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Generating the Pfaffian closure with total Pfaffian functions

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Abstract: Given an o-minimal expansion $\mathcal{R}$ of the real field, we show that the structure obtained from $\mathcal{R}$ by iterating the operation of adding all total Pfaffian functions over $\mathcal{R}$ defines the same sets as the Pfaffian closure of $\mathcal{R}$.

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There are various possibilities for adding Pfaffian objects to o-minimal expansions of the real field and preserving o-minimality. One example is the Pfaffian closure of an o-minimal expansion of the real field, which was shown to be o-minimal by the second author [7]. The purpose of this note is to present a somewhat simpler construction of the Pfaffian closure. Although not as simple as the description in terms of nested leaves obtained by Lion and the second author [5], our construction has the novelty of only using total Pfaffian functions and is reminiscent of the original Pfaffian expansion of the real field constructed by Wilkie [9].

In order to state our result, we need to introduce some terminology. Suppose that $\mathcal{R}$ is an o-minimal expansion of the real field, and that $U \subseteq \mathbb{R}^n$ is an $\mathcal{R}$-definable open subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$. We say that a C^1 function $f : U \to \mathbb{R}$ is *Pfaffian over* $\mathcal{R}$ if there exist $\mathcal{R}$-definable C^1 functions $P_i : U \times \mathbb{R} \to \mathbb{R}$, for $i = 1, \ldots, n$ such that

$$\frac{\partial f}{\partial x_i}(x) = P_i(x, f(x))$$

for all $x \in U$.

Given $n, l \in \mathbb{N}$ such that $l \leq n$, we let G_n^l be the Grassmannian of all linear subspaces of $\mathbb{R}^n$ of dimension l. This is an analytic manifold and is naturally definable in the real field (see [1, 3.4.2]). We also set $G_n = \bigcup_{l=0}^n G_n^l$. Now fix an embedded C^1 submanifold M of $\mathbb{R}^n$ and let $l \leq n$. A C^1 map $d : M \to G_n$ is said to be a *distribution on* M if $d(x) \subseteq T_x M$ for all $x \in M$, where $T_x M$ is the tangent space of M at x. A distribution d is an *l-distribution* if $d(M) \subseteq G_n^l$. Given an l-distribution d on M and an immersed C^1 submanifold V of M, we say that V is an *integral manifold* of d if $T_x V = d(x)$ for all

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$x \in V$. A maximal connected integral manifold is called *leaf* of the distribution. Now suppose that d has codimension one. A leaf L of d is said to be a *Rolle leaf* of d if it is a closed embedded submanifold of M and is such that for all C^1 curves $\gamma : [0,1] \to M$ satisfying $\gamma(0), \gamma(1) \in L$, we have $\gamma'(t) \in d(\gamma(t))$ for some $t \in [0,1]$. A *Rolle leaf over* $\mathcal{R}$ is a Rolle leaf of an $\mathcal{R}$-definable codimension one distribution defined on $\mathbb{R}^n$ for some $n \in \mathbb{N}$. For example, a result due to Khovanskii (see [8, 1.6]) implies that if $f : \mathbb{R}^n \to \mathbb{R}$ is Pfaffian over $\mathcal{R}$, then the graph of f is a Rolle leaf over $\mathcal{R}$.

We can now define the Pfaffian structures involved in our result. Given any o-minimal expansion of the real field $\mathcal{R}$, let $\mathcal{L}(\mathcal{R})$ be the collection of all Rolle leaves over $\mathcal{R}$. Now let $\mathcal{R}_0 = \mathcal{R}$ and, for $i \geq 0$, let $\mathcal{R}_{i+1}$ be the expansion of $\mathcal{R}_i$ by all leaves in $\mathcal{L}(\mathcal{R}_i)$. Let $\mathcal{L}$ be the union of all the $\mathcal{L}(\mathcal{R}_i)$ and let $\mathcal{P}(\mathcal{R})$ be the expansion of $\mathcal{R}$ by all the leaves in $\mathcal{L}$. This structure is called the *Pfaffian closure* of $\mathcal{R}$. The second author showed that it is o-minimal [7].

