Finite-Horizon $\mathcal{H}_\infty$ Control for Discrete Time-Varying Systems with Randomly Occurring Nonlinearities and Fading Measurements

Derui Ding, Zidong Wang, James Lam and Bo Shen

***Abstract*—This paper deals with the $\mathcal{H}_\infty$ control problem for a class of discrete time-varying nonlinear systems with both randomly occurring nonlinearities and fading measurements over a finite-horizon. The system measurements are transmitted through fading channels described by a modified stochastic Rice fading model. The purpose of the addressed problem is to design a set of time-varying controllers such that, in the presence of channel fading and randomly occurring nonlinearities, the $\mathcal{H}_\infty$ performance is guaranteed over a given finite-horizon. The model transformation technique is first employed to simplify the addressed problem, and then the stochastic analysis in combination with the completing squares method are carried out to obtain *necessary and sufficient* conditions of an auxiliary index which is closely related to the finite-horizon $\mathcal{H}_\infty$ performance. Moreover, the time-varying controller parameters are characterized via solving coupled backward recursive Riccati difference equations (RDEs). A simulation example is utilized to illustrate the usefulness of the proposed controller design scheme.**

***Index Terms*—$\mathcal{H}_\infty$ control, finite horizon, fading channels, randomly occurring nonlinearities, recursive Riccati difference equations.**

I. INTRODUCTION

Since networks may greatly decrease the need for hardwiring and the cost of installation as well as implementation, the research on networked systems has been gaining momentum in the past few years. Many important results on filter/controller design problems against network-induced phenomena have been reported in the literature, see [3], [6], [11], [19] and the references therein, where most results have focused on communication delays, packet dropouts and signal quantizations. Unfortunately, another important network-induced phenomenon, namely, channel fading, has not yet received adequate attention in the context of filter/control designs. Roughly speaking, the fading phenomenon can be interpreted as a time-varying random change in the amplitude and phase of the transmitted signal. This kind of unreliable channels stem mainly from multipath propagation (multipath induced fading) and shadowing from obstacles affecting the wave propagation (shadow fading), see [1], [2], [18] for more details. Since the fading phenomenon can seriously degrade the signal quality, an issue of crucial importance is how to design a control system whose performance is insensitive to the effects caused by the fading channels. Very recently, the networked control systems with fading channels have received particular research attention and some preliminary results have been reported in [5], [7], [12], [16] for stability analysis, LQG control and Kalman filter problems.

In networked environments, a large class of nonlinearities can be understood as the additive nonlinear disturbances caused by environmental circumstances [8], [9], [13], [14], [17], where the nonlinear disturbances may occur in a probabilistic way and are randomly changeable in terms of their types and/or intensity, which are customarily referred to as the randomly occurring nonlinearities (RONs). On the other hand, almost all real-world systems are indeed time-varying since the system dynamics may experience constant changes in their structure and parameters caused by temperature, changes of the operating point, aging of components, etc [13]. In recent control literature, the research focus has been largely shifted from conventional linear time-invariant (LTI) systems to nonlinear time-varying ones that can better reflect the reality, see e.g. [4]. Unfortunately, to the best of the authors' knowledge, the finite-horizon $\mathcal{H}_\infty$ control problem for discrete time-varying systems with fading measurements has not been properly investigated so far, not to mention the case where RONs are also involved. It is, therefore, the purpose of this paper to shorten such a gap.

Summarizing the above discussions, it is of both theoretical importance and practical significance to examine how the measurement signals transmitted through fading channels would influence the dynamic behavior of a discrete time-varying system over a finite horizon. Therefore, the objective of this paper is to design an output feedback controller, based on fading measurements, such that the $\mathcal{H}_\infty$ performance is guaranteed over a given finite-horizon for a class of discrete time-varying nonlinear systems. By employing stochastic analysis techniques, the explicit expression of controller gains is characterized in terms of the solution to coupled recursive Riccati difference equations (RDEs). Moreover, a simulation example is provided to show the effectiveness of the proposed control design scheme. *The novelties of this paper lie in the following three aspects: 1) both channel fading and randomly occurring nonlinearities are considered in the design of the output feedback controllers guaranteeing the desired $\mathcal{H}_\infty$ performance; 2) a necessary and sufficient condition is provided for an auxiliary index which is closely related to the desired finite-horizon $\mathcal{H}_\infty$ performance; and 3) a suboptimal controller design scheme is provided by developing a recursive RDE approach.*

Notation: The notation used here is fairly standard except where otherwise stated. $\mathcal{L}_{[0,N]}$ is the space of vector functions over $[0, N]$. I denotes the identity matrix of compatible dimension. A^T, $\|A\|_F$ and $A^\dagger$ denote the transpose, the Frobenius norm and the Moore-Penrose pseudo inverse of a matrix A, respectively. $\mathrm{diag}\{\ldots\}$ stands for a block-diagonal matrix.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following discrete time-varying stochastic system defined on $k \in [0, N]$:

$$\begin{cases} x_{k+1} = A_k x_k + \alpha_k h_k(x_k) + B_k u_k + D_k w_k, \\ z_k = L_k x_k, \end{cases} \tag{1}$$

with measurement

$$y_k = C_k x_k + E_k v_k \tag{2}$$

where $x_k \in \mathbb{R}^{n_x}$ represents the state vector that cannot be observed directly, $y_k \in \mathbb{R}^{n_y}$ is the measurement output, $z_k \in \mathbb{R}^{n_z}$ is the controlled output, and w_k and $v_k \in \mathcal{L}_{[0,N]}$ are the external disturbances. A_k, B_k, C_k, D_k, E_k and L_k are known real-valued time-varying matrices with appropriate dimensions. The stochastic variable α_k is a Bernoulli-distributed white noise sequence taking values of 0 or 1 with the probabilities $\mathrm{Prob}\{\alpha_k = 0\} = 1 - \bar{\alpha}$ and $\mathrm{Prob}\{\alpha_k = 1\} = \bar{\alpha}$. The nonlinear vector-valued function $h_k : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is assumed to be continuously differentiable in x and satisfies the following sector-bound condition

$$(h_k(x) - \Phi_k x)^T (h_k(x) - \Psi_k x) \leq 0, \tag{3}$$

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where Φ_k and Ψ_k are known matrices with compatible dimensions with $\Phi_k > \Psi_k$ for all k.

It follows easily from (3) that

$$\Big[h_k(x) - \Big(\frac{\Phi_k+\Psi_k}{2} + \frac{\Phi_k-\Psi_k}{2}\Big)x\Big]^T \times \Big[h_k(x) - \Big(\frac{\Phi_k+\Psi_k}{2} - \frac{\Phi_k-\Psi_k}{2}\Big)x\Big] \le 0.$$

Then, denoting $\Delta(h_k) = h_k(x) - \frac{1}{2}(\Phi_k+\Psi_k)x$ and $\mathcal{N}_k = \frac{1}{2}(\Phi_k - \Psi_k)$, one has $\Delta^T(h_k)\Delta(h_k) \le x^T\mathcal{N}_k^T\mathcal{N}_k x$, immediately. Therefore, there exists at least a function $\Theta_{k,x}$ satisfying $\Delta(h_k) = \Theta_{k,x}x$ and $\Theta_{k,x}^T\Theta_{k,x} \le \mathcal{N}_k^T\mathcal{N}_k$. Furthermore, the sector-bound condition can be transformed into the sector-bound uncertainties described by

$$h_k(x) = \frac{\Phi_k+\Psi_k}{2}x + F_{k,x}\mathcal{N}_k x_k \tag{4}$$

with $F_{k,x} := \Theta_{k,x}\mathcal{N}_k^{-1}$ satisfying $F_{k,x}^T F_{k,x} \le I$.

We are now in a position to introduce the fading measurements. Let the number of paths, ℓ, be given. Considering the fading channels, the measurement signal received by the controller is described by

$$\tilde{y}_k = \sum_{i=0}^{\ell_k} \vartheta_k^i y_{k-i} + M\xi_k \tag{5}$$

with $\ell_k = \min\{\ell,\ k\}$, where ϑ_k^i $(i = 0,1,\ldots,\ell_k)$ are the channel coefficients which are mutually independent and take values on $[0,1]$ with mathematical expectations $\bar{\vartheta}_i$ and variances $\tilde{\vartheta}_i$. $\xi_k \in l_2([0,+\infty);\mathbb{R}^{n_y})$ is an external disturbance. M is known real-valued matrix with appropriate dimensions. For simplicity, we will set $\{y_k\}_{k\in[-\ell,-1]} = 0$, i.e., $\{x_k\}_{k\in[-\ell,-1]} = 0$ and $\{v_k\}_{k\in[-\ell,-1]} = 0$.

