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# Black Scholes for Portfolios of Options in Discrete Time

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# Black Scholes for Portfolios of Options in Discrete Time: the Price is Right, the Hedge is Wrong\*

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## Abstract

Taking a portfolio perspective on option pricing and hedging, we show that within the standard Black-Scholes-Merton framework large portfolios of options can be hedged without risk in discrete time. The nature of the hedge portfolio in the limit of large portfolio size is substantially different from the standard continuous time delta-hedge. The underlying values of the options in our framework are driven by systematic and idiosyncratic risk factors. Instead of linearly (delta) hedging the total risk of each option separately, the correct hedge portfolio in discrete time eliminates linear (delta) as well as second (gamma) and higher order exposures to the systematic risk factor only. The idiosyncratic risk is not hedged, but diversified. Our result shows that preference free valuation of option portfolios using linear assets only is applicable in discrete time as well. The price paid for this result is that the number of securities in the portfolio has to grow indefinitely. This ties the literature on option pricing and hedging closer together with the APT literature in its focus on systematic risk factors. For portfolios of finite size, the optimal hedge strategy makes a trade-off between hedging linear idiosyncratic and higher order systematic risk.

**Key words:** option hedging; discrete time; portfolio approach; preference free valuation; hedging errors; Arbitrage Pricing Theory.

**JEL Codes:** G13; G12.

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# 1 Introduction

The portfolio approach to asset pricing and asset management has dominated large parts of the financial economics' literature. Early work for linear securities like stocks dates back to Markowitz (1952), Sharpe (1964), Lintner (1965), and Ross (1976). There are two key reasons for the widespread acceptance of the portfolio approach to asset pricing. First, in practice securities are hardly ever held in isolation. Any sensible analysis should therefore incorporate the fact that asset price movements may have a combined effect on individuals' wealth levels. Second, financial securities are subject to different sources of risk, such as common, economy-wide or systematic risk and firm-specific or idiosyncratic risk. By pooling a sufficient number of securities into a portfolio, the latter type of risk can be eliminated, whereas the former cannot. Only systematic risk factors are priced in equilibrium, and usually there is only a small number of them, see for example Chen, Roll, and Ross (1982), or Fama and French (1992,1993). Despite the early attention for the portfolio approach to linear assets, its application to non-linear assets is of much more recent date, see Jarrow, Lando, and Yu (2003) and Björk and Näslund (1998) for applications to portfolios of credit risk instruments and options, respectively.

Notwithstanding the important difference between systematic and idiosyncratic risk following from a portfolio perspective, the dominant approach in the options literature has been based on asset pricing for individual instruments, or on a portfolio of options with a single underlying security, see Mello and Neuhaus (1998). For example, the formula of Black and Scholes (1973) and Merton (1973) for pricing a European call on a stock contains total volatility of the stock, i.e., both its (priced) systematic risk component and its (unpriced) idiosyncratic risk component. Also looking at the hedging portfolio following from the Black-Scholes formula, we see that the total risk rather than the systematic risk only is hedged.

This apparent incongruence is addressed in the current paper, where we adopt the portfolio perspective to option pricing in discrete time. We have two reasons for considering the discrete time framework. First, even though asset prices may move continuously, trading and re-balancing only takes place at discrete time intervals due to for example monitoring and transaction costs, see, e.g., Boyle and Emanuel (1980), Leland (1985), Gilster (1990), Boyle and Vorst (1992), and Mello and Neuhaus (1998). Second, in the continuous time framework underlying the Black-Scholes analysis with asset prices following standard diffusions, no gains are possible by taking a portfolio perspective. In this setting the options are already replicated without risk, and any deviation in pricing would lead to arbitrage opportunities, see

also Björk and Näslund (1998) and Kabanov and Kramkov (1998). This confirms the widespread approach to option pricing based on individual instruments. The portfolio perspective does make a difference if some form of market incompleteness is introduced. Björk and Näslund (1998) and Jarrow et al. (2003) do this by having asset prices follow jump-diffusion processes. In that sense, our approach is related as it introduces market incompleteness through discrete trading moments while asset prices move in continuous time. This incompleteness results in non-zero hedging errors for option replication strategies, which may even be correlated with priced risk factors, see for example Gilster (1990).

In this paper we show that the incompleteness introduced by discrete trading times can be overcome by adopting a portfolio approach to asset pricing similar to the APT literature. By exploiting the cross sectional dimension of the portfolio of underlying values, a unique, preference-free price can be established for a portfolio of options using only the underlying stocks to construct a *static* hedge portfolio. The price of our hedge portfolio is equal to the sum of Black-Scholes prices as derived in the continuous time framework. In particular, the price does not depend on the preferences of the agent that hedges the options. The corresponding hedge portfolio, however, is entirely different from its Black-Scholes counterpart. Whereas the typical delta-hedge in the Black-Scholes framework linearly hedges total risk, the hedge portfolio in our framework hedges linear (delta), second (gamma), and higher order systematic risk only. The idiosyncratic risk is diversified and disappears asymptotically. Therefore it need not be hedged to obtain the correct portfolio price. This provides a closer tie between the different arguments in the standard options pricing and APT literatures as noted above. Static hedging of complex derivatives has been analyzed elsewhere in the literature, see for example Carr, Ellis, and Gupta (1998). The focus there, however, is on statically replicating complex derivatives with simpler derivatives. Here, we concentrate on hedging the (non-linear) systematic risk exposure by holding a static portfolio of stocks.

The set-up of our paper is as follows. In Section 2 we introduce the model and present the main results. Section 3 gives some numerical illustrations and robustness checks of our findings. Section 4 provides concluding remarks. Proofs are gathered in the Appendix.

## 2 Model and main results

Consider a set of  $N$  securities  $S_i$ ,  $i = 1, \dots, N$ , following the multivariate continuous time processes

$$dS = S \odot (\mu dt) + S \odot (\Sigma^{1/2} d\tilde{z}), \quad (1)$$

with  $S = (S_1, \dots, S_N)'$ ,  $\mu = (\mu_1, \dots, \mu_N)'$ ,  $\Sigma = \Sigma^{1/2}(\Sigma^{1/2})'$  a positive definite covariance matrix,  $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_N)'$  an  $N$ -dimensional standard Brownian motion, and  $a \odot b = (a_1 b_1, \dots, a_N b_N)'$  for two  $N$ -dimensional vectors  $a$  and  $b$ . To study the different effects of systematic and idiosyncratic risk in this context, we impose a factor structure on the covariance matrix  $\Sigma$ . In particular, similar to Björk and Näslund (1998) and Jarrow et al. (2003), we set

$$\Sigma = \beta\beta' + \text{diag}(\sigma_1^2, \dots, \sigma_N^2), \quad (2)$$

with  $\beta = (\beta_1, \dots, \beta_N)'$ . This imposes a one-factor structure on the movement of asset prices. Multiple factors can also be accommodated in a straightforward way, but lead to more cumbersome notation. By focusing on a single systematic risk factor only, we are able to pinpoint in closed form the trade-off between hedging systematic versus idiosyncratic option risk at the portfolio level. Using (2), we rewrite (1) for the  $i$ th security as

$$\frac{dS_i}{S_i} = \mu_i dt + \beta_i dz_0 + \sigma_i dz_i, \quad (3)$$

where  $z = (z_0, z_1, \dots, z_N)$  is an  $(N + 1)$ -dimensional standard Brownian motion with  $z(t) \sim N(0, tI)$ . For notational convenience we also define  $\tilde{\sigma}_i^2 = \sigma_i^2 + \beta_i^2$  as the  $i$ th diagonal element of  $\Sigma$  from (2) corresponding to the total risk of security  $S_i$ . In (3),  $z_0$  and  $z_i$  can be interpreted as the systematic and idiosyncratic risk factor, respectively.

Next, consider a portfolio consisting of  $N$  equally weighted short call options on the  $N$  different securities  $S_i$ . Other weightings for the options in the portfolio or other types of option contracts can be included through straightforward generalization.

Björk and Näslund (1998) show that in a perfect markets setting without jump risk, (3) still allows for perfect delta hedging of a portfolio of contingent claims on the  $S_i$ s using the standard Black-Scholes-Merton approach and hedging portfolio. In discrete time however, the standard Black-Scholes hedge is no longer perfect, in the sense that the expected return of the hedge portfolio only vanishes in expectation, and no longer almost surely. In this paper we show that alternative hedge portfolios can be found that provide a

lower hedge error variance in discrete time. These alternative hedge portfolios focus on the higher order exposure to the systematic risk factor rather than on the linear exposure to total risk, i.e., systematic plus idiosyncratic risk.

We proceed in three steps, which are formalised in theorems 1 to 3. First, we derive an expression for the hedge error variance when a standard delta hedge is implemented for each individual call position in the portfolio. Next, we derive a similar expression for the case where an arbitrary portfolio in the underlying values is held to hedge against fluctuations in the option portfolio's values. This allows us to demonstrate how hedge error variances can be reduced by an appropriate choice of the hedge portfolio. Finally, we provide our main theoretical result showing that if the portfolio size grows indefinitely, it becomes possible to construct a perfect static hedge portfolio in finite time by concentrating on the linear and higher order exposures to the systematic risk factor only.

