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Dual-dual formulation for a contact problem with friction

Abstract: A variational inequality formulation is derived for some frictional contact problems from linear elasticity. The formulation exhibits a two-fold saddle point structure and is of dual-dual type, involving the stress tensor as primary unknown as well as the friction force on the contact surface by means of a Lagrange multiplier. The approach starts with the minimization of the conjugate elastic potential. Applying Fenchel's duality theory to this dual minimization problem the connection to the primal minimization problem and a dual saddle point problem is achieved. The saddle point problem possesses the displacement field and the rotation tensor as further unknowns. Introducing the friction force yields the dual-dual saddle point problem. The equivalence and unique solvability of both problems is shown with the help of the variational inequality formulations corresponding to the saddle point formulations, respectively.

Keywords: Contact problems, friction, Fenchel duality, variational inequalities

AMS Mathematics Subject Classification: 35A15, 46A20, 52A40, 74B05, 74M10, 74M15

1 Introduction

The objective of this paper is to derive a dual-dual variational inequality formulation of some frictional contact problems from linear elasticity. The primary unkown of this formulation is the stress tensor which is an important quantity in engineering sciences. As in primal formulations for this kind of problems the displacement field is approximated, the stress tensor has to be computed in a postprocessing which reduces the polynomial order of the corresponding approximation and yields a further source of error. This error-prone computation is avoided within our proposed approach.

The solvability of variational problems having dual-dual form was investigated by Gatica [9]. Based on this work Gatica et al. [11, 13] derived dual-dual mixed formulations for problems in elastostatics and exterior transmission prob-

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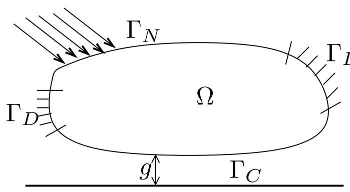
lems using the coupling of finite elements and boundary elements. Dual mixed finite element methods for nonlinear elasticity are presented in Gatica et al. [5, 10]. The latter work presents a formulation having three-fold saddle point structure. Maischak [20] presents a dual approach for a transmission problem with Signorini conditions. For a transmission problem with friction a similar approach is presented by Maischak and Stephan in [21]. Kunisch and Stadler [19] use Fenchel's duality theory to derive the dual problem with the friction force as additional Lagrange multiplier for a contact problem with Tresca friction. Furthermore, Belhachmi et al. [6] present a dual formulation for some unilateral crack problems in elasticity. Here, the authors consider a contact problem where no friction occurs. Note, that the presented work considers both types of variational inequalities. The first type resulting from the normal part and the second type resulting from the friction in the contact conditions, respectively.

The paper is organized as follows. In Section 2 we introduce a model problem and its corresponding dual and primal minimization problems. In Section 2.1 we start with the dual minimization problem which arises from the minimum principle of the conjugate elastic potential and derive its conjugate problem, i.e. the corresponding primal minimization problem. Applying Fenchel's duality theorem, see e.g. [16], we derive a saddle point formulation (11) where the displacement field and the rotation tensor act as Lagrange multipliers. The equivalence between the saddle point problem and the two minimization problems (primal and dual) is proven in Theorem 2.2. The corresponding equivalent dual variational inequality formulation involves a non-differentiable friction functional. Therefore, another Lagrange multiplier, denoted the friction force, is introduced in Section 3. This leads us to another saddle point problem and the equivalent dual-dual variational inequality formulation having two-fold saddle point structure. Theorem 3.3 proves the equivalence of the dual variational inequality problem and the dual-dual variational inequality problem and finally, Theorem 3.4 states existence and uniqueness for both variational inequality problems. Numerical experiments based on our approach are obtained in [1] and will be reported in a forthcoming paper.

2 Dual variational inequality formulation of a frictional contact problem

Let us consider the following contact problem in elasticity with Tresca friction. Assume a linear elastic body occupying the open bounded domain $\Omega \subset \mathbb{R}^d$, where

$d = 2$ or 3. The Lipschitz-boundary $\Gamma := \partial\Omega$ is divided into three disjoint parts, the non-empty Dirichlet boundary Γ_D where we assume homogeneous Dirichlet conditions to hold, the Neumann boundary Γ_N where a prescribed traction is acting on the body and the contact boundary Γ_C where the body is supposed to come into contact with a rigid foundation, see Figure 1. For ease of demonstration we assume that the contact boundary Γ_C and the Neumann boundary Γ_N do not touch, i.e. $\overline{\Gamma_C} \cap \overline{\Gamma_N} = \emptyset$. However by suitable modifications our analysis also applies to the other case (see Remark 2.3). Then the displacement vector field $\mathbf{u}(\mathbf{x})$ in each material point $\mathbf{x} \in \Omega$ satisfies the following PDE:



$$\begin{aligned} -\operatorname{div}\sigma(\mathbf{u}) &= \mathbf{f} && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_D, \\ \sigma(\mathbf{u})\cdot\mathbf{n} &= \mathbf{t}_0 && \text{on } \Gamma_N, \\ u_n \le g;\quad \sigma_n \le 0;\quad (u_n - g)\sigma_n &= 0 && \text{on } \Gamma_C, \\ |\boldsymbol{\sigma}_t| \le \mathcal{F};\quad \boldsymbol{\sigma}_t\cdot\mathbf{u}_t + \mathcal{F}\,|\mathbf{u}_t| &= 0 && \text{on } \Gamma_C. \end{aligned} \tag{1}$$

Fig. 1. Boundary distribution

Here, the stress tensor $\sigma(\mathbf{u})$ is connected to the strain tensor $\varepsilon(\mathbf{u})$ via Hooke's law for linear elasticity, i.e. $\sigma(\mathbf{u}) := \mathbb{C} : \varepsilon(\mathbf{u})$ with the elliptic and symmetric Hooke's tensor of fourth order $\mathbb{C}$, see e.g. Duvaut and Lions [7]. The body force $\mathbf{f} \in \mathbf{L}^2(\Omega)$, the prescribed traction $\mathbf{t}_0 \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$, the positive gap function $g \in H^{\frac{1}{2}}(\Gamma_C)$ and the positive friction function $\mathcal{F} \in L^\infty(\Gamma_C)$ are assumed to be given. On the contact boundary Γ_C we observe the decompositions $\mathbf{u} = u_n\mathbf{n} + \mathbf{u}_t$ and $\sigma\cdot\mathbf{n} = \sigma_n\mathbf{n} + \boldsymbol{\sigma}_t$ of the displacement and the traction into their normal and tangential parts, where $\mathbf{n}$ denotes the unit normal exterior to the boundary Γ and

$$u_n := \mathbf{u}\cdot\mathbf{n}, \qquad \mathbf{u}_t := \mathbf{u} - u_n\mathbf{n}, \qquad \sigma_n := \mathbf{n}^T\cdot\sigma\cdot\mathbf{n}, \qquad \boldsymbol{\sigma}_t := \sigma\cdot\mathbf{n} - \sigma_n\mathbf{n}.$$

The last boundary conditions in the boundary value problem (1) state the friction law of Tresca friction. A more physical law would be the Coulomb friction law, see e.g. Kikuchi and Oden [17, Chapter 10], which reads

$$\begin{aligned} &\text{if } |\boldsymbol{\sigma}_t| < \mu_f\,|\sigma_n|, && \text{then } \mathbf{u}_t = 0 && \text{on } \Gamma_C, \\ &\text{if } |\boldsymbol{\sigma}_t| = \mu_f\,|\sigma_n|, && \text{then } \exists\, s \ge 0 :\ \mathbf{u}_t = -s\,\boldsymbol{\sigma}_t && \text{on } \Gamma_C. \end{aligned} \tag{2}$$

Here $\mu_f \ge 0$ is the friction coefficient, which is assumed to be uniformly Lipschitz on Γ_C. It specifies how strong the body is sticking to the rigid foundation when coming into contact at some point $\mathbf{x} \in \Gamma_C$. The second line in (2) is equivalent to

$$\boldsymbol{\sigma}_t \cdot \mathbf{u}_t = -\mu_f \, |\sigma_n||\mathbf{u}_t| \quad \text{on } \Gamma_C. \tag{3}$$

When considering real life problems the friction coefficient, being not necessarily constant is usually not known exactly since it depends on the material properties of the body Ω and the rigid foundation as well as on the roughness of both materials at each point. The last factor is a local property that changes in each material point. Nevertheless we restrict ourselves to some constant values for the friction coefficient which is sufficient for our purpose.

However, a direct treatment of contact problems with Coulomb friction seems difficult and in some cases even impossible. The first result concerning existence of a solution for a sufficiently small friction coefficient was discovered by Nečas et al. [23]. To approach the Coulomb friction law, Nečas et al. [15, see Chapter 2.5.4] propose a fixed point iteration. We give a short abstract of this approach and refer to the above references for more details.

For given $\mathcal{F}_0 \in L^\infty(\Gamma_C)$ and μ_f, both positive, we compute $\mathcal{F}_{k+1} := \mu_f \, |\sigma_n^k|$ where σ_n^k is the normal stress on Γ_C of the solution of the contact problem (1) with given Tresca friction function $\mathcal{F}_k \geq 0$. We proceed until some stopping criterion is reached. For convenience we drop the index k of the friction function $\mathcal{F}_k$. Note that we demand $\mathcal{F}$ to be in $L^\infty(\Gamma_C)$ and not in $H^{-\frac{1}{2}}(\Gamma_C)$ as we would expect from the definition above. The reason for this assumption will be seen later in this work.

