Almost Sure H_∞ Filtering for Nonlinear Hybrid Stochastic Systems with Mode-Dependent Interval Delays

Huisheng Shu, Hua Yang, Zidong Wang, Fuad E. Alsaadi and Tasawar Hayat

Abstract

In this paper, the problem of almost sure H_∞ filtering is studied for a class of nonlinear hybrid stochastic systems. In the system under investigation, Markovian jumping parameters, mode-dependent interval delays, nonzero exogenous disturbances as well as white noises are simultaneously taken into consideration to better model the real-world systems. Intensive stochastic analysis is carried out to obtain sufficient conditions for ensuring the almost surely exponential stability and the prescribed H_∞ performance for the overall filtering error dynamics. Furthermore, the obtained results are applied to two classes of special hybrid stochastic systems with mode-dependent interval delays, where the desired filter gain is obtained in terms of the solutions to a set of linear matrix inequalities. Finally, two numerical examples are provided to show the effectiveness of the proposed filter design scheme.

Keywords

Almost surely exponential stability, H_∞ filtering, nonlinear hybrid stochastic systems, mode-dependent interval delays

I. Introduction

In the past few decades, much research effort has been paid to the H_∞ filtering problems since its inception in the early 1980s and many research results have been reported in the literature, see e.g. [2, 8, 14, 15, 17, 18, 23, 29, 32, 35] and the references therein. Commonly, the H_∞ filtering theory aims at designing a filter such that the H_∞ norm of the transfer function from the noise to the filtering error is not larger than a desired noise attenuation level [3, 7, 11]. Compared with the popular Kalman filtering approach, the H_∞ filtering method can be utilized to effectively improve the insensitivity against parameter uncertainties and external noises without the knowledge of their statistics. It should be mentioned that the Riccati matrix equation or linear matrix inequality (LMI) approaches, which can be easily handled by using the Matlab software, have been frequently exploited in designing H_∞ filters [6, 21, 27, 31].

It is now well known that nonlinear stochastic systems are a kind of complex systems that play an important role in many branches of science and engineering [1, 20, 22, 26, 28, 34]. As such, the H_∞ filtering problem for

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nonlinear stochastic systems has gained a growing research interest. Different from the linear case, the solution to the nonlinear filtering problem has been given as a non-explicit representation and some common assumptions on nonlinearities, such as Lipschitz-like, ellipsoid-bounded and sector-bounded nonlinearities, have been utilized to facilitate the performance analysis and synthesis. For instance, when the worst-case disturbance is considered in the design procedure, a mixed H_2/H_∞ suboptimal filtering problem has been solved in [34] by minimizing the total energy of the filtering error dynamics. Very recently, the H_∞ filtering problem for a class of general discrete-time nonlinear stochastic system with some network-induced phenomena has been address in [22] and some sufficient conditions have been proposed in the form of Hamilton-Jacobi-Isaacs (HJI) inequalities.

On the other hand, due to component failures or repairs, changing subsystem interconnections and abrupt environmental disturbances, the system structures could be subject to randomly changes and should be modeled as hybrid systems where the typical model is the Markov jumping system. The past few decades have witnessed a significant progress on various filtering problems (e.g. the H_∞ filtering problem for linear/nonlinear hybrid systems) and a great deal of literature has appeared on such a topic, see, e.g. [5,9,30,33]. Furthermore, time-delays exist universally in nature, which could be a source of instability, oscillation and poor performance of systems. Also, for Markov jumping systems, the time-delays for systems governed by different modes may be different, that is, the time-delays may be mode-dependent. Hence, there is a great need to develop new filtering approaches for the stochastic system with Markov jumping and time-delays, and some efforts have recently been made in this regard so far [10, 16, 19, 24, 30]. However, when the nonlinearities and the Markov jumping come together for the systems with time-delays, the filtering problem has become quite involved due primarily to the degree of mathematical complexity where existing filtering techniques cannot be simply applied. It is desirable to consider the effect from these phenomena on the stability analysis and filter design, and this constitutes one of the motivations for the present research.

It should be pointed out that, for systems with guaranteed mean-square stability, the variance of the state process is asymptotically bounded. This means that the system could be well-behaved on the average but the sample state trajectories could have a finite probability of being arbitrarily far from the system's equilibrium point. Therefore, in some cases, the commonly used mean-square measure might be a bit too crude to quantify the dynamic performance and even is unacceptable. For instance, certain performance of the controlled rocket should be guaranteed with the probability 1. Therefore, another type of stability, namely, almost sure stability that describes the system performance from the viewpoint of system sample paths, has attracted considerable attention, see [4, 12, 13, 25] and the references therein. For example, a Razumikhin-type theorem on pth moment input-to-state stability has been developed in [13] for a class of nonlinear stochastic systems with Markovian switching. In [12], a sufficient condition in the form of linear matrix inequalities has been proposed for designing a controller to stabilize almost surely a stochastic system which may not be stabilized in mean-square sense. Unfortunately, to the best of the authors' knowledge, *the H_∞ filtering problem in almost sure sense* for nonlinear hybrid stochastic systems has not been properly investigated so far, not to mention the case where mode-dependent interval delays are also involved. The corresponding challenge we are going to cope with is how we establish a suitable theoretical framework to analyze the H_∞ performance index in the almost sure sense. It is, therefore, the purpose of this paper to shorten such a gap.

Motivated by the above discussions, it would be interesting to develop an approach to analyzing the proposed almost sure H_∞ performance index. For this purpose, we aim to initiate a study on almost sure H_∞ filtering problems for nonlinear hybrid stochastic systems with mode-dependent interval delays (NHSSMDID).

By utilizing the intensive stochastic analysis, we establish some sufficient conditions such that the filtering error dynamics is almost surely exponentially stable and the H_∞ performance constraint is also achieved. The contribution of this paper is mainly threefold: *1) the H_∞ performance combined with the almost surely exponential stability is first utilized to evaluate the system performance; 2) in the plant under consideration, Markovian jumping parameters, mode-dependent interval delays, nonzero exogenous disturbances, as well as white noises are simultaneously taken into consideration to better model the real world systems; and 3) stochastic analysis is carried out to establish a framework to ensure the almost sure H_∞ performance.*

The rest of this paper is organized as follows. In Section II, a class of nonlinear hybrid stochastic systems with mode-dependent interval delays (NHSSMDID) are presented, and some preliminaries are briefly outlined. In Section III, the main results are established in the form of coupled HJI inequalities. These sufficient conditions are then extended to two classes of special hybrid stochastic systems with mode-dependent interval delays in Section IV. Furthermore, two examples are proposed to demonstrate the effectiveness of the obtained results in Section V. Finally, conclusions are drawn in Section VI.

Notation The notation used here is fairly standard unless otherwise specified. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices, and $\mathbb{R}_+ = [0, +\infty)$. $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete probability space with a natural filtration $\{\mathscr{L}\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). M^T represents the transpose of the matrix M. $|\cdot|$ denotes the Euclidean norm. $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation. $\mathbb{P}\{\cdot\}$ means the probability. $C([-\tau, 0]; \mathbb{R}^n)$ denotes the family of all continuous $\mathbb{R}^n$-valued function φ on $[-\tau, 0]$ with the norm $|\varphi| = \sup\{|\varphi(\theta)| : -\tau \leq \theta \leq 0\}$. $C^b_{\mathcal{F}_0}([-\tau, 0); \mathbb{R}^n)$ is the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0); \mathbb{R}^n)$-value random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$. $L^1(\mathbb{R}_+; \mathbb{R}_+)$ denotes the family of functions $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty \lambda(t)dt < \infty$. $\mathcal{K}$ denotes a class of continuous (strictly) increasing functions μ from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $\mu(0) = 0$. $\mathcal{K}_\infty$ denotes a class of functions μ in $\mathcal{K}$ with $\mu(r) \to \infty$ as $r \to \infty$. $L_2(\mathbb{R}_+, \mathbb{R}^p)$ denotes the space of non-anticipative stochastic process $y(t) \in \mathbb{R}^p$ with respect to the filtration $\mathscr{F}_t$ satisfying $|y(t)|^2_{L_2} := \mathbb{E}\int_0^\infty |y(t)|^2 dt < \infty$.

