Codimension formulae for the intersection of fractal subsets of Cantor spaces

Casey Donoven and Kenneth Falconer

January 16, 2015

Abstract

We examine the dimensions of the intersection of a subset E of an m-ary Cantor space $\mathcal{C}^m$ with the image of a subset F under a random isometry with respect to a natural metric. We obtain almost sure upper bounds for the Hausdorff and upper box-counting dimensions of the intersection, and a lower bound for the essential supremum of the Hausdorff dimension. The dimensions of the intersections are typically $\max\{\dim E + \dim F - \dim \mathcal{C}^m, 0\}$, akin to other codimension theorems. The upper estimates come from the expected sizes of coverings, whilst the lower estimate is more intricate, using martingales to define a random measure on the intersection to facilitate a potential theoretic argument.

1 Introduction

The classical codimension formula describes the dimension of the intersection of two manifolds embedded in $\mathbb{R}^n$. More specifically, for manifolds E and F, the dimension of $E \cap \sigma(F)$, where σ is a rigid motion in $\mathbb{R}^n$, is 'often' given by

$$\dim(E \cap \sigma(F)) = \max\{\dim E + \dim F - n, 0\} \tag{1}$$

and 'typically' no more than this value. 'Often' and 'typical' can be made precise in terms of a natural measure on the group of rigid motions on $\mathbb{R}^n$. Dimension formulae for the intersection of one set with what may be regarded as a random image of another have been developed for fractal sets, for various definitions of fractional dimension and for other groups of transformations of $\mathbb{R}^n$. In particular, Mattila [4, 6, 5] obtained fractal codimension formulae in the case of similarities and, under certain restrictions, for isometries, and Kahane [3] for a general class of groups which includes similarites. These formulae have the common pattern of (1).

This paper presents formulae of this type for isometries under a suitable metric of the *m-ary Cantor space*, $\mathcal{C}^m$, defined as the set of infinite words or sequences formed from the symbols $\{1, 2, \ldots, m\}$; thus $\mathcal{C}^m = \{1, 2, \ldots, m\}^{\mathbb{N}}$. We write $x = x_1x_2\ldots$ for a typical member of $\mathcal{C}^m$. We fix $r \in (0, 1)$ and define a metric d on $\mathcal{C}^m$ by

$$d(x_1x_2\ldots,\ y_1y_2\ldots) = r^k, \text{ where } k+1 \text{ is the least integer such that } x_k \neq y_k;$$

then d is an ultrametric which induces the usual topology on the Cantor space.

Although our calculations are entirely in Cantor space, there is a visual geometric interpretation if $r \in (0, 1/m)$ when the Cantor space $\mathcal{C}^m$ may be identified with the m-ary Cantor set C^m as a subset of the real numbers. This may be constructed in an analogous way to the usual middle-third Cantor set, starting with the unit interval and repeatedly replacing each interval by m equally spaced closed subintervals of length ratio r to that of the parent interval and with the end two intervals abutting the ends of the parent interval, see Figure 1. The identification map $\phi : \mathcal{C}^m \to C^m$ is given by $\phi(x_1 x_2 \ldots) = (r + g)\sum_{i=1}^{\infty}(x_i - 1)r^{i-1}$ where g is the gap length between two intervals of the first level of the Cantor set construction. With this identification the metric d on $\mathcal{C}^m$ is equivalent to the Euclidean metric restricted to subsets of C^m. In particular, the Hausdorff and box-counting dimensions of any subset of $\mathcal{C}^m$ defined using the metric d equal the corresponding dimensions with respect to the Euclidean metric on $C^m \subset \mathbb{R}$.

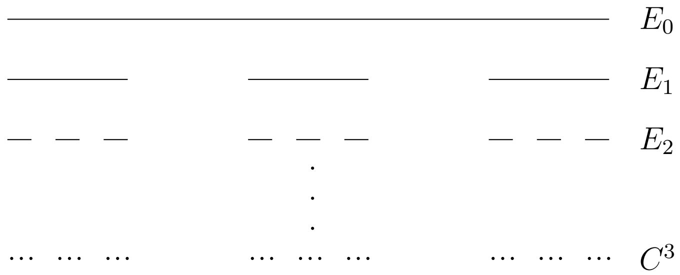

Figure 1: The 3-ary Cantor set with $r = 1/5$

Let $\mathrm{Iso}\,\mathcal{C}^m$ denote the group of isometries of $\mathcal{C}^m$. (With the Cantor set interpretation these isometries may be visualised as combinations of permutations of the construction intervals of the Cantor set at various levels.) The group $\mathrm{Iso}\,\mathcal{C}^m$ and its subgroups continue to be studied intensively, both from group theoretic and dynamical viewpoints, see for example [1]. This paper provides further insight into the geometry of the group. There is a natural invariant probability measure $\mathbf{P}$ on $\mathrm{Iso}\,\mathcal{C}^m$ such that the isometries that induce each admissible permutation of the construction intervals of C^m at a given level have equal probability, see below.

We bring together our main results in the following statement, where $\dim_H$, $\dim_B$ and $\overline{\dim}_B$ denote Hausdorff, box-counting and upper box-counting dimension respectively, see [2] for definitions. Note that $\dim_H \mathcal{C}^m = \dim_B \mathcal{C}^m = -\log m/\log r$.

Theorem 1. *Let $E, F \subset \mathcal{C}^m$ be Borel sets. Then for a random isometry $\sigma \in \mathrm{Iso}\mathcal{C}^m$:*

(i) *almost surely* $\overline{\dim}_B(E \cap \sigma(F)) \le \max\left\{\overline{\dim}_B E + \overline{\dim}_B F + \log m/\log r,\ 0\right\}$,

(ii) *almost surely* $\dim_H(E \cap \sigma(F)) \le \max\left\{\dim_H E + \overline{\dim}_B F + \log m/\log r,\ 0\right\}$,

(iii) $\operatorname{esssup}_{\sigma \in \mathrm{Iso}\mathcal{C}^m}\left\{\dim_H(E \cap \sigma(F))\right\} \ge \max\left\{\dim_H E + \dim_H F + \log m/\log r,\ 0\right\}$.

Parts (i) and (ii) will be obtained using covering arguments. The lower bound (iii) is more complicated, and uses measures defined on E and F to set up a measure martingale that converges almost surely to a measure supported on $E \cap \sigma(F)$. A potential-theoretic argument then gives lower bounds for the dimension.