For the given receiver model (5), we consider the following output feedback controller for the discrete time-varying nonlinear system (1):

$$u_k = K_k\tilde{y}_k = \sum_{i=0}^{\ell} \vartheta_k^i K_k y_{k-i} + K_k\xi_k. \tag{6}$$

Setting $\bar{x}_k = [x_k^T, x_{k-1}^T, x_{k-2}^T, \ldots, x_{k-\ell}^T]^T$ and $\eta_k = [w_k, v_k, v_{k-1}, \ldots, v_{k-\ell}, \xi_k]^T$, we obtain an augmented system from (1) and (6) as follows:

$$\begin{cases} \bar{x}_{k+1} = \big(\mathscr{A}_k + \mathscr{B}_k K_k \bar{\vartheta}\mathscr{C}_k + (\alpha_k - \bar{\alpha})\tilde{\mathscr{A}_k} \\ \qquad + \alpha_k\tilde{\Theta}_k\big)\bar{x}_k + \mathscr{B}_k K_k \vartheta_k \mathscr{C}_k \bar{x}_k \\ \qquad + (\mathscr{B}_k K_k \bar{\upsilon}\mathscr{E}_k + \mathscr{D}_k)\eta_k + \mathscr{B}_k K_k \upsilon_k \mathscr{E}_k \eta_k, \\ z_k = \mathcal{L}_k \bar{x}_k, \end{cases} \tag{7}$$

where $\vartheta_k = [\ (\vartheta_k^0 - \bar{\vartheta}_0)I \quad (\vartheta_k^1 - \bar{\vartheta}_1)I \quad \ldots \quad (\vartheta_k^\ell - \bar{\vartheta}_\ell)I\ ]$,

$$\mathscr{A}_k = \begin{bmatrix} \bar{\mathscr{A}_k} & 0 \\ I & 0 \end{bmatrix}, \quad \tilde{\mathscr{A}_k} = \mathrm{diag}\{\mathscr{M}_k, 0, 0, \ldots, 0\},$$

$$\bar{\mathscr{A}_k} = \Big[\ A_k + \bar{\alpha}\mathscr{M}_k \quad \underbrace{0 \quad 0 \quad \ldots \quad 0}_{\ell-1}\ \Big],\ \bar{\upsilon} = [\ 0 \quad \bar{\vartheta} \quad I\ ],$$

$$\mathscr{B}_k = [\ B_k^T \quad 0 \quad 0 \quad \ldots \quad 0\ ]^T,\ \mathscr{D}_k = \mathrm{diag}\{D_k, 0, 0, \ldots, 0\},$$

$$\mathscr{C}_k = \mathrm{diag}\{C_k, C_{k-1}, \ldots, C_{k-\ell}\},\ \upsilon_k = [\ 0 \quad \vartheta_k \quad 0\ ],$$

$$\mathscr{E}_k = \mathrm{diag}\{0, E_k, E_{k-1}, \ldots, E_{k-\ell}, I\},$$

$$\mathcal{L}_k = [\ L_k \quad 0 \quad 0 \quad \ldots \quad 0\ ],\ \tilde{\Theta}_k = \mathrm{diag}\{\Theta_{k,x}, 0, 0, \ldots, 0\},$$

$$\mathscr{M}_k = (\Phi_k + \Psi_k)/2, \quad \bar{\vartheta} = [\ \bar{\vartheta}_0 I \quad \bar{\vartheta}_1 I \quad \ldots \quad \bar{\vartheta}_\ell I\ ].$$

Our aim in this paper is to design a finite-horizon output feedback controller of the form (6) such that, for the given disturbance attenuation level $\gamma > 0$, the positive definite matrix W and the initial state x_0, the controlled output z_k satisfies the following $\mathcal{H}_\infty$ performance constraint:

$$\mathbb{E}\{\|z_k\|_{[0,N]}^2\} < \gamma^2\|\eta_k\|_{[0,N]}^2 + \gamma^2\mathbb{E}\left\{x_0^T W x_0\right\}, \tag{8}$$

where $\|x_k\|_{[0,N]}^2 := \sum_{k=0}^N \|x_k\|^2$ for any vector sequences x_k.

To cope with the parameter uncertainties in (7), a convenient way is to regard them as one of the sources of the disturbances. Therefore, what we need to do is to reject the influence from all the disturbances to the controlled output according to the prescribed $\mathcal{H}_\infty$ requirement. For this purpose, we rewrite (7) as follows:

$$\begin{cases} \bar{x}_{k+1} = \big(\mathscr{A}_k + \mathscr{B}_k K_k \bar{\vartheta}\mathscr{C}_k\big)\bar{x}_k \\ \qquad + (\alpha_k - \bar{\alpha})\tilde{\mathscr{A}_k}\bar{x}_k + \mathscr{B}_k K_k \vartheta_k \mathscr{C}_k \bar{x}_k \\ \qquad + (\mathscr{B}_k K_k \bar{\theta}\tilde{\mathscr{E}_k} + \tilde{\mathscr{D}_k})\tilde{\eta}_k + \mathscr{B}_k K_k \theta_k \tilde{\mathscr{E}_k}\tilde{\eta}_k, \\ z_k = \mathcal{L}_k \bar{x}_k, \end{cases} \tag{9}$$

where $\tilde{\eta}_k = [\eta_k^T \quad (\varepsilon_k\alpha_k F_{k,x}\mathcal{N}_k x_k)^T]^T$, $\tilde{\mathscr{D}_k} = [\mathscr{D}_k \quad \varepsilon_k^{-1}I]$, $\tilde{\mathscr{E}_k} = \mathrm{diag}\{\mathscr{E}_k, 0\}$, $\bar{\theta} = [\bar{\upsilon} \quad 0]$ and $\theta_k = [\upsilon_k \quad 0]$. Here, ε_k is a positive function representing the scaling of the perturbation, which is introduced to provide more flexibility in the controller design. Furthermore, we introduce the following *auxiliary index*:

$$\mathbb{E}\left\{\|z_k\|_{[0,N]}^2\right\} < \mathbb{E}\big\{\gamma^2\|\tilde{\eta}_k\|_{[0,N]}^2 - \bar{\alpha}\gamma^2\|\varepsilon_k\aleph_k\bar{x}_k\|_{[0,N]}^2\big\} + \gamma^2\mathbb{E}\big\{\bar{x}_0^T W\bar{x}_0\big\}, \tag{10}$$

with $\aleph_k := [\ \mathcal{N}_k \quad 0 \quad \ldots \quad 0\ ]$. It is worth emphasizing that (8) is satisfied if (10) holds, see [10] for more details.

III. Main results

In this section, let us investigate both the analysis and synthesis problems for the $\mathcal{H}_\infty$ controller design of system (1) with fading channels (5). The following four lemmas will be used in deriving our main results, where the proofs of Lemma 2 and Lemma 3 have been moved to the appendices for clarity of presentation.

Lemma 1: [15] Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ be known nonzero matrices with appropriate dimensions. The solution X to $\min_X \|\mathcal{U}X\mathcal{W} - \mathcal{V}\|_F$ is $\mathcal{U}^\dagger\mathcal{V}\mathcal{W}^\dagger$.