II. Problem formulation

In this paper, let $r(t)$ $(t \geq 0)$ be a right-continuous Markov chain taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if} \quad i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if} \quad i = j, \end{cases}$$

where $\triangle > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from mode i to mode j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. It is known that almost all sample paths of $r(\cdot)$ are right-continuous step functions with a finite number of simple jumps in any finite subinterval of $\mathbb{R}_+ := [0, \infty)$.

Let us consider the nonlinear hybrid stochastic systems with mode-dependent interval delays of the form

$$\begin{cases} dx(t) = (f(x(t), x(t-\tau(t, r(t))), t, r(t)) + g(x(t), x(t-\tau(t, r(t))), t, r(t))v(t))dt \\ \qquad\qquad + (h(x(t), x(t-\tau(t, r(t))), t, r(t)) + s(x(t), x(t-\tau(t, r(t))), t, r(t))v(t))dw(t), \\ y(t) = l(x(t), x(t-\tau(t, r(t))), t, r(t)) + k(x(t), x(t-\tau(t, r(t))), t, r(t))v(t), \\ z(t) = m(x(t), x(t-\tau(t, r(t))), t, r(t)), \end{cases} \tag{2.1}$$

with initial data $x(0) = x_0 \in C^b_{\mathcal{F}_0}([-\tau, 0); \mathbb{R}^n)$ and $r(0) = r_0 \in S$, where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $z(t) \in \mathbb{R}^q$ and

$v(t) \in L_2([0\ \infty);\ \mathbb{R}^m)$ are the state vector, the measured output, the state combination to be estimated and the exogenous disturbance, respectively. $\omega(t)$, which is independent on the Markov chain $r(\cdot)$, is a zero-mean one-dimensional Wiener process (Brownian Motion) satisfying $\mathbb{E}[d\omega(t)] = 0$ and $\mathbb{E}[d\omega^2(t)] = t$. The mode-dependent interval delays $\tau(t, r(t)) : \mathbb{R}_+ \times S \to \mathbb{R}_+$, also written as $\tau(t)$ or τ if there is no ambiguity, are known nonnegative differential functions and satisfy the following assumption.

Assumption 1: There are nonnegative numbers l_i, τ_i, δ_i, $\bar{\delta}_i$ and $\bar{\delta}$ such that

$$\begin{aligned} l_i &\leq \tau(t,i) \leq \tau_i, \quad l \leq \tau(t,i) \leq \tau, \\ \tau_t(t,i) = \frac{\partial \tau(t,i)}{\partial t} &\leq \delta_i, \quad \bar{\delta}_i = \delta_i + \gamma_{ii} l_i + \sum_{j \neq i} \gamma_{ij} \tau_j \leq \bar{\delta} < 1 \end{aligned} \tag{2.2}$$

for all $t \geq 0$ and $i \in S$, where $l = \min_{i \in S} l_i$ and $\tau = \max_{i \in S} \tau_i$.

The measurable nonlinear functions $f, g, h, s : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^n$, $l, k : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^p$ and $m : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times S \to \mathbb{R}^q$ are assumed to satisfy $f(0,0,t,i) = 0$, $g(0,0,t,i) = 0$, $h(0,0,t,i) = 0$, $s(0,0,t,i) = 0$, $l(0,0,t,i) = 0$, $k(0,0,t,i) = 0$, $m(0,0,t,i) = 0$ and the global Lipschitz condition, that is, there is a $L > 0$ such that

$$|\phi(x,y,t,i) - \phi(\bar{x},\bar{y},t,i)| \leq L(|x - \bar{x}| + |y - \bar{y}|) \tag{2.3}$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$, $t \geq -\tau$, where the function ϕ could be f, g, h, s, l, k or m.

Remark 1: Under the above assumption, it is easy to verify that the functions f and h satisfy the local Lipschitz condition and the linear growth condition. Therefore, in terms of the well-known existence-and-unique theorem, the system (2.1) with $v(t) = 0$ has a unique solution which can be denoted by $x(t; x_0, r_0)$ for any initial condition $x_0 \in C^b_{\mathcal{F}_0}([-\tau, 0); \mathbb{R}^n)$ and $t \geq -\tau$.

In this paper, we are interested in constructing a filter of the following form for system (2.1):

$$\begin{cases} d\hat{x}(t) = f(\hat{x}(t), \hat{x}(t - \tau(t, r(t))), t, r(t))dt + K(r(t))y(t)dt, \\ \hat{z}(t) = m(\hat{x}(t), \hat{x}(t - \tau(t, r(t))), t, r(t)), \quad \hat{x}(0) = 0, \end{cases} \tag{2.4}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the state, $\hat{z}(t) \in \mathbb{R}^q$ is the output of the filter, and K_i is the filter parameter to be designed.

Denoting $\eta(t) = [x^T(t) \quad \hat{x}^T(t)]^T$ and the filtering error $\tilde{z}(t) = z(t) - \hat{z}(t)$, we obtain the following augmented system

$$\begin{cases} d\eta(t) = (f_e(\eta(t), \eta(t - \tau(t, r(t))), t, r(t)) + g_e(\eta(t), \eta(t - \tau(t, r(t))), t, r(t))v(t))dt, \\ \qquad + (h_e(\eta(t), \eta(t - \tau(t, r(t))), t, r(t)) + s_e(\eta(t), \eta(t - \tau(t, r(t))), t, r(t))v(t))dw(t), \\ \tilde{z}(t) = m(x(t), x(t - \tau(t, r(t))), t, r(t)) - \hat{m}(\hat{x}(t), \hat{x}(t - \tau(t, r(t))), t, r(t)), \quad \hat{x}(0) = 0, \end{cases} \tag{2.5}$$

where

$$\begin{aligned} f_e\ &(\eta(t), \eta(t - \tau(t, r(t))), t, r(t)) \\ &= \begin{bmatrix} f(x(t), x(t - \tau(t, r(t))), t, r(t)) \\ f(\hat{x}(t), \hat{x}(t - \tau(t, r(t))), t, r(t)) + K(r(t))l(x(t), x(t - \tau(t, r(t))), t, r(t)) \end{bmatrix}, \\ g_e\ &(\eta(t), \eta(t - \tau(t, r(t))), t, r(t)) = \begin{bmatrix} g(x(t), x(t - \tau(t, r(t))), t, r(t)) \\ K(r(t))k(x(t), x(t - \tau(t, r(t))), t, r(t)) \end{bmatrix}, \\ h_e\ &(\eta(t), \eta(t - \tau(t, r(t))), t, r(t)) = \begin{bmatrix} h(x(t), x(t - \tau(t, r(t))), t, r(t)) \\ 0 \end{bmatrix}, \\ s_e\ &(\eta(t), \eta(t - \tau(t, r(t))), t, r(t)) = \begin{bmatrix} s(x(t), x(t - \tau(t, r(t))), t, r(t)) \\ 0 \end{bmatrix}. \end{aligned}$$

It can be verified that f_e, g_e, h_e and s_e satisfy the global Lipschitz condition and therefore the system (2.5) has a unique solution denoted by $\eta(t) = \eta(t;\eta_0, r_0)$ on $t \geq -\tau$. In the following, for the sake of notation simplification, denote $K(i) = K_i$ for $i \in S$ and the function $\varphi := \varphi(\eta, \xi, t, i)$ if there is no ambiguity where φ could be f_e, g_e, h_e, s_e, f, g, h, s, l, k or m.