Some basic notation will be used throughout the paper. For each k and each finite word $x_1 x_2 \ldots x_k \in \{1, 2, \ldots, m\}^k$ we associate the *level-k cylinder* $\{x_1 x_2 \ldots x_k y_{k+1} y_{k+2} \ldots : 1 \leq y_i \leq m\}$ which we will generally refer to as an *interval* I, to correspond to the Cantor set interpretation. We write U_k for the set of all kth level intervals. Also, for $A \subset \mathcal{C}^m$ we use $U_k(A)$ to denote the set of kth level intervals that intersect A non-trivially, specifically $U_k(A) = \{I \in U_k : I \cap A \neq \emptyset\}$, so that the intervals of $U_k(A)$ form a cover of A for each k. We will write $|\cdot|$ to denote cardinality, so in particular $|U_k(A)|$ is the number of level k intervals that intersect A. We write $d(A) = \inf\{d(x, y) : x, y \in A\}$ for the *diameter* of a (non-empty) set $A \subset \mathcal{C}^m$, so that $d(I) = r^{-k}$ if I is a kth-level interval.

A convenient way of characterising the isometries $\operatorname{Iso} \mathcal{C}^m$ is using the natural correspondence of $\mathcal{C}$ with the infinite rooted m-ary tree, $\mathcal{T}^m$. The boundary of $\mathcal{T}^m$ is identified with the Cantor space $\mathcal{C}^m$ and the vertices correspond to the intervals or cylinders. Then the group of graph automorphisms of the rooted tree $\mathcal{T}^m$ correspond to the group of isometries $\operatorname{Iso} \mathcal{C}^m$ of the Cantor space. An automorphism acts by 'twisting' the tree at sets of nodes, perhaps infinitely many, rearranging the children of each node into a new permutation, see Figure 2.

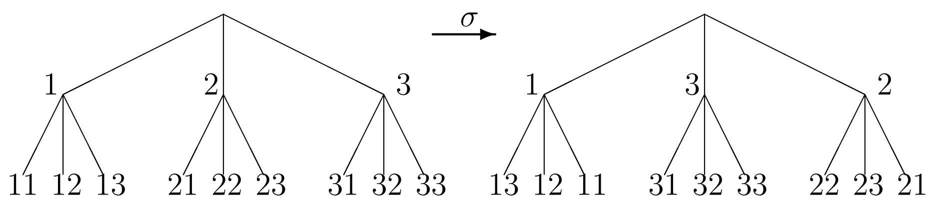

Figure 2: An automorphism σ acting on two levels of the ternary Cantor space

The natural invariant probability space $(\operatorname{Iso} \mathcal{C}^m, \mathcal{F}, \mathbf{P})$ on the isometries of $\mathcal{C}^m$ is defined as follows. For each k let π be an admissible permutation of the intervals of U_k (i.e. one that is achievable by some $\sigma \in \operatorname{Iso} \mathcal{C}^m$) and let $\mathcal{I}_\pi$ be the set of all isometries $\sigma \in \operatorname{Iso} \mathcal{C}^m$ such that $\sigma(I) = \pi(I)$ for all $I \in U_k$. Let $\mathcal{F}_k$ be the finite sigma-field consisting of finite unions of all such $\mathcal{I}_\pi$. We define a probability on $\mathcal{F}_k$ by ascribing equal probability to each $\mathcal{I}_\pi$, so that $\mathbf{P}(\mathcal{I}_\pi) = m^{-k(k+1)/2}$, and extending to $\mathcal{F}_k$. These sigma-fields form an increasing sequence and we define $\mathcal{F} = \mathcal{S}(\bigcup_{k=0}^{\infty} \mathcal{F}_k)$ for the sigma-field generated by their union and extend $\mathbf{P}$ to $\mathcal{F}$ in the usual way. Note that for $I, J \in U_k$ and $\sigma \in \operatorname{Iso} \mathcal{C}^m$, $\mathbf{P}\big(\sigma(I) = J\big) = m^{-k}$.

2 Upper Box Counting Dimension: Upper Bound

In this section, we bound the upper box counting dimension of the intersection of a subset of $\mathcal{C}^m$ with a random image of another subset.

Theorem 2. *Let $E, F \subset \mathcal{C}^m$. Then, almost surely,*

$$\overline{\dim}_B(E \cap \sigma(F)) \leq \max\left\{\overline{\dim}_B E + \overline{\dim}_B F + \frac{\log m}{\log r}, 0\right\}. \tag{2}$$

Proof. First note that

$$U_k\big(E \cap \sigma(F)\big) \subset U_k\big(E\big) \cap U_k\big(\sigma(F)\big) = U_k\big(E\big) \cap \sigma\big(U_k(F)\big)$$

For $k \geq 0$ and $J \in U_k(F)$, consider the indicator function $\chi_J : \mathrm{Iso}\,\mathcal{C}^m \to \{0,1\}$ such that $\chi_J(\sigma) = 1$ when $\sigma(J) \in U_k(E)$. Then

$$\big|U_k\big(E \cap \sigma(F)\big)\big| \leq |U_k\big(E\big) \cap \sigma\big(U_k(F)\big)| = \sum_{J \in U_k(F)} \chi_J(\sigma).$$

A random automorphism σ takes an interval $J \in U_k$ to a particular interval $I \in U_k$ with probability m^{-k}, therefore for all $J \in U_k$

$$\mathbf{E}(\chi_J(\sigma)) = m^{-k}|U_k(E)|.$$

This implies

$$\mathbf{E}\big(\big|U_k\big(E \cap \sigma(F)\big)\big|\big) \leq \sum_{J \in U_k(F)} \mathbf{E}\big(\chi_J(\sigma)\big) = m^{-k}|U_k(E)||U_k(F)|.$$

Assume that $\overline{\dim}_B E + \overline{\dim}_B F + \log m/\log r > 0$, otherwise there is nothing to prove. Take α and β such that $\alpha > \overline{\dim}_B E$ and $\beta > \overline{\dim}_B F$. From the definition of upper box dimension, there exist $c_1, c_2 > 0$ such that, for all $k \geq 0$,

$$|U_k(E)| \leq c_1 r^{-k\alpha} \quad \text{and} \quad |U_k(F)| \leq c_2 r^{-k\beta}.$$