Similarly, we let $\mathcal{L}'(\mathcal{R})$ be the collection of all functions $f : \mathbb{R}^n \to \mathbb{R}$, for all $n \in \mathbb{N}$ that are Pfaffian over $\mathcal{R}$. We define $\mathcal{R}'_i$ and then $\mathcal{P}'(\mathcal{R})$ by mimicking the previous paragraph. The structure $\mathcal{P}'(\mathcal{R})$ is a reduct of $\mathcal{P}(\mathcal{R})$ (by the example above) and it is the purpose of this note to show that they are in fact the same from the point of view of definability.

Theorem 1 *A set $X \subseteq \mathbb{R}^n$ is definable in $\mathcal{P}(\mathcal{R})$ if and only if it is definable in $\mathcal{P}'(\mathcal{R})$.*

If $\mathcal{R}$ admits analytic cell decomposition, then so too does $\mathcal{P}'(\mathcal{R})$ (see [8]) and it follows that in this case, the reduct of $\mathcal{P}'(\mathcal{R})$ in which only analytic functions are added also defines the same sets as $\mathcal{P}(\mathcal{R})$.

Before proving the theorem, we first recall a result from Khovanksii theory that we will repeatedly use in the proof. This originates with Khovanskii's work on pfaffian functions (see [3]) and was adapted to the o-minimal setting by the second author [7] following work of Moussu and Roche [6] and Lion and Rolin [4]. In our proof we shall only ever need to work with a single distribution so we restrict ourselves to Khovanksii theory in this simple setting. We follow the presentation in [8]. Suppose that d is a C^2 distribution on a C^2 manifold $M \subseteq \mathbb{R}^n$ and that $N \subseteq M$ is a C^2 submanifold of M. The *pull-back* of d to N is the distribution d^N on N defined by

$$d^N(x) = T_x N \cap d(x).$$

We say that N is compatible with d if d^N is an l-distribution on N, for some l. From now on, we use the word definable to mean $\mathcal{P}'(\mathcal{R})$-definable. In particular, cell means $\mathcal{P}'(\mathcal{R})$-definable cell. The result we need is as follows (see [8, 3.6]).

Fact 2 *Let d be a definable codimension one distribution on M and suppose that $\mathcal{C}$ is a partition of M into C^2 cells compatible with d. Then there is a $k \in \mathbb{N}$ such that whenever $C \in \mathcal{C}$ and L is a Rolle leaf of d the set $C \cap L$ is a union of at most k Rolle leaves of d^C.*

Given the definition of $\mathcal{P}'(\mathcal{R})$, in order to prove the theorem it suffices to show that if L is a Rolle leaf over $\mathcal{P}'(\mathcal{R})$ then L is definable. For the proof of this, we assume that the reader is familiar with o-minimality (as presented in [2]). First, an easy observation.

Lemma 3 *Suppose that $C \subseteq \mathbb{R}^n$ is an open C^2 cell and that $f : C \to \mathbb{R}$ is Pfaffian over $\mathcal{P}'(\mathcal{R})$. Then f is definable.*

The proof, using a definable diffeomorphism between C and $\mathbb{R}^n$, is left to the reader.

Now suppose that $C \subseteq \mathbb{R}^n$ is a bounded open C^2 cell, and that $\alpha, \beta, \gamma, \delta : C \to \mathbb{R}$ are definable bounded C^2 functions such that

$$\gamma(x) < \alpha(x) < \beta(x) < \delta(x)$$

for all $x \in C$. Let $D = (\alpha, \beta)_C$ and $D' = (\gamma, \delta)_C$ and suppose that d' is a definable integrable n-distribution on D' (for a discussion of integrability in this context, see [8, Section 1]). Suppose that we are given a Rolle leaf L' of d'. Assume that both the graph of α and the graph of β are compatible with d' and let d^α and d^β be the pullbacks of d' to the graphs of α and β respectively. Let d be the restriction of d' to C. By Fact 2, $L' \cap D, L' \cap \operatorname{graph}\alpha$ and $L' \cap \operatorname{graph}\beta$ are finite unions of Rolle leaves of d, d^α and d^β respectively.