Let  $C^i(S_i, t)$  denote the usual Black-Scholes price equation for the price of a call on security  $S_i$ , and let  $C_S^i(S_i, t)$  denote its derivative with respect to  $S_i$ . The standard approach to hedge our option portfolio over a discrete time interval  $\Delta t$  is to construct a (hedge) portfolio consisting of a position  $C_S^i(S_i, t)$  in each of the underlying securities  $S_i$ , and a position  $Q$  in cash, with

$$Q = \frac{1}{N} \sum_i \left( C^i(S_i, t) - C_S^i(S_i, t) S_i \right). \quad (4)$$

We define the hedging error  $\Delta H$  as

$$\begin{aligned} \Delta H = & C_S^i(S_i, t) [S_i(t + \Delta t) - S_i(t)] + Q [e^{r\Delta t} - 1] - \\ & [C^i(S_i(t + \Delta t), t + \Delta t) - C^i(S_i(t), t)]. \end{aligned} \quad (5)$$

For notational convenience, we write  $C^i(S_i(t), t)$  as  $C^i$  from now on. A similar notation is used for its derivatives, e.g.,  $C_S^i$  for the delta of the option. Using the above hedging strategy, the hedge portfolio is the sum of hedge portfolios for the individual positions. We expand the hedge error (5) as a power series in the length of the hedging period  $\Delta t$ , see also Leland (1985) and Mello and Neuhaus (1998). Under the present standard delta hedging strategy, the expected hedging error and its variance take the following form.

**Theorem 1** *Using delta hedging of the individual option positions, the hedging error  $\Delta H$  in (5) satisfies*

$$E[\Delta H] = \mathcal{O}(\Delta t^2),$$



and

$$\begin{aligned} \mathbb{E} [(\Delta H)^2] &= \frac{1}{2} \left[ \frac{1}{N} \sum_i C_{SS}^i S_i^2 \beta_i^2 \right]^2 \Delta t^2 \\ &\quad + \frac{1}{2N^2} \sum_i \left[ (C_{SS}^i S_i^2)^2 (\tilde{\sigma}_i^4 - \beta_i^4) \right] \Delta t^2 + \mathcal{O}(\Delta t^3). \end{aligned} \quad (6)$$

The proof of this theorem can be found from simple generalizations of results in the literature, and follows directly as a special case of Theorem 2. Trivially, for  $N = 1$ , we recover the well-known expression for the hedge error variance as the square of the option's gamma. Clearly, this explicit expression for the variance implies that the return of the option portfolio is no longer replicated without risk in discrete time following the Black-Scholes hedging approach. Consequently, it would seem that there is no unique preference free price for the option portfolio.

The hedge error variance in (6) consists of two terms. The first term reflects the contribution of the systematic risk factor  $z_0$ . This term is of order  $\mathcal{O}(\Delta t^2)$ , which indicates that the systematic component cannot be diversified<sup>1</sup> in a large portfolio context, i.e., is of order  $\mathcal{O}(1)$  in  $N$ . The second term, by contrast, is  $\mathcal{O}(\Delta t^2/N)$  and results from the idiosyncratic risk component of the securities. The delta hedging strategy thus clearly benefits from diversification. The idiosyncratic component of the variance is effectively of order  $\mathcal{O}(1/N)$ . In the limit for large portfolio sizes  $N$  all idiosyncratic risk disappears, and only risk attributable to the common market factor remains. For finite  $N$  the variance is reduced as a result of the correlation between the different underlying securities. The correlation between two individual standard hedging strategies (the off-diagonal terms in the above expression) have also been derived in Mello and Neuhaus (1998).

The variance reduction for large  $N$  can, however, be taken a step further by optimizing over the choice of the hedge portfolio. In this way, we are able to take a more explicit advantage of the correlation structure of the different underlying values. Consider an alternative hedging strategy, where the hedge portfolio contains a fraction  $D^i$  rather than  $C_S^i$  of the  $i$ th security. Keeping the price of the option identical to the usual Black-Scholes price, the cash investment follows directly as

$$Q = \frac{1}{N} \sum_i (C^i - D^i S_i). \quad (7)$$

Using this non-standard hedging strategy, we obtain the following result for the hedge error variance.

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<sup>1</sup>This appears to be supported by the empirical results in for example Gilster (1990).

**Theorem 2** *Using the hedge strategy with  $D^i S_i$  invested in security  $i$ , we can choose the hedge portfolio such that the hedge error  $\Delta H^B$  satisfies  $\mathbb{E}[\Delta H^B] = \mathcal{O}(\Delta t^2)$ . If only market risk is priced, i.e.  $\mu_i = r + \kappa_0 \beta_i$  for all  $i$ , with  $\kappa_0$  the price of systematic risk, then the hedge error variance is given by*

$$\mathbb{E}[(\Delta H^B)^2] = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3, \quad (8)$$

with

$$\mathcal{A}_1 = \frac{1}{N^2} \sum_i (X^i \sigma_i)^2 \Delta t, \quad (9)$$

$$\mathcal{A}_2 = \frac{1}{2} \left[ \frac{1}{N} \sum_i (C_{SS}^i S_i^2 - X^i) \beta_i^2 \right]^2 \Delta t^2, \quad (10)$$

$$\begin{aligned} \mathcal{A}_3 = \frac{1}{N^2} \sum_i \left[ \frac{1}{2} (\tilde{\sigma}_i^4 - \beta_i^4) (C_{SS}^i S_i^2 - X^i)^2 \right. \\ \left. + 2\mu_i \sigma_i^2 (X^i)^2 - 2(\mu_i - r) \sigma_i^2 X^i C_{SS}^i S_i^2 \right] \Delta t^2, \quad (11) \end{aligned}$$

where  $X^i = (D^i - C_S^i) S_i$  denotes the deviation from the standard (delta) hedge portfolio.

All proofs are gathered in the Appendix. We restrict ourselves in Theorem 2 to the case where only market risk is priced. This implies the absence of asymptotic arbitrage in the limit  $N \rightarrow \infty$ , see Björk and Näslund (1998). More formally, if  $\mu_i = r + \kappa_0 \beta_i + \kappa_i \sigma_i$ , with  $\kappa_i$  the price of idiosyncratic risk, the exclusion of asymptotic arbitrage imposes the restriction that the set of  $\{i | \kappa_i \neq 0\}$  has measure zero.

By definition, for the standard hedge portfolio  $X^i = 0$  in Theorem 2 and we recover the expression in (6). Theorem 2 states that the hedge error variance up to order  $\mathcal{O}(\Delta t^2)$  consists of three terms. The term  $\mathcal{A}_2$  reflects the systematic risk component, while the terms  $\mathcal{A}_1$  and  $\mathcal{A}_3$  reflect the idiosyncratic part. It is again easy to see that  $\mathcal{A}_2$  is  $\mathcal{O}(1)$  in  $N$ , while  $\mathcal{A}_1$  and  $\mathcal{A}_3$  are  $\mathcal{O}(1/N)$ . Moreover, the term  $\mathcal{A}_1$  is linear in  $\Delta t$ , whereas  $\mathcal{A}_2$  and  $\mathcal{A}_3$  are quadratic in  $\Delta t$ . For finite portfolio size  $N$ , in the limit  $\Delta t \rightarrow 0$  we should require terms linear in  $\Delta t$  to vanish in order to minimize the hedging risk. This results in the usual allocations  $D^i = C_S^i$  through the expression for  $\mathcal{A}_1$ . In other words, in the continuous time limit the optimal hedge portfolio is the sum of the individual Black-Scholes hedge portfolios.

However, this is no longer the case if we consider the hedge performance for non-infinitesimal values of  $\Delta t$ . In such settings there is a different way to decrease the variance of the hedge portfolio. This is most clearly seen

by focusing on the limiting large portfolio case,  $N \rightarrow \infty$ . We can then ignore the terms  $\mathcal{A}_1$  and  $\mathcal{A}_3$ . Requiring  $\mathbb{E}[\Delta H^B] = \mathcal{O}(\Delta t^2)$  only imposes a single linear restriction on the set of allocations  $D^i$ . The remaining flexibility in the set  $D^i$  can subsequently be used to reduce the variance at higher orders in  $\Delta t$ . In particular, we can select the remaining set of  $D^i$ 's such that  $\mathcal{A}_2$  vanishes as well. This choice is possible as long as not all  $\beta_i$ 's are identical. The intuition for this result is clear. In a large portfolio context, the idiosyncratic risk can be diversified. As a result, in our one-factor model only an exposure to the single systematic risk factor remains. The freedom in the choice of the portfolio composition can then be used to hedge against higher order exposures to this systematic risk component. For example, setting  $\mathcal{A}_2$  to zero annihilates the systematic gamma exposure. By contrast, the standard hedging approach a priori fixes the hedge portfolio composition such that the linear delta exposure to the combined systematic and idiosyncratic risk factors are hedged. Consequently, no freedom remains to hedge the undiversifiable higher order exposures to the systematic component.

For finite portfolio size  $N$  and finite revision time  $\Delta t$  there is a trade-off between hedging all idiosyncratic risk (in the limit  $N\Delta t \rightarrow 0$ , the standard Black-Scholes approach) and hedging market risk only (in the limit  $N\Delta t \rightarrow \infty$ ). Clearly, deviations from the standard hedge approach, both in hedge ratios and in reduction of variance, are larger for larger portfolio sizes and larger revision intervals.