Setting $c = c_1 c_2$, for all $k > 0$,

$$\mathbf{E}\Big(\big|U_k\big(E \cap \sigma(F)\big)\big|\Big) \;\leq\; c r^{-k(\alpha+\beta+\log m/\log r)} = c r^{-kd},$$

where $d = \alpha + \beta + \log m/\log r > 0$. Let $\epsilon > 0$. Then

$$\mathbf{E}\Big(\sum_{k=0}^{\infty} r^{k(d+\epsilon)}\big|U_k\big(E \cap \sigma(F)\big)\big|\Big) \leq \sum_{k=0}^{\infty} c r^{k\epsilon} < \infty.$$

Thus, almost surely, there exists a random $C < \infty$ such that

$$\sum_{k=0}^{\infty} r^{k(d+\epsilon)}\big|U_k\big(E \cap \sigma(F)\big)\big| \;\leq\; C,$$

so

$$\big|U_k\big(E \cap \sigma(F)\big)\big| \;\leq\; C r^{-k(d+\epsilon)}$$

for all $k \geq 0$. When calculating upper box dimension it is enough to consider coverings by intervals of lengths r^{-k} for $0 \leq k < \infty$, so

$$\overline{\dim}_B(E \cap \sigma(F)) \leq d + \epsilon = \alpha + \beta + \log m/\log r + \epsilon.$$

Taking ϵ arbitrarily small and α and β arbitrarily close to $\overline{\dim}_B E$ and $\overline{\dim}_B F$ gives (2). $\square$

Note that a minor variation on this argument shows that $E \cap \sigma(F) = \emptyset$ almost surely if $\overline{\dim}_B E + \overline{\dim}_B F + \log m/\log r < 0$.

3 Hausdorff Dimension: Upper Bound

We will now obtain an upper bound for the Hausdorff dimension of the intersections. We write $\mathcal{H}^s$ for s-dimensional Hausdorff measure, see [6] for its definition and properties. However, rather than work directly with Hausdorff measures, it is convenient to use an equivalent definition based on coverings of subsets of $\mathcal{C}^m$ by intervals or cylinders rather than by arbitrary sets. Let $\mathcal{U} = \bigcup_{k=0}^{\infty} U_k$ denote the collection of intervals and let $d(\cdot)$ denote the diameter of a set with respect to the metric $d(\cdot,\cdot)$. For $s \geq 0$, $\delta > 0$ and $A \subset C^m$, define the δ-premeasures by

$$\mathcal{M}^s_\delta(A) \quad = \quad \inf\Big\{ \sum_{i=1}^{\infty} d(I_i)^s : A \subset \bigcup_{i=1}^{\infty} I_i, d(I_i) \leq \delta \Big\}$$

and let

$$\mathcal{M}^s(A) \quad = \quad \lim_{\delta \to 0} \mathcal{M}^s_\delta(A).$$

Then $\mathcal{M}^s$ is a Borel measure on $\mathcal{C}^m$.

Lemma 3. *For all $A \subset \mathcal{C}^m$, $\mathcal{M}^s(A) = \mathcal{H}^s(A)$. In particular,* $\dim_H(A) = \sup\{s : \mathcal{M}^s(A) > 0\} = \inf\{s : \mathcal{M}^s(A) = 0\}$.

Proof. Cleary $\mathcal{H}^s(A) \leq \mathcal{M}^s(A)$ for all A, since any admissible cover for $\mathcal{M}^s$ is an admissible cover for $\mathcal{H}^s$. For the opposite inequality, note that the diameter of any set $O \subset \mathcal{C}^m$ equals that of the smallest interval I of $\mathcal{U}$ that contains O. Thus replacing any covering set O by the corresponding interval I does not change the diameters involved in the definitions of the measures, so $\mathcal{M}^s(A) \leq \mathcal{H}^s(A)$. $\square$

Theorem 4. *Let $E, F \subset \mathcal{C}^m$. Almost surely*

$$\dim_H(E \cap \sigma(F)) \leq \max\left\{ \dim_H E + \overline{\dim}_B F + \frac{\log m}{\log r}, 0 \right\}. \tag{3}$$

Proof. Take α and β with $\alpha > \dim_H E$ and $\beta > \overline{\dim}_B F$. Then there exists $c > 0$ such that for all $k \geq 0$

$$|U_k(F)| \quad \leq \quad c r^{-k\beta}.$$

By Lemma 3, for all $\delta > 0$ we can find intervals $I_i \in \mathcal{U}$ such that $E \subset \bigcup_i I_i$, $d(I_i) \leq \delta$, and $\sum_i d(I_i)^\alpha \leq 1$. Taking only those intervals I_i that intersect $\sigma(F)$ non-trivially, gives a δ-cover of $E \cap \sigma(F)$ and therefore, for $s > 0$,

$$\mathcal{M}^s_\delta(E \cap \sigma(F)) \quad \leq \quad \sum_i \{d(I_i)^s : \sigma^{-1}(I_i) \cap F \neq \emptyset\}.$$

Taking the expectation,

$$\mathbf{E}\big(\mathcal{M}^s_\delta(E \cap \sigma(F))\big) \quad \leq \quad \sum_i d(I_i)^s \mathbf{P}(\sigma^{-1}(I_i) \cap F \neq \emptyset).$$

If $I_i \in U_k$, then $d(I_i) = r^k$, so

$$\begin{aligned} \mathbf{P}(\sigma^{-1}(I_i) \cap F \neq \emptyset) &= m^{-k}|U_k(F)| \\ &\leq cm^{-k}r^{-k\beta} \\ &= cd(I_i)^{-(\beta+\log m/\log r)}. \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{E}\big(\mathcal{M}^s_\delta(E\cap\sigma(F))\big) &\leq c\sum_i d(I_i)^{s-(\beta+\log m/\log r)} \\ &= c\sum_i d(I_i)^\alpha d(I_i)^{s-(\alpha+\beta+\log m/\log r)} \\ &\leq c\sum_i d(I_i)^\alpha \delta^{s-(\alpha+\beta+\log m/\log r)} \\ &\leq c\delta^{s-(\alpha+\beta+\log m/\log r)} \end{aligned}$$

provided that $s-(\alpha+\beta+\log m/\log r) > 0$. Taking $\delta = 2^{-k}$ and summing,

$$\mathbf{E}\Big(\sum_{k=1}^\infty \mathcal{M}^s_{2^{-k}}(E\cap\sigma(F))\Big) \leq c\sum_{k=1}^\infty 2^{-k(s-(\alpha+\beta+\log m/\log r))} < \infty.$$