For each $i \in S$, let $C^{2,1}(\mathbb{R}^{2n} \times \mathbb{R}_+ \times S; \mathbb{R}_+)$ denote the family of all nonnegative functions $V(\eta, t, i)$ on $\mathbb{R}^{2n} \times \mathbb{R}_+ \times S$ that are twice continuously differentiable in η and once in t. If $V \in C^{2,1}(\mathbb{R}^{2n} \times \mathbb{R}_+ \times S; \mathbb{R}_+)$, define an infinitesimal generator $\mathcal{L}$ of (2.5) from $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}_+ \times S$ to $\mathbb{R}$ by

$$\begin{aligned} \mathcal{LV}(\eta, \xi, t, i) = {} & V_t(\eta, t, i) + V_\eta(\eta, t, i)[f_e + g_e v(t)] \\ & + \frac{1}{2}[h_e + s_e v(t)]^T V_{\eta\eta}(\eta, t, i)[h_e + s_e v(t)] + \sum_{j=1}^{N} \gamma_{ij} V(\eta, t, j), \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} V_t(\eta, t, i) &= \frac{\partial V(\eta, t, i)}{\partial t}, \quad V_{\eta\eta}(\eta, t, i) = \left(\frac{\partial^2 V(\eta, t, i)}{\partial \eta_i \partial \eta_j} \right)_{2n \times 2n}, \\ V_\eta(\eta, t, i) &= \left(\frac{\partial V(\eta, t, 1)}{\partial \eta_i}, \cdots, \frac{\partial V(\eta, t, i)}{\partial \eta_{2n}} \right). \end{aligned}$$

Before formulating the problem to be dealt with in this paper, the following definition is considered.

Definition 1: For the augmented system (2.5) with $v(t) = 0$, $r_0 \in S$ and initial condition $\eta_0 = [x_0^T, 0]^T \in C^b_{\mathcal{F}_0}([-\tau, 0); \mathbb{R}^{2n})$, the trivial solution $\eta(t;\eta_0, r_0)$ is almost surely exponentially stable if there exists a real constant scalar $\lambda > 0$ such that

$$\lim_{t \to \infty} \sup \frac{1}{t} \log(|\eta(t;\eta_0, r_0)|) \leq -\lambda. \tag{2.7}$$

The purpose of this paper is to design an almost sure H_∞ filter of the form (2.4) for the nonlinear hybrid stochastic systems (2.1) with mode-dependent interval delays. More specifically, we are interested in looking for the parameters K_i $(i \in S)$ such that the following requirements are met simultaneously:

a) the augmented system (2.5) with $v(t) \equiv 0$ is almost surely exponentially stable;

b) under the zero initial condition, for a given disturbance attenuation level $\gamma > 0$ and all nonzero $v \in L_2(\mathbb{R}_+;\ \mathbb{R}^q)$, the output error $\tilde{z}(t)$ satisfies

$$|\tilde{z}|^2_{L_2} \leq \gamma^2 |v|^2_{L_2}. \tag{2.8}$$

III. Main Results

In this section, the almost surely exponential stability and the H_∞ performance are analyzed for the augmented system (2.5). A sufficient condition is given to guarantee that the augmented system (2.5) is almost surely exponentially stable and the H_∞ performance is achieved.

Theorem 3.1: For each $i \in S$, let the filter parameter K_i be given. The augmented system (2.5) with $v(t) \equiv 0$ is almost surely exponentially stable if there exist positive scalars $c_1, c_2, \lambda_1, \lambda_2$ and a function $V(\eta, t, i) \in C^{2,1}(\mathbb{R}^{2n} \times [-\tau, \infty) \times S; \mathbb{R}_+)$ such that

$$c_1|\eta|^2 \leq V(\eta, t, i) \leq c_2|\eta|^2, \tag{3.1}$$

$$\mathcal{LV}(\eta, \xi, t, i) \leq -\lambda_1|\eta|^2 + \lambda_2|\xi|^2, \tag{3.2}$$

hold for all $(\eta, t, i) \in \mathbb{R}^{2n} \times [-\tau, \infty) \times S$, and the equation

$$\lambda c_2 - \lambda_1 + \frac{\lambda_2 e^{\lambda\tau}}{1-\bar{\delta}} = 0 \tag{3.3}$$

has a unique root $\lambda > 0$.

Proof: For $\eta_0 \in \mathbb{R}^{2n}$ and $i \in S$, denoting $\eta(t; \eta_0, r_0) = \eta(t)$ and using Itô' formula, we can obtain

$$\mathbb{E}\big[e^{\lambda t} V(\eta(t), t, i)\big] = \mathbb{E}V(\eta_0, 0, i) + \mathbb{E}\int_0^t e^{\lambda s}\big[\lambda V(\eta(s), s, i) + \mathcal{LV}(\eta(s), \eta(s-\tau(s,i)), s, i)\big] ds. \tag{3.4}$$

Note the fact that

$$\begin{aligned} d\tau(s,i) &= \left[\tau_s(s,i) + \sum_{j=1}^{N} \gamma_{ij}\tau(s,j)\right] ds \\ &\leq \left[\delta_i + \gamma_{ii} l_i + \sum_{j\neq i} \gamma_{ij}\tau_j\right] ds \leq \bar{\delta} ds. \end{aligned} \tag{3.5}$$

Taking the above inequality and (3.4) into consideration, we can get that

$$\begin{aligned} &\mathbb{E}\int_0^t e^{\lambda s} |\eta(s-\tau(s,i))|^2 ds \\ &\leq \frac{e^{\lambda\tau}}{1-\bar{\delta}} \mathbb{E}\Big[\int_{-\tau}^0 |\eta(s)|^2 ds + \int_0^t e^{\lambda s}|\eta(s)|^2 ds\Big] \\ &\leq \frac{\tau|\eta_0|^2 e^{\lambda\tau}}{1-\bar{\delta}} + \frac{e^{\lambda\tau}}{1-\bar{\delta}} \mathbb{E}\int_0^t e^{\lambda s}|\eta(s)|^2 ds. \end{aligned} \tag{3.6}$$

Furthermore, it follows from (3.1)-(3.3) and (3.6) that

$$\begin{aligned} &\mathbb{E}\Big(e^{\lambda t} V(\eta(t), t, i)\Big) \\ &\leq c_2|\eta_0|^2 + \mathbb{E}\int_0^t e^{\lambda s}\big(\lambda c_2|\eta(s)|^2 - \lambda_1|\eta(s)|^2 + \lambda_2|\eta(s-\tau(s,i))|^2\big) ds \\ &\leq c_2|\eta_0|^2 + \frac{\lambda_2\tau e^{\lambda\tau}}{1-\bar{\delta}}|\eta_0|^2 + \mathbb{E}\int_0^t e^{\lambda s}\Big[\lambda c_2|\eta(s)|^2 - \lambda_1|\eta(s)|^2 + \frac{\lambda_2 e^{\lambda\tau}}{1-\bar{\delta}}|\eta(s)|^2\Big] ds \\ &\leq c_2|\eta_0|^2 + \frac{\lambda_2\tau e^{\lambda\tau}|\eta_0|^2}{1-\bar{\delta}} \end{aligned} \tag{3.7}$$

which results in

$$\mathbb{E}|\eta(t)|^2 \leq M e^{-\lambda t}, \quad t \geq 0 \tag{3.8}$$

where $M = \frac{c_2 + \lambda_2\tau e^{\lambda\tau}/(1-\bar{\delta})}{c_1}|\eta_0|^2$. Moreover, letting $\varepsilon \in (0, \lambda/2)$ be arbitrary, one has

$$\mathbb{E}|\eta(t)|^2 \leq M e^{-(\lambda-\varepsilon)t}. \tag{3.9}$$

On the other hand, there exists a sufficiently small positive scalar $\delta > 0$ satisfying

$$(3L)^2(\delta^2 + C_p\delta) < \frac{1}{4} \tag{3.10}$$

where C_p is a constant given by the well-known Burkholder-Davis-Gundy inequality. For such a scalar δ, there exists an integer k_0 such that $(k_0 - 1)\delta \geq \tau$.