We can now establish the generic result that in large option portfolios hedging higher order systematic risk is preferable to hedging linear idiosyncratic risk. This is done in the following theorem.

**Theorem 3 (the hedge is wrong)** *Hedging a portfolio of option in the general setting defined earlier, if only market risk is priced, we can choose the allocations  $D^i$  such that*

$$\mathbb{E}[\Delta H^B] = \mathcal{O}(\Delta t^{\frac{n+1}{2}}) + \mathcal{O}(1/N), \quad (12)$$

and

$$\mathbb{E}\left[(\Delta H^B)^2\right] = 0 + \mathcal{O}(\Delta t^{n+1}) + \mathcal{O}(1/N), \quad (13)$$

if  $N \geq n$  and at least  $n$  of the parameters  $\beta_i$  are different and not equal to zero.

If idiosyncratic risk is priced, one can prove a similar result. The number of securities needed to construct the hedge portfolio in that case increases quadratically with  $n$ . Moreover the expected return of this hedging strategy

need no longer be zero, resulting in arbitrage opportunities. This is to be expected, as this market structure, already without any options present, allows for the possibility of arbitrage. The result for the case with idiosyncratic risk priced is presented in Theorem 4 in the Appendix.

The result in Theorem 3 demonstrates that we can construct a riskless hedging strategy for finite time  $\Delta t$  in the limit  $N \rightarrow \infty$  if the  $\beta_i$ s are different. In other words, the risk that is introduced by going to a discrete time set-up vanishes completely: the systematic risk component can be eliminated to any arbitrary order of  $\Delta t$  by choosing the appropriate non-standard hedging strategy. At the same time, the idiosyncratic risk component disappears through diversification. The key to the proof is that the hedge portfolio is chosen such that up to sufficiently high order in  $\Delta t$ , its expectation conditional on the systematic risk component coincides with its unconditional expectation, see (A10). This can be established by matching the higher order characteristics of the systematic risk exposure. To match the two types of expectations, a set of constraints has to be imposed on the portfolio loadings  $D^i$ . All of these constraints are linear in the  $D^i$ s. The coefficients of these constraints involve powers of the systematic volatility  $\beta_i$  and higher order derivatives of the Black-Scholes price (to capture the appropriate curvature), see the Appendix. For this set of constraints to have a solution, the number of securities has to be sufficiently large,  $N \geq n$ . Second, the system of constraints needs to be nonsingular. The latter is ensured by the requirement that at least  $n$  of the  $\beta_i$ s are different and not equal to zero.

Given the entirely different behavior of the standard and portfolio hedging approach, one might wonder whether the Black-Scholes prices are still correct in the present discrete time framework. The next corollary states they are.

**Corollary 1 (the price is right)** *If only market risk is priced and the number of securities diverges ( $N \rightarrow \infty$ ), and if the  $\beta_i$ s are different such that we can set the approximation order  $n$  arbitrarily high ( $n \rightarrow \infty$ ), then the only arbitrage-free price of the options is equal to their Black-Scholes price, except for a set of measure zero.*

Arbitrage opportunities in Corollary 1 are defined as the possibility to gain a return higher than the riskfree rate almost surely. By construction, the hedge portfolio has the same price as Black-Scholes, see (7). Given the result from Theorem 3, this implies that the sum of the options prices equals the sum of Black-Scholes prices. As Theorem 3, however, also holds for every subseries of call options and corresponding underlying values, the set of options with prices different from the Black-Scholes price has measure zero.

### 3 Numerical illustrations

To further illustrate our results and provide more intuition, we discuss several numerical examples. We present results for a finite, but increasing set of  $N$  underlying securities, corresponding to Theorem 2, as well as results for the limiting case  $N \rightarrow \infty$ , corresponding to Theorem 3. As already noted, in order for the result in Theorem 3 to hold, we need heterogeneity in the securities' exposures  $\beta_i$  to systematic risk. At the same time, however, we want to limit the number of free parameters in our numerical experiments. We do so in the following way. First, we set the riskless rate  $r$  to zero. Second, we only use a single indicator for idiosyncratic risk, i.e.,  $\sigma_i^2 \equiv \sigma^2 = c\sigma_m^2$ , where  $\sigma_m$  is the systematic volatility and  $c = 0.5, 1.0, 1.5$ . We normalize all initial stock prices  $S_i$  to unity, and we consider three month at-the-money call option contracts and a hedging frequency of one month,  $\Delta t = 1/12$ . This leaves us with  $N + 2$  parameters: the market volatility  $\sigma_m^2$ , the price of systematic risk  $\kappa_0$ , and the  $\beta_i$ s. To retain comparability across portfolio sizes  $N$  of the portfolio characteristics, we model the  $\beta_i$ s as follows. For a given cumulative distribution function (cdf)  $F$ , we define

$$\beta_i = F^{-1} \left( \frac{2i - 1}{2N} \right) \cdot \sigma_m, \quad (14)$$

with  $F^{-1}$  the inverse of  $F$ , and  $i = 1, \dots, N$ . In this way we allow for heterogeneity across securities without losing comparability over increasing portfolio sizes  $N$ . We set  $F$  to the normal cdf with mean 1 and standard deviation 0.3. We use a market price of systematic risk  $\kappa_0 = 20\%$  and a market volatility of  $\sigma_m = 25\%$ . Simulations for alternative parameter settings and distributional assumptions revealed similar patterns. Figures 1 through 5 present the key patterns. Figures 1 through 4 highlight Theorem 2 for finite portfolio size  $N$ , while Figure 5 illustrates the asymptotic case  $N \rightarrow \infty$  of Theorem 3 and Corollary 1.

Figure 1 presents the hedge error variances for the standard delta hedge and the new 'portfolio' hedge. The new hedge portfolio is optimized by minimizing the hedging error variance as given in Theorem 2 over the stock holdings  $D^i$ . Two effects are clear. First, for the standard hedge, diversification leads to a decrease in hedging error variance for increasing portfolio size  $N$ . From about  $N = 100$  onwards, however, most diversification benefits have materialized and the variance remains stable. This remaining variance is caused by the second order exposure to the systematic risk factor, i.e., the systematic gamma risk. This is also clearly seen in Figure 2, which plots the percentage of the hedge error variance due to the systematic risk component. From about  $N = 100$  onwards, this percentage lies very close to 100%. In the

left-hand plot in Figure 1, less idiosyncratic risk gives rise to higher hedge error variance for large  $N$ . This is due to the parameterization, where the total risk  $\sigma^2 + \beta_i^2$  is lower for lower values of  $\sigma^2/\sigma_m^2$ . Consequently, the gamma term  $C_{SS}$  in (10) is higher for smaller  $\sigma^2$ . As for large  $N$   $\mathcal{A}_2$  dominates  $\mathcal{A}_3$ , this explains the ordering of the curves for large  $N$ . For small  $N$ ,  $\mathcal{A}_3$  is also important and the order of the curves is reversed.

The left-hand panel in Figure 1 further shows that, by contrast, the hedge error variance for the optimal hedge portfolio continues to decrease for larger values of  $N$ . As mentioned earlier, this is due to the fact that the optimal hedge portfolio also protects against higher order exposures to the systematic risk factor. This is seen in the right-hand panel of Figure 2, where the percentage of the variance due to systematic risk exposure is plotted. These percentages decrease rather than increase in  $N$  for  $N$  sufficiently large. In the limit for  $N$  diverging to infinity, the hedge error variance for the optimal portfolio even tends to zero. The percentage of systematic variance is smaller if there is less idiosyncratic risk for given  $\beta_i$ s. This follows again by looking at (9) and (10). Smaller  $\sigma_i$ s, by definition, give rise to a smaller value of  $\mathcal{A}_1$ . As a result, the scope for a reduction in  $\mathcal{A}_2$  by setting  $D^i \propto C_S^i$  without increasing  $\mathcal{A}_1$  too much is larger for lower idiosyncratic risk (and sufficiently large  $N$ ). The right-hand plot in Figure 1 clearly summarizes the results. Hedge error variances can be decreased significantly by adopting a portfolio perspective to hedging options. The benefits are larger if the systematic risk component constitutes the more dominant source of risk, i.e., if  $\sigma^2/\sigma_m^2$  is smaller.

Figure 3 plots the stock holdings  $D^i$  as a function of  $\beta_i$  for various portfolio sizes  $N$ . This allows us to see whether the optimal hedge portfolio overweights small or high  $\beta$  stocks. As the results are very similar for varying ratios of  $\sigma^2$  to  $\sigma_m^2$ , we only present and discuss the case  $\sigma^2/\sigma_m^2 = 1$ . The first thing to note is that the loadings of the standard hedge,  $D^i = C_S^i$ , are relatively stable. They increase with  $\beta_i$ , but their variation is negligible compared to the variation in holdings for the alternative hedge portfolios. The allocations  $D^i$  of the alternative hedge portfolio depend strongly on the market risk exposure of the corresponding underlying security. Stocks with zero systematic risk exposure ( $\beta_i \approx 0$ ) receive a similar loading as in the standard hedge. This is intuitively straightforward as changing the holdings in these securities does not generate a large change in the systematic risk exposure. The optimal hedge overweights high  $\beta$  stocks and possibly negative  $\beta$  stocks, and underweights low and medium  $\beta$  stocks. The latter can even be shorted in substantial amounts for  $N$  sufficiently large.