This implies that, almost surely,

$$\sum_{k=1}^\infty \mathcal{M}^s_{2^{-k}}(E\cap\sigma(F)) < \infty$$

so

$$\mathcal{M}^s(E\cap\sigma(F)) = \lim_{\delta\to 0}\mathcal{M}^s_\delta(E\cap\sigma(F)) = 0.$$

In particular, by Lemma 3, $\dim_H(E\cap\sigma(F)) \leq s$ almost surely, provided that $s > \alpha+\beta+\log m/\log r$. This holds for α and β arbitrarily close to $\dim_H E$ and $\overline{\dim}_B F$, giving (3). $\square$

Again, minor changes to the argument show that $E\cap\sigma(F) = \emptyset$ almost surely if $\dim_H E + \overline{\dim}_B F + \log m/\log r < 0$.

Note that if, as often happens, either E or F is sufficiently regular to have equal Hausdorff and upper box dimensions, then we get $\dim_H$ throughout inequality (3).

4 Hausdorff Dimension: Lower Bound

In this section we obtain a lower bound for the essential supremum of $\dim_H(E\cap\sigma(F))$ where σ is a random isometry. To achieve this we put Frostman-type measures on E and F and define a measure martingale that converges to a measure on $E\cap\sigma(F)$. By examining the s-energy of this measure we obtain a lower bound for the dimension that occurs with positive probability. The bulk of the calculation is devoted to showing that the martingales are $\mathcal{L}^2$-bounded.

Throughout this section, E, F will be Borel subsets of $\mathcal{C}^m$ and $0 < \alpha < \dim_H E$ and $0 < \beta < \dim_H F$. Eventually we will take α and β arbitrarily close to the respective dimensions.

Lemma 5. *There exist probability measures μ and ν, with compact support contained in E and F respectively, and positive constants c_E and c_F such that for all $k \geq 0$ and $I \in U_k$,*

$$\mu(I) \leq c_E r^{k\alpha} \quad \text{and} \quad \nu(I) \leq c_F r^{k\beta}. \tag{4}$$

Proof. By Frostman's Lemma for metric spaces [6, 7], there are probability measures μ and ν, such that $\mu(A) \leq c_E d(A)^\alpha$ and $\nu(A) \leq c_F d(A)^\beta$ for all $A \subset \mathcal{C}^m$. If $I \in U_k$, then $d(I) = r^k$ so the conclusion follows. □

Let $k \in \mathbb{N}$ and let μ and ν be given by Lemma 5. For all $A \in U_k$ and $l \geq k$ define a random variable

$$\tau_l(A) \;=\; m^l \sum_{I \in U_l(A)} \mu(I)\nu(\sigma^{-1}(I)). \tag{5}$$

Note that $\tau_l(A)$ is $\mathcal{F}_l$ measurable, where $\mathcal{F}_l$ is the sigma-field generated by the isometries defined at the lth level, see Section 1. We will show that $\{\tau_l(A), \mathcal{F}_l\}_{l \geq k}$ is an $\mathcal{L}^2$-bounded martingale and that the limits of these martingales give rise to an additive set function on $\mathcal{U} = \bigcup_{k=0}^\infty U_k$ and thus a measure on $\mathcal{C}^m$.

Lemma 6. *Let $A \in U_k$. Then $\{\tau_l(A), \mathcal{F}_l\}_{l \geq k}$ is a non-negative martingale.*

Proof. Let $l \geq k+1$. For each $I \in U_l$, we write $I' \in U_{l-1}$ for the parent interval of I. Then

$$\mathbf{E}\big(\tau_l(A) | \mathcal{F}_{l-1}\big) \;=\; m^l \sum_{I \in U_l(A)} \mu(I) \mathbf{E}\big(\nu(\sigma^{-1}(I)) | \mathcal{F}_{l-1}\big). \tag{6}$$

Conditional on $\mathcal{F}_{l-1}$, $\sigma^{-1}(I)$ is equally likely to be any of the m children of $\sigma^{-1}(I')$ so

$$\mathbf{E}\big(\nu(\sigma^{-1}(I)) | \mathcal{F}_{l-1}\big) \;=\; m^{-1}\nu(\sigma^{-1}(I')).$$

Partitioning the sum (6) over the intervals I' at the $(l-1)$th level gives

$$\begin{aligned} m^l \sum_{I' \in U_{l-1}(A)} \sum_{\substack{I \subset I' \\ I \in U_l}} \mu(I)\mathbf{E}\big(\nu(\sigma^{-1}(I)) | \mathcal{F}_{l-1}\big) &= m^l \sum_{I' \in U_{l-1}(A)} \sum_{\substack{I \subset I' \\ I \in U_l}} m^{-1}\mu(I)\nu(\sigma^{-1}(I')) \\ &= m^{l-1} \sum_{I' \in U_{l-1}(A)} \mu(I')\nu(\sigma^{-1}(I')) \\ &= \tau_{l-1}(A). \end{aligned}$$

Clearly $\tau_l(A) \geq 0$ for all l, so $\{\tau_l(A), \mathcal{F}_l\}_{l \geq k}$ is a non-negative martingale. □

In proving $\mathcal{L}^2$-boundedness, we will need the following inequality.

Lemma 7. *Let $x_1, x_2, \ldots, x_m \geq 0$ be real numbers. Then*

$$m \sum_{i \neq j} x_i x_j \;\leq\; (m-1) \sum_{i,j} x_i x_j \tag{7}$$

Proof. Young's Inequality implies that $x_i x_j \le \frac{1}{2}x_i^2 + \frac{1}{2}x_j^2$ for each pair i and j. By summing over all pairs such that $i \neq j$, we see that

$$\sum_{i\neq j} x_i x_j \;\le\; (m-1)\sum_i x_i^2$$

and therefore

$$\begin{aligned} m\sum_{i\neq j} x_i x_j &= \sum_{i\neq j} x_i x_j + (m-1)\sum_{i\neq j} x_i x_j \\ &\le (m-1)\sum_{i=1}^{m} x_i^2 + (m-1)\sum_{i\neq j} x_i x_j \\ &= (m-1)\sum_{i,j} x_i x_j. \end{aligned}$$

□

Lemma 8. *Assume that $\alpha+\beta > -\log m/\log r$. There is a constant c_0 such that for all $A \in U_k$ and $l \ge k$,*

$$\mathbf{E}\big(\tau_l(A)^2\big) \le c_0 \mu(A) r^{k(\alpha+\beta+\log m/\log r)}. \tag{8}$$

In particular, the martingale $\{\tau_l(A), \mathcal{F}_l\}_{l\ge k}$ is $\mathcal{L}^2$-bounded.