Lemma 4 *Suppose that L is a connected component of $L' \cap D$ and suppose that $\operatorname{graph}\alpha$ is transverse to d'. Then $\operatorname{fr}L \cap \operatorname{graph}\alpha$ is a clopen subset of $L' \cap \operatorname{graph}\alpha$.*

Proof Since L' is a Rolle leaf in D', it is closed in D' and so L is closed in D. So, $\operatorname{fr}L \cap \operatorname{graph}\alpha = \operatorname{cl} L \cap \operatorname{graph}\alpha$ is closed in the graph of α. Using the fact that L' is closed in D' again, we have $\operatorname{cl} L \cap \operatorname{graph}\alpha \subseteq L' \cap \operatorname{graph}\alpha$ and so $\operatorname{fr}L \cap \operatorname{graph}\alpha$ is a closed subset of $L' \cap \operatorname{graph}\alpha$.

We now need to show that $\operatorname{fr}L \cap \operatorname{graph}\alpha$ is open in $L' \cap \operatorname{graph}\alpha$, so let $p \in \operatorname{fr}L \cap \operatorname{graph}\alpha$. Let L_p be the connected component of $L' \cap \operatorname{graph}\alpha$ containing p. By the Frobenius theorem (see [8, Section 1]) there is a neighbourhood U of p and a diffeomorphism $\phi : \mathbb{R}^{n+1} \to U$ such that $\phi^* d' = \ker \mathrm{d}x_{n+1}$ and $\phi(0) = p$. Now, L' is a leaf of d' and $p \in L' \cap U$, so the hyperplane $\mathbb{R}^n \times \{0\}$ is a component of $\phi^{-1}(L' \cap U)$. Since $L' \cap \operatorname{graph}\alpha \cap U$ is a submanifold of $L' \cap U$, we can find an open box B centred at 0 such that $N := \phi^{-1}(L' \cap \operatorname{graph}\alpha \cap U) \cap B$ is connected. Let $B_0 = (\mathbb{R}^n \times \{0\}) \cap B$. Then N is a closed codimension one submanifold of B_0 and so $B_0 \setminus N$ has exactly two components, B_1 and B_2, say. Since $p \in \operatorname{cl} L$, at least one of B_1 or B_2 must be contained in $\phi^{-1}(L \cap U)$. Also, $N = \operatorname{fr}(B_i) \cap B_0$ for each i and so $\phi(N)$ is contained in $\operatorname{fr} L \cap \operatorname{graph}\alpha$. But $\phi(N)$ is open in $L' \cap \operatorname{graph}\alpha$, by our choice of B, and the lemma is proved. □

The following proposition suffices to prove the theorem.

Proposition 5 *Let $L \subseteq \mathbb{R}^n$ be a Rolle leaf over $\mathcal{P}'(\mathcal{R})$. Then L is definable in $\mathcal{P}'(\mathcal{R})$.*

Proof The proof is by induction on n. The $n = 1$ case is trivial, so we assume that $n > 1$ and that the proposition is true for Rolle leaves over $\mathcal{P}'(\mathcal{R})$ contained in $\mathbb{R}^m$ with $m < n$. Thus if $C \subseteq \mathbb{R}^n$ is a C^2 cell of dimension less than n and $V \subseteq C$ is a Rolle leaf of a definable codimension one distribution on C, then V is definable.

Suppose that $L \subseteq \mathbb{R}^n$ is a Rolle leaf over $\mathcal{P}'(\mathcal{R})$. Then L is a closed embedded proper submanifold of $\mathbb{R}^n$, and so there are $p \in \mathbb{R}^n \setminus L$ and $r > 0$ such that $B(p, 2r) \cap L = \emptyset$, where $B(a, \varepsilon)$ is the open ball around a of radius ε. Perhaps after translating and stretching, we may assume that $p = 0$ and that $r = 1$. Let $\phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ be the semialgebraic diffeomorphism $\phi(x) = \frac{x}{\|x\|^2}$. Then $\phi(L)$ is contained in $B(0, 1/2)$ and $\operatorname{cl}(\phi(L)) \subseteq \phi(L) \cup \{0\}$. So, after replacing L by $\phi(L)$, we may assume that L is a Rolle leaf of a definable integrable $(n-1)$-distribution d on $B'(0, 1) := B(0, 1) \setminus \{0\}$, that $L \subseteq B(0, 1/2)$ and that $\operatorname{cl} L \subseteq L \cup \{0\}$.