In what follows, denoting the sequence $k = k_0, k_0+1, \cdots$, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{(k-1)\delta\le t\le k\delta}|\eta(t)|^2\right] &\le 3^2\mathbb{E}|\eta((k-1)\delta)|^2 + 3^2\mathbb{E}\left(\int_{(k-1)\delta}^{k\delta}\left|f_e(\eta(s),\eta(s-\tau(s)),s,r(s))\right|ds\right)^2\\ &\quad + 3^2\mathbb{E}\left[\sup_{(k-1)\delta\le t\le k\delta}\left|\int_{(k-1)\delta}^{t}h_e(\eta(s),\eta(s-\tau(s)),s,r(s))dw(s)\right|^2\right].\end{aligned} \tag{3.11}$$

For the above inequality, noting the local Lipschitz condition and the linear growth condition, one has

$$\begin{aligned}&\mathbb{E}\left(\int_{(k-1)\delta}^{k\delta}|f_e(\eta(s),\eta(s-\tau(s)),s,r(s))|ds\right)^2\\ &\le \mathbb{E}\left(\delta\sup_{(k-1)\delta\le s\le k\delta}|f_e(\eta(s),\eta(s-\tau(s)),s,r(s))|\right)^2\\ &\le 2(L\delta)^2\mathbb{E}\left(\sup_{(k-1)\delta\le s\le k\delta}(|\eta(s)|^2+|\eta(s-\tau(s))|^2)\right).\end{aligned} \tag{3.12}$$

Furthermore, by using the Burkholder-Davis-Gundy inequality, we can obtain

$$\begin{aligned}&\mathbb{E}\left[\sup_{(k-1)\delta\le s\le k\delta}\left|\int_{(k-1)\delta}^{t}|h_e(\eta(s),\eta(s-\tau(s)),s,r(s))dw(s)\right|^2\right]\\ &\le C_p\mathbb{E}\left(\int_{(k-1)\delta}^{k\delta}|h_e(\eta(s),\eta(s-\tau(s)),s,r(s))|^2ds\right)\\ &\le C_p\mathbb{E}\left(\delta\sup_{(k-1)\delta\le s\le k\delta}|h_e(\eta(s),\eta(s-\tau(s)),s,r(s))|^2\right)\\ &\le 2C_pL^2\delta\mathbb{E}\left[\sup_{(k-1)\delta\le s\le k\delta}(|\eta(s)|^2+|\eta(s-\tau(s))|^2)\right].\end{aligned} \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.11) yields

$$\begin{aligned}&\mathbb{E}\left[\sup_{(k-1)\delta\le t\le k\delta}|\eta(t)|^2\right]\\ &\le 3^2Me^{-(\lambda-\varepsilon)(k-1)\delta} + (3L)^2(2\delta^2+2C_p\delta)\mathbb{E}\left[\sup_{(k-1)\delta\le s\le k\delta}(|\eta(s)|^2+|\eta(s-\tau(s))|^2)\right].\end{aligned} \tag{3.14}$$

In light of the fact $(k-1)\delta \ge \tau$, it follows from (3.9) that

$$\mathbb{E}\left[\sup_{(k-1)\delta\le s\le k\delta}|\eta(s-\tau(s))|^2\right] \le Me^{-(\lambda-\varepsilon)((k-1)\delta-\tau)}. \tag{3.15}$$

Furthermore, it follows from (3.10), (3.14) and (3.15) that

$$\mathbb{E}\left[\sup_{(k-1)\delta\le t\le k\delta}|\eta(t)|^2\right] \le 19Me^{-(\lambda-\varepsilon)((k-1)\delta-\tau)}. \tag{3.16}$$

Obviously, by using the well-known Chebyshev's inequality, we have

$$\mathbb{P}\left\{\omega : \sup_{(k-1)\delta\le t\le k\delta}|\eta(t)| > e^{-(\lambda-2\varepsilon)((k-1)\delta-\tau)/2}\right\}$$

$$\leq \frac{\mathbb{E}\left[\sup_{(k-1)\delta\leq t\leq k\delta}|\eta(t)|^2\right]}{e^{-(\lambda-2\varepsilon)((k-1)\delta-\tau)}}$$
$$\leq 19Me^{-\varepsilon((k-1)\delta-\tau)}. \tag{3.17}$$

In what follows, by utilizing the well-known Borel-Cantelli lemma, we can get that for almost all $\omega \in \Omega$,

$$\sup_{(k-1)\delta\leq t\leq k\delta}|\eta(t)| \leq e^{-(\lambda-2\varepsilon)((k-1)\delta-\tau)/2} \tag{3.18}$$

holds for all but finitely many k. Therefore, there exists a $k_1(\omega)$, for all $\omega \in \Omega$ excluding a P-null set, such that (3.18) holds whenever $k \geq \max\{k_0, k_1\}$. Consequently, for almost all $\omega \in \Omega$, when $(k-1)\delta \leq t \leq k\delta$ and $k \geq \max\{k_0, k_1\}$, one has

$$\frac{1}{t}\log(|\eta(t)|) \leq -\frac{(\lambda-2\varepsilon)((k-1)\delta-\tau)}{2t} \leq -\frac{(\lambda-2\varepsilon)((k-1)\delta-\tau)}{2k\delta}$$

which implies

$$\lim_{t\to\infty}\sup\frac{1}{t}\log(|\eta(t)|) \leq -\frac{\lambda-2\varepsilon}{2}. \tag{3.19}$$

It is not difficult to see that the condition (2.7) follows by letting $\varepsilon \to 0$. The proof is complete. ■

Now, let us consider the H_∞ performance of the overall filtering process. In the following theorem, a sufficient condition is obtained to guarantee both the almost surely exponential stability and the H_∞ performance for the augmented system (2.5).

Theorem 3.2: Let the disturbance attenuation level $\gamma > 0$ and the filter parameter $K_i, i \in S$ be given. If there exist positive scalars $c_1, c_2, \lambda_1, \lambda_2$ with $\lambda_1 > \lambda_2/(1-\bar{\delta})$ and a function $V(\eta, t, i) \in C^{2,1}(\mathbb{R}^{2n} \times [-\tau, \infty) \times S; \mathbb{R}_+)$ satisfying the condition (3.3) and the following inequalities:

$$c_1|\eta|^2 \leq V(\eta, t, i) \leq c_2|\eta|^2, \tag{3.20}$$

$$V_t + V_\eta f_e + \frac{1}{2}h_e^T V_{\eta\eta} h_e + \sum_{j=1}^{N}\gamma_{ij}V_j + \Xi + \lambda_1\eta^2 - \lambda_2\xi^2 + |\tilde{z}|^2 < 0, \tag{3.21}$$

$$2\gamma^2 I - s_e^T V_{\eta\eta} s_e > 0, \quad V(0, t, i) = 0, \tag{3.22}$$

where

$$\Xi = \frac{1}{2}\left(V_\eta g_e + h_e^T V_{\eta\eta} s_e\right)\left(2\gamma^2 I - s_e^T V_{\eta\eta} s_e\right)^{-1}\left(g_e^T V_\eta^T + s_e^T V_{\eta\eta} h_e\right), \tag{3.23}$$

then the augmented system (2.5) with $v(t) \equiv 0$ is almost surely exponentially stable and the filtering error $\tilde{z}$ satisfies the H_∞ performance constraint (2.8) for all nonzero exogenous disturbances under the zero-initial condition.