Proof. Let $A \in U_k$. We will first bound $\mathbf{E}\big(\tau_l(A)^2|\mathcal{F}_{l-1}\big)$ in terms of $\tau_{l-1}(A)$ where $l \ge k+1$, to obtain (12) below. As before, we make the convention that $I' \in U_{l-1}$ is the parent interval of $I \in U_l$.

The expectation of $\tau_l(A)^2$ conditional on $\mathcal{F}_{l-1}$ breaks down into three sums:

$$\begin{aligned} \mathbf{E}\big(\tau_l(A)^2|\mathcal{F}_{l-1}\big) &= m^{2l} \sum_{I,J\in U_l(A)} \mu(I)\mu(J)\mathbf{E}\Big(\nu(\sigma^{-1}(I))\nu(\sigma^{-1}(J))\Big|\mathcal{F}_{l-1}\Big) \\ &= m^{2l} \sum_{\substack{I',J'\in U_{l-1}(A)\\ I'\neq J'}} \sum_{\substack{I\subset I'\\ J\subset J'}} \mu(I)\mu(J)\mathbf{E}\Big(\nu(\sigma^{-1}(I))\nu(\sigma^{-1}(J))\Big|\mathcal{F}_{l-1}\Big) \quad (9) \\ &\quad + m^{2l} \sum_{I'\in U_{l-1}(A)} \sum_{\substack{I,J\subset I'\\ I\neq J}} \mu(I)\mu(J)\mathbf{E}\Big(\nu(\sigma^{-1}(I))\nu(\sigma^{-1}(J))\Big|\mathcal{F}_{l-1}\Big) \quad (10) \\ &\quad + m^{2l} \sum_{I'\in U_{l-1}(A)} \sum_{I\subset I'} \mu(I)^2\mathbf{E}\Big(\nu(\sigma^{-1}(I))^2\Big|\mathcal{F}_{l-1}\Big). \quad (11) \end{aligned}$$

We estimate the expectation term in (9), (10), and (11) separately.

Case 1: The sum in (9) is over intervals $I, J \in U_l$ with different parent intervals, $I', J' \in U_{l-1}$ respectively. This affords independence in the calculation of conditional expectation, so

$$\mathbf{E}\Big(\nu(\sigma^{-1}(I))\nu(\sigma^{-1}(J))\Big|\mathcal{F}_{l-1}\Big) \;=\; \mathbf{E}\Big(\nu(\sigma^{-1}(I))\Big|\mathcal{F}_{l-1}\Big)\mathbf{E}\Big(\nu(\sigma^{-1}(J))\Big|\mathcal{F}_{l-1}\Big).$$

Given $\mathcal{F}_{l-1}$, $\sigma^{-1}(I)$ is equally likely to be any one of the m intervals $I_0 \in U_l$ that are children of $\sigma^{-1}(I')$, so

$$\mathbf{E}\Big(\nu(\sigma^{-1}(I))\big|\mathcal{F}_{l-1}\Big) = \sum_{\substack{I_0 \subset \sigma^{-1}(I') \\ I_0 \in U_l}} \frac{\nu(I_0)}{m} = \frac{\nu\big(\sigma^{-1}(I')\big)}{m},$$

with a similar expression for the term involving $\sigma^{-1}(J)$. The expected value in (9) then becomes

$$\mathbf{E}\Big(\nu(\sigma^{-1}(I))\nu(\sigma^{-1}(J))\big|\mathcal{F}_{l-1}\Big) = \frac{\nu(\sigma^{-1}(I'))\nu(\sigma^{-1}(J'))}{m^2}.$$

Case 2: The sum in (10) is over two disjoint intervals with the same parent interval, $I' \in U_{l-1}$. The pair of intervals, $\sigma^{-1}(I)$ and $\sigma^{-1}(J)$, is equally likely to be any of the $m(m-1)$ pairs of distinct children I_0 and J_0 of $\sigma^{-1}(I') \in U_{l-1}$, and using (7),

$$\begin{aligned}
\mathbf{E}\Big(\nu(\sigma^{-1}(I))\nu(\sigma^{-1}(J))\big|\mathcal{F}_{l-1}\Big) &= \sum_{\substack{I_0, J_0 \subset \sigma^{-1}(I') \\ I_0 \neq J_0}} \nu(I_0)\nu(J_0)\frac{1}{m(m-1)} \\
&\leq \sum_{I_0, J_0 \subset \sigma^{-1}(I')} \nu(I_0)\nu(J_0)\frac{1}{m^2} \\
&= \frac{\nu(\sigma^{-1}(I'))^2}{m^2}.
\end{aligned}$$

Case 3: The sum in (11) is over intervals I with parent interval I', and $\sigma^{-1}(I)$ is equally likely to be any of the m children of $\sigma^{-1}(I')$, say I_0. Combining this with the inequality $\nu(I_0) \leq c_F r^{l\beta}$ from (4),

$$\begin{aligned}
\mathbf{E}\Big(\nu(\sigma^{-1}(I))^2\big|\mathcal{F}_{l-1}\Big) &= \sum_{I_0 \subset \sigma^{-1}(I')} \nu(I_0)^2 m^{-1} \\
&\leq \sum_{I_0 \subset \sigma^{-1}(I')} c_F r^{l\beta}\nu(I_0) m^{-1} \\
&= \frac{c_F r^{l\beta}\nu(\sigma^{-1}(I'))}{m}.
\end{aligned}$$