In order to derive a dual variational inequality formulation of problem (1) we consider the dual minimization problem on the Hilbert space

$$\mathbf{X} := \mathbf{H}(\operatorname{div}, \Omega) = \big\{ \tau \in \mathbb{R}^{d \times d} : \tau_{ij} \in L^2(\Omega), \ \sum_{j=1}^{d} \partial_j \tau_{ij} \in L^2(\Omega), \ \forall \, i, j = 1, \dots, d \big\}.$$

The corresponding set of admissible functions is

$$\widetilde{\mathbf{K}} := \{ \tau \in \mathbf{X} : \quad \tau = \tau^T, \ -\operatorname{div} \tau = \mathbf{f} \ \text{ in } \Omega; \ \tau \cdot \mathbf{n} = \mathbf{t}_0 \ \text{ on } \Gamma_N; \\ |\boldsymbol{\tau}_t| \leq \mathcal{F}, \ \tau_n \leq 0 \ \text{ on } \Gamma_C \}.$$

The conjugate energy functional reads

$$\tilde{J}(\tau) := \frac{1}{2} \, \tilde{a}(\tau, \tau) - g(\tau)$$

involving the continuous bilinear form $\tilde{a}(\cdot, \cdot)$ and the continuous linear form $g(\cdot)$ defined by

$$\tilde{a}(\sigma, \tau) := \int_\Omega \sigma : \mathbb{C}^{-1} : \tau \, dx, \qquad g(\tau) := \int_{\Gamma_C} g \tau_n \, ds.$$

The bilinear form $\tilde{a}(\cdot,\cdot)$ is coercive on the subset of all functions $\tau \in \mathbf{X}$ with $\operatorname{div}\tau = 0$, which is a consequence of the definition of $\mathbb{C}^{-1}$, see e.g. Duvaut and Lions [7]. The minimum principle of the conjugate elastic potential states that the stress tensor $\sigma(\mathbf{u})$ of the solution of (1) minimizes $\tilde{J}(\cdot)$ over $\widetilde{\mathbf{K}}$. This leads us to the dual minimization problem.

Find $\sigma \in \widetilde{\mathbf{K}}$ such that

$$\tilde{J}(\sigma) \le \tilde{J}(\tau) \qquad \forall\, \tau \in \widetilde{\mathbf{K}}. \tag{4}$$

The functional $\tilde{J}(\cdot)$ is coercive on $\widetilde{\mathbf{K}}$, strictly convex and continuous and therefore, we have the existence of a unique solution for the dual minimization problem (4), see Ekeland and Témam [8, Proposition 1.2 in Chapter II].

Analogously, we introduce the primal minimization problem corresponding to problem (1).

Find $\mathbf{u} \in \mathbf{K}_g$ such that

$$J(\mathbf{u}) \le J(\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathbf{K}_g. \tag{5}$$

Here, the closed convex subset $\mathbf{K}_g$ is defined by

$$\mathbf{K}_g := \left\{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D,\ v_n \le g \text{ a.e. on } \Gamma_C\right\}.$$

The energy functional $J(\mathbf{v}) := \frac{1}{2}a(\mathbf{v},\mathbf{v}) - L(\mathbf{v}) + j(\mathbf{v})$ involves the coercive (Γ_D has nonzero measure), symmetric and continuous bilinear form $a(\cdot,\cdot)$, the continuous linear form $L(\cdot)$ and the continuous but non-differentiable functional $j(\cdot)$ defined by

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega \varepsilon(\mathbf{u}) : \mathbb{C} : \varepsilon(\mathbf{v})\,dx, \quad L(\mathbf{v}) := \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx + \int_{\Gamma_N} \mathbf{t}_0\cdot\mathbf{v}\,ds, \quad j(\mathbf{v}) := \int_{\Gamma_C} \mathcal{F}|\mathbf{v}_t|\,ds.$$

For the primal minimization problem (5) and the dual minimization problem (4) the following result holds.

Lemma 2.1. *Let $\mathbf{u} \in \mathbf{K}_g$ and $\sigma \in \widetilde{\mathbf{K}}$ be the solutions of the primal minimization problem* (5) *and the dual minimization problem* (4)*, respectively. Then there holds*

$$\sigma = \mathbb{C} : \varepsilon(\mathbf{u}) \quad \text{and} \quad J(\mathbf{u}) + \tilde{J}(\sigma) = 0.$$

The proof follows Nečas et al. [15, see Section 1.1.12] and will be omitted. See [1, Theorem 3.3] for a detailed proof.

2.1 Saddle point formulation

We want to apply the theory of Fenchel's duality to the dual minimization problem (4). This leads us to a variational inequality formulation which may be handled numerically. A detailed introduction to this topic can be found in [8, 16].

The first step is to derive a saddle point formulation that is equivalent to the minimization problems (4) and (5). We state the dual minimization problem (4) as a minimization problem over the space $\mathbf{X}$ in order to apply the duality theory. The operators acting on the stress tensor in the constraints within the convex set $\widetilde{\mathbf{K}}$ lead us to the definition of the space

$$\mathbf{Y} := \mathbf{L}^2(\Omega) \times S \times \mathbf{H}^{-\frac{1}{2}}(\Gamma_N) \times \mathcal{H}^{-\frac{1}{2}}(\Gamma_C) \times H^{-\frac{1}{2}}(\Gamma_C),$$

where the spaces S and $\mathcal{H}^{-\frac{1}{2}}(\Gamma_C)$ depending on the dimension d are defined by

$$S := \begin{cases} L^2(\Omega), & \text{if } d = 2 \\ \mathbf{L}^2(\Omega), & \text{if } d = 3 \end{cases}, \qquad \mathcal{H}^{-\frac{1}{2}}(\Gamma_C) := \left[H^{-\frac{1}{2}}(\Gamma_C)\right]^{d-1}. \tag{6}$$

The choice of the spaces in (6) will be explained next. In the first case the definition results from the definition of the antisymmetric tensor $\textsf{\textbf{as}}(\tau) := \frac{1}{2}(\tau - \tau^T)$ of a tensor $\tau \in \mathbf{X}$. In the two-dimensional case this tensor contains only one term, namely $\frac{1}{2}(\tau_{12} - \tau_{21})$ whereas in the three-dimensional case the antisymmetric tensor contains three terms. Defining the space of antisymmetric tensors $\mathbf{S} := \{\eta \in L^2(\Omega)^{d\times d} : \textsf{\textbf{as}}(\eta) = \eta\}$ and the operator as : $\mathbf{X} \to S$

$$\operatorname{as}(\tau) := \begin{cases} \tau_{12} - \tau_{21}, & \text{if } d = 2, \\ \begin{pmatrix} \tau_{12}-\tau_{21} \\ \tau_{13}-\tau_{31} \\ \tau_{23}-\tau_{32} \end{pmatrix}, & \text{if } d = 3, \end{cases}$$

we make the following observation. For $\tau \in \mathbf{X}$, $\boldsymbol{\eta} \in \mathbf{S}$ and $\eta \in S$ with $\eta = \boldsymbol{\eta}_{12}$ if $d = 2$ and $\eta_1 = \boldsymbol{\eta}_{12}$, $\eta_2 = \boldsymbol{\eta}_{13}$ and $\eta_3 = \boldsymbol{\eta}_{23}$ if $d = 3$ it holds $\boldsymbol{\eta} : \textsf{\textbf{as}}(\tau) = \eta \cdot \operatorname{as}(\tau)$. In the following we will always identify the tensor $\boldsymbol{\eta} \in \mathbf{S}$ with its corresponding component vector $\eta \in S$ as described above.

In the second case we define the unit tangential vectors

$$\mathbf{t} := \begin{pmatrix} n_2 \\ -n_1 \end{pmatrix} \text{ if } d = 2, \qquad \text{and} \qquad \mathbf{t}_1 := \frac{1}{\left|\begin{pmatrix} n_2-n_3 \\ n_3-n_1 \\ n_1-n_2 \end{pmatrix}\right|} \begin{pmatrix} n_2-n_3 \\ n_3-n_1 \\ n_1-n_2 \end{pmatrix}, \quad \mathbf{t}_2 := \mathbf{n}\times\mathbf{t}_1 \quad \text{if } d = 3.$$

For $\mathbf{v} \in \mathbf{H}^1(\Omega)$ and $\tau \in \mathbf{X}$ we then define

$$\begin{aligned} v_t &:= \mathbf{v} \cdot \mathbf{t} \\ \tau_t &:= \boldsymbol{\tau}_t \cdot \mathbf{t} = \mathbf{t} \cdot \tau \cdot \mathbf{n} \end{aligned} \quad \text{if } d = 2, \qquad \begin{aligned} v_{t_1} &:= \mathbf{v} \cdot \mathbf{t}_1,\ v_{t_2} := \mathbf{v} \cdot \mathbf{t}_2 \\ \tau_{t_1} &:= \boldsymbol{\tau}_t \cdot \mathbf{t}_1 = \mathbf{t}_1 \cdot \tau \cdot \mathbf{n} \\ \tau_{t_2} &:= \boldsymbol{\tau}_t \cdot \mathbf{t}_2 = \mathbf{t}_2 \cdot \tau \cdot \mathbf{n} \end{aligned} \quad \text{if } d = 3.$$

If we furthermore define the operator $\mathrm{t} : \mathbf{H}^{-\frac{1}{2}}(\Gamma_C) \to \mathcal{H}^{-\frac{1}{2}}(\Gamma_C)$

$$\mathrm{t}(\mathbf{v}) := \begin{cases} \mathbf{t} \cdot \mathbf{v}, & \text{if } d = 2, \\ \binom{\mathbf{t}_1 \cdot \mathbf{v}}{\mathbf{t}_2 \cdot \mathbf{v}}, & \text{if } d = 3, \end{cases}$$

then we have $v_t = \mathrm{t}(\mathbf{v}_\mathrm{t})$ and $\tau_t = \mathrm{t}(\boldsymbol{\tau}_\mathrm{t})$, if $d = 2$ and $v_{t_i} = \mathrm{t}(\mathbf{v}_\mathrm{t})_i$ and $\tau_{t_i} = \mathrm{t}(\boldsymbol{\tau}_\mathrm{t})_i$ for $i = 1, 2$, if $d = 3$. This leads to $|\mathbf{v}_\mathrm{t}| = |v_t|$, $|\boldsymbol{\tau}_t| = |\tau_t|$ and $\mathbf{v}_t \cdot \boldsymbol{\tau}_t = v_t \tau_t$ in the two-dimensional case and $|\mathbf{v}_t| = |v_{t_1} \mathbf{t}_1 + v_{t_2} \mathbf{t}_2|$, $|\boldsymbol{\tau}_t| = |\tau_{t_1} \mathbf{t}_1 + \tau_{t_2} \mathbf{t}_2|$ and $\mathbf{v}_t \cdot \boldsymbol{\tau}_t = v_{t_1} \tau_{t_1} + v_{t_2} \tau_{t_2}$ in the three-dimensional case.