Proof: In view of the definition of the operator $\mathcal{LV}$, we can get that

$$\begin{aligned}
&\mathcal{L}V(\eta,\xi,t,i)\\
&= V_t + V_\eta(f_e+g_e v) + \frac{1}{2}(h_e+s_e v)^T V_{\eta\eta}(h_e+s_e v) + \sum_{j=1}^{N}\gamma_{ij}V_j\\
&= -(v-(2\gamma^2 I - s_e^T V_{\eta\eta}s_e)^{-1}(g_e^T V_\eta + s_e^T V_{\eta\eta}h_e))^T \cdot \frac{1}{2}(2\gamma^2 I - s_e^T V_{\eta\eta}s_e)\\
&\quad\times (v-(2\gamma^2 I - s_e^T V_{\eta\eta}s_e)^{-1}(g_e^T V_\eta + s_e^T V_{\eta\eta}h_e))\\
&\quad+\frac{1}{2}(g_e^T V_\eta + s_e^T V_{\eta\eta}h_e)^T(2\gamma^2 I - s_e^T V_{\eta\eta}s_e)^{-1}(g_e^T V_\eta + s_e^T V_{\eta\eta}h_e)\\
&\quad+V_t + V_\eta f_e + \frac{1}{2}h_e^T V_{\eta\eta}h_e + \sum_{j=1}^{N}\gamma_{ij}V_j + \gamma^2 v^2.
\end{aligned} \tag{3.24}$$

Then, it follows from (3.21) and (3.22) that

$$\mathcal{L}V + \lambda_1\eta^2 - \lambda_2\xi^2 \le \gamma^2 v^2 - |\tilde{z}|^2. \tag{3.25}$$

In what follows, by integrating from 0 to T of (3.25), we have

$$\mathbb{E}V(\eta(T),T,i) - \mathbb{E}V(\eta(0),0,i) + \lambda_1\int_0^T \eta^2 ds - \lambda_2\int_0^T \xi^2 ds \le \gamma^2\mathbb{E}\int_0^T |v|^2 dt - \mathbb{E}\int_0^T |\tilde{z}|^2 dt. \tag{3.26}$$

Taking $d\tau(s,i) \le \bar{\delta}$ and $\lambda_1 > \lambda_2/(1-\bar{\delta})$ into consideration, we can obtain

$$\lambda_1\int_0^T \eta^2 ds \ge \lambda_2\int_0^T \xi^2 ds. \tag{3.27}$$

Moreover, it follows from $V(\eta(T),T,i)\ge 0$, $V(0,t,i)=0$, (3.26) and (3.27) that

$$\mathbb{E}\int_0^T |\tilde{z}|^2 dt \le \gamma^2\mathbb{E}\int_0^T |v|^2 dt. \tag{3.28}$$

Letting $T\to\infty$, the H_∞ performance constraint in (2.8) is satisfied. Finally, it is not difficult to see that (3.21) implies

$$\mathcal{L}V \le -\lambda_1|\eta|^2 + \lambda_2|\xi|^2.$$

Therefore, it follows from Theorem 3.1 that the augmented system (2.5) with $v\equiv 0$ is almost surely exponentially stable and the proof of is complete. ■

Remark 2: The obtained conditions (3.20)-(3.22) in Theorem 3.2, which are a set of coupled Hamilton-Jacobi-Isaacs (HJI) inequalities, provide the key for the resolution of almost sure H_∞ filtering problems. However, it is well known that there had been no general method to solve such a common second-order nonlinear inequality until now. In order to reduce the computational complexity, the conditions (3.20)-(3.22) can be decoupled into two auxiliary ones that can be solved independently and more easily when a special form of Lyapunov function $V(\eta,t,i)=V_1(x,t,i)+V_2(\hat{x},t,i)$ is taken.

Denote $\hat{\varphi} := \varphi(\hat{x}(t),\hat{x}(t-\tau(t,i)),t,i)$ when there is no ambiguity and the function φ could be f or m. Letting $s\equiv 0$ and $k\equiv 0$ in (2.5), we have the following corollary.

Corollary 1: Let the disturbance attenuation level $\gamma>0$ and the filter parameter $K_i, i\in S$ be given. If there exist positive scalars λ_1,λ_2 with $\lambda_1>\lambda_2/(1-\bar{\delta})$ satisfying the condition (3.3) and the inequality:

$$\Omega_1 + \Omega_2 < 0, \tag{3.29}$$

where

$$\begin{aligned}\Omega_1 &= 2x^T f + h^2 + l^2 + \gamma^{-2} x^T g g^T x + \lambda_1 x^2 - \lambda_2 x^2(t-\tau(t,i)) + 2m^2,\\ \Omega_2 &= \hat{x}^2 + 2\hat{x}\hat{f} + \hat{x}^T K_i K_i^T \hat{x} + \lambda_1 \hat{x}^2 - \lambda_2 \hat{x}^2(t-\tau(t,i)) + 2\hat{m}^2,\end{aligned}$$

then the augmented system (2.5) with $v(t) \equiv 0$, $s \equiv 0$ and $k \equiv 0$ is almost surely exponentially stable and the filtering error $\tilde{z}$ satisfies the H_∞ performance constraint (2.8) for all nonzero exogenous disturbances under the zero-initial condition.

Proof: Define the following Lyapunov function

$$V(\eta, t, i) = x^2 + \hat{x}^2.$$

Under the assumption $s \equiv 0$ and $k \equiv 0$, the condition (3.21) associated with the system (2.5) becomes

$$2x^T f + 2\hat{x}^T \hat{f} + 2\hat{x}^T K_i(y-l) + h^2 + \gamma^{-2} x^T g g^T x + \lambda_1(x^2+\hat{x}^2) - \lambda_2(x^2(t-\tau(t,i)) - \lambda_2\hat{x}^2(t-\tau(t,i))) + |\tilde{z}|^2 < 0. \quad (3.30)$$

Applying the inequality

$$X^T Y + Y^T X \le \varepsilon X^T X + \varepsilon^{-1} Y^T Y, \quad \forall \varepsilon > 0, \quad (3.31)$$

we can obtain

$$2\hat{x}^T K_i l \le \hat{x}^T K_i K_i^T \hat{x} + l^2, \quad |\tilde{z}|^2 = |m - \hat{m}|^2 \le 2m^2 + 2\hat{m}^2.$$

Obviously, we can find that (3.29) is satisfied. Therefore, the proof of Corollary 1 can been accomplished from Theorem 3.2. ■

Remark 3: In this paper, our main idea is to first establish a theoretical framework which is as general as possible in order not to introduce unnecessary conservatism, and then to apply the general results to special class of nonlinearities and/or special class of Lyapunov functions in order to have tractable criteria. In fact, in this paper, all the system parameters and filter parameters are general nonlinear functions without any restrictions/assumptions. This is actually the main contribution of this paper, which can be regarded as a theoretical basis on which some numerically appealing results could be expected when the nonlinearities and Lyapunov functions take special forms.

Remark 4: In this section, the analysis results in Theorem 3.2 and Corollary 1 offer sufficient conditions under which the augmented system (2.5) with $v(t) \equiv 0$ is almost surely exponentially stable and the filtering error satisfies the desired H_∞ performance constraint for all nonzero exogenous disturbances under the zero-initial condition. It should be pointed out that, for the purpose of practical applications, one is more interested in designing a linear filters due to the easy implementation. Therefore, in the next section, the almost sure H_∞ filtering problem with linear filters is discussed for two classes of special hybrid stochastic systems.

IV. Some special cases

To demonstrate that Theorem 3.2 serves as a theoretic basis for the almost sure H_∞ filtering problem of nonlinear hybrid stochastic systems, in this section, we aim to show that Theorem 3.2 can be specialized to the following two special cases: 1) the systems with Lipschitz-type nonlinearities, and 2) linear hybrid systems. The specialized results are described in terms of LMIs, which can be solved by utilizing the Matlab LMI toolbox.

Firstly, denote the matrix associated with the ith mode by $\Gamma_i \triangleq \Gamma(r(t) = i)$ where the matrix Γ could be A, A_d, B, B_d, C, C_d, D_1, D_2, L, F, F_d, H, H_d, A_f or B_f.