Lemma 2: For the external disturbances $\tilde{\eta}_k$ and the initial value $\bar{x}_0$, let $\bar{x}_k$ be the corresponding solution of system (9) defined on $[0,N]$. Then, we have

$$\begin{aligned} &\mathcal{J}_1(\bar{x}_0, \tilde{\eta}_k) \\ &:= \mathbb{E}\left\{\|z_k\|_{[0,N]}^2 - \gamma^2\|\tilde{\eta}_k\|_{[0,N]}^2 + \bar{\alpha}\gamma^2\|\varepsilon_k\aleph_k\bar{x}_k\|_{[0,N]}^2\right\} \\ &= \sum_{k=0}^N \mathbb{E}\left\{\begin{bmatrix}\bar{x}_k \\ \tilde{\eta}_k\end{bmatrix}^T \begin{bmatrix}\mathcal{R}_{k+1}^{11} - \mathcal{P}_k & \mathcal{R}_{k+1}^{12} \\ * & -\mathcal{R}_{k+1}^{22}\end{bmatrix}\begin{bmatrix}\bar{x}_k \\ \tilde{\eta}_k\end{bmatrix}\right\} \\ &\quad + \mathbb{E}\left\{\bar{x}_0^T\mathcal{P}_0\bar{x}_0 - \bar{x}_{N+1}^T\mathcal{P}_{N+1}\bar{x}_{N+1}\right\}. \end{aligned} \tag{11}$$

Furthermore, if $|\mathcal{R}_k^{22}| \ne 0$ for all $k \in [0,N]$, by selecting $\tilde{\eta}_k = (\mathcal{R}_{k+1}^{22})^{-1}(\mathcal{R}_{k+1}^{12})^T\bar{x}_k$ and denoting $\bar{u}_k = K_k\bar{\vartheta}\mathscr{C}_k\bar{x}_k$, one has

$$\begin{aligned} \mathcal{J}_2(\bar{u}_k; \tilde{\eta}_k) &:= \mathbb{E}\{\|z_k\|_{[0,N]}^2 + \|\bar{u}_k\|_{[0,N]}^2\} \\ &= \sum_{k=0}^N \mathbb{E}\left\{\begin{bmatrix}\bar{x}_k \\ \bar{u}_k\end{bmatrix}^T \mathscr{Z}_k^1 \begin{bmatrix}\bar{x}_k \\ \bar{u}_k\end{bmatrix}\right\} \\ &\quad + \mathbb{E}\left\{\bar{x}_0^T\mathcal{Q}_0\bar{x}_0 - \bar{x}_{N+1}^T\mathcal{Q}_{N+1}\bar{x}_{N+1}\right\} \end{aligned} \tag{12}$$

where $\tilde{\alpha} = \bar{\alpha}(1-\bar{\alpha})$, $\{\mathcal{P}_k\}_{0\le k\le N+1}$ and $\{\mathcal{Q}_k\}_{0\le k\le N+1}$ are two families of matrices with partitioning $\mathcal{O}_k = \big[\mathcal{O}_k^{ij}\big]_{\ell+1,\ell+1}$ ($\mathcal{O} := \mathcal{P}$ or $\mathcal{Q}$), and

$$\mathscr{Z}_k^1 = \begin{bmatrix}\Delta_{k+1} + \mathcal{L}_k^T\mathcal{L}_k - \mathcal{Q}_k & \mathcal{S}_{k+1}^1 \\ * & \mathcal{S}_{k+1}^2\end{bmatrix},$$

$$\Pi_{k+1}^1 = \mathrm{diag}\{\tilde{\vartheta}_0, \tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_\ell\} \otimes (K_k^T B_k^T \mathcal{P}_{k+1}^{11} B_k K_k),$$

$$\Pi_{k+1}^2 = [0 \quad \mathrm{diag}\{\tilde{\vartheta}_0, \tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_\ell\} \quad 0 \quad 0] \otimes (K_k^T B_k^T \mathcal{P}_{k+1}^{11} B_k K_k),$$

$$\Pi_{k+1}^3 = \mathrm{diag}\{0, \tilde{\vartheta}_0, \tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_\ell, 0, 0\} \otimes (K_k^T B_k^T \mathcal{P}_{k+1}^{11} B_k K_k),$$

$$\Xi_{k+1}^1 = \mathrm{diag}\{\tilde{\vartheta}_0, \tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_\ell\} \otimes (K_k^T B_k^T \mathcal{Q}_{k+1}^{11} B_k K_k),$$

$$\Xi^2_{k+1} = [0 \quad \text{diag}\{\tilde{\vartheta}_0, \tilde{\vartheta}_1, \dots, \tilde{\vartheta}_\ell\} \quad 0 \quad 0] \otimes (K_k^T B_k^T \mathcal{Q}^{11}_{k+1} B_k K_k),$$
$$\Xi^3_{k+1} = \text{diag}\{0, \tilde{\vartheta}_0, \tilde{\vartheta}_1, \dots, \tilde{\vartheta}_\ell, 0, 0\} \otimes (K_k^T B_k^T \mathcal{Q}^{11}_{k+1} B_k K_k),$$
$$\mathcal{R}^{11}_{k+1} = (\mathscr{A}_k + \mathscr{B}_k K_k \bar{\vartheta} \mathscr{C}_k)^T \mathcal{P}_{k+1} (\mathscr{A}_k + \mathscr{B}_k K_k \bar{\vartheta} \mathscr{C}_k) + \tilde{\alpha} \tilde{\mathscr{A}}_k^T \mathcal{P}_{k+1} \tilde{\mathscr{A}}_k + \mathscr{C}_k^T \Pi^1_{k+1} \mathscr{C}_k + \bar{\alpha} \gamma^2 \varepsilon_k^2 \aleph_k^T \aleph_k + \mathcal{L}_k^T \mathcal{L}_k,$$
$$\mathcal{R}^{12}_{k+1} = (\mathscr{A}_k + \mathscr{B}_k K_k \bar{\vartheta} \mathscr{C}_k)^T \mathcal{P}_{k+1} \times (\mathscr{B}_k K_k \bar{\theta} \tilde{\mathscr{E}}_k + \tilde{\mathscr{D}}_k) + \mathscr{C}_k^T \Pi^2_{k+1} \tilde{\mathscr{E}}_k,$$
$$\mathcal{R}^{22}_{k+1} = \gamma^2 I - (\mathscr{B}_k K_k \bar{\theta} \tilde{\mathscr{E}}_k + \tilde{\mathscr{D}}_k)^T \mathcal{P}_{k+1} \times (\mathscr{B}_k K_k \bar{\theta} \tilde{\mathscr{E}}_k + \tilde{\mathscr{D}}_k) - \tilde{\mathscr{E}}_k^T \Pi^3_{k+1} \tilde{\mathscr{E}}_k,$$
$$\Lambda_{k+1} = (\mathscr{B}_k K_k \bar{\theta} \tilde{\mathscr{E}}_k + \tilde{\mathscr{D}}_k)(\mathcal{R}^{22}_{k+1})^{-1}(\mathcal{R}^{12}_{k+1})^T,$$
$$\Delta_{k+1} = (\mathscr{A}_k + \Lambda_{k+1})^T \mathcal{Q}_{k+1} (\mathscr{A}_k + \Lambda_{k+1}) + \tilde{\alpha} \tilde{\mathscr{A}}_k^T \mathcal{Q}_{k+1} \tilde{\mathscr{A}}_k + \mathscr{C}_k^T \Xi^1_{k+1} \mathscr{C}_k + 2\mathscr{C}_k^T \Xi^2_{k+1} \tilde{\mathscr{E}}_k (\mathcal{R}^{22}_{k+1})^{-1} (\mathcal{R}^{12}_{k+1})^T + \mathcal{R}^{12}_{k+1} (\mathcal{R}^{22}_{k+1})^{-1T} \tilde{\mathscr{E}}_k^T \Xi^3_{k+1} \tilde{\mathscr{E}}_k (\mathcal{R}^{22}_{k+1})^{-1} (\mathcal{R}^{12}_{k+1})^T,$$
$$\mathcal{S}^1_{k+1} = (\mathscr{A}_k + \Lambda_k)^T \mathcal{Q}_{k+1} \mathscr{B}_k, \quad \mathcal{S}^2_{k+1} = \mathscr{B}_k^T \mathcal{Q}_{k+1} \mathscr{B}_k + I.$$

Lemma 3: Given the disturbance attenuation level $\gamma > 0$ and the positive definite matrix W. For the augmented system (9) with any nonzero $\{\tilde{\eta}_k\}_{0 \le k \le N} \in \mathcal{L}_{[0,N]}$, the following two statements are equivalent:

(i) The auxiliary index (10) is satisfied.

(ii) There exists a set of real-valued matrices $\{K_k\}_{0 \le k \le N}$, positive scalars $\{\varepsilon_k\}_{0 \le k \le N}$ and matrices $\{\mathcal{P}_k\}_{0 \le k \le N+1}$ (with the final condition $\mathcal{P}_{N+1} = 0$) such that the following backward recursive RDE:

$$\mathcal{R}^{11}_{k+1} + \mathcal{R}^{12}_{k+1} (\mathcal{R}^{22}_{k+1})^{-1} (\mathcal{R}^{12}_{k+1})^T = \mathcal{P}_k \tag{13}$$

gives

$$\mathcal{R}^{22}_{k+1} > 0 \quad \text{and} \quad \mathcal{P}_0 < \gamma^2 W, \tag{14}$$

where the corresponding matrix parameters are defined in Lemma 2.