Therefore, we define the operator $\Lambda \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ as follows

$$\Lambda\tau = (\Lambda_1\tau, \Lambda_2\tau, \Lambda_3\tau, \Lambda_4\tau, \Lambda_5\tau) := (\operatorname{div}\tau, \operatorname{as}(\tau), \tau \cdot \mathbf{n}, \mathrm{t}(\tau \cdot \mathbf{n}), \tau_n). \tag{7}$$

If we now define the convex set

$$K_{\mathbf{Y}} := \left\{ (\mathbf{v}, \xi, \psi, \mu_t, \mu_n) \in \mathbf{Y} : \mathbf{v} + \mathbf{f} = 0,\ \xi = 0,\ \psi = \mathbf{t}_0,\ |\mu_t| \le \mathcal{F},\ \mu_n \le 0 \right\}$$

then we can state problem (4) as follows.

Find $\sigma \in \mathbf{X}$ such that

$$\tilde{J}(\sigma) + I_K(\Lambda\sigma) \le \tilde{J}(\tau) + I_K(\Lambda\tau) \qquad \forall\, \tau \in \mathbf{X}. \tag{8}$$

Here, the indicator function is defined with respect to the convex set $K_{\mathbf{Y}}$, i.e.

$$I_K(\mathbf{v}, \xi, \psi, \mu_t, \mu_n) := \begin{cases} 0, \text{ if } (\mathbf{v}, \xi, \psi, \mu_t, \mu_n) \in K_{\mathbf{Y}}, \\ \infty, \text{ else.} \end{cases}$$

Following the theory of Fenchel's duality we define the perturbed problem.

Find $\sigma \in \mathbf{X}$ such that for $\mathbf{y} \in \mathbf{Y}$

$$\Phi(\sigma, \mathbf{y}) \le \Phi(\tau, \mathbf{y}) \qquad \forall\, \tau \in \mathbf{X}, \tag{9}$$

where the functional $\Phi : \mathbf{X} \times \mathbf{Y} \to (-\infty, \infty]$ is defined by

$$\Phi(\tau, \mathbf{y}) := \tilde{J}(\tau) + I_K(\Lambda\tau + \mathbf{y}).$$

For $\mathbf{y} = 0$ the perturbed problem (9) is obviously equivalent to the minimization problem (8).

Let us compute the Lagrange functional $L : \mathbf{X} \times \mathbf{Y}' \to \mathbb{R} \cup \{-\infty, \infty\}$ with respect to $\Phi(\cdot, \cdot)$ defined by

$$L(\tau;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) := -\sup_{\mathbf{y}\in\mathbf{Y}}\{\langle(\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n),\mathbf{y}\rangle - \Phi(\tau,\mathbf{y})\}$$
$$= \begin{cases} \frac{1}{2}\,\tilde{a}(\tau,\tau) + B(\mathbf{u},\eta,\boldsymbol{\varphi};\tau) - F(\mathbf{u},\boldsymbol{\varphi}) & \\ + d_{C,t}(\lambda_t,\tau) - j(\lambda_t) + d_{C,n}(\lambda_n,\tau) - g(\tau), & \text{if } \lambda_n \geq 0, \\ -\infty, & \text{else,} \end{cases} \tag{10}$$

with

$$B(\mathbf{u},\eta,\boldsymbol{\varphi};\tau) := \int_\Omega (\mathbf{u}\cdot\operatorname{div}\tau + \eta\cdot\operatorname{as}(\tau))\,dx + \int_{\Gamma_N} \boldsymbol{\varphi}\cdot\tau\cdot\mathbf{n}\,ds,$$
$$F(\mathbf{u},\boldsymbol{\varphi}) := -\int_\Omega \mathbf{u}\cdot\mathbf{f}\,dx + \int_{\Gamma_N} \boldsymbol{\varphi}\cdot\mathbf{t}_0\,ds,$$
$$d_{C,t}(\lambda_t,\tau) := \int_{\Gamma_C} \lambda_t\,\mathrm{t}(\tau\cdot\mathbf{n})\;ds \qquad\text{and}\qquad d_{C,n}(\lambda_n,\tau) := \int_{\Gamma_C} \lambda_n\tau_n\,ds.$$

The result is evident since the supremum is attained for $\Lambda\tau + \mathbf{y} \in K_{\mathbf{Y}}$ and therefore $\mathbf{v} = -(\operatorname{div}\tau + \mathbf{f})$, $\xi = -\operatorname{as}(\tau)$ and $\boldsymbol{\psi} = \mathbf{t}_0 - \tau\cdot\mathbf{n}$. For the tangential part on the contact boundary the supremum in (10) is obviously attained for $|\mu_t + \mathrm{t}(\tau\cdot\mathbf{n})| \leq \mathcal{F}$. Assuming $\lambda_t \neq 0$ and using $(\mu_t + \mathrm{t}(\tau\cdot\mathbf{n}))\lambda_t \leq |\mu_t + \mathrm{t}(\tau\cdot\mathbf{n})||\lambda_t| \leq \mathcal{F}|\lambda_t|$ we observe

$$\langle\lambda_t,\mu_t\rangle_{\Gamma_C} \leq \int_{\Gamma_C} \mathcal{F}|\lambda_t|\,ds - \int_{\Gamma_C} \mathrm{t}(\tau\cdot\mathbf{n})\,\lambda_t\,ds = j(\lambda_t) - d_{C,t}(\lambda_t,\tau).$$

Taking $\mu_t := \frac{\mathcal{F}}{|\lambda_t|}\lambda_t - \mathrm{t}(\tau\cdot\mathbf{n})$ the upper bound for the supremum is achieved. If $\lambda_t = 0$ the supremum is zero as well. For the normal part on the contact boundary the supremum is clearly obtained for $\mu_n \leq -\tau_n$. If $\lambda_n < 0$, then the supremum is infinity. Otherwise, if $\lambda_n \geq 0$ the supremum is attained for $\mu_n = -\tau_n$. We have the following saddle point problem

Find $(\sigma;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) \in \mathbf{X}\times\mathbf{Y}'$ such that for all $(\tau,\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n) \in \mathbf{X}\times\mathbf{Y}'$

$$L(\sigma;\mathbf{v},\xi,\boldsymbol{\psi},\mu_t,\mu_n) \leq L(\sigma;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) \leq L(\tau;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n). \tag{11}$$

The following theorem states equivalence between the saddle point problem (11) and the dual and primal minimization problems (4) and (5). Within the proof we first show, that the primary unknown σ of the saddle point formulation is in the convex set $\widetilde{\mathbf{K}}$. The restriction of the left inequality of the saddle point problem to this convex set leads to the dual minimization problem whereas the right inequality permits the interpretation of the Lagrange multipliers, i.e. the dependence of the multipliers $(\eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n)$ on the displacement $\mathbf{u}$. The second assertion uses the equivalence of the minimization problems to appropriate variational inequality problems.

Theorem 2.2. *The saddle point problem* (11) *is equivalent to the minimization problems* (4) *and* (5) *in the following sense.*

(i) *If* $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \mathbf{Y}'$ *is a saddle point of* (11), *then* $\sigma \in \widetilde{\mathbf{K}}$ *is the unique solution of the minimization problem* (4). *Furthermore, it holds* $\sigma = \mathbb{C} : \varepsilon(\mathbf{u})$ *in* Ω, $\eta = \mathrm{as}(\nabla \mathbf{u})$ *in* Ω, $\mathbf{u} = 0$ *on* Γ_D, $\mathbf{u} + \boldsymbol{\varphi} = 0$ *on* Γ_N, $\mathrm{t}(\mathbf{u}) + \lambda_t = 0$ *on* Γ_C *and* $u_n + \lambda_n = g$ *on* Γ_C.

(ii) *If* $\sigma \in \widetilde{\mathbf{K}}$ *is the unique solution of the dual minimization problem* (4) *and* $\mathbf{u} \in \mathbf{K}_g$ *is the unique solution of the primal minimization problem* (5), *then* $(\sigma; \mathbf{u}, \mathrm{as}(\nabla \mathbf{u}), -\mathbf{u}|_{\Gamma_N}, -\mathrm{t}(\mathbf{u})\,|_{\Gamma_C}, g - u_n|_{\Gamma_C}) \in \mathbf{X} \times \mathbf{Y}'$ *is a saddle point of* (11). *Since* σ *and* $\mathbf{u}$ *are unique, the saddle point is unique as well.*

Proof. (i) Note that the left inequality in (11) holds true for all $\mu_n \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C)$ with $\mu_n < 0$. Without loss of generality we therefore restrict the Lagrange multiplier concerning the normal part on the contact boundary of the saddle point problem to

$$\mu_n \in \widetilde{H}^{\frac{1}{2}}_+(\Gamma_C) := \left\{ \mu \in \widetilde{H}^{\frac{1}{2}}(\Gamma_C) : \mu \geq 0 \ \text{ a.e. on } \Gamma_C \right\}$$

and define

$$\mathbf{Y}'_+ := \mathbf{L}^2(\Omega) \times S \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N) \times \widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C) \times \widetilde{H}^{\frac{1}{2}}_+(\Gamma_C).$$