It may be less clear at first sight why the portfolio loadings take the shape they do in Figure 3. We have argued in Section 2 that the precise shape is due to the optimal hedge portfolio adapting itself to higher order exposure to

the systematic risk factor. This effect can easily be visualized. In Figure 4, we plot the conditional expectation of hedge errors  $E[\Delta H^B|z_0]$ , where the conditioning set contains the systematic risk factor  $z_0$ . The result for the standard delta hedge portfolio shows that hedge errors are most extreme for extreme realizations of  $z_0$ . For large realizations of  $z_0$ , the standard hedge is unable to accomodate the convexity in the option payoff. By overweighting high  $\beta$  stocks and underweighting low  $\beta$  stocks, the optimal hedge becomes more sensitive to extreme realizations of  $z_0$ . This is seen by the reduction in conditionally expected hedging errors over a large range of  $z_0$  outcomes. The increased ability of the optimal hedge to capture the convexity in the option payoffs becomes more apparent for larger portfolio sizes  $N$ . This is also evident from Figure 4.

The conditional expected hedging error can be reduced further by considering the true limiting case  $N \rightarrow \infty$ . We do so in the following way, making explicit use of the expressions used in the proof of Theorem 3. For approximation order  $n = 1, 2, \dots$ , we consider a set of  $n$  different  $\beta_i$ s. Each  $\beta_i$  can be considered as representing a homogenous group of underlying values. The  $\beta_i$ s are constructed as in (14). Using these  $\beta_i$ s, we solve for the  $D_1, \dots, D_n$  such that the hedge error variance is zero up to terms of order  $\mathcal{O}((\Delta t)^{n+1})$  and  $\mathcal{O}(1/N)$ . Because we consider the limiting case  $N \rightarrow \infty$ , we discard the  $\mathcal{O}(1/N)$  terms. This approach yields a linear system of equations, that can easily be solved numerically. The conditional expectation of the value of the option portfolio is given by (see Rubinstein (1984))

$$E[\Delta C_i|z_0] = \tilde{S}_i N(d_{1i}) - e^{-r(T-\Delta t)} K_i N(d_{2i}), \quad (15)$$

where

$$\begin{aligned} \tilde{S}_i &= S_i e^{(\mu_i - \frac{1}{2}\beta_i^2)\Delta t + \beta_i z_0 \sqrt{\Delta t}} \\ d_{1i} &= \frac{\ln(\tilde{S}_i/K_i) + r(T - \Delta t) + \frac{1}{2}\Sigma_i^2}{\Sigma_i} \\ d_{2i} &= d_{1i} - \Sigma_i \\ \Sigma_i^2 &= \tilde{\sigma}_i^2(T - \Delta t) + \sigma_i^2 \Delta t \end{aligned}$$

The conditional expectation of the change in the value of the hedge portfolio is calculated using (A12). The top panels in Figure 5 provide the results in terms of conditionally expected hedging errors up to approximation order  $n = 8$ .<sup>2</sup> Note that  $n = 1$  corresponds to the standard hedge. There is a clear jump in hedge error size when switching from  $n = 1$  to  $n = 2$ . Similar jumps

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<sup>2</sup>This involves the computation of 8th order derivatives of the Black-Scholes price equation.

are seen at every point where  $n$  increases from an odd to an even integer, see the lower graph. This is due to the fact that odd moments of the normal distribution are zero, see also (A13) and below. Also note the change in the scale of the vertical axis between the top-left and top-right panels. To obtain an indication of the pattern of decrease for increasing  $n$ , we compute the expectation of the squared curves in the top panel, augmented with curves for higher values of  $n$ . The natural logarithm of this quantity is plotted in the bottom panel in Figure 5. The decrease of the hedge error size in pairs of  $n$  is again clearly visible. Moreover, the rate of decrease appears exponential given the roughly linear pattern in the bottom panel. This again illustrates the hedge portfolio's ability to match the higher order systematic exposures of the option portfolio if idiosyncratic risk has been eliminated through diversification ( $N \rightarrow \infty$ ). The argument is very similar to the approach taken in Arbitrage Pricing Theory: in the limit of a large number of securities, only the systematic sources of risk in a portfolio of options need to be hedged in discrete time.

We end this section with a few remarks on the computational aspects underlying Figure 5. To compute the optimal stock holdings  $D^i$  for a specific approximation order  $n$ , one needs the  $n$ -th order derivative of the Black-Scholes formula. These derivatives enter the constant term in the linear equations for the  $D^i$ . Though these derivatives can easily be determined recursively, some straightforward manipulation shows that they tend to increase in absolute size with  $n$ . In addition, the coefficients of the  $D^i$ 's in the  $n$ -th equation of the linear system are of the order  $\beta_i^n$ , and thus decrease with  $n$ . These two properties give rise to large  $D^i$  values for increasing approximation order  $n$ . Therefore also the contribution to the variance of the position in each separate security increases. As a result of the correlation between these different securities, however, the total variance tends to zero. Using standard double precision, this cancellation is only numerically tractable for  $n \leq 9$ . The theoretical results remain valid and can clearly be corroborated numerically for  $n \leq 8$ , see Figure 5. Moreover, the main practical implication of our results appears that there is already a significant reduction in error size when switching from the standard delta hedge ( $n = 1$ ) to a hedge incorporating the systematic gamma component ( $n = 2$ ).

## 4 Conclusions

In this paper we have shown that in discrete time hedging risk can be reduced considerably compared to the standard Black-Scholes approach when taking a portfolio perspective. For finite portfolio size one should not focus solely on



minimizing all idiosyncratic risk exposures, but instead consider the trade-off between reducing the linear idiosyncratic risk exposures versus higher order market risk exposures. In the limit of an infinite number of securities, market incompleteness due to discrete trading may even be removed completely. This result is in line with results of Jarrow et al. (2003) who remove jump risk by taking a large portfolio approach. In particular, we showed that if the standard Black-Scholes framework is modified to allow only for discrete trading dates, the Black-Scholes prices for a large portfolio of options still provide the correct price except for a set of options of measure zero. The correct hedge strategy in discrete time, however, is entirely different from the standard Black-Scholes delta hedge: first (delta), second (gamma), and higher order exposures to the systematic risk factors only are hedged at the expense of idiosyncratic risk. The latter is left unhedged, because it can be diversified via the large portfolio context. This ties the literature on option pricing closer to that on APT. In the limiting context of an infinite number of securities to diversify idiosyncratic risk components, only systematic sources of risk in portfolios of options need to be hedged. This contrasts with the continuous time Black-Scholes-Merton framework, where both systematic and idiosyncratic sources of risk are hedged.

Our results have several implications. First, our results suggest an alternative approach to constructing hedges for portfolios of options. Risk factors should be identified at the portfolio level and it may be profitable to hedge higher order exposures to systematic risk factors at the expense of linear exposures to idiosyncratic risk. Second, our results imply that agents managing portfolios of derivatives are at an advantage if their portfolios comprise many different underlying values, subject to these underlying values being influenced by the same systematic risk factors. In such a setting, the maximum benefit can be obtained from diversification. Third, our results suggest further advantages of the portfolio perspective if transaction costs are taken into account. Though a complete analysis of the issue of transaction costs in a portfolio context is beyond the scope of this paper, some patterns are clear. For large portfolios of options, it was shown to be better to hedge higher order terms in the systematic risk components than to hedge the linear exposure to the total risk of all underlying values. If indices are available that are highly correlated with the systematic risk factor, our results may be reinforced by the inclusion of transaction costs. These costs are typically an order of magnitude smaller for index related contracts like options and futures, than for individual stocks. This is especially true if the effects of liquidity and price impact are taken into account. As a result, the costs of setting up and adjusting the hedge portfolio is likely to be smaller if part of the portfolio relates to index contracts. This suggests an interesting line

of future research. If systematic risk is highly correlated with index instruments, systematic gamma risk can be explicitly traded through the use of index options.

To assess whether or not it makes practical sense to actually implement option hedging and pricing decisions on a portfolio basis using index instruments, more insight is needed into the (numerical) balance between the different determinants of the hedge error variance: the portfolio size  $N$ , the magnitude of market risk and idiosyncratic risk, the variation in  $\beta$ s, and transaction costs. Especially the latter may be relevant in a portfolio context, as index instruments usually entail significantly lower transaction costs. A careful analysis of this topic, however, requires a more involved multi-period dynamic programming approach, and is left for future research.