It should be pointed out that Lemma 3 provides a necessary and sufficient condition of the auxiliary index (10). Such a condition serves as a key to solve the addressed stochastic disturbance attenuation problems. In the next stage, we shall proceed to tackle the design problem of the controller (6) such that the closed-loop system (7) satisfies the $\mathcal{H}_\infty$ performance requirement (8).

Theorem 1: For the given disturbance attenuation level $\gamma > 0$ and positive definite matrix W, the closed-loop system (7) satisfies the $\mathcal{H}_\infty$ performance constraint (8) for any nonzero disturbance sequence $\{\eta_k\}_{0 \le k \le N} \in \mathcal{L}_{[0,N]}$ if there exists a set of solutions $\{(\varepsilon_k, \mathcal{P}_k, \mathcal{Q}_k, K_k)\}_{0 \le k \le N}$ with $\varepsilon_k > 0$ satisfying (13) and the following recursive RDE:

$$\Delta_{k+1} + \mathcal{L}_k^T \mathcal{L}_k - \mathcal{S}^1_{k+1} (\mathcal{S}^2_{k+1})^{-1} (\mathcal{S}^1_{k+1})^T = \mathcal{Q}_k, \tag{15}$$

subject to

$$\begin{cases} \mathcal{P}_{N+1} = \mathcal{Q}_{N+1} = 0, & (16a) \\ \mathcal{S}^2_{k+1} > 0,\ \mathcal{R}^{22}_{k+1} > 0,\ \mathcal{P}_0 < \gamma^2 W, & (16b) \\ K_k^* = \arg\min_{K_k} \left\| K_k \bar{\vartheta} \mathscr{C}_k + (\mathcal{S}^2_{k+1})^{-1} (\mathcal{S}^1_{k+1})^T \right\|_F, & (16c) \end{cases}$$

where corresponding matrix parameters are defined in Lemma 2.

Proof: Firstly, if there exists $\{\mathcal{P}_k\}_{0 \le k \le N}$ satisfying (13) and (16b), it can be easily seen from Lemma 3 that the system (9) satisfies the auxiliary index (10) and therefore the pre-specified $\mathcal{H}_\infty$ performance (8) is satisfied for the closed-loop system (7). In this case, the worst-case disturbance can be expressed as $\tilde{\eta}_k^* = (\mathcal{R}^{22}_{k+1})^{-1} (\mathcal{R}^{12}_{k+1})^T \bar{x}_k$.

Next, by employing the worst-case disturbance, we aim to provide a design scheme of the controller parameter K_k. For this purpose, by using the completing squares method, it follows from Lemma 2 that

$$\begin{aligned} &\mathcal{J}_2(\bar{u}_k; \tilde{\eta}_k^*) \\ &= \mathbb{E}\left\{ \bar{x}_0^T \mathcal{Q}_0 \bar{x}_0 - \bar{x}_{N+1}^T \mathcal{Q}_{N+1} \bar{x}_{N+1} \right\} + \sum_{k=0}^N \mathbb{E}\Big\{ \bar{x}_k^T \big(\Delta_{k+1} + \mathcal{L}_k^T \mathcal{L}_k - \mathcal{Q}_k - \mathcal{S}^1_{k+1} (\mathcal{S}^2_{k+1})^{-1} (\mathcal{S}^1_{k+1})^T\big) \bar{x}_k + (\bar{u}_k - \bar{u}_k^*)^T \mathcal{S}^2_{k+1} (\bar{u}_k - \bar{u}_k^*) \Big\} \\ &\le \mathbb{E}\left\{ \bar{x}_0^T \mathcal{Q}_0 \bar{x}_0 - \bar{x}_{N+1}^T \mathcal{Q}_{N+1} \bar{x}_{N+1} \right\} + \sum_{k=0}^N \mathbb{E}\Big\{ \bar{x}_k^T \big(\Delta_{k+1} + \mathcal{L}_k^T \mathcal{L}_k - \mathcal{S}^1_{k+1} (\mathcal{S}^2_{k+1})^{-1} (\mathcal{S}^1_{k+1})^T - \mathcal{Q}_k\big) \bar{x}_k + \left\| K_k \bar{\vartheta} \mathscr{C}_k + (\mathcal{S}^2_{k+1})^{-1} \mathcal{S}^{1T}_{k+1} \right\|_F^2 \left\| \mathcal{S}^2_{k+1} \right\|_F \|\bar{x}_k\|^2 \Big\} \end{aligned} \tag{17}$$

where $\bar{u}_k^* = -(\mathcal{S}^2_{k+1})^{-1} (\mathcal{S}^1_{k+1})^T \bar{x}_k$. Furthermore, the controller parameter K_k can be selected to satisfy (15) and (16c) simultaneously, which ends the proof.

Clearly, it is generally difficult to solve the optimization problem (16c). For the convenience in application, the expression of the parameter K_k can be acquired by using the Moore-Penrose pseudo inverse in the following theorem.

Theorem 2: For the given disturbance attenuation level $\gamma > 0$ and positive definite matrix W, the closed-loop system (7) satisfies the $\mathcal{H}_\infty$ performance constraint (8) for any nonzero disturbance sequence $\{\eta_k\}_{0 \le k \le N} \in \mathcal{L}_{[0,N]}$, if there exists a set of solutions $\{(\varepsilon_k, \lambda_k, \delta_k, \mathcal{P}_k, \mathcal{Q}_k, K_k)\}_{0 \le k \le N}$ (with $\lambda_k > 0$ and $\varepsilon_k > 0$) satisfying the following recursive RDEs:

$$\begin{cases} \mathcal{R}^{11}_{k+1} + \bar{\mathcal{R}}^{12}_{k+1} (\bar{\mathcal{R}}^{22}_{k+1})^{-1} (\bar{\mathcal{R}}^{12}_{k+1})^T = \mathcal{P}_k, & (18a) \\ \bar{\Delta}_{k+1} + \mathcal{L}_k^T \mathcal{L}_k - \bar{\mathcal{S}}^1_{k+1} (\mathcal{S}^2_{k+1})^{-1} (\bar{\mathcal{S}}^1_{k+1})^T = \mathcal{Q}_k, & (18b) \end{cases}$$

subject to

$$\begin{cases} \mathcal{P}_{N+1} = \mathcal{Q}_{N+1} = 0, & (19a) \\ \bar{\mathcal{R}}^{22}_{k+1} > 0,\ \mathcal{P}_0 < \gamma^2 W,\ \mathcal{S}^2_{k+1} > 0, & (19b) \\ K_k^* = \Upsilon^\dagger_{k+1} \Gamma_{k+1} (\bar{\vartheta} \mathscr{C}_k)^\dagger, & (19c) \\ \mathscr{W}_k \le \delta_k I, & (19d) \end{cases}$$

where

$$\mathscr{G}_k = [\, \tilde{\mathscr{D}}_k \quad \lambda_k^{-1} \mathscr{B}_k \,],\ \bar{\Lambda}_{k+1} = \mathscr{G}_k (\bar{\mathcal{R}}^{22}_{k+1})^{-1} (\bar{\mathcal{R}}^{12}_{k+1})^T,$$
$$\bar{\mathcal{R}}^{12}_{k+1} = (\mathscr{A}_k + \mathscr{B}_k K_k \bar{\vartheta} \mathscr{C}_k)^T \mathcal{P}_{k+1} \mathscr{G}_k,\ \mathcal{U} = [\, I \quad 0 \quad 0 \,],$$
$$\bar{\mathcal{R}}^{22}_{k+1} = \gamma^2 I - \mathscr{G}_k^T \mathcal{P}_{k+1} \mathscr{G}_k - \delta_k \mathcal{U}^T \mathcal{U},$$
$$\bar{\mathcal{S}}^1_{k+1} = (\mathscr{A}_k + \bar{\Lambda}_{k+1})^T \mathcal{Q}_{k+1} \mathscr{B}_k,$$
$$\bar{\Delta}_{k+1} = (\mathscr{A}_k + \bar{\Lambda}_{k+1})^T \mathcal{Q}_{k+1} (\mathscr{A}_k + \bar{\Lambda}_{k+1}) + \tilde{\alpha} \tilde{\mathscr{A}}_k^T \mathcal{Q}_{k+1} \tilde{\mathscr{A}}_k + \mathscr{C}_k^T \Xi^1_{k+1} \mathscr{C}_k,$$
$$\Gamma_{k+1} = -(\mathcal{S}^2_{k+1})^{-1} \mathscr{B}_k^T \mathcal{Q}_{k+1} (I + \mathscr{G}_k (\bar{\mathcal{R}}^{22}_{k+1})^{-1} \mathscr{G}_k^T \mathcal{P}_{k+1}) \mathscr{A}_k,$$
$$\Upsilon_{k+1} = I + (\mathcal{S}^2_{k+1})^{-1} \mathscr{B}_k^T \mathcal{Q}_{k+1} \mathscr{G}_k (\bar{\mathcal{R}}^{22}_{k+1})^{-1} \mathscr{G}_k^T \mathcal{P}_{k+1} \mathscr{B}_k,$$
$$\mathscr{W}_k = \gamma^2 \lambda_k \mathscr{E}_k^T \big(\bar{\upsilon}^T K_k^{*T} K_k^* \bar{\upsilon} + \text{diag}\{0, \tilde{\vartheta}_0, \dots, \tilde{\vartheta}_\ell, 0\} \otimes (K_k^{*T} K_k^*) \big) \mathscr{E}_k,$$