Incorporating these three cases in (9)–(11) and using that $\mu(I) \leq c_E r^{l\alpha}$ for every $I \in U_l$,

$$\begin{aligned}
\mathbf{E}\big(\tau_l(A)^2|\mathcal{F}_{l-1}\big) &\leq m^{2l} \sum_{I',J' \in U_{l-1}(A)} \sum_{\substack{I \subset I' \\ J \subset J'}} \mu(I)\mu(J)\frac{\nu(\sigma^{-1}(I'))\nu(\sigma^{-1}(J'))}{m^2} \\
&\quad + m^{2l} \sum_{I' \in U_{l-1}(A)} \sum_{I \subset I'} \mu(I) c_E r^{l\alpha} \frac{c_F r^{l\beta}\nu(\sigma^{-1}(I'))}{m} \\
&= m^{2(l-1)} \sum_{I',J' \in U_{l-1}(A)} \mu(I')\mu(J')\nu(\sigma^{-1}(I'))\nu(\sigma^{-1}(J')) \\
&\quad + c_E c_F r^{l\alpha} r^{l\beta} m^l m^{l-1} \sum_{I' \in U_{l-1}(A)} \mu(I')\nu(\sigma^{-1}(I'))
\end{aligned}$$

$$= \quad \tau_{l-1}(A)^2 + c\tau_{l-1}(A) r^{l(\alpha+\beta+\log m/\log r)}, \tag{12}$$

where $c = c_E c_F$.

We apply this inequality inductively (working backwards) to bound $\mathbf{E}\big(\tau_l(A)^2|\mathcal{F}_k\big)$ where $A \in U_k$. Assume that for some j with $k+1 \leq j \leq l-1$,

$$\mathbf{E}\big(\tau_l(A)^2|\mathcal{F}_j\big) \leq \tau_j(A)^2 + c\tau_j(A) \sum_{i=j+1}^{l} r^{i(\alpha+\beta+\log m/\log r)}; \tag{13}$$

when $j = l-1$ this is just (12). Using the tower property for conditional expectation, inequalities (13), (12) (with j playing the role of l), and that τ_j is a martingale,

$$\begin{aligned}
\mathbf{E}\big(\tau_l(A)^2|\mathcal{F}_{j-1}\big) &= \mathbf{E}\big(\mathbf{E}(\tau_l(A)^2|\mathcal{F}_j)|\mathcal{F}_{j-1}\big) \\
&\leq \mathbf{E}\big(\tau_j(A)^2|\mathcal{F}_{j-1}\big) + c\mathbf{E}\big(\tau_j(A)|\mathcal{F}_{j-1}\big) \sum_{i=j+1}^{l} r^{i(\alpha+\beta+\log m/\log r)} \\
&\leq \tau_{j-1}(A)^2 + c\tau_{j-1}(A) \sum_{i=j}^{l} r^{i(\alpha+\beta+\log m/\log r)},
\end{aligned}$$

for the inductive step. Taking $j = k$ in (13) and recalling that $\alpha+\beta+\log m/\log r > 0$, we conclude that

$$\mathbf{E}\big(\tau_l(A)^2|\mathcal{F}_k\big) \leq \tau_k(A)^2 + c_1\tau_k(A) r^{k(\alpha+\beta+\log m/\log r)}, \tag{14}$$

where c_1 does not depend on l, k or A.

With $A \in U_k$ as before, we take unconditional expectations of this inequality, and use (5) and (4):

$$\begin{aligned}
\mathbf{E}\big(\tau_l(A)^2\big) &\leq \mathbf{E}\big(\tau_k(A)^2\big) + c_1\mathbf{E}\big(\tau_k(A)\big) r^{k(\alpha+\beta+\log m/\log r)} \\
&= m^{2k}\mu(A)^2 \mathbf{E}\big(\nu(\sigma^{-1}(A))^2\big) + c_1 m^k \mu(A) \mathbf{E}\big(\nu(\sigma^{-1}(A))\big) r^{k(\alpha+\beta+\log m/\log r)} \\
&= m^{2k}\mu(A)^2 \sum_{I\in U_k} \nu(I)^2 m^{-k} + c_1 m^k \mu(A) \sum_{I\in U_k} \nu(I) m^{-k} r^{k(\alpha+\beta+\log m/\log r)} \\
&\leq c_E c_F m^k \mu(A) \sum_{I\in U_k} \nu(I) r^{k(\alpha+\beta)} + c_1 \mu(A) r^{k(\alpha+\beta+\log m/\log r)} \\
&\leq c_0 \mu(A) r^{k(\alpha+\beta+\log m/\log r)},
\end{aligned} \tag{15}$$

where $c_0 = c_E c_F + c_1$. □

We now use the τ_l to obtain a limiting measure. First let $A \in \mathcal{S}(U_k)$, the sigma-algebra of subsets of $\mathcal{C}^m$ generated by the kth level intervals, so A is a (finite) union of intervals in U_k. For $l \geq k$ define

$$\tau_l(A) \quad = \quad m^l \sum_{I\in U_l(A)} \mu(I)\nu(\sigma^{-1}(I)).$$

Note that when $A \in U_k$ this coincides with the definition of $\tau_l(A)$ given by (5). For all k and all $A \in \mathcal{S}(U_k)$, $\{\tau_l(A), \mathcal{F}_l\}_{l\geq k}$ is a martingale as a finite sum of martingales. Thus

$\tau_l(A)$ converges almost surely to a random variable on the sigma-field $\mathcal{F} = \mathcal{S}(\bigcup_{k=0}^{\infty} \mathcal{F}_k)$, so we may define, for all $A \in \bigcup_{k=0}^{\infty} \mathcal{S}(U_k)$,

$$\tau(A) = \lim_{l\to\infty} \tau_l(A), \tag{16}$$

the limit existing almost surely for all $A \in \bigcup_{k=0}^{\infty} \mathcal{S}(U_k)$ simultaneously.

Let $A, B \in \bigcup_{k=0}^{\infty} \mathcal{S}(U_k)$ be disjoint, so that $A, B \in \mathcal{S}(U_k)$ for some k. Then, for $l \geq k$, $\tau_l(A \cup B) = \tau_l(A) + \tau_l(B)$. Taking limits gives $\tau(A\cup B) = \tau(A) + \tau(B)$, so almost surely, τ is a finitely additive set function on $\bigcup_{k=0}^{\infty} \mathcal{S}(U_k)$. Since $\{\tau_l(\mathcal{C}^m), \mathcal{F}_l\}_{l\geq 0}$ is a non-negative martingale, $\tau_l(\mathcal{C}^m) < \infty$ almost surely. By the extension theorems, see [8], almost surely τ has a unique extension to $\mathcal{S}\big(\bigcup_{k=0}^{\infty} \mathcal{S}(U_k)\big)$, i.e. τ is a random Borel measure on $\mathcal{C}^m$.