## Appendix

### A Proofs

#### Proof of Theorem 2:

Consider the portfolio consisting of holdings  $D^i$  in each security  $S_i$ , a cash holding given by  $Q$  in (4), and a short position in each option  $C^i$ . The change in value of this portfolio between subsequent revision intervals equals the hedging error  $\Delta H$  and is given by

$$\Delta H = \frac{1}{N} \sum_i D^i \Delta S_i + \Delta Q - \frac{1}{N} \sum_i \Delta C^i. \quad (\text{A1})$$

Expanding this expression, and using the fact that  $\Delta S_i$  is of order  $\mathcal{O}(\Delta t^{1/2})$ , we obtain

$$\begin{aligned} \Delta H = & \frac{1}{N} \sum_i \left[ D^i \Delta S_i + r(C^i - D^i S_i) \Delta t - C_S^i \Delta S_i - \frac{1}{2} C_{SS}^i (\Delta S_i)^2 - C_t^i \Delta t \right. \\ & \left. - \frac{1}{6} C_{SSS}^i (\Delta S_i)^3 - C_{St}^i \Delta S_i \Delta t \right] + \mathcal{O}(\Delta t^2). \end{aligned} \quad (\text{A2})$$

Since the option prices  $C^i$  are still given by the standard Black-Scholes expressions, we can use the Black-Scholes equation

$$C_t^i + \frac{1}{2} \tilde{\sigma}_i^2 C_{SS}^i S_i^2 - r[C^i - C_S^i S_i] = 0 \quad (\text{A3})$$

and its derivative

$$C_{St}^i + \frac{1}{2} \tilde{\sigma}_i^2 C_{SSS}^i S_i^2 + (r + \tilde{\sigma}_i^2) C_{SS}^i S_i = 0 \quad (\text{A4})$$

in order to substitute for  $C_t^i$  and  $C_{St}^i$  respectively. This leads to

$$\begin{aligned} \Delta H = & \frac{1}{N} \sum_i \left[ (D^i - C_S^i) (\Delta S_i - r S_i \Delta t) - \frac{1}{2} C_{SS}^i ((\Delta S_i)^2 - \tilde{\sigma}_i^2 S_i^2 \Delta t) \right. \\ & \left. - \frac{1}{6} C_{SSS}^i ((\Delta S_i)^3 - 3\tilde{\sigma}_i^2 S_i^2 \Delta S_i \Delta t) + (r + \tilde{\sigma}_i^2) C_{SS}^i S_i \Delta S_i \Delta t \right] + \mathcal{O}(\Delta t^2) \end{aligned} \quad (\text{A5})$$

It is now straightforward to compute the expected hedge performance and its variance, using the formulas in the second appendix. We find

$$E[\Delta H] = \frac{1}{N} \sum_i (D^i - C_S^i) S_i (\mu_i - r) \Delta t + \mathcal{O}(\Delta t^2). \quad (\text{A6})$$

The variance of the hedge performance with general hedge ratios  $D^i$  is given by

$$\begin{aligned} \text{var}[\Delta H] &= \left[ \frac{1}{N} \sum_i (D^i - C_S^i) S_i \beta_i \right]^2 \Delta t + \frac{1}{N^2} \sum_i (D^i - C_S^i)^2 S_i^2 \sigma_i^2 \Delta t \\ &+ \frac{1}{2} \left[ \frac{1}{N} \sum_i \left[ (D^i - C_S^i) S_i - C_{SS}^i S_i^2 \right] \beta_i^2 \right]^2 \Delta t^2 \\ &+ 2 \left[ \frac{1}{N} \sum_i (D^i - C_S^i) S_i \beta_i \right] \left[ \frac{1}{N} \sum_j (D^j - C_S^j) S_j \mu_j \beta_j \right] \Delta t^2 \\ &- 2 \left[ \frac{1}{N} \sum_i (D^i - C_S^i) S_i \beta_i \right] \left[ \frac{1}{N} \sum_j C_{SS}^j S_j^2 \beta_j (\mu_j - r) \right] \Delta t^2 \\ &+ \frac{1}{N^2} \sum_i \left[ \frac{1}{2} (\tilde{\sigma}_i^4 - \beta_i^4) \left[ (D^i - C_S^i) S_i - C_{SS}^i S_i^2 \right]^2 \right. \\ &\quad \left. + 2\mu_i \sigma_i^2 (D^i - C_S^i)^2 S_i^2 - 2\sigma_i^2 (\mu_i - r) (D^i - C_S^i) C_{SS}^i S_i^3 \right] \quad (\text{A7}) \end{aligned}$$

These expressions can be simplified considerably if we impose the restriction on the allocations  $D^i$  that the total linear market exposure of the hedge portfolio should vanish,

$$\frac{1}{N} \sum_i (D^i - C_S^i) S_i \beta_i = 0. \quad (\text{A8})$$

This restriction can always be imposed, and is identically satisfied for the standard Black-Scholes hedge. The alternative hedging strategy we propose in this paper also satisfies this constraint. If only market risk is priced, i.e.  $\mu_i = r + \kappa_0 \beta_i$  for all  $i$ , with  $\kappa_0$  the price of market risk, this condition ensures that the expected return on the hedge portfolio (A6) vanishes up to order  $\mathcal{O}(\Delta t^2)$ , or  $E[\Delta H^B] = \mathcal{O}(\Delta t^2)$ , as stated in the theorem. Furthermore, the variance of the hedge portfolio now reduces to

$$\begin{aligned} \text{var}[\Delta H] &= \frac{1}{N^2} \sum_i (D^i - C_S^i)^2 S_i^2 \sigma_i^2 \Delta t \\ &+ \frac{1}{2} \left[ \frac{1}{N} \sum_i \left[ (D^i - C_S^i) S_i - C_{SS}^i S_i^2 \right] \beta_i^2 \right]^2 \Delta t^2 \\ &+ \frac{1}{N^2} \sum_i \left[ \frac{1}{2} (\tilde{\sigma}_i^4 - \beta_i^4) \left[ (D^i - C_S^i) S_i - C_{SS}^i S_i^2 \right]^2 \right. \\ &\quad \left. + 2\mu_i \sigma_i^2 (D^i - C_S^i)^2 S_i^2 - 2\sigma_i^2 (\mu_i - r) (D^i - C_S^i) C_{SS}^i S_i^3 \right] \quad (\text{A9}) \end{aligned}$$

One might be concerned about the positivity of this expression (and similarly (A7)), since it does contain terms that are not *a priori* positive. These terms, however, combine with the terms linear in  $\Delta t$  into complete squares, up to terms of order  $\mathcal{O}(\Delta t^3)$ . ■

**Proof of Theorem 3:** First we show that the difference between the expected value of the hedge error and its expected value conditional on the market process  $z_0$  can be made arbitrary small. For any integer  $n$ , we can choose the allocations  $D^i$  such that

$$\mathbb{E}[\Delta H] = \mathbb{E}[\Delta H|z_0] + \mathcal{O}(\Delta t^{\frac{n+1}{2}}). \quad (\text{A10})$$

where  $z_0$  is  $\mathcal{N}(0,1)$ -distributed and now denotes the realization of the market process. In other words, we can choose the allocations  $D^i$  such that the exposure to the common stochastic factor is cancelled to this order. Moreover, it will turn out that both expectations vanish to this order in  $\Delta t$ .

We make use of an expansion of the conditional expectation  $\mathbb{E}[\Delta H|z_0]$  in powers of  $z_0$  and  $\Delta t$  to derive this preliminary result. Focussing on these parameters  $z_0$  and  $\Delta t$ , we note that in the expansion of the hedging error each factor of  $z_0$  is always accompanied by a factor  $\Delta t^{1/2}$ . In addition, higher powers of  $\Delta t$  arise in the expansions, which means that we can write

$$\begin{aligned} \mathbb{E}[\Delta H|z_0] &= \sum_{\substack{m, n \geq 0 \\ m+n > 0}} \tilde{h}_{mn} \left( z_0 \Delta t^{1/2} \right)^m \Delta t^{n/2} \\ &= \sum_{\ell > 0} \sum_{k=0}^{\ell} h_k^{\ell} z_0^k \Delta t^{\ell/2}. \end{aligned} \quad (\text{A11})$$

The boundary conditions in the above expressions ensure that no constant (independent of  $\Delta t$ ) term appears. The expression on the second line is more convenient for our purposes, and will be used in the remainder of this appendix. Using the fact that, for odd powers  $n$  of  $z_i$ ,  $\mathbb{E}[z_i^n] = 0$ , it follows that  $h_k^{\ell} = 0$  if  $\ell - k$  is odd.

Since our aim is to obtain allocations  $D^i$  such that the exposure to the market stochastic factor  $z_0$  in  $\mathbb{E}[\Delta H|z_0]$  cancels, we need the coefficients  $h_k^{\ell}$  to vanish. We therefore focus explicitly on the terms in  $h_k^{\ell}$  proportional to the allocations  $D^i$ . These terms are equal to

$$\mathbb{E} \left[ \frac{\Delta S_i}{S_i} \middle| z_0 \right] = e^{\beta_i z_0 \Delta t^{1/2} + (\mu_i - \frac{1}{2} \beta_i^2) \Delta t} - 1. \quad (\text{A12})$$

Expanding the above expression in terms of  $z_0$  and  $\Delta t$ , and denoting the contribution to  $h_k^{\ell}$  of the cash and derivative component of the hedge portfolio by  $c_k^{\ell}$ , we obtain

$$h_k^{\ell} = \begin{cases} \frac{1}{N} \sum_{i=1}^N \frac{1}{m! k!} D^i S_i (\mu_i - \frac{1}{2} \beta_i^2)^m \beta_i^k - c_k^{\ell} & \text{if } \ell - k \text{ is even, with } m = \frac{\ell - k}{2}, \\ 0 & \text{if } \ell - k \text{ is odd,} \end{cases} \quad (\text{A13})$$

where  $c_k^{\ell}$  is of course independent of the allocations  $D^i$ .