and the other corresponding matrix parameters are defined as in Lemma 2.

Proof: Denote $\tilde{v}_k = \lambda_k \big(\sum_{i=0}^{\ell} \vartheta_k^i K_k E_{k-i} v_{k-i} + K_k \xi_k \big)$ where $\lambda_k > 0$ is introduced to offer more flexibility in the controller design. Next, selecting

$$\zeta_k = [w_k \quad \tilde{v}_k \quad (\varepsilon_k \alpha_k F_k \mathcal{N}_k x_k)^T]^T,$$

we rewrite (9) as follows:

$$\begin{cases} \bar{x}_{k+1} = \left(\mathscr{A}_k + \mathscr{B}_k K_k \bar{\vartheta}\mathscr{C}_k\right)\bar{x}_k + (\alpha_k - \bar{\alpha})\tilde{\mathscr{A}}_k\bar{x}_k \\ \qquad\quad + \mathscr{B}_k K_k \vartheta_k \mathscr{C}_k \bar{x}_k + \mathscr{G}_k\zeta_k, \\ z_k = \mathcal{L}_k\bar{x}_k. \end{cases} \tag{20}$$

It can be easily seen that the nonzero disturbance sequence $\{\zeta_k\}_{0\leq k\leq N}$ belongs to $\mathcal{L}_{[0,N]}$. On the other hand, it follows from Lemma 1 that (19c) is the solution of the optimization problem

$$\min_{K_k} \left\| \Upsilon_{k+1} K_k \bar{\vartheta}\mathscr{C}_k - \Gamma_{k+1} \right\|_F$$

which can be rewritten as

$$\min_{K_k} \left\| K_k \bar{\vartheta}\mathscr{C}_k + (\mathcal{S}^2_{k+1})^{-1}(\bar{\mathcal{S}}^1_{k+1})^T \right\|_F. \tag{21}$$

According to Theorem 1, if there exists a set of solutions satisfying the recursive RDEs (18a) and (18b) with (19a)-(19d), one has

$$\begin{aligned} \mathbb{E}\left\{\|z_k\|^2_{[0,N]}\right\} < \mathbb{E}\Big\{&\gamma^2\|\zeta_k\|^2_{[0,N]} - \delta_k\|\mathcal{U}\zeta_k\|^2_{[0,N]} \\ &- \bar{\alpha}\gamma^2\|\varepsilon_k\aleph_k\bar{x}_k\|^2_{[0,N]}\Big\} + \gamma^2\mathbb{E}\left\{\bar{x}_0^T W\bar{x}_0\right\}. \end{aligned} \tag{22}$$

Furthermore, in light of (19d), the above inequality yields

$$\begin{aligned} &\mathbb{E}\left\{\|z_k\|^2_{[0,N]}\right\} \\ <\ &\mathbb{E}\Big\{\gamma^2\|\tilde{\eta}_k\|^2_{[0,N]} + \gamma^2\|\tilde{v}_k\|^2_{[0,N]} - \delta_k\|w_k\|^2_{[0,N]} \\ &- \bar{\alpha}\gamma^2\|\varepsilon_k\aleph_k\bar{x}_k\|^2_{[0,N]}\Big\} + \gamma^2\mathbb{E}\left\{\bar{x}_0^T W\bar{x}_0\right\} \\ \leq\ &\mathbb{E}\left\{\gamma^2\|\tilde{\eta}_k\|^2_{[0,N]} - \bar{\alpha}\gamma^2\|\varepsilon_k\aleph_k\bar{x}_k\|^2_{[0,N]}\right\} + \gamma^2\mathbb{E}\left\{\bar{x}_0^T W\bar{x}_0\right\} \end{aligned} \tag{23}$$

which implies that the closed-loop system (7) achieves the $\mathcal{H}_\infty$ performance constraint (8). The proof is complete.

Remark 1: In this paper, we examine how the channel fading and randomly occurring nonlinearities influence the $\mathcal{H}_\infty$ performance over finite-horizon $[0, N]$. It is worth mentioning that the conditions in Lemma 3 and Theorem 1 are obtained mainly by the "completing the square" technique which results in little conservatism. Compared to existing literature, our results have the following three distinguishing features: 1) the system under investigation is in the discrete time-varying form; 2) the technology of model transformation is employed to reduce the complexity of system analysis; and 3) this paper represents one of the first attempts to address both channel fading and RONs for the $\mathcal{H}_\infty$ control problems by using the backward recursive RDEs. Furthermore, in Theorem 2, all the system parameters, the probability for channel coefficients as well as RONs are reflected in the backward recursive RDEs.

Remark 2: In the case of the time-varying systems without RONs, a RDE-based condition of the $\mathcal{H}_\infty$ control can be easily deduced from Theorem 1 as long as the terms $\tilde{\alpha}\mathscr{M}_k$, $\tilde{\alpha}\tilde{\mathscr{A}}_k^T\mathcal{P}_{k+1}\tilde{\mathscr{A}}_k$, $\bar{\alpha}\gamma^2\varepsilon_k^2\aleph_k^T\aleph_k$ and $\tilde{\alpha}\tilde{\mathscr{A}}_k^T\mathcal{Q}_{k+1}\tilde{\mathscr{A}}_k$ are removed from $\mathscr{A}_k$, $\mathcal{R}^{11}_{k+1}$ and Δ_{k+1}. Furthermore, in case of no fading channels, the corresponding results can be obtained from Theorem 1 by setting $\ell = 0$.

IV. Numerical Example

Consider system (1) with the fading measurement (5) with

$$A_k = \begin{bmatrix} 0.42 + \sin(2k-1) & -0.40 \\ -0.40 + e^{-5k} & 0.85 \end{bmatrix},\ B_k = \begin{bmatrix} 0.85 \\ -0.65 \end{bmatrix},$$
$$C_k = [0.65\quad -0.70],\ D_k = [-0.02\quad 0.015]^T,$$
$$E_k = 0.01,\ L_k = [0.20\quad 0.20].$$

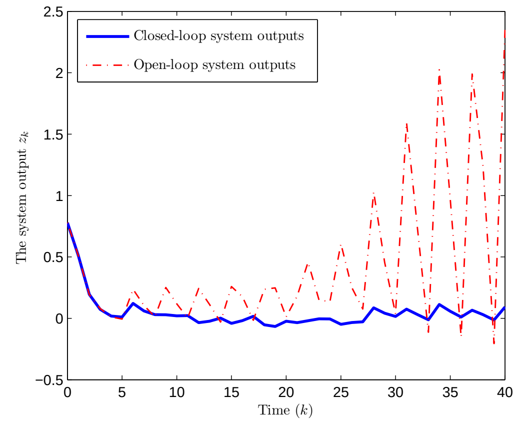

Fig. 1. The system output.