Case 1: We first consider a class of special nonlinear stochastic hybrid systems where nonlinearities is described by Lipschitz conditions. The system is of the following form

$$\begin{cases} dx(t) = \big[A_i x(t) + f(x(t), r(t)) + A_{di}x(t-\tau(t,r(t))) + f_d(x(t-\tau(t,r(t))), r(t)) \\ \qquad\qquad + D_{1i}v(t)\big]dt + \big[B_i x(t) + B_{di}x(t-\tau(t,r(t)))\big]dw(t), \\ y(t) = C_i x(t) + \phi(x(t), r(t)) + C_{di}x(t-\tau(t,r(t))) \\ \qquad\qquad + \phi_d(x(t-\tau(t,r(t))), r(t)) + D_{2i}v(t), \\ z(t) = L_i x(t), \end{cases} \tag{4.1}$$

with initial data $x_0 \in C^b_{\mathcal{F}_0}([-\tau, 0); \mathbb{R}^n)$ and $r_0 \in S$, where $x(t)$, $y(t)$, $z(t)$, $v(t)$, $\omega(t)$ and $\tau(t, r(t))$ are defined similar to those in (2.1); $f(\cdot,\cdot)$, $f_d(\cdot,\cdot)$, $\phi(\cdot,\cdot)$ and $\phi_d(\cdot,\cdot)$ are nonlinear functions which are assumed to satisfy the following conditions:

$$\begin{aligned} |f(x(t), i)|^2 &\leq x^T F_i x, \quad |f_d(x(t-\tau(t,r(t))), i)|^2 \leq x^T(t-\tau(t,r(t)))F_{di}x(t-\tau(t,r(t))), \\ |\phi(x(t), i)|^2 &\leq x^T H_i x, \quad |\phi_d(x(t-\tau(t,r(t))), i)|^2 \leq x^T(t-\tau(t,r(t)))H_{di}x(t-\tau(t,r(t))), \end{aligned}$$

where F_i, F_{di}, H_i and H_{di} are known positive definite matrices with appropriate dimensions for all $i \in S$.

We are interested in constructing a full-order linear filter of the following form for system (4.1):

$$\begin{cases} d\hat{x}(t) = A_f(r(t))\hat{x}(t)dt + B_f(r(t))y(t)dt \\ \hat{z}(t) = L_f(r(t))\hat{x}(t) \end{cases} \tag{4.2}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the state, $\hat{z}(t) \in \mathbb{R}^q$ is the output of the filter, and A_{fi}, B_{fi} and L_{fi} are the filter parameters to be determined. Moreover, the initial values of filters are assume to be $\hat{x}(0) = 0$.

In what follows, we can obtain the augmented system

$$\begin{cases} d\eta(t) = (f_e(\eta(t), \eta(t-\tau(t,r(t))), r(t)) + g_e(\eta(t), \eta(t-\tau(t,r(t))), r(t))v(t))dt \\ \qquad\qquad + h_e(\eta(t), \eta(t-\tau(t,r(t))), r(t))dw(t) \\ \tilde{z}(t) = L(r(t))x(t) - L_f(r(t))\hat{x}(t) \end{cases} \tag{4.3}$$

where

$$\begin{aligned} f_e\ &(\eta(t), \eta(t-\tau(t,r(t))), r(t)) \\ &= \begin{bmatrix} A(r(t))x(t) + f(x(t), r(t)) + A_d(r(t))x(t-\tau(t,r(t))) + f_d(x(t-\tau(t,r(t))), r(t)) \\ A_f(r(t))\hat{x}(t) + B_f(r(t))l(x(t), x(t-\tau(t,r(t))), r(t)) \end{bmatrix}, \\ g_e\ &(\eta(t), \eta(t-\tau(t,r(t))), r(t)) = \begin{bmatrix} D_1^T(r(t))) & D_2^T(r(t))B_f^T(r(t)) \end{bmatrix}, \\ h_e\ &(\eta(t), \eta(t-\tau(t,r(t))), r(t)) = \begin{bmatrix} x^T(t)B^T(r(t)) + x^T(t-\tau(t,r(t)))B_d^T(r(t)) & 0 \end{bmatrix}^T, \\ l\ &(x(t), x(t-\tau(t,r(t))), r(t)) \\ &= C(r(t))x(t) + \phi(x(t), r(t)) + C_d(r(t))x(t-\tau(t,r(t))) + \phi_d(x(t-\tau(t,r(t))), r(t)). \end{aligned}$$

Theorem 4.1: Let the disturbance attenuation level $\gamma > 0$ be given. If there exist positive scalars $\varepsilon_{1i}, \varepsilon_{2i}, \varepsilon_{3i}$,

ε_{4i}, λ_1 and λ_2, and matrices A_{fi}, B_{fi} and L_{fi} satisfying the following LMIs:

$$\begin{bmatrix} \Omega_i & B_{fi}C_i & A_{di} & 0 & 0 & 0 & L_{2i} & B_i \\ * & A_{fi}+A_{fi}^T+\lambda_1 I & B_{fi}C_{di} & 0 & 0 & L_{1i} & 0 & 0 \\ * & * & \varepsilon_{4i}H_{di}-\lambda_2 I & 0 & F_{di} & 0 & 0 & B_{di} \\ * & * & * & -\lambda_2 I & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon_{2i} & 0 & 0 & 0 \\ * & * & * & * & * & \Lambda_{1i} & 0 & 0 \\ * & * & * & * & * & * & \Lambda_{2i} & 0 \\ * & * & * & * & * & * & * & -I \end{bmatrix} < 0, \quad \forall\, i \in S, \tag{4.4}$$

where

$$\begin{aligned} \Omega_i &= A_i + A_i^T + \varepsilon_{1i}I + \varepsilon_{2i}I + \varepsilon_{3i}H_i + \lambda_1 I, \\ L_{1i} &= [B_{fi} \;\; B_{fi} \;\; L_{fi}], \\ \Lambda_{1i} &= \mathrm{diag}\{-\varepsilon_{3i}I, \; -\varepsilon_{4i}I, \; -\sqrt{2}I\}, \\ L_{2i} &= [F_i \;\; L_i \;\; D_{1i}], \\ \Lambda_{2i} &= \mathrm{diag}\{-\varepsilon_{1i}F_i, \; -\sqrt{2}, \; -\gamma^2 I\}, \end{aligned}$$

then the augmented system (4.3) with $v(t) \equiv 0$ is almost surely exponentially stable and the filtering error $\tilde{z}$ satisfies the H_∞ performance constraint (2.8) for all nonzero exogenous disturbances under the zero-initial condition.

Proof: Define the Lyapunov function candidate for system (4.3) as follows

$$V(\eta(t), t, i) = \eta^T(t)\eta(t) = x^T(t)x(t) + \hat{x}^T(t)\hat{x}(t) \quad \forall, i \in S. \tag{4.5}$$