We can now choose the allocations  $D^i$  such that all  $h_k^{\ell}$  vanish for  $k > 0$  and  $\ell \leq n$ , with  $n$  given. If only market risk is priced, this amounts to a set of  $n$  linear constraints on the  $D^i$ . We show this by defining an operator  $\mathcal{K}$  that vanishes when acting on  $\mathbb{E}[\Delta H|z_0]$ . It is given by

$$\mathcal{K} = \frac{\partial}{\partial \Delta t} - \frac{z_0 + 2\kappa_0 \sqrt{\Delta t}}{2\Delta t} \frac{\partial}{\partial z_0} + \frac{1}{2\Delta t} \frac{\partial^2}{\partial z_0^2} - r. \quad (\text{A14})$$

Note that  $\mathcal{K}$  does not depend on idiosyncratic parameters (no index  $i$  is present). We shall first establish that  $\mathcal{K}$  indeed identically vanishes on the conditional expectation of

the hedging error. From the expression for the conditional expected stock values given in equation (A12) we can infer immediately that

$$\mathcal{K} E[S_i(t + \Delta t)|z_0] = (\mu_i - r - \lambda\beta_i) E[S_i(t + \Delta t)|z_0] = 0. \quad (\text{A15})$$

Similarly, we can show that  $\mathcal{K}$  also vanishes when acting on the conditional expectation of the option portfolio. Calculating the conditional expectation and acting with the operator  $\mathcal{K}$  commute, which implies that we can write (note that  $S_i$ ,  $C^i$ ,  $C_S^i$  and  $C_{SS}^i$  are all evaluated at time  $t + \Delta t$ )

$$\begin{aligned} \mathcal{K} E[C^i|z_0] &= E[\mathcal{K} C^i|z_0] \\ &= E \left[ C_t^i + \frac{1}{2} \beta_i^2 C_{SS}^i S_i^2 + C_S^i S_i \left( \mu_i - \kappa_0 \beta_i - \frac{1}{2} \sigma_i^2 + \frac{1}{2} \frac{\sigma_i z_i}{\sqrt{\Delta t}} \right) - r C^i \right] \end{aligned} \quad (\text{A16})$$

Using the explicit form of the conditional expectations in terms of an integral over the idiosyncratic probability density, we can directly establish that

$$E[C_S^i S_i z_i | z_0] = \sigma_i \sqrt{\Delta t} E[C_S^i S_i + C_{SS}^i S_i^2 | z_0]. \quad (\text{A17})$$

Plugging this back in into the previous equation, and using the Black-Scholes equation, yields

$$\mathcal{K} E[C^i|z_0] = (\mu_i - r - \kappa_0 \beta_i) E[C^i S_i | z_0]. \quad (\text{A18})$$

Finally, it follows immediately that  $\mathcal{K}$  vanishes when operating on the cash component of the hedge portfolio. As a result, we established that indeed

$$\mathcal{K} E[\Delta H | z_0] = 0 \quad (\text{A19})$$

if only market risk is priced.

Next we use the representation of  $E[\Delta H | z_0]$  in terms of the expansion coefficients  $h_k^\ell$  as given in (A11), in order to derive the corresponding constraints on the coefficients  $h_k^\ell$ . Those  $h_k^\ell$  that vanish identically as a result of (A19) do not impose any restrictions on the allocations  $D^i$ . We therefore rewrite (A19) as

$$\sum_{\ell > 0} \sum_{k=0}^{\ell} \left[ \frac{1}{2} (\ell - k) h_k^\ell - r h_k^{\ell-2} - \kappa_0 (k+1) h_{k+1}^{\ell-1} + \frac{1}{2} (k+1)(k+2) h_{k+2}^\ell \right] z_0^k \Delta t^{\ell/2-1} = 0, \quad (\text{A20})$$

where we defined  $h_k^0$ ,  $h_k^{-1}$ , and  $h_k^\ell$  with  $k > \ell$  to vanish. As a consequence, each coefficient as denoted by the square brackets has to vanish separately. Using these relations, it follows that at each subsequent order in  $\Delta t^{1/2}$ , there is only one additional constraint, corresponding to  $h_\ell^\ell$ . This coefficient is clearly not constrained by the above formula. Setting it to zero, in combination with setting all  $h_k^k$  to zero for  $k < \ell$ , ensures that  $h_k^m$  vanishes for all  $m \leq \ell$ .

These remaining constraints, corresponding to  $h_\ell^\ell$  with  $\ell \leq n$ , can be satisfied by choosing the allocations  $D^i$  if the system of constraints is non singular, in other words, if the matrix  $B$  with entries  $b_{i\ell} = (\beta_i)^\ell$  ( $i = 1, \dots, N$ ,  $\ell = 1, \dots, n$ ) has  $\text{rank}(B) \geq n$ . This is equivalent to the statement that out of the  $N$  securities we need at least  $n$  securities with different  $\beta_i$ , and  $\beta_i \neq 0$ . This establishes the preliminary result in (A10).

Moreover, setting  $h_\ell^\ell$  to zero for all  $\ell \leq n$  implies that  $E[\Delta H | z_0]$  is zero to order  $\mathcal{O}(\Delta t)^{n+1}$ . In particular also the coefficients  $h_0^{2\ell}$  need to vanish as a result of (A20). The

explicit form of the first few constraints that are not automatically satisfied is given by

$$\begin{aligned} h_1^1 &= \frac{1}{N} \sum_{i=1}^N (D^i - C_S^i) S_i \beta_i \\ h_2^2 &= \frac{1}{2N} \sum_{i=1}^N [(D^i - C_S^i) S_i - C_{SS}^i S_i^2] \beta_i^2. \end{aligned} \quad (\text{A21})$$

Clearly, these constraints correspond to the terms in (A7) that dominate in the large  $N$  limit, at linear and quadratic order in  $\Delta t$  respectively. In general, these constraints are

$$h_\ell^\ell = \frac{1}{\ell! N} \sum_i \left[ D^i S_i - \sum_{m=1}^{\ell} a_\ell^m \frac{\partial^m C^i}{\partial S_i^m} S_i^m \right] \beta_i^\ell, \quad (\text{A22})$$

where the coefficients  $a_\ell^m$  are recursively given by

$$\begin{aligned} a_\ell^1 &= 1, \\ a_\ell^m &= m a_{\ell-1}^m + a_{\ell-1}^{m-1} \quad \text{for } m = 2 \dots \ell - 1, \\ a_\ell^\ell &= 1, \end{aligned} \quad (\text{A23})$$

which will be useful for the numerical results in Section 3.

We next show that the variance of the conditional expectation vanishes up to higher order terms in  $\mathcal{O}(1/N)$ , or

$$E[(\Delta H)^2 | z_0] = E[\Delta H | z_0]^2 + \mathcal{O}(1/N). \quad (\text{A24})$$

This can easily be established by writing

$$(\Delta H)^2 = \frac{1}{N^2} \sum_{ij} \Delta H^i \Delta H^j \quad (\text{A25})$$

and noticing the  $\Delta H^i$  only depends on the stochastic realizations  $z_0$  and  $z_i$ . Taking expectations of all idiosyncratic processes  $z_i$ , we obtain

$$E[\Delta H^i \Delta H^j | z_0] = E[\Delta H^i | z_0] E[\Delta H^j | z_0] + \delta_{ij} f^i(z_0), \quad (\text{A26})$$

where  $f^i(z_0)$  is a (smooth) function of  $z_0$ . Inserting this in (A25) proves (A24).

Finally we can combine the two results above to show that the variance of the unconditional expectation of the hedging error vanishes up to sufficiently high order in  $\Delta t$  and  $1/N$ . We find

$$\begin{aligned} E[(\Delta H)^2] &= E[E[(\Delta H)^2 | z_0]] \\ &= E[E[\Delta H | z_0]^2] + \mathcal{O}(1/N) \\ &= E[(E[\Delta H] + E[\Delta H | z_0] - E[\Delta H])^2] + \mathcal{O}(1/N) \\ &= E[\Delta H]^2 + \mathcal{O}(1/N) + \mathcal{O}(\Delta t^{n+1}), \end{aligned} \quad (\text{A27})$$

where we substituted (A24) in the second equality, and used (A10) in the last equality. ■

We next state an analogous theorem for the case when also idiosyncratic risk is priced, i.e.  $\mu_i = r + \kappa_0 \beta_i + \kappa_i \sigma_i$ . For large enough portfolio size ( $N$ ), and sufficient heterogeneity, now also in the expected returns ( $\mu_i$ ) as well as in exposures to the common market process ( $\beta_i$ ), it is again possible to construct a riskless hedge portfolio. This portfolio gives rise to (additional) arbitrage opportunities.

**Theorem 4** *If both market and idiosyncratic risk are priced, we can choose the allocations  $D^i$  such that for any integer  $n$  the hedge error variance vanishes up to terms of order  $\mathcal{O}(\Delta t^{n+1})$  and terms of order  $\mathcal{O}(1/N)$  if*

$$N \geq N_{\min} = \begin{cases} \frac{1}{4}n(n+2) & \text{if } n \text{ is even,} \\ \frac{1}{4}(n+1)^2 & \text{if } n \text{ is odd,} \end{cases} \quad (\text{A28})$$

and  $\text{rank}(B) \geq N_{\min}$ , where  $B$  is an  $N_{\min} \times N$ -matrix with elements  $b_{ij}$ ,

$$b_{ij} = \left( \mu_j - \frac{1}{2}\beta_j^2 \right)^m \beta_j^k, \quad (\text{A29})$$

with  $j = 1, \dots, N$ ,  $i = 1, \dots, N_{\min}$ ,  $\ell = \lfloor \sqrt{i-3} \rfloor$ ,  $k = \lfloor 2i - \frac{1}{2}\ell^2 \rfloor$ , and  $m = \frac{\ell-k}{2}$ .