Let the nonlinear vector-valued function $f_k(x_k)$ be

$$h_k(x_k) = \begin{cases} \begin{bmatrix} -0.60x_k^1 + 0.30x_k^2 + \tanh(0.30x_k^1) \\ 0.60x_k^2 - \tanh(0.20x_k^2) \end{bmatrix}, \\ \qquad\qquad\qquad\qquad 0 \leq k < 15, \\ \begin{bmatrix} 0.35x_k^1 - \tanh(0.25x_k^1) \\ 0.5x_k^2 \end{bmatrix}, \quad 15 \leq k \leq 41, \end{cases}$$

where x_k^i $(i = 1, 2)$ denotes the i-th element of the system state x_k. The probability of RONs is taken as $\bar{\alpha} = 0.10$. The order of the fading model is $\ell = 2$ and channel coefficients ϑ_k^0, ϑ_k^1 and ϑ_k^2 obey the Gaussian distributions $\mathcal{N}(0.9, 0.1^2)$, $\mathcal{N}(0.2, 0.5^2)$ and $\mathcal{N}(0.2, 0.5^2)$, respectively. Meanwhile, it is easy to see that the constraint (3) can be met with

$$\Phi_k = \begin{cases} \begin{bmatrix} -0.30 & 0.30 \\ 0 & 0.60 \end{bmatrix}, & 0 \leq k < 15, \\ \begin{bmatrix} 0.10 & 0 \\ 0 & 0.50 \end{bmatrix}, & 15 \leq k < 41, \end{cases}$$
$$\Psi_k = \begin{cases} \begin{bmatrix} -0.60 & 0.30 \\ 0 & 0.40 \end{bmatrix}, & 0 \leq k < 15, \\ \begin{bmatrix} 0.35 & 0 \\ 0 & 0.50 \end{bmatrix}, & 15 \leq k < 41. \end{cases}$$

In this example, the $\mathcal{H}_\infty$ performance level γ, positive definite matrix W and time-horizon N are taken as 0.98, diag$\{0.50,\ 0.50\}$ and 40, respectively. Using the given algorithm and Matlab software, the set of solutions to recursive RDEs in Theorem 2 are obtained and the controller gain matrices are shown in Table I, where ε_k, λ_k and δ_k are selected as $\varepsilon_k = 1.0$, $\lambda_k = 2.5$ and $\delta_k = 0.25$, respectively. In the simulation, the exogenous disturbance inputs are selected as

$$w_k = 5\sin(k),\ v_k = 0.8\cos(0.7k),\ \xi_k = 0.48\cos(0.2k).$$

The simulation results are shown in Fig. 1 and Fig. 2, where Fig. 1 plots the output trajectories of the open-loop and closed-loop system, and Fig. 2 depicts the measurement outputs and the received signals by controller, respectively. The simulation results have confirmed that the designed controller performs very well.

It is interesting to see the relationship between the disturbance attenuation level γ and the probability $\bar{\alpha}$. For the same parameters λ_k, ε_k and δ_k, the permitted minimum γ is shown in Table II. It is easy to find that the disturbance attenuation performance deteriorates with increased $\bar{\alpha}$.

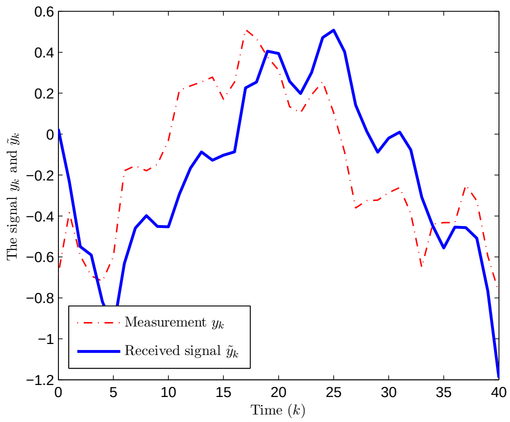

Fig. 2. The measurement signal and the received signal.

TABLE I
THE DISTRIBUTED STATE ESTIMATOR GAIN MATRICES

k	0	1	2	3	4	$\cdots$
K_k	-0.0757	-0.2549	-0.1898	-0.1938	-0.2574	$\cdots$

TABLE II
THE PERMITTED MINIMUM γ

$\bar{\alpha}$	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14
γ	0.94	0.95	0.96	0.98	0.99	1.00	1.02	1.03

V. CONCLUSIONS

In this paper, we have investigated the finite-horizon $\mathcal{H}_\infty$ control problem for a class of discrete time-varying systems with fading channels and randomly occurring nonlinearities. By employing the completing squares method and the stochastic analysis techniques, some sufficient conditions have been provided to ensure that the close-loop system satisfies the $\mathcal{H}_\infty$ performance constraint. Furthermore, the desired controller gains can be obtained by solving two coupled backward recursive RDEs.

APPENDIX I: PROOF OF LEMMA 2

Along the trajectory of system (9), it can be derived that

$$\begin{aligned}&\mathbb{E}\{\bar{x}_{k+1}^T\mathcal{P}_{k+1}\bar{x}_{k+1}-\bar{x}_k^T\mathcal{P}_k\bar{x}_k\}\\
=&\ \mathbb{E}\{\bar{x}_k^T(\mathscr{A}_k+\mathscr{B}_kK_k\bar{\vartheta}\mathscr{C}_k)^T\mathcal{P}_{k+1}(\mathscr{A}_k+\mathscr{B}_kK_k\bar{\vartheta}\mathscr{C}_k)\bar{x}_k\\
&-\bar{x}_k^T\mathcal{P}_k\bar{x}_k+2\bar{x}_k^T(\mathscr{A}_k+\mathscr{B}_kK_k\bar{\vartheta}\mathscr{C}_k)^T\mathcal{P}_{k+1}(\mathscr{B}_kK_k\bar{\theta}\tilde{\mathscr{E}}_k\\
&+\tilde{\mathscr{D}}_k)\tilde{\eta}_k+\bar{x}_k^T\tilde{\alpha}\tilde{\mathscr{A}}_k^T\mathcal{P}_{k+1}\tilde{\mathscr{A}}_k\bar{x}_k+\bar{x}_k^T\mathscr{C}_k^T\vartheta_k^TK_k^T\mathscr{B}_k^T\\
&\times\mathcal{P}_{k+1}\mathscr{B}_kK_k\vartheta_k\mathscr{C}_k\bar{x}_k+2\bar{x}_k^T\mathscr{C}_k^T\vartheta_k^TK_k^T\mathscr{B}_k^T\mathcal{P}_{k+1}\\
&\times\mathscr{B}_kK_k\theta_k\tilde{\mathscr{E}}_k\tilde{\eta}_k+\tilde{\eta}_k^T\tilde{\mathscr{E}}_k^T\theta_k^TK_k^T\mathscr{B}_k^T\mathcal{P}_{k+1}\mathscr{B}_kK_k\theta_k\tilde{\mathscr{E}}_k\tilde{\eta}_k\\
&+\tilde{\eta}_k^T(\mathscr{B}_kK_k\bar{\theta}\tilde{\mathscr{E}}_k+\tilde{\mathscr{D}}_k)^T\mathcal{P}_{k+1}(\mathscr{B}_kK_k\bar{\theta}\tilde{\mathscr{E}}_k+\tilde{\mathscr{D}}_k)\tilde{\eta}_k\}.\end{aligned}\tag{24}$$