Assume $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \mathbf{Y}'_+$ being a saddle point of (11). Then, since $2\lambda_n \in \widetilde{H}^{\frac{1}{2}}_+(\Gamma_C)$, we can insert $(0, 0, 0, 0, 0) \in \mathbf{Y}'_+$ and $(2\mathbf{u}, 2\eta, 2\boldsymbol{\varphi}, 2\lambda_t, 2\lambda_n) \in \mathbf{Y}'_+$ into the left inequality of (11). Noting that

$$j(2\lambda_t) = \int_{\Gamma_C} \mathcal{F}|2\lambda_t|\, ds = 2 \int_{\Gamma_C} \mathcal{F}|\lambda_t|\, ds = 2j(\lambda_t),$$

we conclude after subtracting $L(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n)$

$$B(\mathbf{u}, \eta, \boldsymbol{\varphi}; \sigma) - F(\mathbf{u}, \boldsymbol{\varphi}) + d_{C,n}(\lambda_n, \sigma) + d_{C,t}(\lambda_t, \sigma) - j(\lambda_t) = 0 \qquad (12)$$

and the left inequality of (11) reduces to

$$B(\mathbf{v}, \xi, \boldsymbol{\psi}; \sigma) - F(\mathbf{v}, \boldsymbol{\psi}) + d_{C,n}(\mu_n, \sigma) + d_{C,t}(\mu_t, \sigma) - j(\mu_t) \leq 0$$

for all $(\mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \in \mathbf{Y}'_+$. If we take $(\pm\tilde{\mathbf{v}}, 0, 0, 0, 0) \in \mathbf{Y}'_+$ with $\tilde{\mathbf{v}} \in \mathbf{L}^2(\Omega)$ we observe

$$\int\limits_{\Omega} \tilde{\mathbf{v}} \cdot (\mathbf{f} + \operatorname{div} \sigma)\, dx = 0 \quad \forall\, \tilde{\mathbf{v}} \in \mathbf{L}^2(\Omega)$$

from which we deduce $-\operatorname{div} \sigma = \mathbf{f}$ in Ω. In the same way by inserting $(0, \pm\tilde{\xi}, 0, 0, 0) \in \mathbf{Y}'_+$ for $\tilde{\xi} \in S$ and $(0, 0, \pm\tilde{\boldsymbol{\psi}}, 0, 0) \in \mathbf{Y}'_+$ for $\tilde{\boldsymbol{\psi}} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)$, respectively we have

$$\begin{aligned} \int\limits_{\Omega} \tilde{\xi} \cdot \operatorname{as}(\sigma)\; dx &= 0 \qquad \forall\, \tilde{\xi} \in S, \\ \int\limits_{\Gamma_N} \tilde{\boldsymbol{\psi}} \cdot (\sigma \cdot \mathbf{n} - \mathbf{t}_0)\, ds &= 0 \qquad \forall\, \tilde{\boldsymbol{\psi}} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N) \end{aligned}$$

and we conclude $\sigma = \sigma^T$ in Ω and $\sigma \cdot \mathbf{n} = \mathbf{t}_0$ on Γ_N. For $(0, 0, 0, 0, \mu_n) \in \mathbf{Y}'_+$ with $\mu_n \in \widetilde{H}^{\frac{1}{2}}_+(\Gamma_C)$ we observe

$$d_{C,n}(\mu_n, \sigma) = \int\limits_{\Gamma_C} \mu_n \sigma_n\, ds \leq 0 \quad \forall\, \mu_n \in \widetilde{H}^{\frac{1}{2}}_+(\Gamma_C)$$

and thus $\sigma_n \leq 0$ a.e. on Γ_C. Finally, if we insert $(0, 0, 0, \tilde{\mu}_t, 0) \in \mathbf{Y}'_+$ with $\tilde{\mu}_t \in \widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C)$ such that $\mathrm{t}(\sigma \cdot \mathbf{n})\, \tilde{\mu}_t = |\mathrm{t}(\sigma \cdot \mathbf{n})||\tilde{\mu}_t|$ we have

$$d_{C,t}(\tilde{\mu}_t, \sigma) - j(\tilde{\mu}_t) = \int\limits_{\Gamma_C} (\mathrm{t}(\sigma \cdot \mathbf{n})\, \tilde{\mu}_t - \mathcal{F}|\tilde{\mu}_t|)\, ds = \int\limits_{\Gamma_C} (|\sigma_t| - \mathcal{F})|\tilde{\mu}_t|\, ds \leq 0$$

and since $|\mu_t| \geq 0$ on Γ_C, we have $|\sigma_t| \leq \mathcal{F}$ a.e. on Γ_C, which means $\sigma \in \widetilde{\mathbf{K}}$. Using (12) the right inequality of the saddle point problem reduces to

$$\begin{aligned} \tilde{J}(\sigma) = \frac{1}{2}\tilde{a}(\sigma, \sigma) - g(\sigma) \leq \tilde{J}(\tau) &+ B(\mathbf{u}, \eta, \boldsymbol{\varphi}; \tau) - F(\mathbf{u}, \boldsymbol{\varphi}) \\ &+\; d_{C,n}(\lambda_n, \tau) + d_{C,t}(\lambda_t, \tau) - j(\lambda_t) \end{aligned} \qquad \forall\, \tau \in \mathbf{X}.$$

Restricting τ to the convex set $\widetilde{\mathbf{K}}$ we have $B(\mathbf{u}, \eta, \boldsymbol{\varphi}; \tau) - F(\mathbf{u}, \boldsymbol{\varphi}) = 0$. Furthermore, with $\tau_n \leq 0$, $\lambda_n \geq 0$ and $|\tau_t| \leq \mathcal{F}$ on Γ_C there holds

$$d_{C,n}(\lambda_n, \tau) \leq 0 \qquad \text{and} \qquad d_{C,t}(\lambda_t, \tau) - j(\lambda_t) \leq 0.$$

Now the right inequality of the saddle point problem (11) restricted to $\widetilde{\mathbf{K}}$ reads

$$\tilde{J}(\sigma) \leq \tilde{J}(\tau) + d_{C,n}(\lambda_n, \tau) + d_{C,t}(\lambda_t, \tau) - j(\lambda_t) \leq \tilde{J}(\tau) \qquad \forall\, \tau \in \widetilde{\mathbf{K}},$$

which means $\sigma \in \widetilde{\mathbf{K}}$ is the solution of the dual minimization problem (4). Additionally the right inequality in (11) states that σ minimizes the functional $L(\cdot; \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n)$ in $\mathbf{X}$. The linear forms that only act on Lagrange multipliers can be regarded as constants within this minimization problem and so we have for all $\tau \in \mathbf{X}$

$$\frac{1}{2}\tilde{a}(\sigma,\sigma) - g(\sigma) + B(\mathbf{u},\eta,\boldsymbol{\varphi};\sigma) + d_{C,n}(\lambda_n,\sigma) + d_{C,t}(\lambda_t,\sigma)$$
$$\leq \frac{1}{2}\tilde{a}(\tau,\tau) - g(\tau) + B(\mathbf{u},\eta,\boldsymbol{\varphi};\tau) + d_{C,n}(\lambda_n,\tau) + d_{C,t}(\lambda_t,\tau).$$

Now $\mathbf{X}$ as a Hilbert space is evidently closed and convex and the above functional is Gâteaux differentiable with respect to τ. Therefore using the theory of variational inequalities (see e.g. Kinderlehrer and Stampacchia [18]) the above minimization problem is equivalent to the following variational inequality problem of finding $\sigma \in \mathbf{X}$ such that

$$\tilde{a}(\sigma,\tau-\sigma) - g(\tau-\sigma) + B(\mathbf{u},\eta,\boldsymbol{\varphi};\tau-\sigma) + d_{C,n}(\lambda_n,\tau-\sigma) + d_{C,t}(\lambda_t,\tau-\sigma) \geq 0 \qquad \forall\, \tau \in \mathbf{X}. \tag{13}$$

Choosing $\tau = \pm\phi + \sigma$ with $\phi \in [C_0^\infty]^{2\times 2} \cap \mathbf{X}_s$ with the subspace $\mathbf{X}_s \subset \mathbf{X}$ of symmetric tensors, the bilinear form concerning η and the terms on the boundaries vanish and we have

$$\int_\Omega \sigma : \mathbb{C}^{-1} : \phi\, dx + \int_\Omega \mathbf{u} \cdot \operatorname{div} \phi\, dx = 0 \qquad \forall\, \phi \in [C_0^\infty]^{2\times 2} \cap \mathbf{X}_s.$$

Integrating by parts and using the symmetry of ϕ in the right integral leads to

$$\int_\Omega \mathbf{u} \cdot \operatorname{div} \phi\, dx = -\int_\Omega \nabla\mathbf{u} : \phi\, dx = -\int_\Omega \nabla\mathbf{u} : \frac{1}{2}(\phi + \phi^T)\, dx$$
$$= -\int_\Omega \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) : \phi\, dx = -\int_\Omega \varepsilon(\mathbf{u}) : \phi\, dx$$

and so the above equation reads

$$\int_\Omega (\sigma : \mathbb{C}^{-1} - \varepsilon(\mathbf{u})) : \phi\, dx = 0 \qquad \forall\, \phi \in [C_0^\infty]^{2\times 2} \cap \mathbf{X}_s,$$

which means $\sigma = \mathbb{C} : \varepsilon(\mathbf{u})$ in Ω. If we do not require ϕ to be symmetric in the above choice of τ, then again integrating by parts and using $\sigma : \mathbb{C}^{-1} = \varepsilon(\mathbf{u})$ the variational inequality reduces to

$$\int_\Omega \left[\varepsilon(\mathbf{u}) - \nabla\mathbf{u} + \boldsymbol{\eta}\right] : \phi\, dx = \int_\Omega (\boldsymbol{\eta} - \frac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^T)) : \phi\, dx = 0 \qquad \forall\, \phi \in [C_0^\infty]^{2\times 2}$$

and therefore $\boldsymbol{\eta} = \frac{1}{2}(\nabla \mathbf{u} - \nabla \mathbf{u}^T)$ in Ω and accordingly $\eta = \mathrm{as}(\nabla \mathbf{u})$. Next we observe by choosing $\tau \in \mathbf{X}_s$ and using $\sigma : \mathbb{C}^{-1} = \varepsilon(\mathbf{u})$

$$\int_\Omega \varepsilon(\mathbf{u}) : (\tau - \sigma)\, dx + \int_\Omega \mathbf{u} \cdot \mathrm{div}(\tau - \sigma)\, dx = \int_\Gamma \mathbf{u} \cdot (\tau - \sigma) \cdot \mathbf{n}\, ds.$$