**Proof of Theorem 4:** This proof is mostly analogous to that of Theorem 3. The difference is that, if idiosyncratic risk is also priced, it is no longer possible to construct an operator similar to  $\mathcal{K}$  that annihilates the conditional expectation of the hedging error. As a result, we need a larger number (increasing quadratically with  $n$ ) of securities  $S_i$  in order to satisfy equation (A10). More specifically, whereas in the previous theorem we only had to cancel the nontrivial constraints corresponding to  $h_\ell^\ell$  with  $\ell \leq n$ , we now need to choose the allocations  $D^i$  in order to cancel all  $h_k^\ell$  with  $\ell \leq n$  and  $k > 0$ . Note that we do not need  $h_0^\ell$  to vanish in order for (A10) to be valid.

The number of constraints can be found by straightforwardly counting the number of nontrivial  $h_k^\ell$  in (A13), leading to the minimal required number of securities to satisfy these constraints being equal to

$$N_{\min} = \begin{cases} \frac{1}{4}n(n+2) & \text{if } n \text{ is even,} \\ \frac{1}{4}(n+1)^2 & \text{if } n \text{ is odd.} \end{cases} \quad (\text{A30})$$

In order for this set of constraints to be non singular, we need in addition that the matrix  $B$  with elements  $b_{ij} = (\mu_i - \frac{1}{2}\beta_i^2)^m \beta_i^k$ , with  $i = 1, \dots, N$ ,  $j = 1, \dots, N_{\min}$ ,  $\ell = \lfloor \sqrt{j-3} \rfloor$ ,  $k = \lfloor 2j - \frac{1}{2}\ell^2 \rfloor$ , and  $m = \frac{\ell-k}{2}$ , has  $\text{rank}(B) \geq N_{\min}$ . This can be straightforwardly deduced from (A13).

The remainder of the proof is identical to that of the previous theorem. Note that in this case there is no constraint on the coefficients  $h_0^\ell$ . Therefore the expected value of the hedge return no longer needs to vanish, giving rise to arbitrage opportunities. ■

## B Useful identities

The following expressions are used throughout the computations. They are given without proof, but they can straightforwardly be derived from equation (1). The discrete time analogue of this expression is

$$\frac{\Delta S_i}{S_i} = e^{(\beta_i z_0 + \sigma_i z_i)\Delta t^{1/2} + (\mu_i - \frac{1}{2}\tilde{\sigma}_i^2)\Delta t} - 1 \quad (\text{B31})$$

where the stochastic realizations  $z_0, z_i$  are independently  $\mathcal{N}(0, 1)$  distributed. Taking expectations of the product of powers of (B31) gives rise to

$$\mathbb{E} \left[ \left( \frac{\Delta S_i}{S_i} \right)^k \left( \frac{\Delta S_j}{S_j} \right)^l \right] = \sum_{m=0}^k \sum_{n=0}^l (-1)^{k+l+m+n} \binom{k}{m} \binom{l}{n} \exp \left[ \left( m(\mu_i - \frac{1}{2}\tilde{\sigma}_i^2) + n(\mu_j - \frac{1}{2}\tilde{\sigma}_j^2) \right) \right]$$

$$+ \frac{1}{2}(m\beta_i + n\beta_j)^2 + \frac{1}{2}m^2\sigma_i^2 + \frac{1}{2}n^2\sigma_j^2 + mn\sigma_i^2\delta_{ij} \Big) \Delta t \Big]. \quad (\text{B32})$$

Expanding the above expressions in powers of  $\Delta t$  up to the relevant order for the calculations in the proof of Theorem 2, gives rise to

$$\begin{aligned} \text{E} \left[ \frac{\Delta S_i}{S_i} \right] &= \mu_i \Delta t \\ \text{E} \left[ \frac{\Delta S_i}{S_i} \frac{\Delta S_j}{S_j} \right] &= [\beta_i\beta_j + \sigma_i^2\delta_{ij}] \Delta t \\ &\quad + \left[ \mu_i\mu_j + \frac{1}{2}\beta_i^2\beta_j^2 + \beta_i\beta_j(\mu_i + \mu_j) \right. \\ &\quad \left. + \left( \frac{1}{2}(\tilde{\sigma}_i^4 - \beta_i^4) + 2\mu_i\sigma_i^2 \right) \delta_{ij} \right] \Delta t^2 \\ \text{E} \left[ \frac{\Delta S_i}{S_i} \left( \frac{\Delta S_j}{S_j} \right)^2 \right] &= [\mu_i\tilde{\sigma}_j^2 + \beta_i^2\beta_j^2 + 2\beta_i\beta_j(\mu_j + \tilde{\sigma}_j^2) \\ &\quad + (\tilde{\sigma}_i^4 - \beta_i^4 + 2\sigma_i^2(\mu_i + \tilde{\sigma}_i^2)) \delta_{ij}] \Delta t^2 \\ \text{E} \left[ \frac{\Delta S_i}{S_i} \left( \frac{\Delta S_j}{S_j} \right)^3 \right] &= 3 [\beta_i\beta_j + \sigma_i^2\delta_{ij}] \tilde{\sigma}_j^2 \Delta t^2 \\ \text{E} \left[ \left( \frac{\Delta S_i}{S_i} \right)^2 \left( \frac{\Delta S_j}{S_j} \right)^2 \right] &= [\tilde{\sigma}_i^2\tilde{\sigma}_j^2 + 2\beta_i^2\beta_j^2 + 2(\tilde{\sigma}_i^4 - \beta_i^4)\delta_{ij}] \Delta t^2 \end{aligned} \quad (\text{B33})$$

for the expectation of products of powers of these processes.

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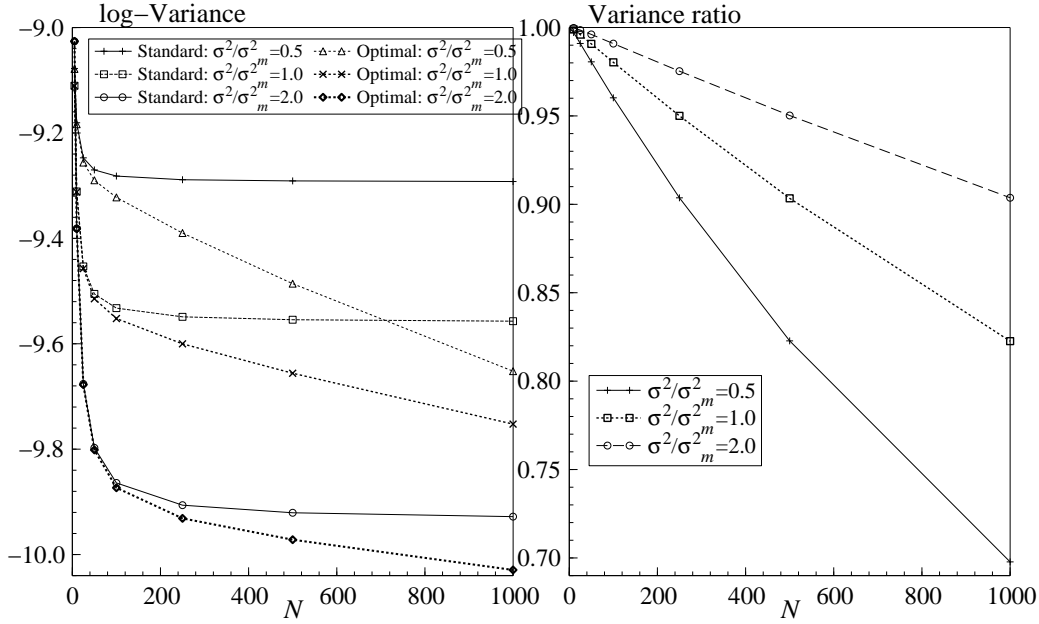


Figure 1: Variances

For a portfolio of  $N$  at-the-money calls, the left panel depicts the hedge error variances of the standard delta hedge portfolio as well as the variances of the optimal hedge portfolio. The right panel shows the ratio of these variance. In both panels these results are shown as a function of the portfolio size  $N$ . The systematic risk exposures of the underlying securities are given by  $\beta_i/\sigma_m \sim 1+0.3\Phi^{-1}((2i-1)/(2N))$ ,  $\Phi^{-1}$  the inverse standard normal c.d.f., and  $i = 1, \dots, N$ . The idiosyncratic variance is set to  $\sigma^2 = c\sigma_m^2$  for  $c = 0.5, 1, 2$ , the market variance to  $\sigma_m = 25\%$ , the riskfree rate to  $r = 0$ , and the market risk premium to  $\kappa_0 = 20\%$ . The options have a maturity of 3 months ( $T - t = 0.25$ ) and the holding period is one month ( $\Delta t = 1/12$ ).