Taking (24) into consideration, it follows that

$$\begin{aligned}&\mathbb{E}\{\|z_k\|_{[0,N]}^2\}=\sum\nolimits_{k=0}^N\mathbb{E}\left\{\bar{x}_k^T\mathcal{L}_k^T\mathcal{L}_k\bar{x}_k\right\}+\mathbb{E}\{\bar{x}_0^T\mathcal{P}_0\bar{x}_0\\
&-\bar{x}_{N+1}^T\mathcal{P}_{N+1}\bar{x}_{N+1}\}+\sum\nolimits_{k=0}^N\mathbb{E}\Big\{\bar{x}_k^T\Big[(\mathscr{A}_k\\
&+\mathscr{B}_kK_k\bar{\vartheta}\mathscr{C}_k)^T\mathcal{P}_{k+1}\left(\mathscr{A}_k+\mathscr{B}_kK_k\bar{\vartheta}\mathscr{C}_k\right)\\
&+\bar{\alpha}\tilde{\mathscr{A}}_k^T\mathcal{P}_{k+1}\tilde{\mathscr{A}}_k+\mathscr{C}_k^T\Pi_{k+1}^1\mathscr{C}_k-\mathcal{P}_k\Big]\bar{x}_k\\
&+2\bar{x}_k^T\Big[(\mathscr{A}_k+\mathscr{B}_kK_k\bar{\vartheta}\mathscr{C}_k)^T\mathcal{P}_{k+1}(\mathscr{B}_kK_k\bar{\theta}\tilde{\mathscr{E}}_k+\tilde{\mathscr{D}}_k)\\
&+\mathscr{C}_k^T\Pi_{k+1}^2\tilde{\mathscr{E}}_k\Big]\tilde{\eta}_k+\tilde{\eta}_k^T\Big[(\mathscr{B}_kK_k\bar{\theta}\tilde{\mathscr{E}}_k+\tilde{\mathscr{D}}_k)^T\mathcal{P}_{k+1}\\
&\times(\mathscr{B}_kK_k\bar{\theta}\tilde{\mathscr{E}}_k+\tilde{\mathscr{D}}_k)+\tilde{\mathscr{E}}_k^T\Pi_{k+1}^3\tilde{\mathscr{E}}_k\Big]\tilde{\eta}_k\Big\}\\
=&\ \mathbb{E}\{\bar{x}_0^T\mathcal{P}_0\bar{x}_0-\bar{x}_{N+1}^T\mathcal{P}_{N+1}\bar{x}_{N+1}\}+\mathbb{E}\{\gamma^2\|\tilde{\eta}_k\|_{[0,N]}^2\\
&-\gamma^2\|\varepsilon_k\aleph_k\bar{x}_k\|_{[0,N]}^2\}+\sum_{k=0}^N\mathbb{E}\{[\bar{x}_k^T\ \tilde{\eta}_k^T]\mathscr{X}_k^3[\bar{x}_k^T\ \tilde{\eta}_k^T]^T\}\end{aligned}\tag{25}$$

with $\mathscr{X}_k^2=\begin{bmatrix}\mathcal{R}_{k+1}^{11}-\mathcal{P}_k & \mathcal{R}_{k+1}^{12}\\ * & -\mathcal{R}_{k+1}^{22}\end{bmatrix}$.

Similarly, noticing that $\bar{u}_k=K_k\bar{\vartheta}\mathscr{C}_k\bar{x}_k$, one has

$$\begin{aligned}&\mathbb{E}\Big\{\bar{x}_{k+1}^T\mathcal{Q}_{k+1}\bar{x}_{k+1}-\bar{x}_k^T\mathcal{Q}_k\bar{x}_k\Big\}\\
=&\ \mathbb{E}\Big\{\bar{x}_k^T\Big(\mathscr{A}_k^T\mathcal{Q}_{k+1}\mathscr{A}_k+\tilde{\alpha}\tilde{\mathscr{A}}_k^T\mathcal{Q}_{k+1}\tilde{\mathscr{A}}_k+\mathscr{C}_k^T\Xi_{k+1}^1\mathscr{C}_k\\
&-\mathcal{Q}_k\Big)\bar{x}_k+2\bar{x}_k^T\mathscr{A}_k^T\mathcal{Q}_{k+1}\mathscr{B}_k\bar{u}_k+\bar{u}_k^T\mathscr{B}_k^T\mathcal{Q}_{k+1}\mathscr{B}_k\bar{u}_k\\
&+2\bar{x}_k^T\Big[\mathscr{A}_k^T\mathcal{Q}_{k+1}(\mathscr{B}_kK_k\bar{\theta}\tilde{\mathscr{E}}_k+\tilde{\mathscr{D}}_k)+\mathscr{C}_k^T\Xi_{k+1}^2\tilde{\mathscr{E}}_k\Big]\tilde{\eta}_k\\
&+2\bar{u}_k^T\mathscr{B}_k^T\mathcal{Q}_{k+1}(\mathscr{B}_kK_k\bar{\theta}\tilde{\mathscr{E}}_k+\tilde{\mathscr{D}}_k)\tilde{\eta}_k\\
&+\tilde{\eta}_k^T\Big[(\mathscr{B}_kK_k\bar{\theta}\tilde{\mathscr{E}}_k+\tilde{\mathscr{D}}_k)^T\mathcal{Q}_{k+1}(\mathscr{B}_kK_k\bar{\theta}\tilde{\mathscr{E}}_k+\tilde{\mathscr{D}}_k)\\
&+\tilde{\mathscr{E}}_k^T\Xi_{k+1}^3\tilde{\mathscr{E}}_k\Big]\tilde{\eta}_k\Big\}.\end{aligned}$$

Moreover, under $|\mathcal{R}_{k+1}^{22}|\neq 0$ for all $k\in[0,N]$, by selecting $\tilde{\eta}_k=(\mathcal{R}_{k+1}^{22})^{-1}(\mathcal{R}_{k+1}^{12})^T\bar{x}_k$, it is easy to obtain that

$$\begin{aligned}&\mathbb{E}\{\|z_k\|_{[0,N]}^2\}\\
=&\sum_{k=0}^N\mathbb{E}\{\|z_k\|^2+\|\bar{u}_k\|^2-\|\bar{u}_k\|^2\}+\mathbb{E}\{\bar{x}_0^T\mathcal{Q}_0\bar{x}_0\\
&-\bar{x}_{N+1}^T\mathcal{Q}_{N+1}\bar{x}_{N+1}\}+\sum_{k=0}^N\mathbb{E}\{\bar{x}_k^T\Delta_{k+1}\bar{x}_k\\
&+2\bar{x}_k^T(\mathscr{A}_k+\Lambda_{k+1})\mathcal{Q}_{k+1}\mathscr{B}_k\bar{u}_k+\bar{u}_k^T\mathscr{B}_k^T\mathcal{Q}_{k+1}\mathscr{B}_k\bar{u}_k\}\\
=&\ \mathbb{E}\left\{\bar{x}_0^T\mathcal{Q}_0\bar{x}_0-\bar{x}_{N+1}^T\mathcal{Q}_{N+1}\bar{x}_{N+1}\right\}-\sum_{k=0}^N\mathbb{E}\Big\{\|\bar{u}_k\|^2\Big\}\\
&+\sum_{k=0}^N\mathbb{E}\Big\{\bar{x}_k^T(\Delta_{k+1}+\mathcal{L}_k^T\mathcal{L}_k)\bar{x}_k+2\bar{x}_k^T(\mathscr{A}_k+\Lambda_{k+1})^T\\
&\times\mathcal{Q}_{k+1}\mathscr{B}_k\bar{u}_k+\bar{u}_k^T(\mathscr{B}_k^T\mathcal{Q}_{k+1}\mathscr{B}_k+I)\bar{u}_k\Big\}\\
=&\ \mathbb{E}\left\{\bar{x}_0^T\mathcal{Q}_0\bar{x}_0-\bar{x}_{N+1}^T\mathcal{Q}_{N+1}\bar{x}_{N+1}\right\}-\sum_{k=0}^N\mathbb{E}\Big\{\|\bar{u}_k\|^2\Big\}\\
&+\sum_{k=0}^N\mathbb{E}\left\{\begin{bmatrix}\bar{x}_k\\ \bar{u}_k\end{bmatrix}^T\mathscr{X}_k^3\begin{bmatrix}\bar{x}_k\\ \bar{u}_k\end{bmatrix}\right\}\end{aligned}\tag{26}$$

with $\mathscr{X}_k^3=\begin{bmatrix}\Delta_{k+1}+\mathcal{L}_k^T\mathcal{L}_k-\mathcal{Q}_k & (\mathscr{A}_k+\Lambda_{k+1})^T\mathcal{Q}_{k+1}\mathscr{B}_k\\ * & \mathscr{B}_k^T\mathcal{Q}_{k+1}\mathscr{B}_k+I\end{bmatrix}$.

Obviously, equalities (11) and (12) are guaranteed by (25) and (26), respectively. Therefore, the proof is complete.