If we take $\tau \in \mathbf{X}_s$ in (13) with $\tau \cdot \mathbf{n} = \sigma \cdot \mathbf{n}$ on $\Gamma_N \cup \Gamma_C$ and $\tau \cdot \mathbf{n} = \pm \boldsymbol{\psi} + \sigma \cdot \mathbf{n}$ on Γ_D for some $\boldsymbol{\psi} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_D)$ we get with $\langle \cdot, \cdot \rangle$ denoting the respective dual pairing

$$\langle \mathbf{u}, \boldsymbol{\psi} \rangle_{\Gamma_D} = 0 \qquad \forall\, \boldsymbol{\psi} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_D)$$

which leads to $\mathbf{u} = 0$ on Γ_D. In the same way by choosing $\tau \in \mathbf{X}_s$ with $\tau \cdot \mathbf{n} = \sigma \cdot \mathbf{n}$ on Γ_C and $\tau \cdot \mathbf{n} = \pm \boldsymbol{\psi} + \sigma \cdot \mathbf{n}$ on Γ_N for some $\boldsymbol{\psi} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$ we get

$$\langle (\mathbf{u} + \boldsymbol{\varphi}), \boldsymbol{\psi} \rangle_{\Gamma_N} = 0 \qquad \forall\, \boldsymbol{\psi} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_N)$$

and so $\mathbf{u} = -\boldsymbol{\varphi}$ on Γ_N. Next choosing $\tau \in \mathbf{X}_s$ with $\tau \cdot \mathbf{n} = \sigma \cdot \mathbf{n}$ on Γ_N, $\tau \cdot \mathbf{n} = \pm \boldsymbol{\psi} + \sigma \cdot \mathbf{n}$ on Γ_C for some $\boldsymbol{\psi} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_C)$ where $\mathrm{t}(\boldsymbol{\psi}) = 0$ and $\psi_n = \tilde{\psi} \in H^{-\frac{1}{2}}(\Gamma_C)$ we get

$$\langle (u_n + \lambda_n - g), \tilde{\psi} \rangle_{\Gamma_C} = 0 \qquad \forall\, \tilde{\psi} \in H^{-\frac{1}{2}}(\Gamma_C),$$

from which we deduce $u_n + \lambda_n = g$ on Γ_C. Finally, we choose $\tau \in \mathbf{X}_s$ with $\tau \cdot \mathbf{n} = \sigma \cdot \mathbf{n}$ on $\Gamma_N \cup \Gamma_D$, $\tau \cdot \mathbf{n} = \pm \boldsymbol{\psi} + \sigma \cdot \mathbf{n}$ on Γ_C for some $\boldsymbol{\psi} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_C)$ with $\psi_n = 0$ and $\mathrm{t}(\boldsymbol{\psi}) = \tilde{\psi} \in \mathcal{H}^{-\frac{1}{2}}(\Gamma_C)$ to get

$$\langle (\mathrm{t}(\mathbf{u}) + \lambda_t), \tilde{\psi} \rangle_{\Gamma_C} = 0 \qquad \forall\, \tilde{\psi} \in \mathcal{H}^{-\frac{1}{2}}(\Gamma_C),$$

which states $\mathrm{t}(\mathbf{u}) = -\lambda_t$ on Γ_C and concludes the first assertion.

(ii) To prove the second assertion we let $\sigma \in \widetilde{\mathbf{K}}$ be the solution of the dual minimization problem (4) and $\mathbf{u} \in \mathbf{K}_g$ the solution of the primal minimization problem (5). Then due to Lemma 2.1 we have $\sigma = \mathbb{C} : \varepsilon(\mathbf{u})$ in Ω. Since $\sigma \in \widetilde{\mathbf{K}}$ we have

$$B(\mathbf{v}, \xi, \boldsymbol{\psi}; \sigma) - F(\mathbf{v}, \boldsymbol{\psi}) = 0 \qquad \forall\, (\mathbf{v}, \xi, \boldsymbol{\psi}) \in \mathbf{L}^2(\Omega) \times S \times \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N).$$

Furthermore, the primal minimization problem (5) is equivalent to the variational inequality problem of finding $\mathbf{u} \in \mathbf{K}_g$ such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) - L(\mathbf{v} - \mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) \geq 0 \qquad \forall\, \mathbf{v} \in \mathbf{K}_g.$$

Letting $\mathbf{v} = \mathbf{u} + \tilde{\mathbf{v}}$ with $\tilde{\mathbf{v}} \in \mathbf{K}_0$, then $\mathbf{v} \in \mathbf{K}_g$ and we have

$$\int_\Omega \sigma : \varepsilon(\tilde{\mathbf{v}})\, dx - L(\tilde{\mathbf{v}}) + j(\mathbf{u} + \tilde{\mathbf{v}}) - j(\mathbf{u}) \geq 0 \qquad \forall\, \tilde{\mathbf{v}} \in \mathbf{K}_0.$$

Using $\sigma \in \widetilde{\mathbf{K}}$ and Green's formula the above inequality reduces to

$$\int_{\Gamma_C} \tilde{v}_n \sigma_n \, ds + \int_{\Gamma_C} \mathrm{t}(\tilde{\mathbf{v}}) \, \mathrm{t}(\sigma \cdot \mathbf{n}) \, ds + \int_{\Gamma_C} \mathcal{F}|\mathrm{t}(\mathbf{u} + \tilde{\mathbf{v}})| \, ds - \int_{\Gamma_C} \mathcal{F}|\mathrm{t}(\mathbf{u})| \, ds \geq 0 \qquad \forall \, \tilde{\mathbf{v}} \in \mathbf{K}_0.$$

Now choosing $\tilde{\mathbf{v}} \in \mathbf{K}_0$ with $\tilde{v}_n = 0$ and $\mathrm{t}(\tilde{\mathbf{v}}) = \pm \, \mathrm{t}(\mathbf{u})$ on Γ_C we have using $\lambda_t = -\,\mathrm{t}(\mathbf{u})$

$$d_{C,t}(\lambda_t, \sigma) - j(\lambda_t) = -(d_{C,t}(\mathrm{t}(\mathbf{u}), \sigma) + j(\mathbf{u})) = -\int_{\Gamma_C} \mathrm{t}(\mathbf{u}) \, \sigma_t \, ds - \int_{\Gamma_C} \mathcal{F}|\mathrm{t}(\mathbf{u})| \, ds = 0.$$

Analogously to the above argumentation we can write the dual minimization problem (4) as a variational inequality problem of finding $\sigma \in \widetilde{\mathbf{K}}$ such that

$$\tilde{a}(\sigma, \tau - \sigma) - g(\tau - \sigma) \geq 0 \qquad \forall \, \tau \in \widetilde{\mathbf{K}}.$$

If we now define

$$\mathcal{K}_n := \big\{ \mathbf{v} \in \mathbf{H}^1(\Omega) : \operatorname{div} \varepsilon(\mathbf{v}) = 0 \text{ in } \Omega, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D, \, \varepsilon(\mathbf{v}) \cdot \mathbf{n} = \mathbf{0} \text{ on } \Gamma_N, \\ \mathrm{t}(\varepsilon(\mathbf{v}) \cdot \mathbf{n}) = 0 \text{ and } \varepsilon(\mathbf{v})_n \leq -\sigma_n \text{ on } \Gamma_C \big\},$$

we have $\tau = \sigma + \varepsilon(\mathbf{v}) \in \widetilde{\mathbf{K}}$ for all $\mathbf{v} \in \mathcal{K}_n$. Integrating by parts in the above inequality and using $\sigma : \mathbb{C}^{-1} = \varepsilon(\mathbf{u})$ and $\mathbf{v} \in \mathcal{K}_n$ leads to

$$\begin{aligned} \tilde{a}(\sigma, \tau - \sigma) = \int_{\Omega} \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \, dx &= -\int_{\Omega} \mathbf{u} \cdot \operatorname{div} \varepsilon(\mathbf{v}) \, dx + \int_{\Gamma_N \cup \Gamma_C} \mathbf{u} \cdot \varepsilon(\mathbf{v}) \cdot \mathbf{n} \, ds \\ &= \int_{\Gamma_C} u_n \, \varepsilon(\mathbf{v})_n \, ds \geq g(\varepsilon(\mathbf{v})) \qquad \forall \, \mathbf{v} \in \mathcal{K}_n. \end{aligned}$$

Noting that $\sigma_n \leq 0 \leq -\sigma_n$ we can choose $\mathbf{v} \in \mathcal{K}_n$ with $\varepsilon(\mathbf{v})_n = \pm \sigma_n$ on Γ_C in the above inequality and using $\lambda_n = g - u_n$ we arrive at

$$d_{C,n}(\lambda_n, \sigma) = -\int_{\Gamma_C} (u_n - g) \sigma_n \, ds = 0.$$

Since $\sigma \in \widetilde{\mathbf{K}}$ we observe for $\mu_t \in \widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C)$ and $\mu_n \in \widetilde{H}^{\frac{1}{2}}_{+}(\Gamma_C)$ with $\mathcal{F} \geq 0$

$$\mathrm{t}(\sigma \cdot \mathbf{n}) \, \mu_t \leq |\mathrm{t}(\sigma \cdot \mathbf{n})||\mu_t| \leq \mathcal{F}|\mu_t| \quad \text{ and } \quad \sigma_n \mu_n \leq 0$$

and therefore we can state the left inequality of the saddle point problem (11)

$$\begin{aligned} L(\sigma; \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) &= \tilde{J}(\sigma) + B(\mathbf{v}, \xi, \boldsymbol{\psi}; \sigma) - F(\mathbf{v}, \boldsymbol{\psi}) + d_{C,n}(\mu_n, \sigma) \\ &\quad + d_{C,t}(\mu_t, \sigma) - j(\mu_t) \leq \tilde{J}(\sigma) = L(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n). \end{aligned}$$

To prove the right inequality we first observe for $\tau \in \mathbf{X}$

$$L(\sigma;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) - L(\tau;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) = -\frac{1}{2}\tilde{a}(\sigma-\tau,\sigma-\tau) + \tilde{a}(\sigma,\sigma-\tau)$$
$$-g(\sigma-\tau) + B(\mathbf{u},\eta,\boldsymbol{\varphi};\sigma-\tau) + d_{C,n}(\lambda_n,\sigma-\tau) + d_{C,t}(\lambda_t,\sigma-\tau).$$

