mathcal{E} \cap U$ is not empty. Let $y \in \mathcal{E} \cap U$ and let $\varphi(y) < t < \infty$. Then there is a $z \in \mathcal{E} \cap U$ with $\|z\| = \varphi(z) = t$, because otherwise the set $\{y \in \mathcal{E} \cap U : \|y\| < t\}$ would be a bounded, non-empty, clopen subset of $\mathcal{E}$, in violation of Theorem 1. Thus $\{\varphi(y) : y \in \mathcal{E} \cap U\}$ is an unbounded non-empty subinterval of $[0, \infty)$. We may conclude that for each $x \in C$ the set $\{x\} \times [\varphi(x), \infty]$ is contained in the closure of G_φ. The proof is complete. □

We now consider some examples. Theorem 3 in combination with Corollary 2 shows that, for example, the following closed subgroup of $(\ell^p, +)$ is homeomorphic to complete Erdős space:

$$\{z \in \ell^p : nz_n \in \mathbb{Z} \text{ for each } n \in \mathbb{N}\}.$$

This result also follows from [**8**].

Another representation of complete Erdős space is

$$\{z \in \ell^2 : z_n \in \{0\} \cup \{1/i : i \in \mathbb{N}\} \text{ for each } n \in \mathbb{N}\},$$

which was featured by Erdős in [**9**].

Corollary 4. *If every E_n is a zero-dimensional closed subset of $\mathbb{R}$ such that $\mathcal{E} \neq \emptyset$, then $\mathcal{E} \times \mathfrak{E}_{\mathrm{c}}$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}}$.*

Proof. Apply Theorem 3 to the sequence $(E'_n)_{n=1}^\infty$, where $E'_{2k-1} = E_k$ and $E'_{2k} = \{0\} \cup \{1/i : i \in \mathbb{N}\}$ for $k \in \mathbb{N}$. □

Consider also the following choice: $E_n = \{0, 1/n\}$, for $n \in \mathbb{N}$. If $p = 1$ then, by Theorems 1 and 3 and the well-known fact $\sum_{n=1}^\infty 1/n = \infty$, we have that $\mathcal{E}$ is homeomorphic to complete Erdős space and we might call this minimal representation *harmonic Erdős space*. Interestingly, if $p > 1$, then it is easily verified that $\mathcal{E}$ is a Cantor set. This example also provides us with an elegant and concrete model for the Lelek fan, as follows. Let $C = \{0, 1\}^{\mathbb{N}}$ be the Cantor set and define $\varphi : C \to [0, \infty]$ by

$$\varphi(x) = \sum_{n=1}^\infty \frac{x_n}{n} \quad \text{for } x = (x_1, x_2, \dots) \in C.$$

By the proof of Theorem 3, φ is a Lelek function and L_φ/∞ is a Lelek fan, the *harmonic Lelek fan.*

Remark 5. Note that the proofs in this note can easily be adapted to work also for the quasi-Banach spaces ℓ^p, $p < 1$.

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