Proposition 9. *The support of τ is contained in $E \cap \sigma(F)$, with $\tau(\mathcal{C}^m) < \infty$ almost surely and $\tau(\mathcal{C}^m) > 0$ with positive probability. Moreover, for all $k \geq 0$ and $A \in U_k$,*

$$\mathbf{E}\big(\tau(A)^2\big) \leq c_0 \mu(A) r^{k(\alpha+\beta+\log m/\log r)}. \tag{17}$$

Proof. Let $x \notin E \cap \sigma(F)$ but $x \in \mathcal{C}^m$. Since μ and ν have support on compact subsets of E and F respectively, either $x \notin \operatorname{supp}(\mu)$ or $\sigma^{-1}(x) \notin \operatorname{supp}(\nu)$. Without loss of generality, assume $x \notin \operatorname{supp}(\mu)$. Then there exists an open neighborhood of x that does not intersect $\operatorname{supp}(\mu)$, which we may take to be an interval $A \in U_k$ for some k. Then by (5), for all $l \geq k$, $\tau_l(A) = 0$, so $\tau(A) = 0$ and x is not in the support of τ.

Since $\{\tau_l(\mathcal{C}^m), \mathcal{F}_l\}_{l\geq 0}$ is a non-negative martingale $0 \leq \tau(\mathcal{C}^m) < \infty$ almost surely, and, since it is $\mathcal{L}^2$-bounded, $\tau(\mathcal{C}^m) > 0$ with positive probability. Since $\mathcal{L}^2$-bounded martingales converge in $\mathcal{L}^2$, (17) follows from (8).

$\square$

The s-energy of a measure υ is defined as $I_s(\upsilon) = \displaystyle\int\int \frac{\mathrm{d}\upsilon(x)\mathrm{d}\upsilon(y)}{d(x,y)^s}$. We use the following variation of the potential theoretic method to bound the Hausdorff dimension of $E \cap \sigma(F)$, see [2, Section 4.3] and [6, Chapter 8].

Theorem 10. *Let F be a Borel subset of $\mathcal{C}^m$ and υ a measure with support in F and $0 < \upsilon(F) < \infty$. If $I_s(\upsilon) < \infty$, then $\dim_H(F) \geq s$.*

To use this theorem, we find the expected value of $I_s(\tau)$, where τ is the random measure on $E \cap \sigma(F)$ constructed above.

Lemma 11. *Let $0 < s < \alpha + \beta + \log m/\log r$. Then*

$$\mathbf{E}\Big(\int\int \frac{\mathrm{d}\tau(x)\mathrm{d}\tau(y)}{d(x,y)^s}\Big) < \infty.$$

Proof. For $x, y \in \mathcal{C}^m$, we write $x \wedge y$ for the smallest interval I such that $x, y \in I$. We split the integral up into domains $\{x, y : x \wedge y \in I\}$ for each $I \in \mathcal{U}$ and then use (17).

$$\mathbf{E}\Big(\int\int \frac{\mathrm{d}\tau(x)\mathrm{d}\tau(y)}{d(x,y)^s}\Big) \;\leq\; \mathbf{E}\Big(\sum_{k=0}^{\infty}\sum_{I\in U_k}\int\int_{x\wedge y = I} \frac{\mathrm{d}\tau(x)\mathrm{d}\tau(y)}{d(x,y)^s}\Big)$$

$$
\begin{aligned}
&\leq \sum_{k=0}^{\infty} \sum_{I \in U_k} \mathbf{E}\Big(r^{-ks} \int\int_{x \wedge y = I} \mathrm{d}\tau(x) \mathrm{d}\tau(y) \Big) \\
&\leq \sum_{k=0}^{\infty} r^{-sk} \sum_{I \in U_k} \mathbf{E}\big(\tau(I)^2\big) \\
&\leq c_0 \sum_{k=0}^{\infty} r^{-sk} \sum_{I \in U_k} \mu(I) r^{k(\alpha+\beta+\log m/\log r)} \\
&\leq c_0 \sum_{k=0}^{\infty} r^{k(\alpha+\beta+\log m/\log r - s)} \\
&< \infty,
\end{aligned}
$$

since $\alpha + \beta + \log m/\log r - s > 0$. $\square$

Our final theorem now follows from the potential theoretic characterization of Hausdorff dimension.

Theorem 12. *Let E and F be Borel subsets of $\mathcal{C}^m$. For all $\epsilon > 0$,*

$$
\dim_H(E \cap \sigma(F)) > \dim_H E + \dim_H F + \frac{\log m}{\log r} - \epsilon \tag{18}
$$

with positive probability.

Proof. Let $0 < \alpha < \dim_H E$, $0 < \beta < \dim_H F$ and $0 < s < \alpha + \beta + \log m/\log r$. From Lemma 11, the s-energy of τ, $I_s(\tau)$, is finite almost surely. Provided that $\tau(\mathcal{C}^m) > 0$, which happens with positive probability by Proposition 9, then by Theorem 10

$$
\dim_H(E \cap \sigma(F)) \geq s.
$$

By choosing α and β sufficiently close to $\dim_H E$ and $\dim_H F$ and s close to $\alpha + \beta + \log m/\log r$, we obtain (18) for any given $\epsilon > 0$. $\square$

We may rephrase Theorem 12 as follows, with the case of equality coming from Theorem 4.

Corollary 13. *Let E and F be Borel subsets of $\mathcal{C}^m$. Then*

$$
\operatorname{esssup}_{\sigma \in \mathrm{Iso}\mathcal{C}^m} \{\dim_H(E \cap \sigma(F))\} \geq \dim_H E + \dim_H F + \frac{\log m}{\log r}.
$$

Equality holds if either $\dim_H E = \overline{\dim}_B E$ or $\dim_H F = \overline{\dim}_B F$.

It is natural to ask whether the lower bound in Corollary 13 occurs with positive probability rather than just as an essential supremum. The following example shows that this is not true in general.