APPENDIX II: PROOF OF LEMMA 3

(ii) $\Rightarrow$ (i). For non-negative definite matrices $\{\mathcal{P}_k\}_{0\leq k\leq N+1}$ satisfying the recursive RDE (13), it follows from Lemma 2 that

$$\begin{aligned}&\mathbb{E}\left\{\|z_k\|_{[0,N]}^2\right\}-\mathbb{E}\left\{\gamma^2\|\tilde{\eta}_k\|_{[0,N]}^2-\bar{\alpha}\gamma^2\|\varepsilon_k\aleph_k\bar{x}_k\|_{[0,N]}^2\right\}\\
=&\ \mathbb{E}\left\{\bar{x}_0^T\mathcal{P}_0\bar{x}_0-\bar{x}_{N+1}^T\mathcal{P}_{N+1}\bar{x}_{N+1}\right\}+\sum_{k=0}^N\mathbb{E}\Big\{\bar{x}_k^T(\mathcal{R}_{k+1}^{11}\\
&-\mathcal{P}_k)\bar{x}_k+2\bar{x}_k^T\mathcal{R}_{k+1}^{12}\tilde{\eta}_k-\tilde{\eta}_k^T\mathcal{R}_{k+1}^{22}\tilde{\eta}_k\Big\}\end{aligned}$$

$$= \sum_{k=0}^{N} \mathbb{E}\Big\{\bar{x}_k^T\big(\mathcal{R}_{k+1}^{11} - \mathcal{P}_k + \mathcal{R}_{k+1}^{12}(\mathcal{R}_{k+1}^{22})^{-1}(\mathcal{R}_{k+1}^{12})^T\big)\bar{x}_k - (\tilde{\eta}_k - \tilde{\eta}_k^*)^T\mathcal{R}_{k+1}^{22}(\tilde{\eta}_k - \tilde{\eta}_k^*)\Big\} + \mathbb{E}\left\{\bar{x}_0^T\mathcal{P}_0\bar{x}_0 - \bar{x}_{N+1}^T\mathcal{P}_{N+1}\bar{x}_{N+1}\right\}$$

where $\tilde{\eta}_k^* = (\mathcal{R}_{k+1}^{22})^{-1}(\mathcal{R}_{k+1}^{12})^T\bar{x}_k$.

Since $\mathcal{R}_k^{22} > 0$ and $\mathcal{P}_0 < \gamma^2 W$, for any nonzero $\{\tilde{\eta}_k\}_{0\le k\le N} \in \mathcal{L}_{[0,N]}$, it can be derived from the final condition $\mathcal{P}_{N+1} = 0$ that

$$\begin{aligned} &\mathbb{E}\left\{\|z_k\|_{[0,N]}^2\right\} - \gamma^2\mathbb{E}\{\bar{x}_0^T W\bar{x}_0\} \\ &\quad - \mathbb{E}\{\gamma^2||\tilde{\eta}_k||_{[0,N]}^2 - \bar{\alpha}\gamma^2||\varepsilon_k\aleph_k\bar{x}_k||_{[0,N]}^2\} \\ &< \mathbb{E}\left\{\|z_k\|_{[0,N]}^2\right\} - \mathbb{E}\{\bar{x}_0^T\mathcal{P}_0\bar{x}_0\} \\ &\quad - \mathbb{E}\{\gamma^2||\tilde{\eta}_k||_{[0,N]}^2 - \bar{\alpha}\gamma^2||\varepsilon_k\aleph_k\bar{x}_k||_{[0,N]}^2\} \\ &= -\mathbb{E}\Big\{\sum_{k=0}^{N}(\tilde{\eta}_k - \tilde{\eta}_k^*)^T\mathcal{R}_{k+1}^{22}(\tilde{\eta}_k - \tilde{\eta}_k^*)\Big\} < 0. \end{aligned} \tag{27}$$

(i) $\Rightarrow$ (ii). We proceed to show that "if (ii) is not true, then (i) is also not true". For convenience, let us provide an expression for the condition and conclusion of this proposition, respectively.

• The IF statement (i.e., the condition that (ii) is not true): Due to $\mathcal{R}_{N+1}^{22} = \gamma^2 I > 0$, $\mathcal{P}_N$ can be calculated from the recursion (13). It is easy to see that, by the same procedure, the recursion RDE (13) can be solved backward when $|\mathcal{R}_{k+1}^{22}| \neq 0$ for all $k \in [0, N-1]$. It means that the recursion RDE (13) fails if there exists some k_0 satisfying $|\mathcal{R}_{k_0+1}^{22}| = 0$, which fails without the condition (14).

In short, in terms of (14) and the backward recursion character of (13), the "*if statement*" can be divided into three cases:

a) $\mathcal{R}_{k+1}^{22} > 0$ for all $k \in [0, N-1]$, but the initial condition $\mathcal{P}_0 < \gamma^2 W$ can't be satisfied;
b) there exists a k_0 such that $|\mathcal{R}_{k_0+1}^{22}| = 0$ and $\mathcal{R}_{k+1}^{22} > 0$ ($k_0 < k < N$). It means that $\mathcal{R}_{k_0+1}^{22}$ has at least one zero eigenvalue;
c) there exists a k_0 such that i) $\mathcal{R}_{k_0+1}^{22}$ is neither positive semi-definite nor positive definite; ii) $\mathcal{R}_{k+1}^{22} > 0$ ($k_0 < k < N$), that is, $\mathcal{R}_{k_0+1}^{22}$ has at least one negative eigenvalue.

Furthermore, combining b) and c), one has that, for some k_0, $\mathcal{R}_{k+1}^{22} > 0$ ($k_0 < k < N$) and $\mathcal{R}_{k_0+1}^{22}$ has at least one zero or negative eigenvalue denoted as $\lambda_{k_0} \le 0$.

• The THEN statement (i.e., the conclusion that (i) is not true): There exists $(\bar{x}_0, \tilde{\eta}) \neq 0$ such that

$$\mathbb{E}\left\{\|z_k\|_{[0,N]}^2\right\} \ge \mathbb{E}\left\{\gamma^2||\tilde{\eta}_k||_{[0,N]}^2 - \bar{\alpha}\gamma^2||\varepsilon_k\aleph_k\bar{x}_k||_{[0,N]}^2\right\} + \gamma^2\mathbb{E}\left\{\bar{x}_0^T W\bar{x}_0\right\}. \tag{28}$$

First, denote

$$\mathcal{J}(\bar{x}_0, \tilde{\eta}) := \mathbb{E}\Big\{\|z_k\|_{[0,N]}^2 - \gamma^2||\tilde{\eta}_k||_{[0,N]}^2 + \bar{\alpha}\gamma^2||\varepsilon_k\aleph_k\bar{x}_k||_{[0,N]}^2\Big\} - \gamma^2\mathbb{E}\left\{\bar{x}_0^T W\bar{x}_0\right\}. \tag{29}$$

Case a): We can choose $\tilde{\eta}_k = \tilde{\eta}_k^*$, and then obtain

$$\begin{aligned} &\mathcal{J}(\bar{x}_0, \tilde{\eta}) \\ &= \sum_{k=0}^{N}\mathbb{E}\big\{\bar{x}_k^T\big(\mathcal{R}_{k+1}^{11} - \mathcal{P}_k + \mathcal{R}_{k+1}^{12}(\mathcal{R}_{k+1}^{22})^{-1}(\mathcal{R}_{k+1}^{12})^T\big)\bar{x}_k \\ &\quad - (\tilde{\eta}_k - \tilde{\eta}_k^*)^T\mathcal{R}_{k+1}^{22}(\tilde{\eta}_k - \tilde{\eta}_k^*)\big\} - \gamma^2\mathbb{E}\big\{\bar{x}_0^T W\bar{x}_0\big\} \\ &\quad + \mathbb{E}\big\{\bar{x}_0^T\mathcal{P}_0 x_0 - \bar{x}_{N+1}^T\mathcal{P}_{N+1}\bar{x}_{N+1}\big\} \\ &= \mathbb{E}\big\{\bar{x}_0^T(\mathcal{P}_0 - \gamma^2 W)\bar{x}_0\big\}. \end{aligned} \tag{30}$$

Obviously, there always exists a $\bar{x}_0 \neq 0$ satisfying $\mathcal{J}(\bar{x}_0, \tilde{\eta}) \ge 0$, even if $\mathcal{P}_0 - \gamma^2 W$ has at least one non-negative eigenvalue.

Case b) and **Case c)**: We assume that there exists a non-positive eigenvalue of $\mathcal{R}_{k+1}^{22}$ at time k_0, and design the special sequence $(\bar{x}_0, \tilde{\eta}) \neq 0$ as follows:

$$\bar{x}_0 = 0 \quad \text{and} \quad \tilde{\eta}_k = \begin{cases} \psi_{k_0}, & k = k_0, \\ \tilde{\eta}_k^*, & k_0 < k \le N, \\ 0, & 0 \le k < k_0, \end{cases} \tag{31}$$

where ψ_{k_0} is the eigenvector of $\mathcal{R}_{k_0+1}^{22}$ with respect to λ_{k_0}. For the purpose of simplicity, denote $\tilde{\eta} := \{\tilde{\eta}_k\}_{0\le k\le N}$.

The rest of the proof follows readily from that of Lemma 2 in [4].

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