Integrating by parts and using $\boldsymbol{\eta} = \frac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^T) = \nabla\mathbf{u} - \varepsilon(\mathbf{u})$ leads to

$$\begin{aligned}
&L(\sigma;\,\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) - L(\tau;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n)\\
&= -\frac{1}{2}\tilde{a}(\sigma-\tau,\sigma-\tau) + \tilde{a}(\sigma,\sigma-\tau) - \int_\Omega \nabla\mathbf{u} : (\sigma-\tau)\,dx\\
&\quad + \int_\Omega (\nabla\mathbf{u} - \varepsilon(\mathbf{u})) : (\sigma-\tau)\,dx + \int_{\Gamma_N} (\mathbf{u}+\boldsymbol{\varphi})\cdot(\sigma-\tau)\cdot\mathbf{n}\,ds\\
&\quad + \int_{\Gamma_C} (\mathrm{t}(\mathbf{u}) + \lambda_t)\,\mathrm{t}((\sigma-\tau)\cdot\mathbf{n})\;ds + \int_{\Gamma_C} (u_n - g + \lambda_n)(\sigma_n - \tau_n)\,ds \qquad \forall\,\tau\in\mathbf{X}.
\end{aligned}$$

Finally, using the definitions for $\boldsymbol{\varphi}, \lambda_n$ and λ_t the boundary integrals vanish and with $\varepsilon(\mathbf{u}) = \mathbb{C}^{-1} : \sigma$ we can show the right inequality of the saddle point problem (11)

$$L(\sigma;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) - L(\tau;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) = -\frac{1}{2}\tilde{a}(\sigma-\tau,\sigma-\tau) \le 0 \qquad \forall\,\tau\in\mathbf{X},$$

where the inequality is due to the ellipticity of the bilinear form $\tilde{a}(\cdot,\cdot)$. □

To conclude this section we have the following dual variational inequality problem, being equivalent to the saddle point formulation (11).

Find $(\sigma;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n) \in \mathbf{X}\times\mathbf{Y}'_+$ such that

$$\begin{aligned}
\tilde{a}(\tau,\sigma) + B(\mathbf{u},\eta,\boldsymbol{\varphi};\tau) + d_{C,t}(\lambda_t,\tau) + d_{C,n}(\lambda_n,\tau) &= g(\tau) && \forall\,\tau\in\mathbf{X}\\
B(\mathbf{v},\xi,\boldsymbol{\psi};\sigma) &= F(\mathbf{v},\boldsymbol{\psi}) && \forall\,(\mathbf{v},\xi,\boldsymbol{\psi})\in Z\\
d_{C,t}(\mu_t-\lambda_t,\sigma) - j(\mu_t) + j(\lambda_t) &\le 0 && \forall\,\mu_t\in\widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C)\\
d_{C,n}(\mu_n-\lambda_n,\sigma) &\le 0 && \forall\,\mu_n\in\widetilde{H}^{\frac{1}{2}}_+(\Gamma_C)
\end{aligned} \tag{14}$$

with $Z := \mathbf{L}^2(\Omega)\times S\times\widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N)$.

Remark 2.3. *If the contact boundary Γ_C and the Dirichlet boundary Γ_D have positive distance, then we deal with $H^{\frac{1}{2}}(\Gamma_C)$ for λ_t and λ_n. The above theory is* *still valid, since we can regard the Lagrange multipliers on the boundary parts as one function on* $\Gamma_\Sigma := \Gamma_C \cup \Gamma_N$. *In this case we decompose* $\tilde{\boldsymbol{\lambda}} \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_\Sigma)$ *into the corresponding parts on the boundaries, where we approximate* $\tilde{\boldsymbol{\lambda}}|_{\Gamma_N}$ *with some* $\boldsymbol{\varphi} \in \overline{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N) := \{\chi|_{\Gamma_N} : \chi \in \widetilde{\mathbf{H}}^{\frac{1}{2}}(\Gamma_\Sigma)\}$ *and take* $\lambda_t, \lambda_n \in H^{\frac{1}{2}}(\Gamma_C)$ *with* $\lambda_t \mathbf{t} + \lambda_n \mathbf{n} = \tilde{\boldsymbol{\lambda}}|_{\Gamma_C}$. *Then inspection of the proof shows that a corresponding modification of Theorem 2.2 remains valid in this case with*

$$\overline{\mathbf{Y}}' := \mathbf{L}^2(\Omega) \times S \times \overline{\mathbf{H}}^{\frac{1}{2}}(\Gamma_N) \times \mathcal{H}^{\frac{1}{2}}(\Gamma_C) \times H^{\frac{1}{2}}(\Gamma_C).$$

instead of $\mathbf{Y}'$.

3 Dual-dual variational inequality formulation of a frictional contact problem

Since the j-functional in the saddle point formulation (11) is non-differentiable we introduce another Lagrange multiplier in order to approximate the unit direction vector of the tangential displacement vector λ_t on the contact boundary Γ_C. In 2D this corresponds to the sign of λ_t. First, we have to restrict the new Lagrange multiplier to the support of the friction function $\mathcal{F}$. We set

$$A_C := \overline{\operatorname{supp} \mathcal{F}} \qquad \text{and} \qquad \nu \lambda_t = |\lambda_t| \quad \text{a.e. on } A_C,$$

with

$$\nu \in \Lambda := \{\kappa \in [L^\infty(A_C)]^{d-1} : |\kappa| \leq 1, \text{ a.e. on } A_C\}$$

and define the bilinear form

$$q(\kappa, \mu) := \int\limits_{\Gamma_C} \mathcal{F} \kappa \mu \, ds \qquad \text{for } \kappa \in \Lambda,\ \mu \in \widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C).$$

Defining $\widetilde{\mathbf{X}} := \mathbf{X} \times \Lambda$ we consider the following saddle point formulation. Find $(\sigma, \nu; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \widetilde{\mathbf{X}} \times \mathbf{Y}'_+$ such that for all $(\tau, \kappa; \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \in \widetilde{\mathbf{X}} \times \mathbf{Y}'_+$

$$\widetilde{L}(\sigma, \nu; \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) \leq \widetilde{L}(\sigma, \nu; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \leq \widetilde{L}(\tau, \kappa; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \tag{15}$$

where

$$\widetilde{L}(\tau, \kappa; \mathbf{v}, \xi, \boldsymbol{\psi}, \mu_t, \mu_n) := \frac{1}{2} \tilde{a}(\tau, \tau) - g(\tau) + B(\mathbf{v}, \xi, \boldsymbol{\psi}; \tau) - F(\mathbf{v}, \boldsymbol{\psi}) + d_{C,t}(\mu_t, \tau) - q(\mu_t, \kappa) + d_{C,n}(\mu_n, \tau).$$

Analogously to Section 2 we have the saddle point formulation (15) being equivalent to the following dual-dual variational inequality problem.

Find $(\sigma,\nu;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n)\in\widetilde{\mathbf{X}}\times\mathbf{Y}'_+$ such that

$$\begin{array}{lcll}
\tilde{a}(\tau,\sigma)+B(\mathbf{u},\eta,\boldsymbol{\varphi};\tau)+d_{C,t}(\lambda_t,\tau)+d_{C,n}(\lambda_n,\tau) & = & g(\tau) & \forall\,\tau\in\mathbf{X}\\
B(\mathbf{v},\xi,\boldsymbol{\psi};\sigma) & = & F(\mathbf{v},\boldsymbol{\psi}) & \forall\,(\mathbf{v},\xi,\boldsymbol{\psi})\in Z\\
d_{C,t}(\mu_t,\sigma)-q(\mu_t,\nu) & = & 0 & \forall\,\mu_t\in\widetilde{\mathcal{H}}^{\frac12}(\Gamma_C)\\
d_{C,n}(\mu_n-\lambda_n,\sigma) & \le & 0 & \forall\,\mu_n\in\widetilde{H}^{\frac12}_+(\Gamma_C)\\
q(\kappa-\nu,\lambda_t) & \le & 0 & \forall\,\kappa\in\Lambda
\end{array} \tag{16}$$

Remark 3.1. *Note that the last inequality in* (16) *induces*

$$\nu\lambda_t=|\lambda_t|\quad a.e.\ \ on\ A_C, \tag{17}$$

since taking $\kappa=\frac{\lambda_t}{|\lambda_t|}|_{A_C}\in\Lambda$ *we get*

$$\int\limits_{\Gamma_C}\mathcal{F}(|\lambda_t|-\nu\lambda_t)\,ds\le 0\ .$$

But as $\nu\in\Lambda$ *we have* $|\lambda_t|-\nu\lambda_t\ge 0$ *and thus*

$$\int\limits_{\Gamma_C}\mathcal{F}(|\lambda_t|-\nu\lambda_t)\,ds\ge 0\qquad\Rightarrow\qquad\int\limits_{\Gamma_C}\mathcal{F}(|\lambda_t|-\nu\lambda_t)\,ds=0.$$

Since the friction function $\mathcal{F}$ *is positive on* A_C *we conclude* (17). *If* $\lambda_t=0$ *on some part* $\Gamma_C^{st}\subset A_C$ *we are in the situation where the body is sticking on the rigid foundation. Then from* (17) *we have that* $\nu\in\Lambda$ *can be chosen arbitrarily. But in this case, we have from the third equation in* (16) *by taking* $\mu_t\in\widetilde{\mathcal{H}}^{\frac12}(\Gamma_C)$ *with* $\operatorname{supp}(\mu_t)\subset\Gamma_C^{st}$

$$0=\int\limits_{\Gamma_C^{st}}\mu_t(\mathrm{t}(\sigma\cdot\mathbf{n})-\mathcal{F}\nu)\,ds\quad\Rightarrow\quad\mathcal{F}\nu=\mathrm{t}(\sigma\cdot\mathbf{n})\quad on\ \Gamma_C^{st}. \tag{18}$$

Finally, if $\lambda_t\neq 0$ *we conclude from* (17) *and* (18)

$$\nu=\frac{\lambda_t}{|\lambda_t|}=\frac{\mathrm{t}(\sigma\cdot\mathbf{n})}{|\mathrm{t}(\sigma\cdot\mathbf{n})|}\quad\wedge\quad\nu\,\mathrm{t}(\sigma\cdot\mathbf{n})=|\mathrm{t}(\sigma\cdot\mathbf{n})|=\mathcal{F}. \tag{19}$$

Corollary 3.2. *For the normal stress* σ_n *and the tangential stress* $\mathrm{t}(\sigma\cdot\mathbf{n})$ *on* Γ_C *we conclude*

$$\sigma_n \le 0 \qquad \textit{and} \qquad |\sigma_t| \le \mathcal{F} \qquad \textit{on}\ \Gamma_C.$$

Proof. The first assertion follows from (16), by choosing $\mu_n = 0$ and $\mu_n = 2\lambda_n$ in the first inequality of (16) and using the fact, that $\mu_n \ge 0$ on Γ_C. The second assertion follows due to Remark 3.1. □

The next theorem shows the equivalence of the two variational inequality problems (14) and (16). The proof uses Remark 3.1 and the Hahn-Banach theorem.