Then, we can obtain that

$$\begin{aligned} & V_t + V_\eta f_e + \frac{1}{2}h_e^T V_{\eta\eta} h_e + \sum_{j=1}^{N} \gamma_{ij}V_j + \Xi + \lambda_1\eta^2 - \lambda_2\xi^2 + |\tilde{z}|^2 \\ = {} & 2x^T A_i x + 2x^T f(x,i) + 2x^T A_{di}x(t-\tau) + 2x^T f_d(x(t-\tau),i) + 2\hat{x}^T A_{fi}\hat{x} \\ & + 2\hat{x}^T B_{fi}(C_i x + \phi(x,i) + C_{di}x(t-\tau) + \phi_d(x(t-\tau),i)) + \gamma^{-2}x^T D_{1i}D_{1i}^T x \\ & + [B_i x + B_{di}x(t-\tau)]^T[B_i x + B_{di}x(t-\tau)] + \lambda_1 x^T x + \lambda_1 \hat{x}^T\hat{x} \\ & - \lambda_2 x^T(t-\tau)x(t-\tau) - \lambda_2\hat{x}^T(t-\tau)\hat{x}(t-\tau) + 2x^T L_i^T L_i x + 2\hat{x}^T L_{fi}^T L_{fi}\hat{x} \\ \le {} & x^T[A_i + A_i^T + \varepsilon_{1i}I + \varepsilon_{1i}^{-1}F_i + \varepsilon_{2i}I + \varepsilon_{3i}H_i + \lambda_1 I + 2L_i^T L_i + \gamma^{-2}D_{1i}D_{1i}^T]x \\ & + \hat{x}^T(B_{fi}C_i + C_i^T B_{fi}^T)x + \hat{x}^T[A_{fi} + A_{fi}^T + \lambda_1 I + \varepsilon_{3i}^{-1}B_{fi}B_{fi}^T + \varepsilon_{4i}^{-1}B_{fi}B_{fi}^T + 2L_{fi}^T L_{fi}]\hat{x} \\ & + 2x^T A_{di}x(t-\tau) + x^T(B_{fi}C_{di} + C_{di}^T B_{fi}^T)x(t-\tau) + x^T(t-\tau)[\varepsilon_{2i}^{-1}F_{di} + \varepsilon_{4i}H_{di} - \lambda_2 I]x(t-\tau) \\ & + [B_i x + B_{di}x(t-\tau)]^T[B_i x + B_{di}x(t-\tau)] - \lambda_2\hat{x}^T(t-\tau)\hat{x}(t-\tau). \end{aligned}$$

According to the Schur Complement Lemma, it follows directly from (4.4) that the right side of the inequality in (4.6) is less than zero for any $[x^T(t) \;\; \hat{x}^T(t) \;\; x^T(t-\tau) \;\; \hat{x}^T(t-\tau)] \neq 0$. Furthermore, it is easy to verify that (3.20) and (3.22) are true. As such, we can conclude from Theorem 3.2 that the augmented system (4.3) is almost surely exponentially stable and the filtering error $\tilde{z}$ satisfies the H_∞ performance requirement, which completes the proof. ■

Case 2: When the system (4.1) reduces to linear case of the form

$$\begin{cases} dx(t) = [A(r(t))x(t) + A_d(r(t))x(t-\tau(t,r(t))) + D_1(r(t))v(t)]dt \\ \qquad\qquad + [B(r(t))x(t) + B_d(r(t))x(t-\tau(t,r(t)))]dw(t), \\ y(t) = C(r(t))x(t) + C_d(r(t))x(t-\tau(t,r(t))) + D_2(r(t))v(t), \\ z(t) = L(r(t))x(t), \end{cases} \tag{4.6}$$

by applying the same filter (4.2) and Theorem 4.1, we have the following corollary.

Corollary 2: Let the disturbance attenuation level $\gamma > 0$ be given. If there exist positive scalars λ_1, λ_2, and matrices A_{fi}, B_{fi}, L_{fi} satisfying the following LMIs:

$$\begin{bmatrix} A_i + A_i^T + \lambda_1 I & B_{fi}C_i & A_{di} & 0 & 0 & L_2 & B_i \\ * & A_{fi} + A_{fi}^T + \lambda_1 I & B_{fi}C_{di} & 0 & L_{fi} & 0 & 0 \\ * & * & -\lambda_2 I & 0 & 0 & 0 & B_{di} \\ * & * & * & -\lambda_2 I & 0 & 0 & 0 \\ * & * & * & * & -\sqrt{2}I & 0 & 0 \\ * & * & * & * & * & \Lambda_2 & 0 \\ * & * & * & * & * & * & -I \end{bmatrix} < 0, \quad \forall\, i \in S, \tag{4.7}$$

where $L_2 = [L_i \quad D_{1i}]$ and $\Lambda_2 = \mathrm{diag}\{-\sqrt{2}, \ -\gamma^2 I\}$, then the augmented system (4.6) with $v(t) \equiv 0$ is almost surely exponentially stable and the filtering error $\tilde{z}$ satisfies the H_∞ performance constraint (2.8) for all nonzero exogenous disturbances under the zero-initial condition.

V. Numerical Examples

In this section, two numerical examples are employed to demonstrate the theory presented in this paper. For this purpose, let $\omega(t)$ be a scalar Wiener process, $\gamma(t)$ be a right-continuous Markov chain which is independent on $\omega(t)$ and takes values in $S = \{1, 2\}$, the step size be $\triangle = 0.001$.

Example 1: **Almost sure H_∞ filtering with filter form (2.4)**

Let the generator Γ be

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -0.8 & 0.8 \\ 0.5 & -0.5 \end{pmatrix}.$$

Consider NHSSMDID in the form of (2.1) with parameters

$$\begin{aligned} & f(x(t), x(t-\tau(t,1)), t, 1) = -3x(t) + \frac{1}{2}\sin(x(t)) + \frac{1}{3}\sin(x(t-\tau(t,1))), \\ & g(x(t), x(t-\tau(t,1)), t, 1) = 1, \quad h(x(t), x(t-\tau(t,1)), t, 1) = \frac{1}{2}x(t), \quad s(x(t), x(t-\tau(t,1)), t, 1) = 0, \\ & l(x(t), x(t-\tau(t,1)), t, 1) = 2x(t), \quad m(x(t), x(t-\tau(t,1)), t, 1) = \frac{1}{2}x(t); \\ & f(x(t), x(t-\tau(t,2)), t, 2) = -\frac{5}{2}x(t) + \frac{1}{3}\sin(x(t)) + \frac{1}{3}\sin(x(t-\tau(t,2))), \\ & g(x(t), x(t-\tau(t,2)), t, 2) = \frac{1}{2}, \quad h(x(t), x(t-\tau(t,2)), t, 2) = \frac{1}{3}x(t), \quad s(x(t), x(t-\tau(t,2)), t, 2) = 0, \\ & l(x(t), x(t-\tau(t,2)), t, 2) = \frac{3}{2}x(t), \quad m(x(t), x(t-\tau(t,2)), t, 2) = \frac{1}{2}x(t). \end{aligned}$$

The disturbance attenuation level is $\gamma = \sqrt{1.24}$. We choose the filter parameters as $K_1 = 1$ and $K_2 = \frac{3}{2}$. It is not difficult to verify that the conditions in Corollary 1 are satisfied.

Example 2: **Almost sure H_∞ filtering with filter form (4.2)**

Let the generator Γ be

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -0.7 & 0.7 \\ 0.3 & -0.3 \end{pmatrix}.$$

Consider NHSSMDID in the form of (2.1) with

$$\begin{aligned}
&f(x(t), x(t-\tau(t,1)), t, 1) = -2x(t) + \frac{1}{3}\sin(x(t)) + \frac{1}{3}x(t-\tau(t,1)),\\
&g(x(t), x(t-\tau(t,1)), t, 1) = \frac{1}{5}x(t) - \sin(x(t)), \quad h(x(t), x(t-\tau(t,1)), t, 1) = -\frac{1}{2}x(t),\\
&s(x(t), x(t-\tau(t,1)), t, 1) = 0;\\
&l(x(t), x(t-\tau(t,1)), t, 1) = \frac{1}{2}x(t), \quad m(x(t), x(t-\tau(t,1)), t, 1) = x(t);\\
&f(x(t), x(t-\tau(t,2)), t, 2) = -\frac{3}{2}x(t) + \sin(x(t)) + \frac{1}{3}\sin(x(t-\tau(t,2))),\\
&g(x(t), x(t-\tau(t,2)), t, 2) = 1, \quad h(x(t), x(t-\tau(t,2)), t, 2) = \frac{2}{3}x(t), \quad s(x(t), x(t-\tau(t,2)), t, 2) = 0,\\
&l(x(t), x(t-\tau(t,2)), t, 2) = \frac{3}{4}x(t), \quad m(x(t), x(t-\tau(t,2)), t, 2) = \frac{1}{2}x(t).
\end{aligned}$$

Letting the disturbance attenuation level be $\gamma = \sqrt{0.74}$, we can construct a filter in the form of (2.4) with

$$, A_{f1} = \frac{2}{5}, \quad B_{f1} = 2, \quad A_{f2} = \frac{1}{2}, \quad B_{f2} = 3.$$

It is not difficult to verify that A_{fi} and B_{fi} $(i = 1, 2)$ satisfy the conditions of Theorem 4.1 with $V(\eta, t, r(t)) = x^2(t) + \hat{x}^2(t)$. Therefore, it follows from Theorem 4.1 that the filter of the form (4.2) is an almost sure H_∞ filter.