Example 14. *For all $0 < \alpha, \beta < -\log m/\log r$ with $\alpha + \beta + \log m/\log r > 0$ there exist Borel sets E and F in $\mathcal{C}^m$ such that $\dim_H E = \dim_B E = \alpha$ and $\dim_H F = \dim_B F = \beta$ and*

$$
\mathbf{P}\Big\{ \dim_H(E \cap \sigma(F)) \geq \dim_H E + \dim_H F + \frac{\log m}{\log r} \Big\} = 0.
$$

Proof. For each integer $i > 1/\alpha$, choose some interval $I_i \in U_i$ and construct a Borel set $E_i \subset I_i$ such that $\dim_H E_i = \dim_B E_i = \alpha - 1/i$. We may do this using a Cantor-type construction starting with I_i but varying slightly the number of children intervals at each stage to get the required dimension. In doing so we may further ensure that $|U_k(E_i)| \leq r^{-k\alpha} = m^{-k\alpha \log r/\log m}$ for all $k \geq i$. Let $E = \bigcup_{i>1/\alpha} E_i$, so $\dim_H E = \alpha$.

In the same way, for $j > 1/\beta$, let $F = \bigcup_{j>1/\beta} F_j$, where $F_j \subset I_j$ for some $I_j \in U_j$ and $\dim_H F_j = \dim_B F_j = \beta - 1/j$, with $|U_k(F_j)| \leq m^{-k\beta \log r/\log m}$ for all $k \geq j$. Thus $\dim_H F = \beta$.

By Theorem 2 or Theorem 4, for each $i > 1/\alpha, j > 1/\beta$,

$$\dim_H(E_i \cap \sigma(F_j)) \leq \max\left\{\alpha + \beta + \log m/\log r - 1/i - 1/j, 0\right\}$$

with probability 1. Let $\epsilon > 0$. Since $E \cap \sigma(F) = \bigcup_{i>1/\alpha} \bigcup_{j>1/\beta} E_i \cap \sigma(F_j)$,

$$\begin{aligned}
&\mathbf{P}\big(\dim_H(E \cap \sigma(F)) > \alpha + \beta + \log m/\log r - \epsilon\big)\\
&\quad\leq \sum_{1/i+1/j<\epsilon} \mathbf{P}\big(\dim_H(E_i \cap \sigma(F_j)) > \alpha + \beta + \log m/\log r - \epsilon\big)\\
&\quad\leq \sum_{1/i+1/j<\epsilon} \mathbf{P}\big(E_i \cap \sigma(F_j) \neq \emptyset\big)\\
&\quad\leq \sum_{j\geq i>1/\epsilon} \mathbf{P}\big(E_i \cap \sigma(F_j) \neq \emptyset\big) + \sum_{i\geq j>1/\epsilon} \mathbf{P}\big(E_i \cap \sigma(F_j) \neq \emptyset\big) \qquad (19)
\end{aligned}$$

For $j \geq i$, by construction E_i is contained in at most $m^{-j\alpha \log r/\log m}$ intervals of U_j, so

$$\mathbf{P}\big(E_i \cap \sigma(F_j) \neq \emptyset\big) \leq \mathbf{P}\big(E_i \cap \sigma(I_j) \neq \emptyset\big) \leq m^{-j\alpha \log r/\log m}\big/m^j = m^{-j(1+\alpha \log r/\log m)}.$$

Since $1 + \alpha \log r/\log m > 0$, the left hand sum of (19) is at most

$$\sum_{i>1/\epsilon} \sum_{j\geq i} m^{-j(1+\alpha \log r/\log m)} \leq c_1 \sum_{i>1/\epsilon} m^{-i(1+\alpha \log r/\log m)} \leq c_2 m^{-(1+\alpha \log r/\log m)/\epsilon},$$

where, provided that ϵ is sufficiently small, c_1 does not depend on i and ϵ and c_2 does not depend on ϵ. With a similar estimate of $c_3 m^{-(1+\beta \log r/\log m)/\epsilon}$ for the right hand sum of (19) we conclude that

$$\lim_{\epsilon\to 0} \mathbf{P}\big(\dim_H(E \cap \sigma(F)) > \alpha + \beta + \log m/\log r - \epsilon\big) = 0.$$

□

Nevertheless, if E and F are of positive Hausdorff measure in their dimensions the lower bound is attained with positive probability.

Proposition 15. *Let E and F be Borel subsets of $\mathcal{C}^m$ and suppose that $\mathcal{H}^\alpha(E) > 0$ and $\mathcal{H}^\beta(F) > 0$ where $\alpha = \dim_H E$ and $\beta = \dim_H F$. Then*

$$\mathbf{P}\left\{\dim_H(E \cap \sigma(F)) \geq \dim_H E + \dim_H F + \frac{\log m}{\log r}\right\} > 0. \qquad (20)$$

Proof. In this case, the inequalities (4) of Lemma 5 hold for suitable constants c_E and c_F with α and β are actually equal to the dimensions of E and F. The argument of Section 4 then goes through without the need to approximate these dimensions. The probability for which (18) holds is just the probability that $\tau(\mathcal{C}^m) > 0$ which does not depend on $\epsilon > 0$, so taking ϵ arbitraily small gives (20). □

References

[1] L. Bartholdi, R. Grigorchuk and V. Nekrashevych. From fractal groups to fractal sets. In *Fractals in Graz 2001*, pp 25–118, Trends Math., Birkhäuser, Basel, 2003.

[2] K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications.* John Wiley & Sons, Hoboken, NJ, 3rd ed. 2014.

[3] J.-P. Kahane. Sur la dimension des intersections. In *Aspects of mathematics and its applications*, volume 34 of *North-Holland Math. Library*, pp 419–430. North-Holland, Amsterdam, 1986.

[4] P. Mattila. Hausdorff dimension and capacities of intersections of sets in n-space. *Acta Math.*, 152: 77–105, 1984.

[5] P. Mattila. On the Hausdorff dimension and capacities of intersections. *Mathematika*, 32: 213–217, 1986.

[6] P. Mattila. *Geometry of sets and measures in Euclidean spaces.* Cambridge University Press, Cambridge, 1995.

[7] C. A. Rogers. *Hausdorff Measures.* Cambridge University Press, Cambridge, Reprint 1970.

[8] S. J. Taylor. *Introduction to Measure and Integration.* Cambridge University Press, Cambridge, 1998.