Theorem 3.3. *The variational inequality problems* (14) *and* (16) *are equivalent in the following sense. If* $(\sigma;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n)\in\mathbf{X}\times\mathbf{Y}'_+$ *is a solution of* (14), *then* $(\sigma,\nu;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n)\in\widetilde{\mathbf{X}}\times\mathbf{Y}'_+$ *solves* (16) *with* $\nu := \frac{\lambda_t}{|\lambda_t|}|_{A_C}\in\Lambda$. *On the other hand if* $(\sigma,\nu;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n)\in\widetilde{\mathbf{X}}\times\mathbf{Y}'_+$ *is a solution of* (16), *then* $(\sigma;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n)$ *solves* (14).

Proof. Let $(\sigma,\nu;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n)\in\widetilde{\mathbf{X}}\times\mathbf{Y}'_+$ be the solution of (16). To prove that $(\sigma;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n)\in\mathbf{X}\times\mathbf{Y}'_+$ is a solution of (14), we only have to show the inequality in (14) concerning the tangential displacement on the contact boundary. With $\nu\in\Lambda$ we have

$$-q(\nu,\mu_t) = -\int_{\Gamma_C}\mathcal{F}\nu\mu_t\,ds \ge -\int_{\Gamma_C}\mathcal{F}|\mu_t|\,ds = -j(\mu_t) \quad \forall\,\mu_t\in\widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C).$$

Using (17) we deduce $q(\lambda_t,\nu) = j(\lambda_t)$ and the third equation in (16) finally leads to

$$\begin{aligned}0 = d_{C,t}(\mu_t,\sigma) - q(\mu_t,\nu) &= d_{C,t}(\mu_t-\lambda_t,\sigma) - q(\mu_t,\nu) + q(\lambda_t,\nu)\\ &\ge d_{C,t}(\mu_t-\lambda_t,\sigma) - j(\mu_t) + j(\lambda_t) \qquad \forall\,\mu_t\in\widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C).\end{aligned}$$

But this is just the first inequality in (14).

On the other hand let $(\sigma;\mathbf{u},\eta,\boldsymbol{\varphi},\lambda_t,\lambda_n)\in\mathbf{X}\times\mathbf{Y}'_+$ be the solution of (14). For $\sigma\in\mathbf{X}$ fixed, we have $d_\sigma(\mu) := d_{C,t}(\mu,\sigma)$ is a continuous linear functional on $\widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C)$. Choosing $\mu_t = 0$ and $\mu_t = 2\lambda_t$ in the first inequality of (14) we get $d_{C,t}(\lambda_t,\sigma) - j(\lambda_t) = 0$ and the inequality reduces to

$$d_{C,t}(\mu_t,\sigma) - j(\mu_t) \le 0 \qquad \forall\,\mu_t\in\widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C) \tag{20}$$

Taking $\mu_t = \pm\,\mu\in\widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C)$ in (20) we get

$$|d_{C,t}(\mu,\sigma)| \le j(\mu) \quad \forall\,\mu\in\widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C).$$

As $\mathcal{F} \in L^\infty(\Gamma_C) \subset L^1(\Gamma_C)$ we can define the mapping

$$\pi : \widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C) \to L^1(\Gamma_C), \quad \mu \mapsto \mathcal{F}\mu.$$

Taking into account the positivity of $\mathcal{F}$ we have

$$|d_{C,t}(\mu,\sigma)| \leq j(\mu) = \int\limits_{\Gamma_C} \mathcal{F}|\mu|\, ds = \int\limits_{\Gamma_C} |\mathcal{F}\mu|\, ds = \|\pi\mu\|_{L^1(\Gamma_C)} \quad \forall\, \mu \in \widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C),$$

which is a seminorm on $L^1(\Gamma_C)$ and therefore sublinear. Since $\widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C) \subset L^1(\Gamma_C)$ the assumptions of the Hahn-Banach theorem, see e.g. Yosida [25, Chapter IV], are fulfilled and we have the existence of some linear functional $\tilde{d}_\sigma$ on $L^1(\Gamma_C)$ which is an extension of d_σ such that

$$\tilde{d}_\sigma(\mu) \leq \|\pi\mu\|_{L^1(\Gamma_C)} \quad \forall\, \mu \in L^1(\Gamma_C).$$

For $\mu \in L^1(\Gamma_C)$ we have the existence of $\nu \in L^\infty(\Gamma_C)$, the dual of $L^1(\Gamma_C)$, with $\|\nu\|_{L^\infty(\Gamma_C)} \leq 1 \ \Rightarrow \nu \in \Lambda$ and

$$\|\mu\|_{L^1(\Gamma_C)} = \langle \mu, \nu \rangle.$$

Here $\langle\cdot,\cdot\rangle$ denotes the duality product. This can be seen very easily from the definition of the L^1-norm as the supremum of the duality product over all dual functions $\kappa \in L^\infty(\Gamma_C)$

$$\|\mu\|_{L^1(\Gamma_C)} := \sup_{\|\kappa\|_{L^\infty(\Gamma_C)} \leq 1} \langle \mu, \kappa \rangle.$$

The supremum is assumed for $\nu = \frac{\mu}{|\mu|} \in L^\infty(\Gamma_C)$. Since $\tilde{d}_\sigma$ is an extension of d_σ we have

$$d_\sigma(\mu) = d_{C,t}(\mu,\sigma) \leq \langle \nu, \pi\mu \rangle = \int\limits_{\Gamma_C} \mathcal{F}\nu\mu\, ds = q(\mu,\nu) \quad \forall\, \mu \in \widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C).$$

If we take $\mu = \pm\mu_t \in \widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C)$ we finally arrive at

$$d_{C,t}(\mu_t,\sigma) = q(\mu_t,\nu) \quad \forall\, \mu_t \in \widetilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_C).$$

Finally, due to the choice of ν we have

$$\nu\lambda_t = |\lambda_t| \geq \kappa\lambda_t \quad \Rightarrow \quad q(\lambda_t, \kappa - \nu) \leq 0 \quad \forall\, \kappa \in \Lambda,$$

which completes the proof. □

The above derivations permit us to state existence and uniqueness results of the variational inequality problems (14) and (16).

Theorem 3.4. **(i)** *There exists exactly one solution* $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \mathbf{Y}'_+$ *of the dual variational inequality problem* (14)*.*

(ii) *There exists exactly one solution* $(\sigma, \nu; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \widetilde{\mathbf{X}} \times \mathbf{Y}'_+$ *of the dual-dual variational inequality problem* (16)*.*

Moreover, Theorem 3.3 shows how the solutions of the two problems are related.

Proof. In Section 2 we have seen that the dual variational inequality problem (14) is equivalent to the saddle point problem (11). Due to Theorem 2.2 we have the equivalence of the saddle point problem (11) with the primal minimization problem (5) and the dual minimization problem (4). Since both minimization problems are uniquely solvable we have that the saddle point problem (11) as well as the dual variational inequality problem (14) are uniquely solvable.

The second statement follows directly from Theorem 3.3 and the existence and uniqueness of the dual variational inequality problem (14). Note that the uniqueness of the additional Lagrange multiplier ν follows from the restriction to the support of the friction function $\mathcal{F}$ as explained in Remark 3.1. □

Remark 3.5. *In the case* $\Gamma_C \cap \Gamma_D = \emptyset$*, i.e. when* $\overline{\Gamma_C} \cap \overline{\Gamma_N} \neq \emptyset$*, we modify our setting as in Remark 2.3. Further we take* $\nu \in \tilde{\Lambda} := \{\kappa \in \tilde{H}^{-1/2}(A_C) : |\kappa| \leq 1, \ a.e. \text{ on } A_C\}$*. Then the modification of Theorem 3.4 reads: There exists exactly one solution* $(\sigma; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \overline{\mathbf{Y}}'_+$ *of the dual variational inequality problem* (14) *with* $\boldsymbol{\psi} \in \overline{\mathbf{H}}^{1/2}(\Gamma_N), \mu_t \in H^{\frac{1}{2}}(\Gamma_C), \mu_n \in H^{\frac{1}{2}}_+(\Gamma_C)$*. There exists exactly one solution* $(\sigma, \nu; \mathbf{u}, \eta, \boldsymbol{\varphi}, \lambda_t, \lambda_n) \in \mathbf{X} \times \tilde{\Lambda} \times \overline{\mathbf{Y}}'_+$ *of the dual-dual variational inequality problem* (16)*.*

4 Numerical experiment

In order to confirm that the dual-dual variational inequality problem (16) is useful for applications a numerical example is presented next. The discrete variational inequality problem corresponding to (16) reads.