In simulation, the time-varying delays are assume to be $v(t) = \exp(-t/4)$, $\tau(t,1) = 1 + \sin(t)/3$ and $\tau(t,2) = 1 + \cos(t)/6$. The simulation results are presented in Fig. 1∼Fig. 4, where Fig. 1 and Fig. 3 display the real state x and its estimation $\hat{x}$, and Fig. 2 and Fig. 4 plot the filtering error $\hat{z}$, which confirm that the H_∞ performance constraint (2.8) is well achieved.

VI. Conclusions

In this paper, the almost sure H_∞ filtering problem has been investigated for nonlinear hybrid stochastic systems with mode-dependent interval delays. In the plant under discussion, Markovian jumping parameters, mode-dependent interval delays, nonzero exogenous disturbances, as well as white noises are simultaneously taken into consideration to better model the real world systems. By utilizing the stopping time method combined with martingale inequalities, some sufficient conditions have been proposed to guarantee that the resulting augmented system is almost surely exponentially stable and the prescribed H_∞ performance is satisfied. Furthermore, the obtained results have been applied to two classes of special hybrid stochastic systems, where the desired filter gains have been obtained by solving a set of LMIs. Finally, numerical examples have been provided to show the effectiveness of the proposed filter design scheme.

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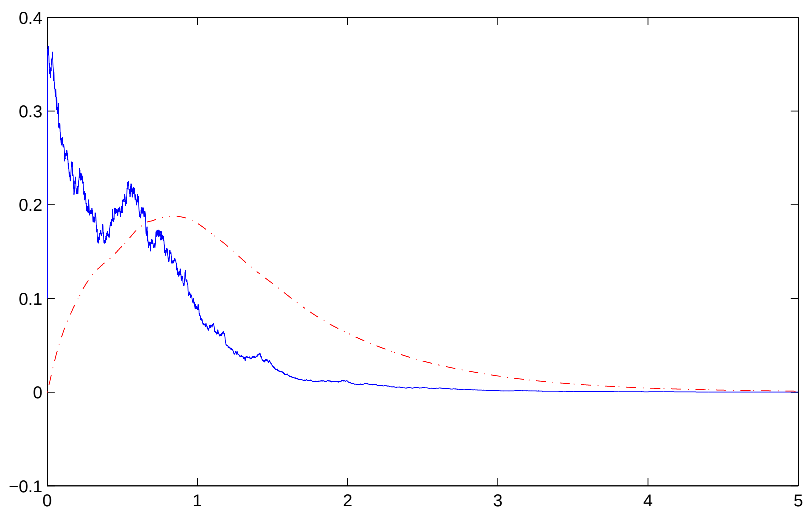

Fig. 1. x (solid) and $\hat{x}$ (dashed) of Example 1.

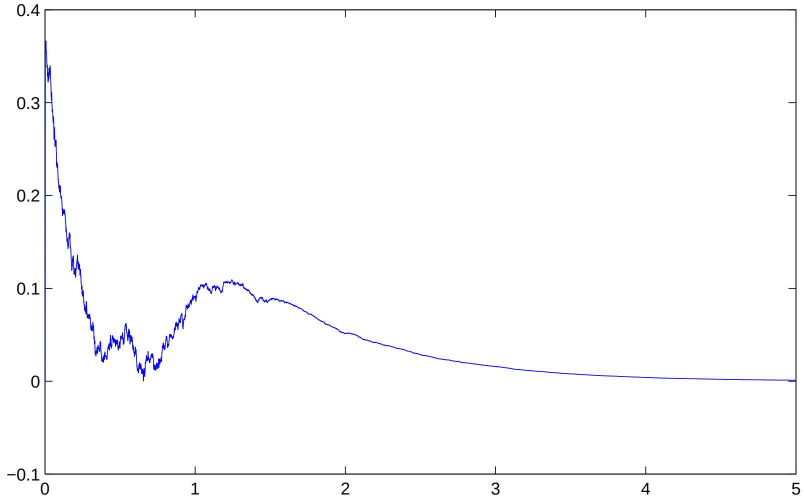

Fig. 2. Filtering error $\tilde{z}$ of Example 1.

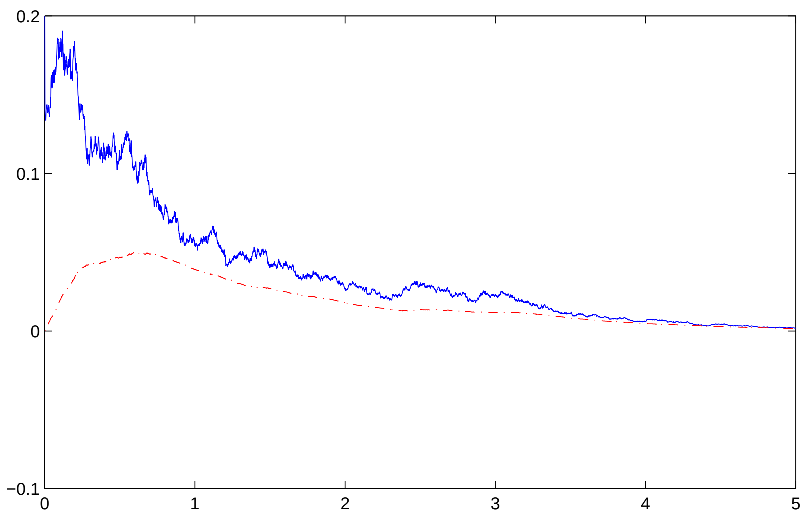

Fig. 3. x (solid) and $\hat{x}$ (dashed) of Example 2.

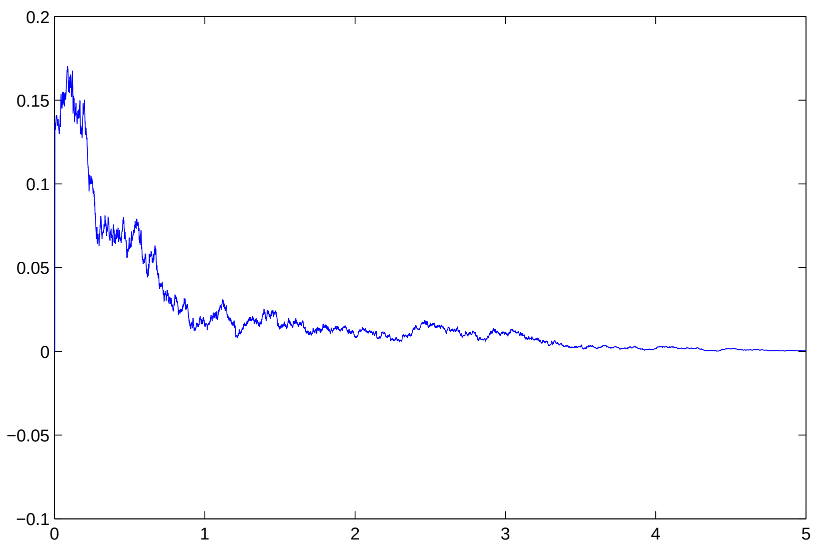

Fig. 4. Filtering error $\tilde{z}$ of Example 2.