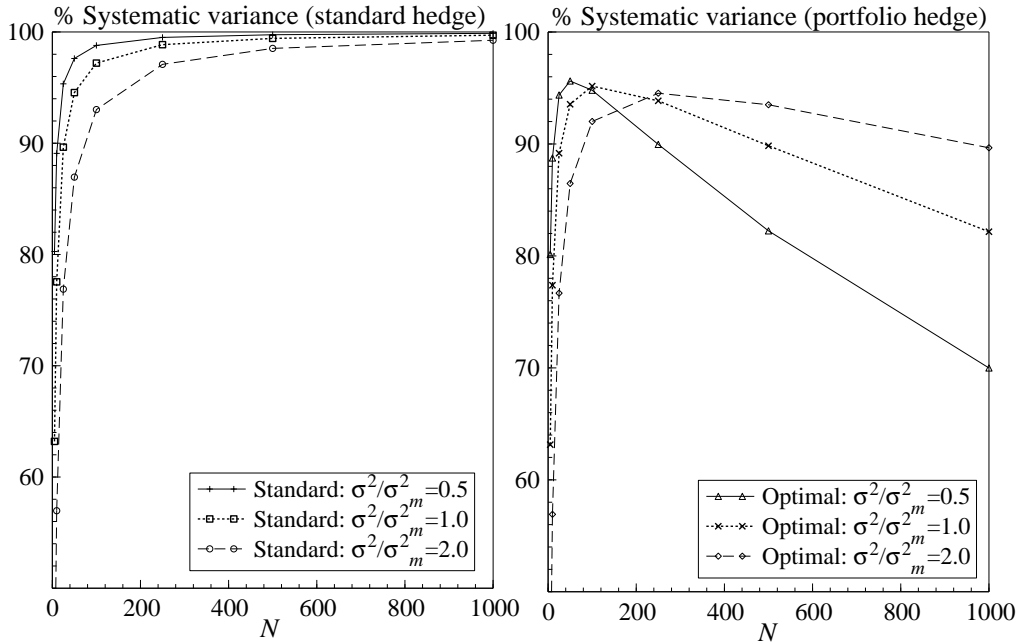


Figure 2: Variances decomposition

For a portfolio of  $N$  at-the-money calls on different underlying values, this figure contains the percentage of the hedge error variances due to systematic risk. It is defined via the hedge error variance formula in Theorem 2 as  $\mathcal{A}_2/(\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3)$ . The percentage is plotted as a function of the portfolio size  $N$ . Parameters are chosen as in Figure 1.

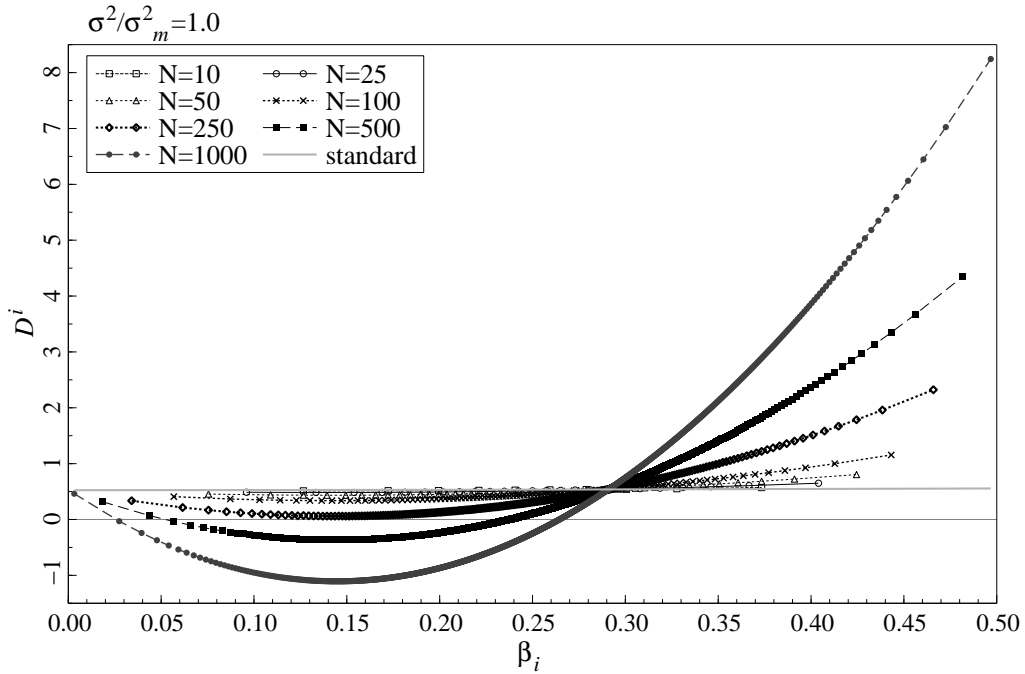


Figure 3: Hedge portfolio stock holdings

This figure contains the optimal portfolio holdings  $D^i$  as a function of the systematic risk exposures  $\beta_i$  for various portfolio sizes  $N$ . Parameters are chosen as in Figure 1. The  $D^i$ 's are obtained by optimizing the expression for hedge error variance in Theorem 2, which correct to order  $\mathcal{O}(\Delta t^2)$ . For comparison, we also plot the standard hedge portfolio holdings  $D^i = C_S^i$  (standard).

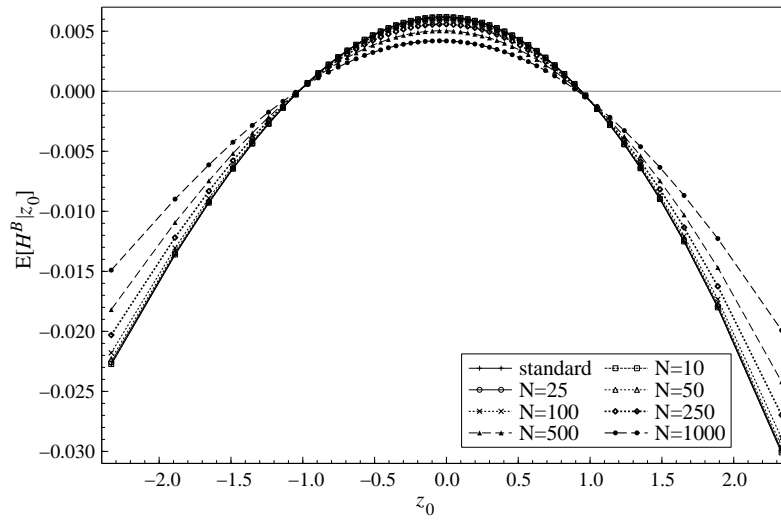


Figure 4: Conditional expected hedging error

The horizontal axis gives the realization of the systematic risk component  $z_0$ . The vertical axis gives the conditional (on  $z_0$ ) expectation of the hedge error  $\Delta H^B$ . The stock holdings in the hedge portfolio are the ones presented in Figure 3 and are obtained by minimizing the expression for the hedge error variance in Theorem 2.

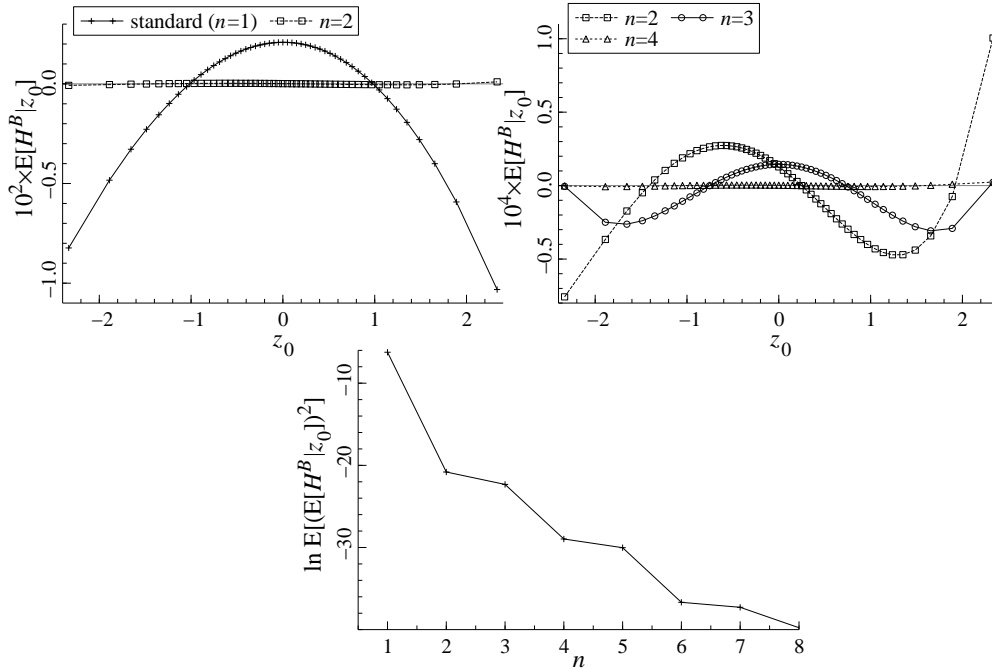


Figure 5: Hedge errors for asymptotic portfolios ( $N \rightarrow \infty$ )

The horizontal axis gives the realization of the systematic risk component  $z_0$ . The vertical axis in the top panels gives the conditional (on  $z_0$ ) expectation of the hedge error  $\Delta H^B$ . The stock holdings in the hedge portfolio are obtained by setting the hedge error variance equal to zero up to the  $n$ -th order  $\Delta t$  using the formulas in the proof of Theorem 3. Note the different vertical scale between the top-left and top-right panel. The bottom panel holds the log of the expectation of the squared conditionally expected hedge errors as presented in the top panels.