Find $(\sigma^h, \nu^{\hat{h}}; \mathbf{u}^h, \eta^h, \boldsymbol{\varphi}^{\tilde{h}}, \lambda_t^{\tilde{h}}, \lambda_n^{\tilde{h}}) \in \widetilde{\mathbf{X}}_h \times \widetilde{\mathbf{Y}}_h^+$ such that

$$\begin{aligned}
&\tilde{a}(\sigma^h, \tau) + B(\mathbf{u}^h, \eta^h, \boldsymbol{\varphi}^{\tilde{h}}; \tau) + d_{C,t}(\lambda_t^{\tilde{h}}, \tau) + d_{C,n}(\lambda_n^{\tilde{h}}, \tau) &&= g(\tau) &&\forall \tau \in \mathbf{X}_h \\
&B(\mathbf{v}, \xi, \boldsymbol{\psi}; \sigma^h) &&= F(\mathbf{v}, \boldsymbol{\psi}) &&\forall (\mathbf{v}, \xi, \boldsymbol{\psi}) \in Z_h \\
&d_{C,t}(\mu_t, \sigma^h) - q(\mu_t, \nu^{\hat{h}}) &&= 0 &&\forall \mu_t \in C_{\tilde{h}} \\
&d_{C,n}(\mu_n - \lambda_n^{\tilde{h}}, \sigma^h) &&\leq 0 &&\forall \mu_n \in C_{\tilde{h}}^+ \\
&q(\lambda_t^{\tilde{h}}, \kappa - \nu^{\hat{h}}) &&\leq 0 &&\forall \kappa \in \Lambda_{\hat{h}},
\end{aligned} \tag{21}$$

where $\widetilde{\mathbf{X}}_h := \mathbf{X}_h \times \Lambda_{\hat{h}}$, $\widetilde{\mathbf{Y}}_h^+ := Z_h \times C_{\tilde{h}} \times C_{\tilde{h}}^+$ and $Z_h := \mathbf{M}_h \times S_h \times \mathbf{N}_{\tilde{h}}$. Within the formulation we choose the PEERS elements, see Arnold et al. [3], to approximate the triple $(\sigma, \mathbf{u}, \eta)$. For the displacement on the Neumann and contact boundary we choose continuous hat functions, and the friction force on the active part of the contact boundary is approximated by piecewise constant functions. The finite element discretization (21) of the dual-dual formulation is based on different mesh sizes for the primal variables and Lagrange multipliers, which is indicated by the different parameters $h, \tilde{h}$ and $\hat{h}$. The necessity of different mesh sizes results from discrete inf-sup conditions which guarantee the unique solvability of discrete system (see also Babuska and Gatica [4], Gatica and Maischak [12], Schroeder [24]). Lemma 3.28 and Lemma 3.29 in Andres [1] give the following discrete inf-sup conditions.

For the bilinear form
$\hat{B}(\tau^h, (\mathbf{v}^h, \xi^h, \boldsymbol{\psi}^{\tilde{h}}, \mu_t^{\tilde{h}}, \mu_n^{\tilde{h}})) = B(\tau^h, \mathbf{v}^h, \xi^h, \boldsymbol{\psi}^{\tilde{h}}) + d_{C,t}(\mu_t^{\tilde{h}}, \tau^h) + d_{C,n}(\mu_n^{\tilde{h}}, \tau^h)$
there exists $0 < C_0 < 1, \beta_1 > 0$ independent of h and $\tilde{h}$ such that for all $h \leq C_0 \tilde{h}$ and $(\mathbf{v}^h, \xi^h, \boldsymbol{\psi}^{\tilde{h}}, \mu_t^{\tilde{h}}, \mu_n^{\tilde{h}}) \in \widetilde{\mathbf{Y}}_h$

$$\sup_{0 \neq \tau^h \in \mathbf{X}_h} \frac{\hat{B}(\tau^h, (\mathbf{v}^h, \xi^h, \boldsymbol{\psi}^{\tilde{h}}, \mu_t^{\tilde{h}}, \mu_n^{\tilde{h}}))}{\|\tau^h\|_{\mathbf{X}}} \geq \beta_1 \|(\mathbf{v}^h, \xi^h, \boldsymbol{\psi}^{\tilde{h}}, \mu_t^{\tilde{h}}, \mu_n^{\tilde{h}})\|_{\mathbf{Y}'}. \tag{22}$$

For the bilinear form $q(\cdot, \cdot)$ there exists $0 < \hat{C}_0 < 1$ and $\beta_2 > 0$, where β_2 depends on $\mathcal{F}$ such that for all $\tilde{h} \leq \hat{C}_0 \hat{h}$ and for all $\kappa \in L_{\hat{h}}$

$$\sup_{\mu \in C_{\tilde{h}}} \frac{q(\mu, \kappa)}{\|\mu\|_{\widetilde{H}^{\frac{1}{2}}(\Gamma_C)}} \geq \beta_2 \|\kappa\|_{-\frac{1}{2}, \hat{h}} \tag{23}$$

with the mesh dependent norm $\|\cdot\|_{-\frac{1}{2}, \hat{h}}$. A detailed discussion on the discretization of problem (16) can be found in Andres [1] and Andres et al. [2].

Let us consider the domain $\Omega := [-4, 4] \times [-1, 1]$ with boundary Γ divided into the Dirichlet part $\Gamma_D := \{-4, 4\} \times [-1, 1]$, the Neumann part $\Gamma_N := [-4, 4] \times \{1\}$ and the contact part $\Gamma_C := [-4, 4] \times \{-1\}$. We choose Young's modulus

$E := 200\,000$ and Poisson's ratio $\nu := 0.25$ which leads to the Lamé coefficients $\lambda = \mu = 80\,000$. The friction coefficient is $\mu_f = 0.5$. Note, that we have solved the contact problem with Coulomb friction, see (2), using the fixed point algorithm for a series of contact problems with Tresca friction that is discussed in Section 2. The volume body force is set to zero, furthermore we assume, that the body is subject to the boundary traction

$$\mathbf{t}_0 = \begin{cases} \begin{pmatrix} 0 \\ -3000\,(1-\frac{x^2}{2}+\frac{x^4}{16}) \end{pmatrix}, & \text{if } x_1 \in [-2,2], \\ 0, & \text{else}, \end{cases}$$

on the Neumann boundary Γ_N. The body is fixed at the Dirichlet boundary Γ_D. On the contact boundary we assume the body Ω to come into contact with a rigid foundation which has the positive distance

$$g := \begin{cases} \dfrac{|x_1+3|}{10}, & \text{if } x_1 \le -2, \\ 0.1, & \text{if } x_1 \in (-2,2), \\ \dfrac{|x_1-3|}{10}, & \text{if } x_1 \ge 2. \end{cases}$$

The discrete problem (21) is solved with a nested Uzawa-type algorithm, see Andres [1] for details. The following figures show the approximated solutions for a scheme of about 250 000 degrees of freedom. The computation took ten minutes on an 8 core computing system with 2.93 GHz and 48GB. Each core uses two Intel Nehalem X5570 processors.

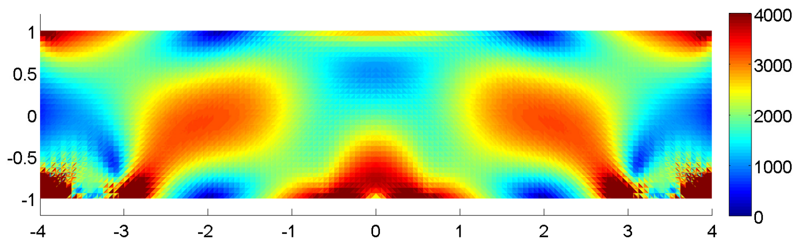


Fig. 2. Von Mises equivalent stress in Ω.

In Figure 2 the von Mises equivalent stress for plane strain is illustrated. Using the theory of Nečas and Hlaváček [22, Section 10.2] and the yield criterion of Han and Reddy [14, Section 3.3] we can derive the following equation for the von Mises equivalent stress for plane strain

$$\sigma_0 := \Big((\nu^2 - \nu + 1)[\sigma_{11}^2 + \sigma_{22}^2] + (2\nu^2 - 2\nu - 1)\sigma_{11}\sigma_{22} + 3\sigma_{12}\Big)^{\frac{1}{2}},$$

where ν denotes Poisson's ratio. Note, that we have used $\frac{3}{2}(\sigma_{12}^h + \sigma_{21}^h)$ instead of $3\sigma_{12}$ in the above equation, as the approximated stress tensor does not have to be symmetric. The singularities of the equivalent stress in the corners of the domain are due to the change of the boundary conditions and will be neglected in the following discussion.

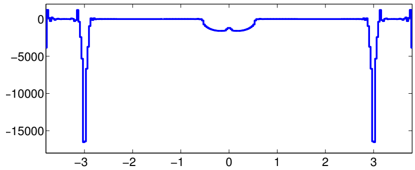


Fig. 3. Normal stress σ_n^h on Γ_C.

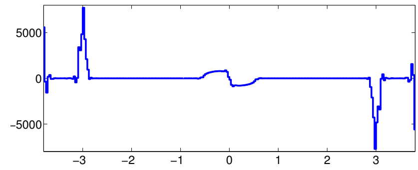


Fig. 4. Tangential stress σ_t^h on Γ_C.

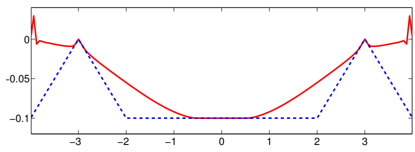


Fig. 5. Normal displacement $\lambda_n^{\tilde{h}} - g$ (red).

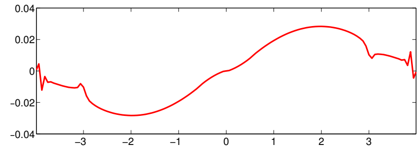


Fig. 6. Tangential displacement $\lambda_t^{\tilde{h}}$ on Γ_C.

Figures 3 and 4 show the normal stress and the tangential stress on the contact boundary, respectively. We can see, that the inequalities of Corollary 3.2 are fulfilled. Here, the absolute value of the tangential stress does not exceed $\mu_f \sigma_n^h$. Furthermore we observe a small zone on Γ_C around the point $P = (0, -1.0)$, where the body is sticking on the obstacle. This observation is verified by Figure 6, where the tangential displacement, i.e. $\lambda_t^{\tilde{h}}$, is shown.

Finally, Figure 5 shows the normal displacement $-u_n = \lambda_n^{\tilde{h}} - g$ on the contact boundary, where we have chosen the equation using the Lagrange multiplier $\lambda_n^{\tilde{h}}$ and the gap function g according to Theorem 2.2. For a better demonstration of those areas on Γ_C, that are in contact, the rigid foundation is also illustrated in dashed blue lines.

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