Let Π_{n-1} be the projection onto the first $n-1$ coordinates. For each coordinate permutation σ on $\mathbb{R}^n$, the set $B_\sigma = \{x \in B'(0, 1) : \Pi_{n-1}|_{\sigma^*(d(\sigma^{-1}(x)))} \text{ has rank } n-1\}$ is open and together these sets cover $B'(0, 1)$. So it suffices to show that $L \cap B_\sigma$ is definable for each σ. Fix σ, which we may assume to be the identity. Let $\mathcal{C}$ be a C^2 cell decomposition of $B'(0, 1)$ compatible with $B_{\mathrm{id}}, B'(0, 1/2)$ and d. We show that $C \cap L$ is definable for each cell $C \in \mathcal{C}$ such that $C \subseteq B_{\mathrm{id}}$. If $C \in \mathcal{C}$ is not open then $L \cap C$ is definable, by Fact 2 and the inductive hypothesis. So, suppose that $C \in \mathcal{C}$ is open and that $C \subseteq B_{\mathrm{id}}$. Let N be a component of $L \cap C$. Since N is a Rolle leaf of $d|_C$ and C is a cell, N is the graph of a function $f : \Pi_{n-1}(N) \to \mathbb{R}$. Let $\alpha, \beta : \Pi_{n-1}(C) \to \mathbb{R}$ be the functions such that $\operatorname{graph}\alpha$ and $\operatorname{graph}\beta$ are the two cells in $\mathcal{C}$ forming the 'bottom' and 'top' of C. Then the graph of α is compatible with d and so it is either tangent to d or transverse to d. Since $\operatorname{graph}\alpha$ is connected, if it is tangent to d, then either $\operatorname{graph}\alpha \subseteq L$ or $L \cap \operatorname{graph}\alpha = \emptyset$. If the graph of α is transverse to d then by Fact 2 and the inductive hypothesis, $L \cap \operatorname{graph}\alpha$ is definable. By

Lemma 4, $\mathrm{fr}N \cap \mathrm{graph}\alpha$ is a clopen subset of $L \cap \mathrm{graph}\alpha$ and so $\mathrm{fr}N \cap \mathrm{graph}\alpha$ is also definable. This all also holds with the graph of β in place of the graph of α. Since N is bounded and the graph of a continuous function, $x \in \mathrm{fr}\Pi_{n-1}(N)$ if and only if there is a y such that $(x, y) \in \mathrm{fr}N$. So the set $\mathrm{fr}\Pi_{n-1}(N) \cap \Pi_{n-1}(C)$ is definable. Let $\mathcal{D}$ be a cell decomposition of $\Pi_{n-1}(C)$ compatible with $\mathrm{fr}\Pi_{n-1}(N) \cap \Pi_{n-1}(C)$. Then for each $D \in \mathcal{D}$ we either have $D \subseteq \Pi_{n-1}(N)$ or $D \cap \Pi_{n-1}(N) = \emptyset$. For each non-open cell $D \in \mathcal{D}$ such that $D \subseteq \Pi_{n-1}(N)$, let $E_D = (\alpha|_D, \beta|_D)_D$. Take a cell decomposition of E_D compatible with d. Let E' be a cell in this decomposition such that $\mathrm{graph} f|_D \cap E'$ is non-empty. Then by Fact 2, $\mathrm{graph} f|_D \cap E'$ is either a finite union of Rolle leaves of the pullback of d to E' and so definable by the inductive hypothesis, or is equal to E' (in the case that E' is tangent to d). So the graph of $f|_D$ is definable. Finally, for each open cell $D \in \mathcal{D}$ such that $D \subseteq \Pi_{n-1}(N)$, the restriction of f to D is Pfaffian over $\mathcal{P}'(\mathcal{R})$ and so is definable by Lemma 3. So N is definable, as required.

□

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